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A combining method for the approximate solution of spatial segregation limit of reaction-diffusion systems

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Abstract In this paper, we concern ourselves with the study of a class of stationary states for reaction-diffusion systems with densities having disjoint supports. Major contribution of this work is computing the numerical solution of problem as the rate of interaction between two different species tend to infinity. The main difficulty is the nonlinearity nature of problem. To do so, an efficient iterative method is proposed by hybrid of the radial basis function (RBF) collocation and finite difference (FD) methods to approximate the solution. Numerical results with good accuracies are achieved where the shape parameter is carefully selected. Finally, some numerical examples are given to illustrate the good performance of the method.

Keywords. Free boundary problems, Two-phase membrane, One phase obstacle problem, Segregation, Finite difference method, Multiquadric radial basis functions.

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1. Introduction

In recent years the spatial segregation for reaction-diffusion systems has been widely studied in the literature, see [4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 19]. An interesting problem is the existence of spatially inhomogeneous solutions for competition models of Lotka-Volterra type, see [17, 23, 24]. Understanding of the interaction between different species has developed as a central problem in the study of population ecology. The presence of strong interactions of competitive type produces, to the limit, the spatial segregation of the densities. In other words, in the limiting configurations all the populations survive, but have disjoint support, we refer the reader to [21, 22, 25]. In the limiting configuration as the competition rate tends to infinity, led to a free boundary problem (FBP). An FBP is a partial differential equation that in which some part of the boundary is not known, but is to be determined so that some extra boundary condition is satisfied. The segment of the boundary of domain which

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is not known at the outset of the problem is called the free boundary. Then, both the free boundary and the solution of the differential equation should be determined.
The aim of this work is to present an iterative method for approximating the numerical solution of following two problems. But here the proposed method is lack of convergence results.

Let Ω be a connected and bounded-open subset of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega$ and let r be a fixed integer. Consider the steady states of r competing species coexisting in the same area Ω. Here u_i represents the population density of the ith species and f_i prescribe the internal dynamic of u_i. Assume further that the boundary data $\phi_i \in W^{1,\infty}(\partial\Omega)$ are positive functions having disjoint supports, namely $\phi_i.\phi_j = 0$, for $i \neq j$.

Problem I: Consider the minimization problem

$$Minimize \;\; E(u_1, u_2, \ldots, u_r) = \int_\Omega \sum_{i=1}^{r} (\frac{1}{2}|\nabla u_i|^2 + f_i u_i) dx,$$

over the set

$$S = \{(u_1, u_2, \ldots, u_r) \in (W^{1,2}(\Omega))^r : \; u_i \geq 0, \; u_i.u_j = 0, \; u_i = \phi_i \; on \; \partial\Omega\}.$$

We assume that f_i is uniformly continuous and $f_i \geq 0$.
The special cases of **Problem I** are one phase obstacle problem and two phase membrane problem with $r = 1$ and $r = 2$, respectively. In the following is briefly given an introduction of these two methods.

- Two phase membrane problem ($r = 2$): Assume that $g \in W^{1,2}(\Omega)$ and takes both positive and negative values over $\partial\Omega$, and $\lambda^{\pm} : \Omega \to \mathbb{R}$ are positive Lipschitz-continuous functions. Consider the following functional

$$Minimize \;\; E(v) = \int_\Omega [\frac{1}{2}|\nabla v|^2 + \lambda^+ \max(v, 0) - \lambda^- \min(v, 0)] dx.$$

Set $u_1 = \max(v, 0)$, $u_2 = \min(v, 0)$, $g_1 = \max(g, 0)$, $g_2 = \min(g, 0)$ and substitute them in the above functional. So we have

$$Minimize \;\; E(u_1, u_2) = \int_\Omega (\frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2 + \lambda^+ u_1 + \lambda^- u_2) dx, \tag{1.1}$$

over the set

$$S = \{(u_1, u_2) \in (W^{1,2}(\Omega))^2 : \; u_i \geq 0, \; u_1.u_2 = 0, \; u_i = g_i \; on \; \partial\Omega \; i = 1, 2\}.$$

Applying the Euler-Lagrange equation on the functional (1.1), one can obtain the following equivalent FBP (see [36])

$$\begin{cases} \Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}}, & in \; \Omega; \\ u = g, & on \; \partial\Omega, \end{cases} \tag{1.2}$$

where χ_A denotes the characteristic function of the set A. The boundary

$$\Gamma(u) = \partial\{x \in \Omega : u(x) > 0\} \cup \partial\{x \in \Omega : u(x) < 0\} \cap \Omega,$$

is called the free boundary.

- One phase obstacle problem ($r = 1$): Consider the following functional

$$Minimize \;\; E(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + fu)dx, \tag{1.3}$$

over the set

$$S = \{u \in W^{1,2}(\Omega) : \; u \geq 0, \; u = \phi \; on \; \partial\Omega\}.$$

Similar to the two phase membrane problem the Euler-Lagrange equation corresponding to the functional (1.3) is

$$\begin{cases} \Delta u = f\chi_{\{u>0\}}, & in \; \Omega; \\ u(x) = \phi(x), & on \; \partial\Omega; \\ u(x) = |\nabla u(x)| = 0, & in \; \Omega \backslash \{u > 0\}. \end{cases} \tag{1.4}$$

