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Solving brachistochrone problem via scaling functions of Daubechies wavelets

Azad Kasnazani*
Department of Applied Mathematics,
University of Kurdistan, Sanandaj, Iran.
E-mail: A.Kasnazani@uok.ac.ir

Amjad Alipanah
Department of Applied Mathematics,
University of Kurdistan, Sanandaj, Iran.
E-mail: A.Alipanah@uok.ac.ir

Abstract In this paper, we proposed an effective method based on the scaling function of Daubechies wavelets for the solution of the brachistochrone problem. An analytic technique for solving the integral of Daubechies scaling functions on dyadic intervals is investigated and these integrals are used to reduce the brachistochrone problem into algebraic equations. The error estimate for the brachistochrone problem is proposed and the numerical results are given to verify the effectiveness of our method.

Keywords. Daubechies wavelets, Scaling function, Brachistochrone problem, Error analysis, Numerical results.

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1. Introduction

For the last three decades wavelet theory has attracted a lot of interests because of the advantages of it over former theories specially in better approximating functions that have discontinuous and sharp peaks. Various kinds of wavelets were generated, among them, Daubechies wavelets which discovered by Ingrid Daubechies in 1988, have attracted tremendous attention due to their significant properties in numerical analysis. They are orthogonal wavelets, compactly supported, the degree of their vanishing moment is not limited and the most important property of the Daubechies wavelets is that their regularity increases linearly with their support width [5]. These characteristics make Daubechies wavelets a good candidate for solving Calculus of Variation (CV) problems. The wavelet application in CV and optimal control problems have been discussed in some papers. For example Razzaghi and Yousefi used Legendre wavelets for the solution of nonlinear problems in the Calculus of Variations [16], the Haar wavelet is used for solving CV problems by Zarebnia and Barandak Imcheh [22] and Karimi et al. used Haar wavelet for solving optimal control problems [10]. But as we know the Haar wavelet has the simplest linear discontinuous shape

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* corresponding.

and the precession of its approximation is not sufficient. In the Legendre wavelet when the order of Legendre polynomials increases, in practice, due to the large polynomial oscillation, precision decreases. Daubechies wavelets and their scaling functions have no explicit form which causes serious difficulties in computations. According to our information, so far these wavelets have not been used for solving CV and optimal control problems. In this paper, we overcame this problem and it has been explained that how these wavelets can be applied for solving CV problems.
Historically, the brachistochrone problem is the first CV problem that Johann Bernoulli introduced in 1696 [8, 19]. In recent years a great deal of research has focused on approximating the solution for the brachistochrone problem. For example the gradient method [2], successive sweep algorithm [2, 3], the classic Chebshev method [9], multistage Monte Carlo method [20], rationalized Haar functions [15] and nonclassical pseudospectral method [1, 12].
The purpose of this paper is to introduce a new method for solving the brachistochrone problem based on scaling functions of Daubechies wavelets. With the use of the features of Daubechies scaling functions, many parts of the problem are accurately calculated and for the first time an error analysis has been done.
In this paper, first the integrals of Daubechies scaling functions on dyadic intervals (i.e., the first point and last point of interval is dyadic point) are computed exactly. Then, the unknown function in the brachistochrone problem and its first derivative are projected in the Daubechies scaling functions space with the unknown coefficients and next the main integral in the brachistochrone problem is approximated by the simpson method. After that by applying the necessary conditions for finding extremum, the brachistochrone problem is transformed into algebraic equations.
This paper is organized as follows. In Section 2, we briefly deal with a description of Daubechies wavelets and evaluation of Daubechies scaling functions at dyadic points. In Section 3, a procedure for obtaining integrals which contain scaling function of Daubechies wavelets on dyadic intervals is presented. In Section 4, The method for solving the brachistochrone problem is put forwarded. This is followed by an investigating of error analysis and in the rest of this section, numerical results are shown.

