Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 2, 2021, pp. 375-392
DOI:10.22034/cmde.2020.33660.1544

Partial eigenvalue assignment of descriptor fractional discrete-time linear systems by parametric state feedback

Sakineh Bigom Mirassadi
Faculty of Mathematical Sciences,
Shahrood University of Technology, Shahrood, Iran.
E-mail: s.mirassadi@shahroodut.ac.ir

Hojjat Ahsani Tehrani*
Faculty of Mathematical Sciences,
Shahrood University of Technology, Shahrood, Iran.
E-mail: hahsani@shahroodut.ac.ir

Abstract In this paper, we present a nonlinear parametric method to stabilize descriptor fractional discrete-time linear system practically. Parametric methods with the free parameters can be adjusted to obtain better performance responses like minimum norm in state feedback. The aim is assigning desirable eigenvalues to obtain satisfactory responses by forward state feedback and forward and propositional state feedback in new systems with large matrices. However, finding the solution to nonlinear parametric equations makes some errors. In partial eigenvalue assignment, just a part of the open-loop spectrum of the standard linear systems is reassigned, while leaving the rest of the spectrum invariant. The size of matrices, state, and input vectors are decreased and the stability is kept. At the end, summary and conclusions are proposed and the convergence of state vectors in the descriptor fractional discrete-time system to zero is also shown by figures in a numerical example. Our method is also compared with another method with one of orthogonality relations in our article and example.

Keywords. Descriptor fractional discrete-time, Nonlinear equations, Parametric state feedback, Partial eigenvalue assignment.

2010 Mathematics Subject Classification. 34A08, 93B52, 93D15.

1. Introduction

As numerous studies show, fractional-order models can depict the physical plant better than the classical integer-order ones. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes like viscoelastic systems, chaotic synchronization, electromagnetic systems, electrical circuits theory, fractances, mechatronics systems, signal processing, and chemical mixing [1, 2, 8, 13, 16, 17]. Descriptor fractional systems describe a large class of systems, which are not only theoretically interesting, but they also have a great importance in practice. It is fair to say that descriptor fractional models

Received: 23 May 2019 ; Accepted: 27 January 2020.
∗ corresponding author.

give a more complete class of dynamical models than the conventional state-space systems.
In this article, the descriptor fractional discrete-time linear system is converted to the standard descriptor model with unlimited delay in state by the fractional derivative definition whose control is impossible. Decreasing the sequence of coefficients of delays and defining a new state vector may help us obtain a standard descriptor discrete-time linear system, but with large matrices. We may find several methods for stability of just positive standard and descriptor systems. Some of them were derived by the use of Drazin inverse [5] and Shuffle algorithm [9] in which some initial conditions like having full row rank matrices in every performed algorithm and finding index of Shuffle and Drazin in pointed papers are necessary.
The free parameters which do not affect the time-optimality can be adjusted to obtain better performance responses. We compare some methods via parametric forward state matrix, parametric forward and propositional state matrix, partial eigenvalue assignment by parametric forward and propositional state feedback, and partial eigenvalue assignment using orthogonality relations to stabilize the standard descriptor discrete-time linear systems. To gain forward and propositional state feedback matrices, two standard linear systems need to exist. Assigned nonzero arbitrary eigenvalue to the first standard system and assigning inverted the desired eigenvalue for the standard descriptor system to the second one, desired eigenvalues are assigned to the standard descriptor linear system. Using forward and propositional state feedback matrices may not need a full rank open-loop matrix in the standard descriptor systems when forward state feedback matrix is used. Reassigning undesired eigenvalues of open-loop spectrums in new systems with smaller sizes of matrices such that other eigenvalues unchanged is well-done by the use of partial eigenvalue assignment. Likewise, we do not deal with some sufficient conditions like no having eigenvalues near zero and being distinct eigenvalues using orthogonality relations [15].

2. Statement of the problem

Consider the descriptor fractional discrete-time linear system described by

$$E\Delta^{\alpha}x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}^+ = \{0, 1, 2, \cdots\}, \tag{2.1}$$

where α is fractional-order difference of state vector and $0 < \alpha < 1$, $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are state and input vectors, the matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$ are known constant matrices with $rank(E) < n, rank(B) = m$ which $1 \leq m \leq n$, and x_0 is a nonzero definite vector.

Definition 2.1. The descriptor fractional system (2.1) is called asymptotically stable if and only if $\lim_{k\to\infty} x_k = 0$ for any $x_0 \in \mathbb{R}^n$.

Definition 2.2. The Grunwald-Letnikov fractional derivative with fractional-order α is defined by,

$$_{GL}D^{\alpha}_{a,t}x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{i=0}^{[\frac{t-a}{h}]} (-1)^i \binom{\alpha}{i} x(t - ih), \tag{2.2}$$

where [.] means the integer part, $\alpha \in \mathbb{R}^+$, and

$$\binom{\alpha}{i} = \begin{cases} 1 & \text{for} \quad \text{i}= 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} & \text{for} \quad \text{i}= 1,\ 2,\ ... \end{cases} \tag{2.3}$$

Definition 2.3. The fractional difference of the order $\alpha \in \mathbb{R}^+$ with zero initial point in discrete-time systems is defined by [4]

$$\Delta^\alpha x(t_k) = \Delta^\alpha x_k = \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} x_{k-i}. \tag{2.4}$$

Theorem 2.4. *For $n \in \mathbb{N}, 0 < \alpha < 1$ we have* [12]

$$D^{n+\alpha} x(t) = D^n D^\alpha x(t). \tag{2.5}$$

We can easily assume $0 < \alpha < 1$ by this theorem.

Using the definition 2.3, we may write the equations (2.1) in the form

$$Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{k} c_i E x_{k-i} + Bu_k, \tag{2.6}$$

which

$$c_i = c_i(\alpha) = (-1)^i \binom{\alpha}{i+1}, \quad i = 1, 2, \cdots, k \tag{2.7}$$

and $A_\alpha = A + \alpha E$. Also $\binom{\alpha}{i+1}$ is defined by (2.3).
Note that the equation (2.6) describes a linear discrete-time descriptor system with unlimited delay in state. To make the control of this system possible, we should change it to standard descriptor linear system. Although the converted standard descriptor linear systems may have large matrices, but stability of them is proved [3].