Problem II: Consider the following system of r differential equations

$$\begin{cases} -d_i\Delta u_i(x) = -\kappa u_i(x)\sum_{j\neq i}^r a_{i,j}u_j(x) + \lambda f_i(x, u_i(x)), & in \; \Omega; \\ u_i(x) \geq 0, & in \; \Omega; \; i = 1, 2, \dots, r \\ u_i(x) = \phi_i(x), & on \; \partial\Omega, \end{cases} \tag{1.5}$$

where d_i and λ are positive numbers. The positive constants a_{ij} and κ represent the interaction between the population u_i and u_j and the rate of interaction between two species, respectively. As κ tends to the infinity, then competition-diffusion systems shows a limiting configuration with segregated state. The limit problem turns out to be an FBP. Without loss of generality, we condider $a_{i,j} = 1$. The objective of this work is to obtain the numerical solution of system (1.5) as κ goes to infinity. Here we investigate the **Problem II** for the below cases

- $f_i(x, u_i(x)) = 0$. In this case the uniqueness of the limiting configuration is given in [18] with $r = 3$ and the numerical approximation of this system is given in [17].
- $f_i(x, u_i(x)) = u_i(1 - u_i)$. This case is a steady state of the following auxiliary system when the boundary values are time independent

$$\begin{cases} \frac{d}{dt}u_i - d_i\Delta u_i = -\kappa u_i\sum_{j\neq i}^m a_{i,j}u_j + \lambda u_i(1 - u_i), & in \; \Omega \times (0, \infty); \\ u_i(x, 0) = u_{i,0}(x), & in \; \Omega; \quad i = 1, 2, \dots, r \\ u_i(x, t) = \phi_i(x), & on \; \partial\Omega \times (0, \infty). \end{cases} \tag{1.6}$$

Therefore, FBPs have an important roles in the variety of applications. The FBPs have been studied from different viewpoints and there are numerous papers about the theoretical behavior and numerical solutions of the elliptic FBPs, two-phase membrane problem, one-phase obstacle problem and variational inequalities, see [1, 2, 3, 9, 11, 13, 15, 20, 26, 28, 27, 29, 32, 33, 35, 38].

The major difficulty of obtaining the numerical solution of the FBPs is the nonlinearity nature of problem due to the unknown free surface. In this study, we propose an efficient iterative method to compute the numerical solution of **Problem I** and

Problem II. Also, this method can be applied for the FBP of the form

$$\begin{cases} \Delta u + f^{+}(u) = 0, & in\ \Omega; \\ u = g, & on\ \partial\Omega; \end{cases} \tag{1.7}$$

where,

$$f^{+}(u) = \begin{cases} \lambda^{+}u, & if\ u > 0; \\ 0 & if\ u \leq 0. \end{cases}$$

This method is based on combination of the finite difference and the radial basis function collocation methods.

In the next section, we give a brief introduction on the RBF collocation method. In the Sections 3 and 4, a new method with hybrid of the RBF collocation and the FD methods is presented to approximate the solutions of the problems I and II, respectively. Numerical examples are given in the Section 5 and indicated the high accuracy of the new method. Finally some concluding remarks are presented in the Section 6 .

2. RBF collocation method

In recent years, there has been a growing interest in research of different variant of meshless methods. In a meshless method a set of scattered nodes are used instead of meshing the domain of the problem. Recently, the RBFs procedure [30], is known as a powerful tool for the scattered data interpolation problem. The use of RBFs as a meshless method for numerical solution of partial differential equations is based on the collocation scheme. Due to the collocation technique, this method does not need to evaluate any integral. RBF collocation methods are interesting due to relative ease of implementation, high convergence rate and flexibility with regards to the enforcement of arbitrary boundary conditions. The main advantage of numerical procedures which use RBFs over traditional techniques is the meshless property of these methods. In what follows, we introduce the RBF collocation method.

Let R be the Euclidean distance between a fixed point $\bar{x} \in \mathbb{R}^n$ and any $x \in \mathbb{R}^n$, i.e., $R = ||x - \bar{x}||_2$. A radial function $\varphi(x) = \varphi(||x - \bar{x}||_2)$ depends only on the distance between x and fixed point $\bar{x}$. Some well-known radial basis functions are Gausian, Hardy Multiquadric, Inverse Multiquadric and Inverse Quadric. One of the RBFs that is of our interest, is Hardy Multiquadric radial function $\varphi(R) = \sqrt{R^2 + c^2}$, where c is known as the "shape parameter", and describes the relative width of the RBF functions about their centers. In practice, tuning of this parameter can dramatically effect the quality of the solution obtained. However, increasing the value of the shape parameter makes the collocation matrices significantly more ill-conditioned and smaller values of this parameter, produce the better approximation. Schaback in [34], describes this phenomenon as the uncertainty relation; better conditioning is associated with worse accuracy, and worse conditioning is associated with improved accuracy. Accordingly, much research has been directed toward finding effective methods of optimization, see [31, 37]. Despite research done by many scientists to develop algorithms for selecting the values of c which produce most accurate interpolation, the optimal choice of shape parameter is still an open question.