2. Daubechies scale function

Daubechies wavelets are generated from the scaling function $\phi(x)$ with the compact support $[0, 2p-1]$ where p is the number of vanishing moments. These wavelets such as other wavelets have the ability to represent a function in different levels of resolution. The scaling function $\phi(x)$ and Daubechies wavelet $\psi(x)$ satisfy their two scale relations or dilation equation:

$$\phi(x) = \sum_{k=0}^{2p-1} h_k \phi(2x-k), \tag{2.1}$$

$$\psi(x) = \sum_{k=0}^{2p-1} g_k \phi(2x-k).$$

In [17], the exact values of h_k are evaluated by imposing some useful properties of daubechies wavelets such as orthogonality and ability to exactly represent polynomials of degree smaller than p. In [11], it is proven that for a fixed p, there exists only one linear independent scaling function ϕ , which satisfies in Eq. (2.1) and if we denote $D = 2p$ then the parameter D is called the wavelet genus or Daubechies number [5, 14]. The following relation between h_k and g_k is hold [6]

$$g_k = (-1)^k h_{D-k-1}.$$

In order to achieve representation of functions in Hilbert space $L^2(\mathbb{R})$ at different levels of resolution [6, 21], one can use the translated dilations of the scaling function, defined as:

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k), \qquad j, k \in \mathbb{Z}.$$

The set of orthogonal functions $\phi_{j,k}$ for particular j, generates the subspace V_j. The vector subspaces V_j satisfy the following conditions [4, 6]
1. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$,
2.$\cap_{j\in\mathbb{Z}} V_j = \{0\}$,
3. $V_j \subset V_{j+1}, \quad j \in \mathbb{Z}$.

For each j, since V_j is a proper subspace of V_{j+1}, there is some space in V_{j+1} called W_j, which when combined with V_j gives us V_{j+1}. This space W_j is called the *wavelet subspace* and is orthogonal complementary to V_j in V_{j+1}, and so

$$V_{j+1} = V_j \oplus W_j,$$

where $\oplus$ represents a *direct sum.* So

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j.$$

Let $\{\psi(. - k)|\ k \in \mathbb{Z}\}$, be an orthonormal basis of the subspace W_0, which $\psi(x)$ is known as mother wavelet. Then

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \qquad k \in \mathbb{Z},$$

is an orthonormal basis for W_j. Generally, scaling functions of Daubechies wavelet do not have a closed form. Instead, one can use Eq. (2.1) to determining the scaling function $\phi(x)$ in all dyadic points. The complete explanation of this procedure is named cascade algorithm and is introduced in [13, 14].

3. Integration of Daubechies scaling function on dyadic interval

As previously mentioned, the support of Daubechies scaling function with p vanishing moments is $[0, D-1]$ where $D = 2p$. In this section, first, the integral of Daubechies scaling function on integer intervals is computed. Then by using a recursion relation (2.1) the integral of Daubechies scaling function on every dyadic interval, in arbitrary resolution will be computed.

3.1. **Computing the integral of Daubechies scaling function on integer intervals.** For simplicity we define

$$A_{[a,b]} = \int_a^b \phi(x)dx. \tag{3.1}$$

Then by applying Eq. (2.1) to the scaling function ϕ, the following results is obtained:

$$\begin{aligned} A_{[0,i]} &:= \int_0^i \sum_{k=0}^{D-1} h_k \phi(2x-k)dx, \qquad i \in \mathbb{Z} \\ &= \frac{1}{2}\sum_{k=0}^{D-1} h_k \int_{-k}^{2i-k} \phi(t)dt \\ &= \frac{1}{2}\sum_{k=0}^{D-1} h_k A_{[0,2i-k]}, \end{aligned}$$

or

$$A_{[0,i]} = \frac{1}{2}\sum_{k=0}^{D-1} h_k A_{[0,2i-k]}. \tag{3.2}$$

It is clearly the following equations are held:

$$A_{[0,i]} = 1, \quad \text{if} \quad i \geq D-1, \tag{3.3}$$

$$A_{[0,i]} = 0, \quad \text{if} \quad i \leq 0. \tag{3.4}$$

Eqs. (3.2)-(3.4) can be written as a system of linear equations $C\mathbf{x} = b$, where the entries of matrices C, $\mathbf{x}$ and b are given as

$$C_{i,j} = \begin{cases} 2-h_i & \text{for } i = j, \\ -h_{2i-j} & \text{for } i \neq j, \end{cases}$$

$$b_i = \begin{cases} 0 & i \leq p-1, \\ \sum_{k=0}^{2(i-p)+1} h_k & i \geq p, \end{cases}$$

$$x_i = \mathbf{A}_{[0,i]}, \qquad 1 \leq i \leq D-2,$$

hence the unknown parameters $A_{[0,i]}$ can be computed by solving the above linear system. For example linear system of equations (3.2)-(3.4), when $D = 4$ and $D = 6$ are given as follows:
for $D = 4$

$$\begin{bmatrix} 2-h_1 & -h_0 \\ -h_3 & 2-h_2 \end{bmatrix} \begin{bmatrix} A_{[0,1]} \\ A_{[0,2]} \end{bmatrix} = \begin{bmatrix} 0 \\ h_0+h_1 \end{bmatrix},$$