3. Stability of descriptor fractional discrete-time linear systems

The coefficients c_i in (2.7) strongly decrease for increasing i when $0 < \alpha < 1$. Assuming $c_i = 0$ for $i > L$ the system (2.6) is changed to a descriptor linear system with L delays [4]

$$Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{L} c_i E x_{k-i} + Bu_k. \tag{3.1}$$

Now by defining the new state vector $X_k \in \mathbb{R}^{\bar{n}}$

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-L} \end{bmatrix}, \tag{3.2}$$

which $\bar{n} = n(L+1)$, we may convert the time delay descriptor system (3.1) to a standard descriptor system

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}U_k, \tag{3.3}$$

where

$$\bar{A} = \begin{bmatrix} A_\alpha & c_1E & c_2E & \cdots & c_LE \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}, \tag{3.4}$$

where $U_k = u_k \in \mathbb{R}^m$ is the input vector, $\bar{E}, \bar{A} \in \mathbb{R}^{\bar{n}\times\bar{n}}, \bar{B} \in \mathbb{R}^{\bar{n}\times m}$, and $rank(\bar{E}) < \bar{n}$.

Definition 3.1. The descriptor fractional system (2.1) is called practically stable if and only if the time delay system (3.1) or equivalently the system (3.3) is asymptotically stable [4].

3.1. **Eigenvalue assignment with forward state feedback law.** Consider system (3.3) by forward state feedback law

$$U_k = F'_f X_{k+1}. \tag{3.5}$$

The aim is to design the forward state feedback (3.5) which produces a closed-loop system of (3.3) with the satisfactory response by assigning desirable eigenvalues $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$.
To establish the proposed results, consider the following assumptions

$$I)rank[\bar{E}|\bar{B}] = \bar{n}, \quad II)rank[\bar{A}] = \bar{n}, \quad III)rank[\bar{B}] = m. \tag{3.6}$$

If assumption (I) holds, then there exists F'_f such that [3]

$$rank[\bar{E} - \bar{B}F'_f] = \bar{n}. \tag{3.7}$$

Substituting feedback (3.5) into the equation (3.3), one can write

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}F'_fX_{k+1} \quad \Rightarrow \quad (\bar{E} - \bar{B}F'_f)X_{k+1} = \bar{A}X_k$$

. therefore

$$X_{k+1} = (\bar{E} - \bar{B}F'_f)^{-1}\bar{A}X_k, \tag{3.8}$$

is a standard linear system which is well-defined by (3.7).

Theorem 3.2. *The standard descriptor discrete-time linear system (3.8) is asymptotically stable if and only if eigenvalues of* $(\bar{E} - \bar{B}F'_f)^{-1}\bar{A}$ *lie in the unit disk* [7].

Lemma 3.3. *Consider a matrix* $M \in \mathbb{R}^{n\times n}$ *with* $rank(M) = n$ *and the eigenvalues equal to* $\lambda_1, \lambda_2, \cdots, \lambda_n$*. Then, the eigenvalues of* M^{-1} *are* $\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1}$ [10, 11].

Theorem 3.4. *Define the matrices* N, M *as*

$$N = \bar{A}^{-1}\bar{E}, \quad M = -\bar{A}^{-1}\bar{B}, \tag{3.9}$$

such that the pair of (M, N) *be controllable. Also let* F'_f *be state feedback matrix, such that* $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$ *is the set of eigenvalues of the closed-loop system*

$$\begin{cases} z_{k+1} = Nz_k + Mw_k, \\ \quad w_k = F'_f z_k, \end{cases} \tag{3.10}$$

where $\lambda_i \in \mathbb{C}$ *and* $\lambda_i \neq 0$, $i = 1, 2, \cdots, \bar{n}$, *are arbitrarily assigned. Then for this gained* F'_f, *the desired spectrum* $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ *is the eigenvalues of the controlled system (3.3) with forward feedback (3.5) and also, the condition (3.7) holds.*

Proof. Considering that (M, N) is controlled, then one can find a state feedback matrix F'_f such that the controlled system (3.10) given by

$$z_{k+1} = (N + MF'_f)z_k \tag{3.11}$$

has eigenvalues equal to $\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}$. Now by (3.9) note that:

$$N + MF'_f = \bar{A}^{-1}(\bar{E} - \bar{B}F'_f) \tag{3.12}$$

so

$$(N + MF'_f)^{-1} = (\bar{E} - \bar{B}F'_f)^{-1}\bar{A}. \tag{3.13}$$

The closed-loop matrices of systems (3.10) and (3.3) via feedback law (3.5) are inverse of each other by (3.8), (3.11), (3.12), and (3.13). Therefore (3.7) holds and the set of eigenvalue of closed-loop system (3.3) with feedback law (3.5) is equal to $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ by lemma 3.3. □

Remark 3.5. By the definitions (3.4), the matrices $\bar{E}$ and $\bar{A}$ in system (3.3) are singular because $rank(E) < n$ is the necessary condition in the descriptor fractional discrete-time linear system (2.1) and the matrix including last n columns and first n rows of $\bar{A}$, i.e. $[c_L E]$, is not full rank. So the method in subsection 3.1 can not help us stabilize the system (2.1).

The method based on using forward state feedback when $\bar{A}$ is singular, i.e. the condition (II) in (3.6) is not satisfied is not applicable. This problem is removed in next subsection.

3.2. **Eigenvalue assignment with forward and propositional state feedback law.** When we use the forward and propositional state feedback instead of the forward state feedback, we do not need the condition of being full rank of matrix $\bar{A}$ in system (3.3). It is excellent for using forward and propositional state feedback.