Now let us consider a PDE in the form of

$$\begin{cases} Lu = f, & in\ \Omega; \\ u = g, & on\ \partial\Omega. \end{cases} \tag{2.1}$$

Both given f and $g : \mathbb{R}^n \to R$ are sufficiently smooth. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N$ be a given set of scattered nodal points in the domain and on the boundary. The unknown solution u is approximated via

$$u(x) \approx \tilde{u}(x) = \sum_{i=1}^{N} \alpha_i \varphi_i(x),$$

where $\varphi_i(x) = \varphi(||x - \bar{x}_i||_2)$, $i = 1, 2, \ldots, N$ are constructed on the set of nodal points and α_i, $i = 1, 2, \ldots, N$ are the unknown coefficients to be determined. For the spatial discretization, we consider $x_1, x_2, \ldots, x_N$ nodal points, where $x_1, x_2, \ldots, x_M$ are located in the domain and $x_{M+1}, x_{M+2}, \ldots, x_N$ on the boundary of the problem. After substituting $\tilde{u}$ in the PDE (2.1), the unknown scalars α_i are chosen so that the following problem is interpolated at the points x_j, $1 \leq j \leq N$, as follows

$$\begin{cases} L\tilde{u}_j = f_j, & j = 1, 2, \ldots, M, \\ \tilde{u}_j = g_j, & j = M+1, M+2, \ldots, N, \end{cases} \tag{2.2}$$

where $f_j = f(x_j)$, $g_j = g(x_j)$ and $\tilde{u}_j = \tilde{u}(x_j)$, which results in the following linear system of equations

$$A\alpha = b, \tag{2.3}$$

where $A_{j,i} = L(\varphi_i(x_j))$, $b_j = f_j$, $j = 1, 2, \ldots, M$ and $A_{j,i} = \varphi_i(x_j)$, $b_j = g_j$, $j = M+1, M+2, \ldots, N$ and $i = 1, 2, \ldots, N$ and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$. It can be shown that the matrix A is invertible for distinct set of scattered points in the case of multiquadric but, it is ill-conditioned. For a fixed number of interpolation points the condition number of A depends on the shape parameter c. So, in practice the values of c must be adjusted with the number of interpolating points in order to produce a well-conditioned interpolation matrix. Therefore, we choose experimentally a good value of the shape parameter c which yields the least residual norm.

3. Numerical approximation for the **Problem I**

In this section, an iterative method which is based on the combination of the RBF collocation and the FD methods is introduced to obtain the approximate solution of **Problem I**. For concreteness, we first present our proposed method for the special case of the **Problem I** with $r = 2$, i.e., two-phase boundary problem (1.2) and later expand it for the general case, i.e., an arbitrary value of r. Similarly, one can apply the proposed method for the one phase obstacle problem, too. Also, this method is capable for obtaining the approximate solution of the FBP (1.7) which will be described. For this means, first we need to introduce some basic notations.

Consider an uniform mesh on $\Omega \subset \mathbb{R}^2$ and for simplicity assume that $\Omega = [-1, 1] \times [-1, 1]$ and $\Delta x = \Delta y = h$. Let

$$p_{i,j} = (-1 + (i-1)h, -1 + (j-1)h), \quad i, j = 1, \ldots, m, \quad h = \frac{2}{m-1}, \ N = m^2.$$

Set

$$x_l = p_{i,j}, \quad i, j = 1, \ldots, m,$$

and

$$\bar{u}_l = \frac{1}{4}[u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1})], \;\; i, j = 2, \ldots, m-1, \quad (3.1)$$

where $l = j + (i-1)m$ and $\bar{u}_l = \bar{u}(x_l)$. Note that for the one-dimensional case, we define

$$p_i = -1 + (i-1)h, \;\; i = 1, \ldots, m, \;\; h = \frac{2}{m-1}, \;\; N = m,$$

$$x_l = p_l, \quad l = 1, \ldots, m,$$

$$\bar{u}_l = \frac{1}{2}[u(p_{l-1}) + u(p_{l+1})], \;\; l = 2, \ldots, m-1. \quad (3.2)$$

Consider x_l, $l = 1, 2, \ldots, N$, as collocation points. Let M of them are located in the domain and $N - M$ of them on the boundary of the problem. The unknown solution u can be approximated by a linear combination of the form

$$u(x) \approx \tilde{u}(x) = \sum_{i=1}^{N} \alpha_i \varphi_i(x), \quad (3.3)$$

where $\varphi_i(x) = \sqrt{c^2 + ||x - \bar{x}_i||^2}$ is the Multiquadric RBF and $\bar{x}_i$, $i = 1, 2, \ldots, N$, are the centers of RBF. Also, α_i, $i = 1, 2, \ldots, N$ are the unknown coefficients to be determined. Note that these centers and collocation points may or may not the share common points. Here, we consider both of them the same.

The major difficulty in solving the FBPs numerically is the nonlinearity nature of problem due to the unknown free surface. Hence, we present an iterative method which is based on combination of the RBF collocation and the FD methods to obtain an approximate solution of this problem. This method has been described as follows.

For $N - M$ nodal points are located on the boundaries ($x_l \in \partial\Omega$), the Dirichlet boundary condition is imposed by

$$\tilde{u}_l^{k+1} = g_l, \quad (3.4)$$

at the iteration $k+1$. For the nodes located in the interior of the domain, we present two methods **A** and **B** for the two-phase problem (1.2) and the free boundary problem (1.7), respectively.

