Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 2, 2021, pp. 393-398
DOI:10.22034/cmde.2019.32914.1533

Solving some stochastic differential equation using Dirichlet distributions

Hawre Hadad
Department of Statistics, Science and Research branch,
Islamic Azad University, Tehran, Iran.
E-mail: hawre.hadad@srbiau.ac.ir

Hazhir Homei*
Department of Statistics, Faculty of Mathematical Science,
University of Tabriz, P.O.Box 51666–16471, Tabriz, Iran.
E-mail: homei@tabrizu.ac.ir

Mohammad Hassan Behzadi
Department of Statistics, Science and Research branch,
Islamic Azad University, Tehran, Iran.
E-mail: behzadi@srbiau.ac.ir

Rahman Farnoosh
School of Mathematics,
Iran University of Science and Technology, Tehran, Iran.
E-mail: rfarnoosh@iust.ac.ir

Abstract Stochastic linear combinations of some random vectors are studied where the distribution of the random vectors and the joint distribution of their coefficients have Dirichlet distributions. A method is provided for calculating the distribution of these combinations which has been studied before. Our main result is the same as but from a different point of view.

Keywords. Stochastic Linear Combination, Dependent Components, Lifetime.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Identifying the distribution of the randomly weighted averages of various distributions are in the interest of researchers. The main motivation for writing this note comes from [1].

For given random variables $X_1, \cdots, X_n$, the distribution of the stochastic linear combination $Z = \sum_{i=1}^{n} W_i X_i$ is used for the problems in lifetime, stochastic matrices, and neural networks. Also, other applications are in sociology and biology.

Let X_i $(1 \leq i \leq n)$ be the lifetime measured in a lab and $0 \leq W_i \leq 1$ be the random effect of the environment on it; so $W_i X_i \leq X_i$ and thus $\sum_{i=1}^{n} W_i X_i$ is the average lifetime in the environment (see [4]). Recently, several authors have focused on computing the lifetime of systems in the real conditions. Indeed, the randomly

Received: 28 April 2019 ; Accepted: 27 January 2020.
* corresponding.

linear combination of random vectors have been used in different problems including traditional portfolio selection models, relationship between attitudes and behavior, the number of cancer cells in tumor biology, stream flow in hydrology, branching processes, infinite particle systems and probabilistic algorithms, vehicle speed, and lifetime (cf. [1], and the references therein). Therefore, numerous researchers payed their attention to find the distribution of lifetime. By accepting all the assumptions of [1], we want to obtain the result [1] in a different approach.

The inner product of two random vectors was introduced in [2, 3] and the exact distribution of this product was investigated for some random vectors with Beta and Dirichlet distributions. In this paper, a new generalization for the inner product of two random vectors is introduced. For a random vector $\mathbf{W}' = \langle W_1, \cdots, W_n \rangle$ and a vector $\underline{\mathbf{X}} = \langle \mathbf{X}_1, \cdots, \mathbf{X}_n \rangle$ of random vectors (each $\mathbf{X}_i$ being k-dimensional) the inner product of $\mathbf{X}$ and $\mathbf{W}$ is essentially the linear transformation of $\mathbf{W}$ under the $k \times n$ matrix $\underline{\mathbf{X}}$ which is $\mathbf{Z} = \sum_{i=1}^{n} W_i \mathbf{X}_i$. We assume that $\mathbf{W}$ is independent of $\mathbf{X}_1, \cdots, \mathbf{X}_n$ and all of which are with Dirichlet distribution. To identify the distribution of random linear combination, $\mathbf{Z} = \sum_{i=1}^{n} W_i \mathbf{X}_i$, moment method is used (see e.g. [1, 3, 4, 5, 6, 8, 10] and the references therein).

In this paper, a way is introduced to identify the distribution of randomly linear combinations (of Dirichlet distributions) which has been obtained by solving some specific differential equations in [1].

2. The Method

The method for identifying the distribution of the randomly linear combination is given in the sequel, which has been studied before in [10].

Theorem 2.1. *Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be independent k-variate random vectors with $Dirichlet(\alpha^{(1)}), Dirichlet(\alpha^{(2)}), \cdots, Dirichlet(\alpha^{(n)})$ distributions corspondingly, for some k-dimensional vectors $\alpha^{(j)} = \left\langle \alpha_1^{(j)}, \cdots \alpha_k^{(j)} \right\rangle, (j = 1, \cdots, n)$. Let also the random vector $\mathbf{W} = \langle W_1, \cdots, W_n \rangle$ be independent from $\mathbf{X}_1, \cdots, \mathbf{X}_n$ all of which are with the Dirichlet distribution*

$$Dirichlet(\sum_{j=1}^{k} \alpha_j^{(1)}, \sum_{j=1}^{k} \alpha_j^{(2)}, \cdots, \sum_{j=1}^{k} \alpha_j^{(n)}).$$

Then the distribution of randomly linear combination $\mathbf{Z} = \sum_{i=1}^{n} W_i \mathbf{X}_i$ is as follows,

$$Dirichlet(\sum_{i=1}^{n} \alpha_1^{(i)}, \sum_{i=1}^{n} \alpha_2^{(i)}, \cdots, \sum_{i=1}^{n} \alpha_k^{(i)}).$$