Consider system (3.3) by forward and propositional state feedback law

$$U_k = F_f X_{k+1} + F_p X_k. \tag{3.14}$$

The aim is to design the forward and propositional state feedback (3.14) which produces a closed-loop system of (3.3) with the satisfactory response by assigning desirable eigenvalues $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$.
To establish the proposed results, consider the following assumptions

$$I) rank[\bar{E}|\bar{B}] = \bar{n}, \quad II) rank[\bar{B}] = m.$$

If assumption (I) holds, then there exists F_f such that [3]

$$rank[\bar{E} - \bar{B}F_f] = \bar{n}. \tag{3.15}$$

Substituting feedback (3.14) into the equation (3.3), one can write

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}F_f X_{k+1} + \bar{B}F_p X_k \quad \Rightarrow \quad (\bar{E} - \bar{B}F_f)X_{k+1} = (\bar{A} + \bar{B}F_p)X_k$$

, therefore

$$X_{k+1} = (\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p)X_k, \tag{3.16}$$

is a standard linear system which is well-defined by (3.15).

Theorem 3.6. *The standard descriptor discrete-time linear system (3.16) is asymptotically stable if and only if eigenvalues of* $(\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p)$ *lie in the unit disk* [7].

Obtaining propositional and forward state feedbacks F_p and F_f, first, the propositional feedback matrix F_p is obtained by assigning non-zero arbitrary eigenvalues to the closed-loop of system

$$\begin{cases} q_{k+1} = \bar{A}q_k + \bar{B}v_k, \\ v_k = F_p q_k. \end{cases} \tag{3.17}$$

Then, we obtain the forward state feedback matrix F_f, by assigning $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$ to the system (3.19), where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$ are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$, and $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ is the set of desired eigenvalues for the standard descriptor system (3.3) via state feedback (3.14).

Theorem 3.7. *Define the matrices* N, M *as*

$$N = (\bar{A} + \bar{B}F_p)^{-1}\bar{E}, \quad M = -(\bar{A} + \bar{B}F_p)^{-1}\bar{B}, \tag{3.18}$$

such that the pair of (M, N) *be controllable. Also let* F_f *be state feedback matrix, such that* $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$ *is the set of eigenvalues of the closed-loop system*

$$\begin{cases} z_{k+1} = Nz_k + Mw_k, \\ w_k = F_f z_k, \end{cases} \tag{3.19}$$

where $\lambda_i \in \mathbb{C}$ *and* $\lambda_i \neq 0$, $i = 1, 2, \cdots, \bar{n}$, *are arbitrarily assigned. Then for this gained* F_f, *the desired spectrum* $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ *includes the eigenvalues of the controlled system (3.3) with forward and propositional feedback (3.14) and also, the condition (3.15) holds.*

Proof. Considering that (M, N) is controlled, then one can find a state feedback matrix F_f such that the controlled system (3.19) given by

$$z_{k+1} = (N + MF_f)z_k, \tag{3.20}$$

has eigenvalues equal to $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$. Now by (3.18) note that:

$$N + MF_f = (\bar{A} + \bar{B}F_p)^{-1}(\bar{E} - \bar{B}F_f), \tag{3.21}$$

so

$$(N + MF_f)^{-1} = (\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p). \tag{3.22}$$

The closed-loop matrices of systems (3.20) and (3.3) via feedback law (3.14) are inverse of each other by (3.8), (3.20), (3.21), and (3.22). Therefore (3.15) holds and the set of eigenvalues of closed-loop system (3.3) with feedback law (3.14) is equal to $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ by lemma 3.3. □

3.3. **Eigenvalue assignment by nonlinear parametric similarity transformation.** In this subsection, we use the method of parametric similarity transformation to compute the forward state feedback matrices F'_f and F_f in subsections 3.1 and 3.2. Our assignment procedure is composed of two stages. First, we obtain a primary state feedback matrix Φ which assigns all the eigenvalues of closed-loop system to zero. Then, we produce a state feedback matrix F which assigns all the closed-loop system eigenvalues in desired region. Consider controllable standard system

$$\begin{cases} x_{k+1} = A_1 x_k + B_1 u_k, \\ u_k = F x_k \end{cases} \tag{3.23}$$

and the state transformation

$$x_k = T\tilde{x}_k, \tag{3.24}$$

where T can be obtained by elementary similarity operations as described in [10, 11]. Substituting (3.24) into (3.23) yields

$$\tilde{x}_{k+1} = T^{-1}A_1T\tilde{x}_k + T^{-1}B_1u_k.$$

It is noted that the transformation matrix T is invertible. In this way,

$$\tilde{A}_1 = T^{-1}A_1T, \quad \tilde{B}_1 = T^{-1}B_1, \tag{3.25}$$

are in a compact canonical form known as vector companion form:

$$\tilde{A}_1 = \begin{bmatrix} & G_0 & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} S_0 \\ 0_{n-m,m} \end{bmatrix}. \tag{3.26}$$

Here G_0 is a $m \times n$ matrix and S_0 is a $m \times m$ upper triangular matrix.

The state feedback matrix which assigns all the eigenvalues to zero, for the transformed pair $(\tilde{B}_1, \tilde{A}_1)$, is then chosen as

$$\tilde{\Phi} = -S_0^{-1}G_0, \tag{3.27}$$

which results in the primary state feedback matrix for the pair (B_1, A_1) defined as

$$\Phi = \tilde{\Phi}T^{-1}. \tag{3.28}$$

The transformed closed-loop matrix

$$\tilde{\Gamma}_0 = \tilde{A}_1 + \tilde{B}_1 \tilde{\Phi}, \tag{3.29}$$

assumes a compact Jordan form with zero eigenvalues

$$\tilde{\Gamma}_0 = \begin{bmatrix} & 0_{m,n} & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}. \tag{3.30}$$

If $\tilde{A}_\lambda$ is any matrix in vector companion form, i.e.