Method A

Consider the two-phase problem (1.2). It is clear that in the interior points of domain, this problem is equivalent with the following problem

$$\begin{cases} \Delta u = \lambda^+, & if\ u > 0; \\ \Delta u = -\lambda^-, & if\ u < 0; \\ u = 0, & \text{otherwise}. \end{cases} \quad (3.5)$$

By applying the FD discritization on the system (3.5), this equation transfers to

$$\begin{cases} u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=\lambda_l^+h^2, & if\ \bar{u}_l-\frac{\lambda_l^+h^2}{4}>0;\\ u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=-\lambda_l^-h^2, & if\ \bar{u}_l+\frac{\lambda_l^-h^2}{4}<0;\\ u(p_{i,j})=0, & \text{otherwise;} \end{cases} \tag{3.6}$$

where $\lambda_l^+=\lambda^+(x_l)$, $\lambda_l^-=\lambda^-(x_l)$. Now we can combine the equation with the RBF collocation method and obtain an iterative method at the iteration k for the each node located in the domain ($x_l\ in\ \Omega$) as

$$\begin{cases} \tilde{u}_l^{k+1}=0, & if\ \hat{u}_l^k\le 0\ and\ \hat{\hat{u}}_l^k\ge 0;\\ \Delta\tilde{u}^{k+1}(x)\mid_{x=x_l}=\lambda_l^+\chi_{\hat{u}_l^k>0}-\lambda_l^-\chi_{\hat{\hat{u}}_l^k<0}, & \text{otherwise;} \end{cases} \tag{3.7}$$

where $\tilde{u}_l^{k+1}=\tilde{u}^{k+1}(x_l)=\sum_{i=1}^{N}\alpha_i^{k+1}\varphi_i(x_l)$,

$$\hat{u}_l^k=\bar{u}_l^k-\frac{\lambda_l^+h^2}{4},\quad \hat{\hat{u}}_l^k=\bar{u}_l^k+\frac{\lambda_l^-h^2}{4}$$

and

$$\bar{u}_l^k=\frac{1}{4}[u^k(p_{i-1,j})+u^k(p_{i+1,j})+u^k(p_{i,j-1})+u^k(p_{i,j+1})],\quad k=0,1,\dots, \tag{3.8}$$
$$i,j=2,3,\dots,m-1,\quad l=j+(i-1)m.$$

System (3.7), can be reduced to the following simple form

$$(\chi_{\hat{u}_l^k\le 0}\cdot\chi_{\hat{\hat{u}}_l^k\ge 0})\tilde{u}_l^{k+1}+(\chi_{\hat{u}_l^k>0}+\chi_{\hat{\hat{u}}_l^k<0})\Delta\tilde{u}^{k+1}(x)\mid_{x=x_l}=\lambda_l^+\chi_{\hat{u}_l^k>0}-\lambda_l^-\chi_{\hat{\hat{u}}_l^k<0},\ x_l\in\Omega.$$

Putting equations (3.4) and (3.7) together results in a linear system of equations. By choosing a proper initial guess and solving this linear system at each iteration, the approximate solution of the two-phase problem is obtained. An sketch of this iterative method is given in the Algorithm 3.1.

Method B

Consider the FBP (1.7). In the interior points of domain, this problem is equivalent with

$$\begin{cases} \Delta u=-\lambda^+u, & if\ u>0;\\ \Delta u=0, & if\ u<0;\\ u=0, & \text{otherwise.} \end{cases} \tag{3.9}$$

By using the FD discritization, the system (3.9) can be written as

$$\begin{cases} u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=-\lambda_l^+h^2u(p_{i,j}), & if\ \frac{\bar{u}_l}{1-\frac{\lambda_l^+h^2}{4}}>0;\\ u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=0, & if\ \bar{u}_l<0;\\ u(p_{i,j})=0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Let $\bar{M}=\max_{x_l\in\Omega}\sqrt{\lambda_l^+}$. It is easy to show that for $h<2/\bar{M}$ one can obtain $1-\frac{\lambda_l^+h^2}{4}>0$, for every $x_l\ in\ \Omega$. Thus the system (3.10), is reduced to

$$\begin{cases} u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=-\lambda_l^+h^2u(p_{i,j}), & if\ \bar{u}_l>0;\\ u(p_{i-1,j})+u(p_{i+1,j})+u(p_{i,j-1})+u(p_{i,j+1})-4u(p_{i,j})=0, & if\ \bar{u}_l<0;\\ u(p_{i,j})=0, & if\ \bar{u}_l=0. \end{cases} \tag{3.11}$$

Hence, For each node located in the domain (x_l in Ω), we have the following iterative procedure by combining the FD and RBF collocation methods

$$\begin{cases} \tilde{u}_l^{k+1} = 0, & if\ \bar{u}_l^k = 0; \\ \Delta \tilde{u}^{k+1}(x) \mid_{x=x_l} = -\lambda_l^+ \tilde{u}_l^{k+1} \chi_{\bar{u}_l^k > 0}, & \text{otherwise}; \end{cases} \tag{3.12}$$

where $k = 0, 1, 2, \ldots$.
System (3.12), can be transferred to the following simple form

$$(\chi_{\bar{u}_l^k=0})\tilde{u}_l^{k+1} + (\chi_{\bar{u}_l^k>0} + \chi_{\bar{u}_l^k<0})\Delta \tilde{u}^{k+1}(x) \mid_{x=x_l} = -\lambda_l^+ \tilde{u}_l^{k+1} \chi_{\bar{u}_l^k>0}, \quad x_l \in \Omega.$$

Putting equations (3.4) and (3.12) together results in a linear system of equations. By choosing a proper initial guess and solving this linear system at each iteration, the approximate solution of the FBP is obtained.
Two above methods are summarized as the following algorithm.