and for $D = 6$

$$\begin{bmatrix} 2-h_1 & -h_0 & 0 & 0 \\ -h_3 & 2-h_2 & -h_1 & -h_0 \\ -h_5 & -h_4 & 2-h_3 & -h_2 \\ 0 & 0 & -h_5 & 2-h_4 \end{bmatrix} \begin{bmatrix} A_{[0,1]} \\ A_{[0,2]} \\ A_{[0,3]} \\ A_{[0,4]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h_0+h_1 \\ \sum_{i=0}^{3} h_i \end{bmatrix}.$$

Obviously matrices of coefficients in these linear systems are diagonally dominant matrices and so computing $A_{[0,i]}$ encounters no difficulty. In the case $D = 4$ the answers are
$A_{[0,1]} = .8496793685588857$ and $A_{[0,2]} = 1.016346035225553$.

3.2. **Computing the integrals of Daubechies scaling function on dyadic rational intervals.** In the previous subsection, the values of $A_{[a,b]}$, exactly computed when a and b are integers, and now we obtain a recursion relation for computing $A_{\left[l2^{-j},(l+1)2^{-j}\right]}$, when $j \in \mathbb{N}$ and $0 \le l \le (D-1)2^j - 1$ as

$$\begin{aligned} A_{\left[l2^{-j},(l+1)2^{-j}\right]} &= \int_{l2^{-j}}^{(l+1)2^{-j}} \phi(x)dx \\ &= \int_{l2^{-j}}^{(l+1)2^{-j}} \left(\sum_{k=0}^{D-1} h_k \phi(2x-k) \right) dx \\ &= \sum_{k=0}^{D-1} \frac{1}{2} h_k \int_{l2^{1-j}-k}^{(l+1)2^{1-j}-k} \phi(t)dt \\ &= \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{\left[l2^{1-j}-k,(l+1)2^{1-j}-k\right]}. \end{aligned}$$

By using this recursion relation, the values of $A_{[a,b]}$ are computed when a and b are dyadic rational. The values of $A_{[a,b]}$ for a Daubechies scaling function for $D = 4$ are given in Table 1.

TABLE 1. The values of $A_{[a,b]}$ for Daubechies when $D = 4$

$A_{[0,\frac{1}{4}]} = 9.9095205e(-2)$	$A_{[1,\frac{5}{4}]} = 1.7456594e(-1)$	$A_{[2,\frac{9}{4}]} = -2.3661145e(-2)$
$A_{[\frac{1}{4},\frac{2}{4}]} = 1.9107569e(-1)$	$A_{[\frac{5}{4},\frac{6}{4}]} = 5.3104960e(-2)$	$A_{[\frac{9}{4},\frac{10}{4}]} = 5.8193442e(-3)$
$A_{[\frac{2}{4},\frac{3}{4}]} = 2.4938899e(-1)$	$A_{[\frac{6}{4},\frac{7}{4}]} = -1.0216272e(-3)$	$A_{[\frac{10}{4},\frac{11}{4}]} = 1.6326381e(-3)$
$A_{[\frac{3}{4},1]} = 3.1011948e(-1)$	$A_{[\frac{7}{4},2]} = -5.9982607e(-2)$	$A_{[\frac{11}{4},3]} = -1.3687209e(-4)$

4. Brachistochrone problem

The brachistochrone problem may be formulated as minimizing the performance index J,

$$J = \int_0^1 \left[\frac{1 + (y'(x))^2}{1 - y(x)} \right]^{1/2} dx, \tag{4.1}$$

$$y(0) = 0, \quad y(1) = -0.5. \tag{4.2}$$

As is well known, the exact solution to the brachistochrone problem is a cycloid given by the following parametric equations [7].

$$y = 1 - \frac{\beta}{2}(1 + \cos 2\alpha), x = \frac{c}{2} + \frac{\beta}{2}(2\alpha + \sin 2\alpha),$$

with given boundary conditions, the integration constants are found to be

$$\beta = 1.6184891, c = 2.7300631.$$

4.1. **Solution of the brachistochrone problem by the scaling functions of Daubechies wavelets.** In this section, we use the scaling functions of Daubechies wavelets to approximate the solution of brachistochrone problem. In order to use Daubechies scaling functions, with a suitable linear transformation, we change the integration interval $[0, 1]$ in Eq. (4.1) into $[0, D - 1]$. Let $P_j(y')$ denotes the operator that orthogonally projects y' onto V_j, so