$$\tilde{A}_\lambda = \begin{bmatrix} & G_\lambda & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}, \tag{3.31}$$

with the eigenvalue spectrum $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ containing a set of self conjugate eigenvalues, then as shown in [10, 11]

$$\tilde{F} = S_0^{-1}(-G_0 + G_\lambda), \tag{3.32}$$

is the feedback matrix which assigns the eigenvalue spectrum to the closed-loop matrix $\tilde{\Gamma} = \tilde{A}_1 + \tilde{B}_1 \tilde{F}$, and F may then be obtained by

$$F = \tilde{F} T^{-1}. \tag{3.33}$$

Note that G_λ is an $m \times n$ parametric matrix in the form:

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix}. \tag{3.34}$$

To obtain the nonlinear system of equations relating the parameters of G_λ, the characteristic polynomial of $\tilde{A}_\lambda$ must be obtained. Thus, let

$$det(\tilde{A}_\lambda - \lambda I) = P_n(\lambda) = 0, \tag{3.35}$$

where

$$P_n(\lambda) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n), \tag{3.36}$$

is the characteristic polynomial of the closed-loop system. Since it is required that the zeros of this polynomial lie in the set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, it is clear that

$$P_n(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n). \tag{3.37}$$

Remark 3.8. Obtaining the roots of the characteristic polynomial we may consider following equations

$$\begin{cases} a_1 = -\sum_{i=1}^n \lambda_i, \\ a_2 = \sum_{i,j=1,i<j}^n \lambda_i\lambda_j, \\ \vdots \\ a_k = (-1)^k \sum_{i_1,i_2,\dots i_k=1, i_1<i_2<\dots<i_k}^n \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \\ \vdots \\ a_n = (-1)^n \prod_i^n \lambda_i. \end{cases} \tag{3.38}$$

Now by equating the coefficients of the characteristic polynomial as (3.38), the following nonlinear system of equations is obtained:

$$\begin{cases} f_1(g_{11},\cdots,g_{mn}) = a_1, \\ f_2(g_{11},\cdots,g_{mn}) = a_2, \\ \vdots \\ f_n(g_{11},\cdots,g_{mn}) = a_n, \end{cases} \tag{3.39}$$

where $g_{ij}, i = 1,\cdots,m, j = 1,\cdots,n$, are the elements of G_λ. In this way, a nonlinear system of n equations with $n \times m$ unknowns is obtained. By choosing $n \times (m-1)$ unknowns arbitrarily it is then possible to solve the system. It is clear that for one case the result may be linear parameters. So one may say gained linear parameters controller is the subset of nonlinear parameters controllers.

In general, solving nonlinear parametric equations (3.39) is difficult and makes some problems, especially for large n and m. In next section, two methods to reduce the dimension of large scale matrices to small scale ones are displayed.

4. Partial eigenvalue assignment

In this section, we present the existence and uniqueness theorem and a nonlinear parametric algorithm to find the state feedback matrices in standard systems. The aim of partial eigenvalue assignment is reassigning undesired eigenvalues of open-loop spectrums in new system with smaller sizes of matrices such that other eigenvalues unchanged. Therefore the stability in partial eigenvalue assignment for the standard descriptor system is kept by reassigning eigenvalues in the unit disk and unchanging the remain of eigenvalues in the standard system (3.19). Also we present some sufficient conditions to be exist in another algorithm are not necessary for parametric algorithm of eigenvalue assignment.

4.1. Existence and uniqueness.

Theorem 4.1. *(Eigenvector criterion of controllability). The standard system (3.23) or, equivalently, the matrix pair (B_1, A_1) is controllable with respect to the eigenvalue λ of A_1 if $y^H B_1 \neq 0$ for all $y \neq 0$ such that $y^H A_1 = \lambda y^H$* [6].

Definition 4.2. The standard system (3.23) or the matrix pair (B_1, A_1) is partially controllable with respect to the subset $\{\lambda_1,\cdots,\lambda_p\}$ of the spectrum of A_1 if it is controllable with respect to each of the eigenvalues $\lambda_j, j = 1,\cdots,p$.

Definition 4.3. The standard system (3.23) or the matrix pair (B_1, A_1) is completely controllable if it is controllable with respect to every eigenvalue of A_1.

Theorem 4.4. *(Existence and uniqueness for eigenvalue assignment problem). The eig-envalue assignment problem for the pair (B_1, A_1) is solvable for any arbitrary set $S = \{\mu_1,\cdots,\mu_p\}$ if and only if (B_1, A_1) is completely controllable. The solution is unique if and only if the system is a single-input system (that is, if B_1 is a vector). In the multi-input case, there are infinitely many solutions, whenever a solution exists* [6].

Theorem 4.5. *(Existence and uniqueness for partial eigenvalue assignment problem). Let* $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_p; \lambda_{p+1}, \cdots, \lambda_n\}$ *be the diagonal matrix containing the eigenvalues* $\lambda_1, \cdots, \lambda_n$ *of* $A_1 \in \mathbb{C}^{n\times n}$*. Assume that the sets* $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_n\}$ *are disjoint. Let the eigenvalues* $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ *to be changed to* $\{\mu_1, \mu_2, \cdots, \mu_p\}$ *and the remaining eigenvalues stay invariant. Then the partial eigenvalue assignment problem for the pair* (B_1, A_1) *is solvable for any choice of the closed-loop eigenvalues* $\{\mu_1, \mu_2, \cdots, \mu_p\}$ *if and only if the pair* (B_1, A_1) *is partially controllable with respect to the set* $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$*. The solution is unique if and only if the system is a completely controllable single-input system. In the multi-input case, and in the single-input case when the system is not completely controllable, there are infinitely many solutions, whenever a solution exists* [6].

4.2. **Eigenvalue assignment algorithm using orthogonality relations.** There exists an algorithm for partial eigenvalue assignment using orthogonality relations in [15] as follows

Inputs:

(I) $\{M_k, M_{k-1}, \cdots, M_0\}$ are $n \times n$ real non-symmetric constant matrices.

(II) b is an n-vector and $D = diag(\mu_1, \cdots, \mu_p)$ closed under complex conjugation.