Algorithm 3.1.

step 1: Choose an initial guess as $u_l^0 = \begin{cases} 0, & if\ x_l\ in\ \Omega; \\ g_l, & if\ x_l\ on\ \partial\Omega \end{cases}$

step 2: For $k = 0, 1, 2, \ldots$, until convergence, Do
step 3: Compute $\bar{u}_l^k$ from equation (3.8)
step 4: Solve the linear system obtained from method A or B
step 5: Set the approximate solution $\tilde{u}^k(x) = \sum_{i=1:N}^{N} \alpha_i^k \varphi_i(x)$, where
$\alpha^k = (\alpha_1^k, \alpha_2^k, \ldots, \alpha_N^k)^T$ is the solution obtained from step 4
step 6: Put $u^k = \tilde{u}^k$
step 7: EndDo

Now consider the **Problem I**. The above results can be generalized for an arbitrary value of r as follows.
Set $\phi_{i,l} = \phi_i(x_l)$, $f_{i,l} = f_i(x_l)$ and $u_{i,l} = u_i(x_l)$ for $i = 1, 2, \ldots, r$. Let

$$\bar{u}_{w,l}^k = \frac{1}{4}[u_w^k(p_{i-1,j}) + u_w^k(p_{i+1,j}) + u_w^k(p_{i,j-1}) + u_w^k(p_{i,j+1})],$$
$$k = 0, 1, \ldots; i, j = 2, 3, \ldots, m-1; \quad l = j + (i-1)m,$$

and

$$\hat{u}_{w,l}^k = \bar{u}_{w,l}^k - \frac{f_{w,l}h^2}{4} - \sum_{p \neq w} \bar{u}_{p,l}^k,$$

at the iteration k with $w = 1, 2, \ldots, r$. For the nodal points located on the boundaries ($x_l \in \partial\Omega$), the Dirichlet boundary condition is imposed by

$$\tilde{u}_{i,l}^{k+1} = \phi_{i,l}, \quad i = 1, 2, \ldots, r, \tag{3.13}$$

and for nodes which are located in the interior of the domain, we have

$$\begin{cases} \tilde{u}_{i,l}^{k+1} = 0, & if\ \hat{u}_{i,l}^k \leq 0; \quad i = 1, 2, \ldots, r. \\ \Delta \tilde{u}_i^{k+1}(x) \mid_{x=x_l} = f_{i,l}, & \text{otherwise}; \end{cases} \tag{3.14}$$

Putting equations (3.13) and (3.14) together results in the r linear system of equations. The new iterative method for an arbitrary value of r is described in the following algorithm.

Algorithm 3.2.

step 1: Choose an initial guess as $u^0_{i,l} = \begin{cases} 0, & if\ x_l\ in\ \Omega; \quad i = 1, 2, \ldots, r, \\ \phi_{i,l}, & if\ x_l\ on\ \partial\Omega \end{cases}$

step 2: For $k = 0, 1, 2, \ldots$, until convergence, Do
step 3: Compute $\hat{u}^k_{i,l}$, for $i = 1, 2, \ldots, r$
step 4: Solve the linear systems obtained from (3.13) and (3.14)
step 5: Set the approximate solution $\tilde{u}^k_i(x) = \sum_{j=1:N}^{N} \alpha^k_{i,j}\varphi_j(x)$, where $\alpha^k_i = (\alpha^k_{i,1}, \alpha^k_{i,2}, \ldots, \alpha^k_{i,N})^T, i = 1, 2, \ldots, r$ are the solutions obtained from step 4
step 6: Put $u^k = \tilde{u}^k$
step 7: EndDo

4. Numerical approximation for the **Problem II**

In this section we will explain how to apply the new method to obtain the numerical solution of **Problem II** as κ tends to ∞. At first, we consider the case $f_i(x, u_(x)) = u_i(1-u_i)$ with two components. Then this method can be generalized for the general case with r component. After that, we will explain that in which way the method can be modified for the case $f_i(x, u_(x)) = 0$.

Consider the following problem

$$d_2\Delta v - d_1\Delta u = \lambda u(1-u)\chi_{u>0} - \lambda v(1-v)\chi_{v>0}. \tag{4.1}$$

Similar to the **Method A** and the **Method B** in the Section 3, we apply our method for solving (4.1). It is easy to show that by employing the second order, centred, finite difference scheme on equation (4.1), we obtain quadratic equation with respect to $u(x_i, y_j)$ and $v(x_i, y_j)$. With regarding the properties of solutions, free boundaries and segregated state of ϕ_i in **Problem II**, we drive our scheme.

Method C

Set

$$\hat{u}_l = \frac{\lambda - \frac{4d_1}{h^2} + \sqrt{(\lambda - \frac{4d_1}{h^2})^2 - 4\lambda(\frac{4d_2}{h^2}\bar{v}^k_l - \frac{4d_1}{h^2}\bar{u}^k_l)}}{2\lambda} \tag{4.2}$$

and

$$\hat{v}_l = \frac{-\lambda + \frac{4d_2}{h^2} + \sqrt{(\lambda - \frac{4d_2}{h^2})^2 + 4\lambda(\frac{4d_2}{h^2}\bar{v}^k_l - \frac{4d_1}{h^2}\bar{u}^k_l)}}{2\lambda}. \tag{4.3}$$

The boundary conditions of the **Problem II** are imposed similar to **Method A**. But for the nodes located in the interior of the domain by combining the RBF collocation and the FD methods, we have

$$\begin{cases} \tilde{u}^{k+1}_l = 0, & if\ \hat{u}^k_l \le 0; \\ \Delta\tilde{u}^{k+1}(x)\mid_{x=x_l} = u^k_l(1-u^k_l), & \text{otherwise}; \end{cases} \tag{4.4}$$

and

$$\begin{cases} \tilde{v}^{k+1}_l = 0, & if\ \hat{v}^k_l \le 0; \\ \Delta\tilde{v}^{k+1}(x)\mid_{x=x_l} = v^k_l(1-v^k_l), & \text{otherwise}. \end{cases} \tag{4.5}$$