$$y' \simeq P_j(y'),$$

and so

$$P_j(y') = \sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}(x), \tag{4.3}$$

where k_1 and k_2 are such that the last point in the support of ϕ_{j,k_1} is smaller than 0 and the first point in the support of ϕ_{j,k_2} is bigger than $D - 1$, so

$$k_1 = 2^j a + 2 - L, \quad and \quad k_2 = 2^j(D - 1) - 1.$$

In order to approximating y in Eq. (4.1), we integrate $P_j(y')$ in Eq. (4.3)

$$\int_0^x P_j(y')dt = \sum_{k=k_1}^{k_2} a_{j,k} \int_0^x \phi_{j,k}(t)dt,$$

hence

$$y(x) \simeq \sum_{k=k_1}^{k_2} a_{j,k} \int_0^x \phi_{j,k}(t)dt + y(0).$$

Let $H_{j,k}(x)$, be defined as follows:

$$H_{j,k}(x) = \int_0^x \phi_{j,k}(t)dt, \tag{4.4}$$

then we have

$$y(x) \simeq \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(x) + y(0). \tag{4.5}$$

From Eqs. (2.1) - (3.1) and (4.4) for dyadic x_is we get

$$\begin{aligned} H_{j,k}(x_i) &= \int_a^{x_i} 2^{\frac{j}{2}} \phi_(2^j t - k)dt \\ &= 2^{-\frac{j}{2}} \int_{-k}^{2^j x_i - k} \phi_(s)ds \\ &= 2^{-\frac{j}{2}} A_{[-k, 2^j x_i - k]}, \end{aligned}$$

or simplicity

$$H_{j,k}(x_i) = 2^{-\frac{j}{2}} A_{[-k, 2^j x_i - k]}. \tag{4.6}$$

Hence values of $H_{j,k}(x_i)$ analytically will be computed when x_is are dyadic points and so no error occurs in approximating y from y'. For simplicity if the integrant in Eq. (4.1) is written as $F(x, y, y')$ then by substituting Eqs. (4.3) and (4.5) in Eq. (4.1) we get

$$J[y] = \int_0^{D-1} F\left(x, \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(x), \sum_{k=k_1}^{k_2} a_{j,k} \phi_{jk}(x)\right) dx.$$

Let x_i be a dyadic point in $[0, D-1]$ and by using (4.6)

$$F_i = F\Big(x_i, \sum_{k=k_1}^{k_2} 2^{-\frac{j}{2}} a_{j,k} A_{[-k, 2^j x_i - k]}, \sum_{k=k_1}^{k_2} a_{j,k} \phi_{jk}(x_i)\Big), \quad i = 1...n,$$

and composite Simpson integration such that nodes be dyadic points we have

$$J\left(a_{j,k_1}, \ldots, a_{j,k_2}\right) = \frac{h}{3}\Big[F_0 - F_n + \sum_{i=1}^{\frac{n}{2}} (4F_{2i-1} + 2F_{2i})\Big], \tag{4.7}$$

where $n = 2^{5+2j}$ and $h = n^{-1}$.

Therefore $J[y]$ transforms into function $J(a_{j,k_1}, \ldots, a_{j,k_2})$ and brachistochrone problem transforms into finding coefficients $a_{j,k_1}, \ldots, a_{j,k_2}$ such that function $J(a_{j,k_1}, \ldots, a_{j,k_2})$ is minimum. Now the boundary conditions (4.2) will also be expanded by Daubechies scaling function,

$$y(0) = \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(0) + y(0), \tag{4.8}$$

$$y(D-1) = \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(D-1) + y(0). \tag{4.9}$$

From Eq. (4.4) it is clear that

$$H_{j,k}(0) = 0, \quad j, k \in \mathbb{Z},$$

and by using Lagrange multipliers technique [7], for Eqs. (4.7) and (4.9) we get

$$J^* = J(a_{j,k_1}, \dots, a_{j,k_2}) + a_{j,k_2+1}\Big(\sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(D-1) + y(0) - y(D-1)\Big).$$

The necessary conditions for obtaining the extremum of the above equation is

$$\frac{\partial J^*}{\partial a_{j,i}} = 0, \quad i = k_1, \dots, k_2 + 1. \tag{4.10}$$

Eequations (4.10) give us $k_2 - k_1 + 1$ nonlinear equations with $k_2 - k_1 + 1$ unknowns, which can be solved for the unknowns $a_{j,k}, k = k_1, \cdots, k_2 + 1$ by using Newtons iterative method.