Proof. We find the general moments $(s_1, s_2, \cdots, s_k)$ of Z as fallow:

$$E(\prod_{j=1}^{k} (\sum_{i=1}^{n} W_j X_{ij})^{s_j}) = E(\prod_{j=1}^{k} (\sum_{h_j} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}} \prod_{i=1}^{n} (W_j X_{ij})^{h_{ij}})) \quad (2.1)$$

where $\sum_{h_j}$ denotes summation over all nonnegative integers $h_j = (h_{1j}, h_{2j}, \cdots, h_{nj})$ subject to

$$\sum_{i=1}^{n} h_{ij} = s_j, \quad (j = 1, 2, \cdots, n).$$

Equation (2.1) can be rearranged as

$$= E(\sum_{h_1} \cdots \sum_{h_k} (\prod_{j=1}^{k} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}} \prod_{j=1}^{k} \prod_{i=1}^{n} (W_i X_{ij})^{h_{ij}}))$$

$$= E(\sum_{h_1} \cdots \sum_{h_k} (\prod_{j=1}^{k} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}} (\prod_{i=1}^{n} W_i^{h_{i.}}) \prod_{j=1}^{k} \prod_{i=1}^{n} X_{ij}^{h_{ij}}))$$

where $h_{i.} = \sum_{j=1}^{k} h_{ij}$ and we have,

$$= \sum_{h_1} \cdots \sum_{h_k} (\prod_{j=1}^{k} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}} E(\prod_{i=1}^{n} W_i^{h_{i.}}) E(\prod_{j=1}^{k} \prod_{i=1}^{n} X_{ij}^{h_{ij}})), \tag{2.2}$$

By using the Dirichlet distribution, we have

$$E(\prod_{i=1}^{n} W_i^{h_{i.}}) = \frac{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)})}{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)} + \sum_{j=1}^{k} s_j)} \prod_{i=1}^{n} \frac{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)} + h_{i.})}{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)})}$$

Also, we have

$$E \prod_{j=1}^{k} \prod_{i=1}^{n} X_{ij}^{h_{ij}} = \prod_{i=1}^{n} E \prod_{j=1}^{k} X_{ij}^{h_{ij}},$$

and by using the dirichlet distribution, we have

$$E(\prod_{j=1}^{k} X_{ij}^{h_{ij}}) = \frac{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)})}{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)} + h_{i.})} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j^{(i)} + h_{ij})}{\Gamma(\alpha_j^{(i)})}$$

So, by using (2.2)

$$= \sum_{h_1} \cdots \sum_{h_k} \prod_{j=1}^{k} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}} (\frac{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)})}{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)} + \sum_{j=1}^{k} s_j)}$$
$$\prod_{i=1}^{n} \frac{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)} + h_{i.})}{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)})} (\prod_{i=1}^{n} \frac{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)})}{\Gamma(\sum_{j=1}^{k} \alpha_j^{(i)} + h_{i.})} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j^{(i)} + h_{ij})}{\Gamma(\alpha_j^{(i)})})$$
$$= \frac{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)})}{\Gamma(\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j^{(i)} + \sum_{j=1}^{k} s_j)} \sum_{h_1} \cdots \sum_{h_k} \prod_{j=1}^{k} \binom{s_j}{h_{1j}, h_{2j}, \cdots, h_{nj}}$$
$$\prod_{j=1}^{k} \prod_{i=1}^{n} \frac{\Gamma(\alpha_j^{(i)} + h_{ij})}{\Gamma(\alpha_j^{(i)})}.$$

By considering the fact that the sum of the Dirichlet-multimonial distribution on its support equals to one, we have

$$= \frac{\Gamma(\sum_{i=1}^{n}\sum_{j=1}^{k}\alpha_j^{(i)})}{\Gamma(\sum_{i=1}^{n}\sum_{j=1}^{k}\alpha_j^{(i)}+\sum_{j=1}^{k}s_j)}\prod_{j=1}^{k}\frac{\Gamma(\sum_{i=1}^{n}\alpha_j^{(i)}+s_j)}{\Gamma(\sum_{i=1}^{n}\alpha_j^{(i)})}$$

which is the general moment of the k-variate

$$Dirichlet(\sum_{i=1}^{n}\alpha_1^{(i)},\sum_{i=1}^{n}\alpha_2^{(i)},\cdots,\sum_{i=1}^{n}\alpha_k^{(i)})$$

distribution, and since Z is a bounded random variable, its distribution is uniquely determined by its moments. Thus the proof is complete. □

3. Some Applications in Stochastic Differential Equations by c-Characteristic Function

The following proof is suggested by a reviewer of the journal *Communications in Statistics: Theory and Methods* in which the paper [1], is published:
“ Jiang, Dickey and Kuo [7] define the multivariate c-characteristic function to solve many problems that are difficult to manage using the traditional characteristic function or moment generating function.