Now the generalization of the method for an arbitrary value of r is as follows. Let

$$\hat{u}_{i,l}^k = \frac{-\lambda + \frac{4d_i}{h^2} + \sqrt{(-\lambda + \frac{4d_i}{h^2})^2 - 4\lambda(\sum_{p\neq i} \frac{4d_p}{h^2}\bar{u}_{p,l}^k - \frac{4d_i}{h^2}\bar{u}_{i,l}^k)}}{2\lambda}.$$

For the nodes located in the interior of the domain, we have

$$\begin{cases} \tilde{u}_{i,l}^{k+1} = 0, & if\ \hat{u}_{i,l}^k \leq 0; \quad i = 1, 2, \ldots r, \\ \Delta \tilde{u}_i^{k+1}(x) \mid_{x=x_l} = u_{i,l}^k(1 - u_{i,l}^k), & \text{otherwise}. \end{cases} \tag{4.6}$$

The algorithm for this case is given as follows.

Algorithm 4.1.

Choose an initial guess as $u_{i,l}^0 = \begin{cases} 0, & if\ x_l\ in\ \Omega; \quad i = 1, 2, \ldots, r, \\ \phi_{i,l}, & if\ x_l\ on\ \partial\Omega \end{cases}$

step 2: For $k = 0, 1, 2, \ldots$, until convergence, Do
step 3: Compute $\hat{u}_{i,l}^k$, for $i = 1, 2, \ldots, r$
step 4: Solve the linear systems obtained from (3.13) and (4.6)
step 5: Set the approximate solution $\tilde{u}_i^k(x) = \sum_{j=1:N}^{N} \alpha_{i,j}^k \varphi_j(x)$, where

$$\alpha_i^k = (\alpha_{i,1}^k, \alpha_{i,2}^k, \ldots, \alpha_{i,N}^k)^T.$$

$i = 1, 2, \ldots, r$ are the solutions obtained from step 4
step 6: Put $u^k = \tilde{u}^k$.
step 7: EndDo

Now, we can apply a similar method for **Problem II** when $f_i(x, u_i(x)) = 0$ and $\kappa \to \infty$.
Method D
Set

$$\hat{u}_{i,l}^k = \bar{u}_{i,l}^k - \sum_{p\neq i} \bar{u}_{p,l}^k, \quad i = 1, 2, \ldots, r.$$

For the interior nodes of domain we have

$$\begin{cases} \tilde{u}_{i,l}^{k+1} = 0, & if\ \hat{u}_{i,l}^k \leq 0; \quad i = 1, 2, \ldots r, \\ \Delta \tilde{u}_i^{k+1}(x) \mid_{x=x_l} = 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

5. Numerical experiments

In this section, some different numerical examples are presented to illustrate the effectiveness of the new method to obtain the numerical solution of the FBPs. Examples 5.1-5.4 and 5.5-5.6 show the numerical solutions of **Problem I** and **Problem II** for different values of r with a good accuracy, respectively. For the tests reported in this section, we adopt experimentally a good value of the parameter c for new method which yields the least residual norm. A good value of parameter c for the examples 5.1 and 5.2 is depicted in Figure 1. All the numerical experiments were computed with some MATLAB codes.

Example 5.1. For the first test consider the following one-dimensional two-phase equation

$$\begin{cases} u'' = 8\chi_{\{u>0\}} - 8\chi_{\{u<0\}}, & x \in (-1, 1) \\ u(1) = 1, \quad u(-1) = -1, \end{cases}$$

with the exact solution

$$u^*(x) = \begin{cases} 4x^2 - 4x + 1, & 0.5 \le x \le 1, \\ 0, & -0.5 \le x \le 0.5, \\ -4x^2 - 4x - 1, & -1 \le x \le -0.5. \end{cases}$$

As it can be seen from Figure 1, a good value of c for this example is almost 0.6 with $m = 21$ that yields the least RMS residual norm, where

$$||r||_{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} r^2(x_i)}.$$

We apply Algorithm 3.1 to obtain the approximate solution of this problem. Numerical results are shown in Table 1 for equidistance collocation points with $m = 21, 121$ and 201. In this table

$$||e||_\infty = \max_{1\le i\le N} |u(x_i) - u^*(x_i)|$$

and

$$e_{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(u(x_i) - u^*(x_i))^2}.$$

As it can be seen from this table, the numerical solution is obtained with a good accuracy. Apply Algorithm 3.1 for this problem the final solution is obtained after 4 iterations which is depicted in the Figure 2, with $m = 21$.

TABLE 1. Numerical results for Example 5.1.

m	c	Iteration	$\|\|e\|\|_\infty$	e_{RMS}
21	0.6	4	$2.4e-03$	$9.2345e-04$
121	0.09	29	$1.7453e-04$	$8.6393e-05$
201	0.09	49	$2.8768e-05$	$1.2547e-05$

Example 5.2. [15] Consider the following problem

$$\begin{cases} u'' = 2\chi_{\{u>0\}} - \chi_{\{u<0\}}, & x \in (-1,1), \\ u(1) = 1, \quad u(-1) = -1. \end{cases}$$

According to Figure 1, a good value of the shape parameter for this example, is almost $c = 0.4$. We apply the Algorithm 3.1 for obtaining the approximate solution of this problem with $m = 11$ and $c = 0.4$. The final solution of this example is depicted in the Figure 3.