Output:

The feedback vectors $\{f_i\}_{i=1}^k$ such that the spectrum of modified matrix polynomial

$$P(\lambda) = M_k\lambda^k + (M_{k-1} - bf_1^T)\lambda^{k-1} + (M_0 - bf_k^T)$$

, is $\{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_{kn}\}$, where $\{\lambda_{p+1}, \cdots, \lambda_{kn}\}$ are the last $kn-p$ eigenvalues of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda M_1 + M_0$.

Assumptions:

(I) M_k is nonsingular matrix.

(II) The sets $\{\mu_1, \cdots, \mu_p\}$, $\{\lambda_1, \cdots, \lambda_p\}$ are distinct and closed under complex conjugation, where $\{\lambda_1, \cdots, \lambda_{kn}\}$ are the eigenvalues of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda M_1 + M_0$.

(III) $\Lambda_1 = diag(\lambda_1, \cdots, \lambda_p)$

Step 1. Obtain the first p eigenvalues $\{\lambda_1, \cdots, \lambda_p\}$ of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda M_1 + M_0$ that need to be reassigned and the corresponding left eigenvectors $Y_1 = (y_1, y_2, \cdots, y_p)$.

Step 2. Compute the explicit expression for β

$$\beta_j = \frac{1}{b^T\bar{y}_j}\frac{\mu_j - \lambda_j}{\lambda_j}\prod_{i=1, i\neq j}^{p}\frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}, \quad j = 1, \cdots, p.$$

Step 3. Form

$$f_i = \sum_{j=1}^{i}[M_{k-i+j}^T\bar{Y}_1\Lambda_1^j]\beta^T, f_k = -M_0^T\bar{Y}_1\beta^T, i = 1, \cdots, k-1, \quad \beta^T \in \mathbb{C}^p.$$

By Step 2, it is clear that sufficient conditions for the existence of β, and consequently for a solution to the partial pole assignment problem to be exist are:

(1) No $\lambda_j, j = 1, \cdots, p$ vanishes,

(2) The $\{\lambda_i\}_{i=1}^p$ are distinct,

(3) The vector b must be not orthogonal to $\bar{y}_j, j = 1, \cdots, p$.

The above sufficient conditions are necessary to have an efficient algorithm. In next subsection, we propose a useful algorithm without any need for these conditions.

4.3. **Parametric algorithm of partial eigenvalue assignment.** Following algorithm presents a parametric method of partial eigenvalue assignment on the standard system (3.23).

Inputs:
(a) The $n \times n$ matrix A_1.
(b) The $n \times m$ control matrix B_1.
(c) The set $\{\mu_1, \mu_2, \cdots, \mu_p\}$, closed under complex conjugation.
(d) The self-conjugate subset $\{\lambda_1, \cdots, \lambda_p\}$ of the spectrum $\{\lambda_1, \cdots, \lambda_n\}$ of the matrix A_1 and the associated right eigenvector set $\{y_1, \cdots, y_p\}$.
Output:
The real feedback matrix F such that the spectrum of the closed-loop matrix A_1+B_1F is $\{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_n\}$.
Assumptions:
(a) The matrix pair (B_1, A_1) is partially controllable with respect to the eigenvalues $\{\lambda_1, \cdots, \lambda_p\}$.
(b) The sets $\{\lambda_1, \cdots, \lambda_p\}, \{\lambda_{p+1}, \cdots, \lambda_n\}$, and $\{\mu_1, \cdots, \mu_p\}$ are disjoint.
Step 1. Form

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_p), \quad Y_1 = (y_1, \cdots, y_p).$$

Step 2. Find feedback K such that $eig(\Lambda_1 + Y_1^H B_1 K) = \{\mu_1, \cdots, \mu_p\}$.
Step 2.1. Calculate S_0, G_0, Φ by transformation matrix T and elementary similarity operations on the pair of (B_1, A_1).
Step 2.2. Define the $m \times p$ parametric matrix G_λ in the form

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1p} \\ g_{21} & g_{22} & \cdots & g_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m1} & g_{m2} & \cdots & g_{mp} \end{bmatrix}.$$

Step 2.3. Solve the following nonlinear system of equations

$$\begin{cases} f_1(g_{11}, \cdots, g_{mp}) = a_1, \\ f_2(g_{11}, \cdots, g_{mp}) = a_2, \\ \vdots \\ f_p(g_{11}, \cdots, g_{mp}) = a_p, \end{cases}$$

which $a_k = (-1)^k \sum_{i_1, \cdots, i_k=1, i_1 < \cdots < i_k}^{p} \lambda_{i_1} \cdots \lambda_{i_k}, k = 1, \cdots, p$.

Step 2.4. Form $K = S_0^{-1}(-G_0 + G_\lambda)T^{-1}$.

Step 3. Form $F = KY_1^H$. Now we have $eig(A_1 + B_1 F) = \{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_n\}$.

In the next section of the paper an example is presented in order to compare the numerical results obtained by our methods.

5. Illustrative example

The following example is given to investigate all methods presented by this article. In case (a) until case (d), eigenvalue assignment by parametric forward state feedback, eigenvalue assignment by parametric forward and propositional state feedback, partial eigenvalue assignment using orthogonality relations, and partial eigenvalue assignment by parametric forward and propositional state feedback are focused.

Example 5.1. The stabilization of the descriptor fractional discrete-time linear system

$$E\Delta^{0.6}x_{k+1} = Ax_k + Bu_k,$$

where

$$E = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & -3 \\ 0 & -2 & 6 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & -4 \\ 3 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix},$$

is examined. The matrices of system (3.3) is obtained by

$$\bar{A} = \begin{bmatrix} 2.6 & 3 & 0.8 & 0.12 & 0 & 0.36 & -0.06 & 0 & -0.17 \\ -0.4 & -0.8 & -5.8 & 0.12 & 0.24 & -0.36 & -0.06 & -0.11 & 0.17 \\ 3 & -0.2 & -1.4 & 0 & -0.24 & 0.72 & 0 & 0.11 & -0.34 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Case (a). Consider the subsections 3.1 and 3.3. The eigenvalue assignment via parametric forward state feedback is not applicable. The pair (M, N) may not be defined because of singularity of the matrix $\bar{A}$. Here we have $rank(\bar{A}) = 8 < 9$.