4.2. **Error analysis.** By defining E and $\hat{E}$ as below:

$$E := \|y - P_j(y)\|_\infty,$$

$$\hat{E} := \|y' - P_j(y')\|_\infty,$$

it can be concluded as follows [18]

$$E = O(2^{-j}), \quad \hat{E} = O(2^{-j}). \tag{4.11}$$

Lemma 4.1. *Let* $F(y, y') := \left[\frac{1+(y'(t))^2}{1-y(t)}\right]^{\frac{1}{2}}$ *and* $y(0) = 0, \quad y(1) = -0.5$. *Then*

$$\sqrt{\frac{2}{3}} \leq F(y, y') \leq \sqrt{1 + (y')^2(0) + g}, \tag{4.12}$$

where g is acceleration due to gravity.

Proof. In the brachistochrone problem one should find the shape of the curve down, which a bead sliding from rest and accelerated by gravity will slip (without friction) from point $(0, 0)$ to $(1, -0.5)$ in the least time. So it is clearly that $-0.5 \leq y(t) \leq 0$, for all $t \in [0, 1]$. Therefore

$$\frac{2}{3} \leq \frac{1}{1 - y(t)} \leq 1. \tag{4.13}$$

In addition $y'(t)$ is the velocity of the mass in time t, and we have that

$$(y'(t))^2 \leq (y'(0))^2 + g, \tag{4.14}$$

So from Eqs. (4.13)-(4.14) we conclude that

$$\sqrt{\frac{2}{3}} \leq F(y, y') \leq \sqrt{1 + (y'(0))^2 + g}.$$

□

Theorem 4.2. *Let J be the functional given in Eq. (4.1), then*

$$\Big|J[y]-J[P_j(y)]\Big| \leq \frac{\sqrt{3}}{2\sqrt{2}}\left(1+y'^2(0)+g\right)E+\frac{3\sqrt{3}}{2\sqrt{2}}\hat{E}\left[\sqrt{y'^2(0)+g}+\frac{\hat{E}}{2}\right]. \quad (4.15)$$

Proof. For simplicity, let $X(t)$ and $X'(t)$ be the approximation of $y(t)$, $y'(t)$ respectively, and be given as follows

$$X(t)=\sum_{k=k_1}^{k_2} a_{j,k}\int_0^x \phi_{j,k}(t)dt+y(0), \quad X'(t)=\sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}(x),$$

therefore we have

$$J[y]-J[X]=\int_0^1\left[\left(\frac{1+y'^2}{1-y}\right)^{\frac{1}{2}}-\left(\frac{1+X'^2}{1-X}\right)^{\frac{1}{2}}\right]dt, \quad (4.16)$$

by simplifying the above equation we have

$$\Big|J[y]-J[X]\Big|=\int_0^1\left|\frac{(y-X)+(y'-X')(y'+X')+(X')^2y-(y')^2X}{\big(F(y,y')+F(X,X')\big)(1-y)(1-X)}\right|dt. \quad (4.17)$$

Now suppose that $y(t)=X(t)+\varepsilon(t)$, where $\varepsilon(t)$ is the error of approximation by Daubechies scaling function and it is clear that

$$|\varepsilon(t)|\leq E, \quad |\varepsilon'(t)|\leq \hat{E}. \quad (4.18)$$

From Eq. (4.18) we have

$$(X'(t))^2y(t)-(y'(t))^2X(t)=\varepsilon(t)\left(y'(t)\right)^2+y(t)\varepsilon'(t)\left(\varepsilon'(t)-2y'(t)\right), \quad (4.19)$$

also from lemma 4.1 and Eq. (4.13) and this fact that $\varepsilon(t)$ is small, we have the following relation

$$\frac{1}{\left(F(y,y')+F(X,X')\right)|(1-y)(1-X)|}\leq\frac{\sqrt{3}}{2\sqrt{2}}. \quad (4.20)$$

So from Eqs. (4.15) - (4.20) we have

$$\Big|J[y]-J[X]\Big|\leq\frac{\sqrt{3}}{2\sqrt{2}}\int_0^1|\varepsilon(t)|\left(1+y'^2(t)\right)+|\varepsilon'(t)||\left(1-y)(2y'-\varepsilon'(t))|\right)dt, \quad (4.21)$$

therefore by using Eq. (4.14) the above equation can be expressed in the following form

$$|J[y]-J[X]|\leq\frac{\sqrt{3}}{2\sqrt{2}}\left(1+y'^2(0)+g\right)E+\frac{3\sqrt{3}}{2\sqrt{2}}\hat{E}\left[\sqrt{y'^2(0)+g}+\frac{\hat{E}}{2}\right].$$

□

According to Eq. (4.11) values of E and $\hat{E}$ decrease, when resolution j increases and so Theorem 4.2 shows that the error of our method is limited.