If $u = (u_1, \cdots, u_k)$ is a random vector, its multivariate c-characteristic function is defined as $g(t_1, \cdots, t_k; u, c) = E\left\{(1 - it_1u_1 - \cdots - it_ku_k)^{-c}\right\}$ where c is a positive real number. See Definition 2.1 in [7]. when c is fixed, Jiang, Dickey and Kuo ([7], Lemma 2.2) show that there is a one-to-one correspondence between random vector and its multivariate c-characteristic function. Moreover, if $u \sim Dirichlet(\alpha_1, \cdots, \alpha_k)$ and $c = \alpha_1 + \cdots + \alpha_k$, then its multivariate c-characteristic function is given by

$$g(t_1, \cdots, t_k; u, c) = E\left\{\left(1 - i\sum_{j=1}^{k}t_ju_j\right)^{-c}\right\} = \prod_{j=1}^{k}(1 - it_j)^{-\alpha_j}.$$

See Corollary 3.4 in [7].

Now, we provide a simpler proof for Theorem 2.1, by the above identity only. Let $Z = (Z_1, \cdots, Z_k) = \sum_{j=1}^{n} W_j\mathbf{X}_j$ where $\mathbf{X}_j = (X_{j1}, \cdots, X_{jk})$ for $1 \le j \le n$. It can be shown that

$$1 - it_1Z_1 - \cdots - it_kZ_k = 1 - i\sum_{j=1}^{n}W_j\sum_{r=1}^{k}t_rX_{jr}.$$

Let $c = \sum_{j=1}^{n} \sum_{r=1}^{k} \alpha_r^{(j)}$. Then, the multivariate c-characteristic function of Z is

$$\begin{aligned} g(t_1, \cdots, t_k; Z, c) &= E\left\{(1 - it_1 u_1 - \cdots - it_k u_k)^{-c}\right\} \\ &= EE\left\{\left(1 - i\sum_{j=1}^{n} W_j \sum_{r=1}^{k} t_r X_{jr}\right)^{-c} |\mathbf{X}_1, \cdots, \mathbf{X}_n\right\} \\ &= \prod_{j=1}^{n} E\left\{\left(1 - i\sum_{r=1}^{k} t_r X_{jr}\right)^{-\sum_{r=1}^{k}\alpha_r^{(j)}}\right\} \\ &= \prod_{j=1}^{n}\prod_{r=1}^{k}(1 - it_r)^{-\alpha_r^{(j)}} = \prod_{r=1}^{k}(1 - it_r)^{-\sum_{j=1}^{n}\alpha_r^{(j)}} \end{aligned}$$

Therefore,

$$Z \sim Dirichlet\left(\sum_{j=1}^{n}\alpha_1^{(j)}, \cdots, \sum_{j=1}^{n}\alpha_k^{(j)}\right).\text{''}$$

We will employ some closely related argument in our future research papers.

4. Conclusions

The main result of this paper (Theorem 2.1) shows that our results cannot be generalized by changing the parameters of the distribution. So, we will investigate two topics in two future research works: (1) the obtained results cannot be generalized for arbitrary parameters, and we will propose one other method for obtaining more and better results; and (2) by using the simulation and the approximation methods, the results will be generalized, as was previously done by Homei and Nadarajah ([6]).

5. Acknowledgements

The authors would like to thank the referees for their constructive comments and suggestions.

References

[1] H. Homei, *The stochastic linear combination of Dirichlet distributions*, Communications in Statistics: Theory and Methods, to appear , (2019).

[2] H. Homei, *Characterizations of Arcsin and Related Distributions Based on a New Generalized Unimodality*, Communications in Statistics: Theory and Methods, *46* (2017a), 1024–1030.

[3] H. Homei, *Randomly Weighted Averages: a multi-variate case*, The Book of the Abstracts of the 11th Seminar on Probability and Stochastic Processes, Qazvin, Iran, (30–31 August 2017), Imam Khomeini International University, 49.

[4] H. Homei, *A Novel Extension of Randomly Weighted Averages*, Statistical Papers, *56* (2015), 933–946 .

[5] H. Homei, *Randomly Weighted Averages with Beta Random Proportions*, Statistics and Probability Letters, *82* (2012), 1515–1520.

[6] H. Homei and S. Nadarajah, *On Products and Mixed Sums of Gamma and Beta Random Variables Motivated by Availability*, Methodology and Computing in Applied Probability, *20* (2018), 799–810.

[7] T. j. Jiang, J. M. Dickey, and K. L. Kuo, *A new multivariate transform and the distribution of a random functional of a FergusonDirichlet process*, Stochastic Processes and Their Applications, *111*(1) (2004), 77–95.

[8] N. L. Johnson, S. Kotz, *Randomly weighted averages: Some aspects and extensions*, The American Statistician, *44* (1990), 245–249.

[9] W. Van Assche, *A random variable uniformly distributed between two independent random variables*, Sankhy a: The Indian Journal of StatisticsSeries A *49* (1987), 207–211.

[10] N. A. Volodin, S. Kotz, and N. L. Johnson, *Use of moments in distribution theory: A multivariate case*, Journal of Multivariate Analysis, *46*(1) (1993), 112–119.