Now consider the standard systems (3.17) and (3.19) by propositional and forward state feedbacks respectively. Only obtaining the forward feedback matrix F_f is displayed by the propositional state feedback matrix F_p as

$$F_p = \begin{bmatrix} -0.48 & 10.33 & -31.08 & 1.01 & -3.82 & 14.13 & 0.01 & 0.6 & -1.76 \\ -1.27 & -1.57 & 4.42 & -0.12 & 0.69 & -2.42 & 0 & -0.15 & 0.42 \end{bmatrix}.$$

by assigning all eigenvalues to 0.1.

The pair of (M, N) is obtained by

$$N = 10^8 \times$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.09 & 0.79 & -2.09 & 0.01 & -0.03 & 0.12 & 0 & 0 & -0.01 \\ -0.11 & -0.89 & 2.35 & -0.01 & 0.03 & -0.14 & 0 & 0 & 0.01 \\ -0.03 & -0.29 & 0.78 & 0 & 0.01 & -0.04 & 0 & 0 & 0 \end{bmatrix},$$

$$M = 10^7 \times \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.99 & 6.92 \\ -1.12 & -7.79 \\ -0.37 & -2.59 \end{bmatrix}.$$

Case (b). Consider the subsections 3.2 and 3.3. The eigenvalue assignment via parametric forward and propositional state feedback is solved by given pair (M, N) and F_p. We assign $\{\pm 2.5, \pm 2.5, \pm 2.5, \pm 2.5, 2.5\}$ in the system (3.19), to be assigned $\{\pm 0.4, \pm 0.4,$

$\pm 0.4, \pm 0.4, 0.4\}$ in system (3.3) and obtain feedback F_f.

Consider the matrix G_λ as

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} & g_{19} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} & g_{29} \end{bmatrix}.$$

Nonlinear parametric equations are as follow:

$$\begin{cases} g_{11} + g_{22} = 2.5 \\ g_{13} - g_{24} + g_{21}g_{12} - g_{11}g_{22} = 12.5 \\ g_{15} + g_{26} + g_{23}g_{12} + g_{21}g_{14} - g_{13}g_{22} - g_{24}g_{11} = -31.25 \\ g_{17} + g_{28} + g_{23}g_{14} + g_{21}g_{16} - g_{22}g_{15} - g_{24}g_{13} - g_{26}g_{11} + g_{25}g_{12} = 0 \\ g_{19} + g_{27}g_{12} + g_{23}g_{16} + g_{21}g_{18} - g_{22}g_{17} - g_{24}g_{15} - g_{26}g_{13} - g_{28}g_{11} \\ \qquad + g_{25}g_{14} = 0 \\ g_{29}g_{12} + g_{27}g_{14} - g_{22}g_{19} - g_{24}g_{17} - g_{26}g_{15} - g_{28}g_{13} + g_{25}g_{16} + g_{23}g_{18} \\ \qquad = -488.28 \\ g_{29}g_{14} + g_{27}g_{16} + g_{25}g_{18} - g_{24}g_{19} - g_{26}g_{17} - g_{28}g_{15} = 1.22 \times 10^3 \\ g_{29}g_{16} + g_{27}g_{18} - g_{26}g_{19} - g_{28}g_{17} = 1.52 \times 10^3 \\ g_{29}g_{18} - g_{28}g_{19} = -3.81 \times 10^3 \end{cases}$$

The forward state feedback matrix

$$F_f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} & f_{19} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} & f_{27} & f_{28} & f_{29} \end{bmatrix},$$

where

$$\begin{aligned} f_{11} &= 231.58g_{11} + 122193.04g_{12} + 0.47g_{13} - 0.17g_{14} + 1788.7g_{18} - 1621.07g_{21} \\ &\quad - 855351.79g_{22} - 3.3g_{23} + 1.24g_{24} - 12520.93g_{28} - 0.37 \times 10^9, \\ f_{12} &= 209.36g_{11} + 121396.82g_{12} + 0.49g_{13} - 0.17g_{14} - 1192.46g_{16} \\ &\quad + 1073.34g_{18} - 1465.56g_{21} - 849778.26g_{22} - 3.44g_{23} + 1.2g_{24} \\ &\quad + 8347.28g_{26} - 7513.41g_{28} - 0.39 \times 10^9, \\ f_{13} &= -10.54g_{11} - 38341.99g_{12} - 0.21g_{13} + 0.04g_{14} + 3577.4g_{16} \\ &\quad + 1549.84g_{18} + 73.81g_{21} + 268394.12g_{22} + 1.51g_{23} - 0.3g_{24}, \\ &\quad - 25041.86g_{26} - 10848.91g_{28} + 0.18 \times 10^9, \\ f_{14} &= -3.26g_{12} + 0.02g_{24} + 223554.29, \end{aligned}$$