Example 5.3. [15] The third test is the following 2D two-phase problem

$$\Delta u = 1\chi_{\{u>0\}} - 1\chi_{\{u<0\}}, \quad (x,y) \in (-1,1)^2,$$

$$g(x,y) = \begin{cases} (\frac{1-x}{2})^2, & -1 \le x \le 1, \ \ y = 1, \\ y^2, & 0 \le y \le 1, \ \ x = -1, \\ -y^2, & -1 \le y \le 0, \ \ x = -1, \\ -(\frac{1-x}{2})^2, & -1 \le x \le 1, \ \ y = -1, \\ 0, & -1 \le x \le 1, \ \ x = 1. \end{cases}$$

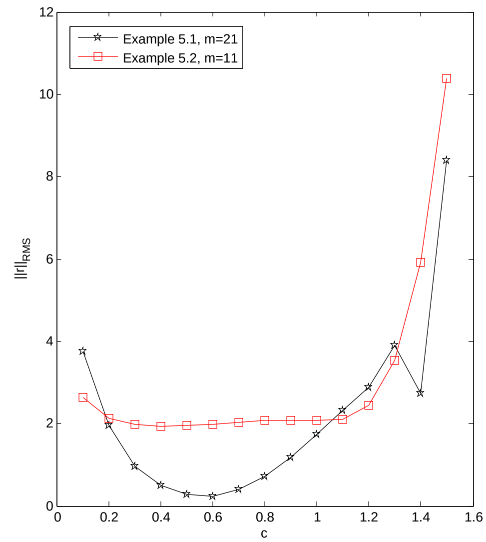

FIGURE 1. The $||r||_{RMS}$ versus different values of c for Example 5.1 and Example 5.2.

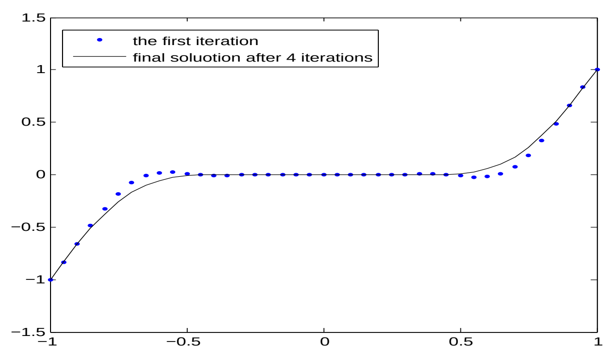

FIGURE 2. The convergence history of the new method for Example 5.1 with $m = 21$ and $c = 0.6$.

We apply Algorithm 3.1 to obtain the numerical solution of this example with $m = 31$ and $c = 0.4$. The final solution obtained after 14 iterations. In the Figure 4, the obtained solution in different level sets and the numerical solution are shown.

Example 5.4. Consider the following problem

$$\Delta u = -u\chi_{\{u>0\}}, \quad (x, y) \in (-4, 4)^2,$$

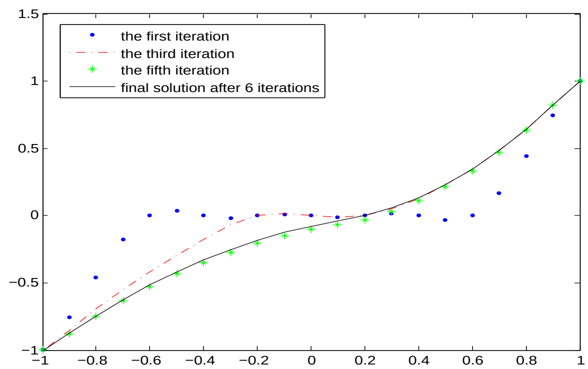

FIGURE 3. The convergence history of the new method for Example 5.2 with $m = 11$ and $c = 0.4$.

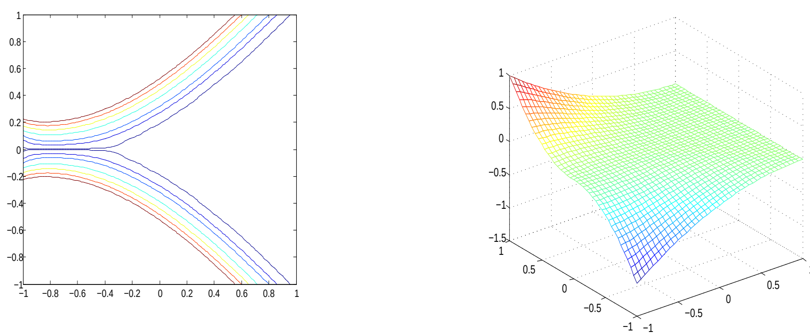

FIGURE 4. the level sets of solution (left), the numerical solution (right) with $m = 31$ and $c = 0.4$ for Example 5.3.

with the analytical solution

$$u(x, y) = \begin{cases} J_0(r), & if\ r < r_c, \\ A \ln \frac{r_c}{r}, & if r \geq r_c, \end{cases}$$

where $r^2 = x^2 + y^2$ and $r_c \approx 2.404826$ is the first zero of $J_0(r)$ and $A \approx 1.248459$.
By applying Algorithm 3.1 to obtain the numerical solution of this problem, after 6 iterations, we obtain $||e||_\infty = 5.2586e - 04$ and $e_{RMS} = 2.0659e - 04$ with $m = 31$ and $c = 0.4$. The numerical solution of problem, the level set of solution and the error solution are depicted in Figure 5. As it can be seen from the figures, the numerical results shows a good accuracy of the new method.