4.3. **Numerical results.** All computational efforts in this work has carried out using Maple software in 16 decimal digits. Using Daubechies scaling functions with two vanishing moments, the computational results are summarized in Figure 1, Table 2 and Table 3. Also the numerical results obtained via other methods given in literatures are given in Table 4. The relative error for resolution j is defined as

$$E_j = \frac{|J_{exact} - J_j|}{J_{exact}}, \tag{4.22}$$

where J_{exact} is 0.9984981482937085306776 [1] and so relative error and absolute error are very close. From Figure 1 and Table 2, it is obviously observed that despite of using the weakest Daubechies scaling function with 2 vanishing moments when resolution j increases, the numerical results are excellent in comparison with other methods given in Table 4.

Table 2. Numerial results of the presented method for $y'(0)$ in the brachistochrone problem

j	$y'(0)$
0	-0.7503150486
1	-0.7852018649
2	-0.7860920771
3	-0.7864785072
4	-0.7864450197
5	-0.7864402049
6	-0.7864407452
7	-0.7864407974

Table 3. Numerial results of the presented method for J in the brachistochrone problem

j	J	E_j
0	0.9981985988492	$3.0e(-4)$
1	0.9984132759511	$8.5e(-5)$
2	0.9984944538506	$3.7e(-6)$
3	0.9984973594802	$7.9e(-7)$
4	0.9984980863868	$6.2e(-8)$
5	0.9984981402059	$8.1e(-9)$
6	0.9984981478644	$4.3e(-10)$
7	0.9984981482328	$6.1e(-11)$

TABLE 4. Numerical results of other methods for the brchistochrone problem.

Methods	$y'(0)$	error for J
Gradient Method[2]	-0.7832273	$0.7e(-6)$
Successive sweep method [2, 3]	-0.7834292	$0.8e(-6)$
Chebyshev solution[9]	-0.7864406	$0.1e(-6)$
Legendre wavelets[15]	-0.7864408	$0.1e(-10)$
Nonclassical pseudospectral[1, 12]	-0.78644079	$0.1e(-11)$

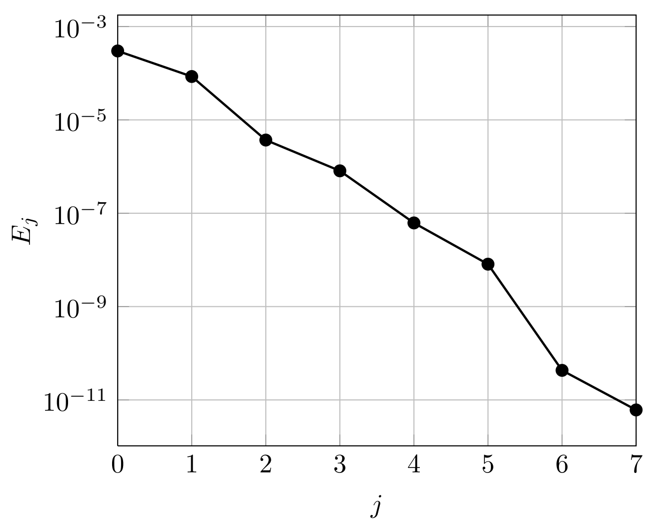

FIGURE 1. Relative errors E_j, for different resolution $j = 0, 1, \cdots, 7$.

5. Conclusion

In this paper, we use an analytic method for computing the integration of Daubechies scaling functions. In this technique, y and its derivation are projected in Daubechies scaling function space. We applied a direct method for solving the brachistochrone problem and this well-known problem is completely solved by the new proposed procedure. It is believed that the approach is a powerful and efficient direct method and by considering the error analysis of the problem and the numerical results, if we apply higher vanishing moments and higher resolutions, much better results can be approachable. The method of using the Daubechies scaling function basis to solve brachistochrone problem reduces it to nonlinear algebraic equations. One of the great advantages of our method is that Daubechies wavelets are compactly supported and increasing in vanishing moments will yield smoother wavelets. So our method for approximating curves with discontinuities and sharp peaks is a good choice. The only disadvantage of our method is that the computations are available only in dyadic points and dyadic intervals.

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