$$
\begin{aligned}
f_{15} &= -18.31g_{11} - 9777.49g_{12} - 0.03g_{13} - 1192.45g_{14} + 3577.4g_{16} \\
&\quad + 1549.84g_{18} + 128.22g_{21} + 68442.51g_{22} + 0.26g_{23} + 8347.2g_{24} \\
&\quad - 0.85g_{26} + 1001.67g_{28} + 0.3 \times 10^8, \\
f_{16} &= 54.95g_{11} + 29323.68g_{12} + 0.11g_{13} + 3577.36g_{14} - 0.36g_{16} + 429.28g_{18} \\
&\quad - 384.67g_{21} - 205265.88g_{22} - 0.79g_{23} - 25041.54g_{24} + 2.55g_{26} \\
&\quad - 3005.02g_{28} - 0.91 \times 10^8, \\
f_{17} &= 0.02g_{22} - 23.98, \\
f_{18} &= 8.64g_{11} + 3369.67g_{12} + 0.01g_{13} + 0.11g_{14} + 66.77g_{18} - 60.51g_{21} \\
&\quad - 23587.7g_{22} - 0.12g_{23} - 0.8g_{24} - 467.44g_{28} - 0.14 \times 10^8, \\
f_{19} &= -25.93g_{11} - 10109.05g_{12} - 0.05g_{13} - 0.34g_{14} - 200.33g_{18} + 181.55g_{21} \\
&\quad + 70763.45g_{22} + 0.37g_{23} + 2.42g_{24} + 1402.34g_{28} + 0.42 \times 10^8, \\
f_{21} &= 231.58g_{21} + 122193.04g_{22} + 0.47g_{23} - 0.17g_{24} + 1788.7g_{28} + 0.53 \times 10^{10}, \\
f_{22} &= 209.36g_{21} + 121396.82g_{22} + 0.49g_{23} - 0.17g_{24} - 1192.46g_{26} + 1073.34g_{28} \\
&\quad + 0.56 \times 10^{10}, \\
f_{23} &= -10.54g_{21} - 38341.99g_{22} - 0.21g_{23} + 0.04g_{24} + 3577.4g_{26} + 1549.84g_{28} \\
&\quad - 0.25 \times 10^8, \\
f_{24} &= -3.26g_{22} - 31936.34, \\
f_{25} &= -18.31g_{21} - 9777.49g_{22} - 0.03g_{23} - 1192.45g_{24} + 0.12g_{26} - 143.09g_{28} \\
&\quad - 0.43 \times 10^7, \\
f_{26} &= 54.95g_{21} + 29323.68g_{22} + 0.1g_{23} + 3577.36g_{24} - 0.36g_{26} + 429.28g_{28} \\
&\quad + 0.13 \times 10^8, \\
f_{27} &= -0.004g_{22} + 3.42, \\
f_{28} &= 8.64g_{21} + 3369.67g_{22} + 0.01g_{23} + 0.11g_{24} + 66.77g_{28} + 0.2 \times 10^7, \\
f_{29} &= -25.93g_{21}10109.05g_{22} - 0.05g_{23} - 0.34g_{24} - 200.33g_{28} - 0.6 \times 10^7,
\end{aligned}
$$

is obtained by

$$
F_f = 10^8 \times \begin{bmatrix} -3.84 & -4.02 & 1.81 & 0 & 0.31 & -0.93 & 0 & -0.14 & 0.42 \\ 0.54 & 0.57 & -0.25 & 0 & -0.04 & 0.13 & 0 & 0.02 & -0.06 \end{bmatrix},
$$

and elements of matrix G_λ are as
$g_{11} = -4.31, g_{12} = -9.18, g_{13} = -1.7, g_{14} = -4.03, g_{15} = 13.21, g_{16} = -14.9,$
$g_{17} = 44.13, g_{18} = -111.46, g_{19} = 65.42, g_{21} = 1.69, g_{22} = 6.81, g_{23} = 4.13,$
$g_{24} = -0.32, g_{25} = -9.04, g_{26} = -9.9, g_{27} = -27.69, g_{28} = 48, g_{29} = 6.04.$

As it is shown in Fig 1 the variables $x_i(t), i = 1, 2, 3$ converge to zero and the eigenvalues of the closed-loop matrix of the standard descriptor system (3.3) are in the unit disk.

Case (c). Consider the method in subsection 4.2. Because $eig(N) = \{64.37 \pm 85.18i,$

$-81.93, 15.68 \pm 11.72i, 13.72, -2.53, 0.61, 0\}$, we reassign $p = 2$ eigenvalues $\{10, 10\}$ instead of $\{0, 0.61\}$ while leaving the other eigenvalues unchanged. The first sufficient conditions for the solution of the partial eigenvalue assignment using orthogonality relations is not existed. $\{\lambda_1, \lambda_2\}$ should not vanish while $\{\lambda_1, \lambda_2\} = \{0, 0.61\}$. Therefore this method can not be used in this example, too.

Case (d). Consider the method in subsection 4.3. Similar to case (c), because $eig(N) = \{64.37 \pm 85.18i, -81.93, 15.68 \pm 11.72i, 13.72, -2.53, 0.61, 0\}$, we need to reassign $\{10, 10\}$ instead of $\{0, 0.61\}$ while leaving the other eigenvalues unchanged (Fig 2). So by using partial eigenvalue assignment on new pair $(Y_1^H M, \Lambda_1)$

$$Y_1^H M = \begin{bmatrix} 0 & 0.2 \\ -0.04 & -0.12 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0.61 & 0 \\ 0 & 0 \end{bmatrix}.$$

and considering

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

nonlinear parametric equations are as follow:

$$\begin{cases} g_{11} + g_{22} = 20 \\ -g_{12}g_{21} + g_{11}g_{22} = 100. \end{cases}$$

The matrix feedback

$$K_f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

where

$$\begin{aligned} f_{11} &= -13.95g_{11} + 4.65g_{12} + 8.64, \\ f_{12} &= -22.05g_{11} - 0.52g_{12}, \\ f_{21} &= -13.95g_{21} + 4.65g_{22} - 2.88, \\ f_{22} &= -22.05g_{21} - 0.52g_{22}, \end{aligned}$$