Example 5.5. Consider the **Problem II** with $r = 4$, $f_1 = 8$, $f_2 = 6$, $f_3 = 2$ and $f_4 = 1$. Let $\Omega = [-1, 1]$ and boundary values of ϕ_i, $i = 1, 2, 3, 4$, be as follows

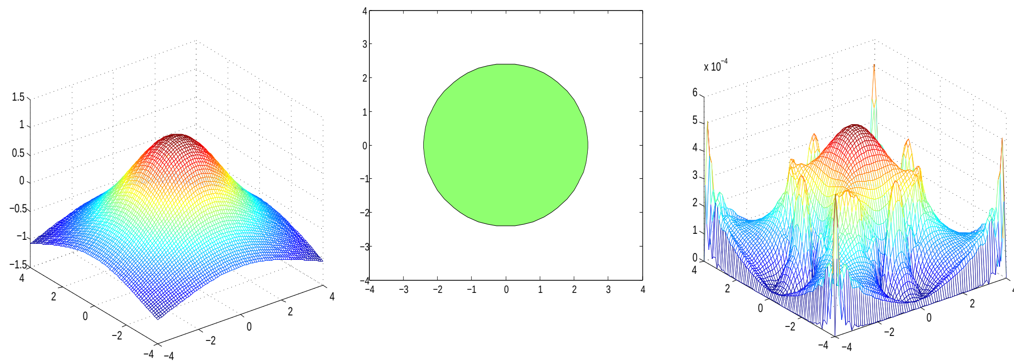

FIGURE 5. The numerical solution of problem (left), the level set of solution (middle), the error solution (right) for Example 5.4.

$$\phi_1(x,y) = \begin{cases} 1-x^2, & x \in [-1,1], y=1, \\ 0, & elsewhere, \end{cases}$$

$$\phi_2(x,y) = \begin{cases} 1-y^2, & y \in [-1,1], x=1, \\ 0, & elsewhere, \end{cases}$$

$$\phi_3(x,y) = \begin{cases} 1-x^2, & x \in [-1,1], y=-1, \\ 0, & elsewhere, \end{cases}$$

$$\phi_4(x,y) = \begin{cases} 1-y^2, & y \in [-1,1], x=-1, \\ 0, & elsewhere, \end{cases}$$

We apply Algorithm 3.2 with $c = 0.05$ and $m = 21$. The contours of solution and the surface of $u_1 + u_2 + u_3 + u_4$ are depicted in Figure 6.

Example 5.6. For the last test, consider the equation (4.1) with $\Omega = [0,1]$, $\lambda = 1$, $d_1 = 1.5$ and $d_2 = 1$. The steady boundary values for $u(x,y,t)$ and $v(x,y,t)$ are defined by

$$\phi(x,0,t) = \begin{cases} 0.5-2.5x, & x \in [0,.2], \\ 0, & x \in [0.2,1], \end{cases} \quad \phi(0,y,t) = 0.5,$$

$$\phi(x,1,t) = \begin{cases} 0.5-\frac{5}{8}x, & x \in [0,.8], \\ 0, & x \in [0.8,1], \end{cases} \quad \phi(1,y,t) = 0,$$

$$\psi(x,0,t) = \begin{cases} 0, & x \in [0,.2], \\ -\frac{1}{8}+\frac{5}{8}x, & x \in [0.2,1], \end{cases} \quad \psi(0,y,t) = 0,$$

$$\psi(x,1,t) = \begin{cases} 0, & x \in [0,.8], \\ -2+2.5x, & x \in [0.8,1], \end{cases} \quad \psi(1,y,t) = 0.5.$$

We apply Algorithm 4.1 with $c = 0.1$ and $m = 21$. The surface of $u+v$ is depicted in Figure 7.

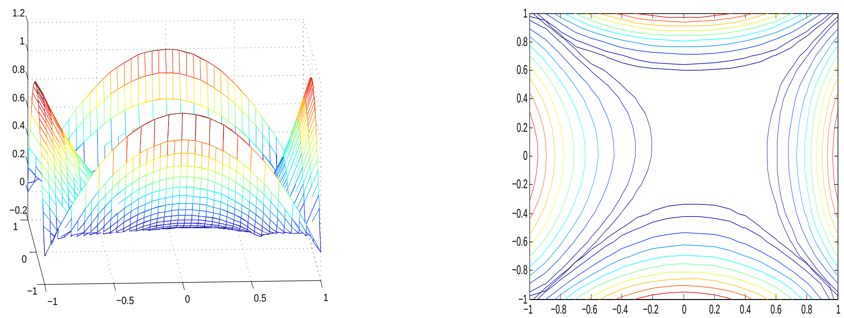

FIGURE 6. The surface $u_1 + u_2 + u_3 + u_4$ (left), the contours of solution (right) for Example 5.5.

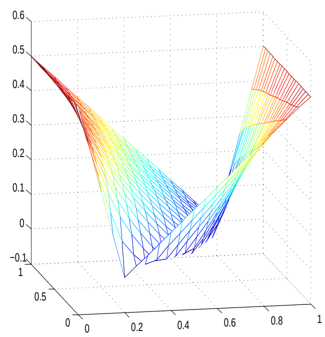

FIGURE 7. The surface $u + v$ for Example 5.6.

6. Conclusion

The major difficulty in solving FBPs is the nonlinearity nature of the problem. In this paper an efficient iterative method is proposed for obtaining the numerical solution of the FBPs. This method is based on RBF collocation method and finite difference scheme. The most important advantage of this method is its mesh free nature. Numerical studies demonstrated, by choosing a good shape parameter, the new method is fast, accurate and convergent.

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