is obtained by

$$K_f = \begin{bmatrix} -130.94 & -220.54 \\ 43.65 & -5.21 \end{bmatrix}.$$

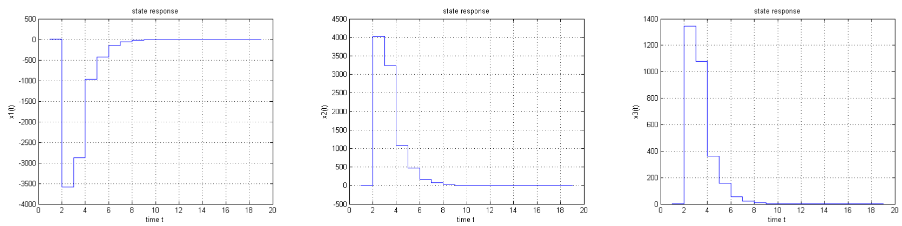
and the elements of matrix G_λ are as

$$g_{11} = 10, g_{12} = 0, g_{21} = 0, g_{22} = 10.$$

Also F_f is obtained as

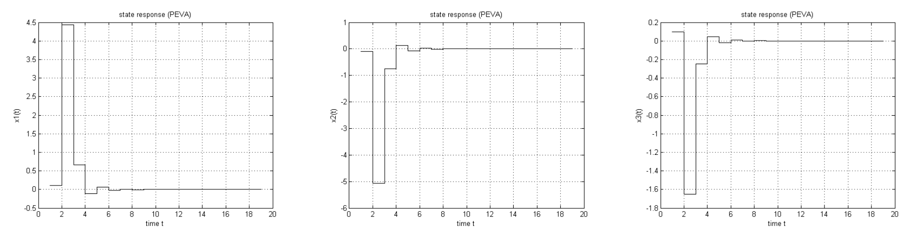
$$F_f = \begin{bmatrix} 28.84 & -79.99 & 308.52 & -10.62 & 31.77 & -123.42 & 0.28 & -4.16 & 13.24 \\ 4.73 & 14.64 & -30.15 & 1.45 & -4.79 & 18.26 & -0.03 & 0.88 & -2.76 \end{bmatrix}.$$

The eigenvalues of the closed-loop matrix of the standard system (3.19) and the standard descriptor system (3.3) are $\{64.37 \pm 85.18i, -81.93, 15.68 \pm 11.72i, 13.72, -2.53, 10, 10\}, \{-0.395, 0.04 \pm 0.03i, -0.012, 0.005 \pm 0.007i, 0.072, 0.1, 0.1\}$ respectively. The figures show that the input variables $x_i(t), i = 1, 2, 3$ in case (d) (Fig 2) better converged to zero from case (b) (Fig 1).



(a) (b) (c)

Figure 1. $x_i(t)$ in case (b)



(a) (b) (c)

Figure 2. $x_i(t)$ in case (c)

6. Conclusions

Some methods for stabilization and control of descriptor fractional discrete-time linear systems are compared. Assigning desired eigenvalues in unit disk to the converted standard descriptor discrete-time linear system is done by eigenvalue assignment with parametric forward state feedback. This method needs an impossible sufficient condition for some examples where it is possible by eigenvalue assignment with parametric forward and propositional state feedback. Solving nonlinear equations may make error for large matrices. Decreasing dimensions of matrices and the number of nonlinear parametric equations partial eigenvalue assignment may be used. The partial eigenvalue assignment algorithm using orthogonality relations is not doable for reassigning indistinct and vanished eigenvalues. But in partial eigenvalue assignment, we can reassign undesired indistinct and even zero eigenvalues while leaving the rest of the spectrum invariant. Also the eigenvalues of closed-loop matrix in last method lie in desired region and convergence to zero is better for vectors $x_i(t), i = 1, \cdots, n$. The results presented in this article are also applicable in stabilization of delayed, two-dimensional, and positive fractional systems. The subject of minimum norm of nonlinear parametric feedback matrices is remarkable, too. An extension of these considerations for continuous-time descriptor fractional linear systems is still an open problem.

References

[1] R. Bagley and R. Calico, *Fractional-order state equations for the control of viscoelastically damped structures*, J. Guidance, Control Dynamics, *14* (1991), 304–311.

[2] M. Boutayeb, M. Darouach, and H. Rafaralahy, *Generalized statespace observers for chaotic synchronization and secure communication*, IEEE Trans. Circ. Syst. I: Fund. Theory and Appli., *49* (2002), 345–349.

[3] A. Bunse-Gerstner, N. Nichols, and V. Mehrmaqnn, *Regularization of descriptor systems by derivative and proportional state feedback*, SIAM J. Matrix Anal, 1992.

[4] M. Buslowicz and A. Ruszewski, *Necessary and sufficient conditions for stability of fractional discrete-time linear state-space systems*, Bull. Pol. Acad. Sci. Tech. Sci., *61* (2013), 779–786.

[5] S. L. Campbell, C. D. Meyer, and N. J. Rose, *Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients*, SIAM Journal on Applied Mathematics, *31* (1976), 411–425.

[6] B. N. Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Academic Press, New York, 2002.

[7] A. Halanay and V. Rasvan, *Stability and stable oscillations in discrete-time systems*, Gordon and Breach Science Publisher, Australia, 21–23, 2000.

[8] O. Heaviside, *Electromagnetic theory*, New York, Chelsea Publishing Company, 3rd edO, 1971.

[9] T. Kaczorek, *Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm*, Archives of Control Sciences, *21* (2011), 287–298.

[10] S. M. Karbassi and D. J. Bell, *Parametric time-optimal control of linear discrete-time systems by state feedback-part 1: Regular kronecker invariants*, International journal of control, *57* (1993a), 817–830.

[11] S. M. Karbassi and D. J. Bell, *Parametric time-optimal control of linear discrete-time systems by state feedback-part 2: Irregular kronecker invariants*, International journal of control, *57* (1993b), 831–883.

[12] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[13] M. Nakagava and K. Sorimachi, *Basic characteristics of a fractance device,* IEICE Trans. Fundamentals E75-A, (1992), 1814–1818.

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[15] M. A Ramadan and E. A. El-Sayed, *Partial eigenvalue assignment problem of high order control systems using orthogonality relations*, Computers and Mathematics with Applications, *59* (2010), 1918–1928.

[16] H. Sun, A. Abdelwahad, and B. Onaral, *Linear approximation of transfer function with a pole of fractional-order*, IEEE Trans. Aut. Contr., *29* (1984), 441–444.

[17] Ch. Ch. Tseng, *Design of FIR and IIR fractional-order Simpson digital integrators*, Signal Processing, *87* (2007), 1045–1057.