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Exact solutions and numerical simulations of time-fractional Fokker-Plank equation for special stochastic process

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Abstract In this paper, a type of time-fractional Fokker-Planck equation (FPE) of the Ornstein-Uhlenbeck process is solved via Riemann-Liouville and Caputo derivatives. An analytical method based on symmetry operators is used for finding reduced form and exact solutions of the equation. A numerical simulation based on the Müntz-Legendre polynomials is applied in order to find some approximated solutions of the equation.

Keywords. Fokker-Plank equation, Riemann Liouville derivative, Caputo derivative, Lie point symmetry, Müntz-Legendre polynomial.

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1. Introduction

When one is confronted with a complicated system of PDEs or FDEs (fractional differential equations) arising from some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutions can be used as a models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behavior of more general types of solutions. Thus, an analytic and powerful method is needed for this purpose. Symmetry analysis of differential equations is a method based on finding some differential operators (vector fields) called symmetries. These operators are the largest local group of transformations acting on the independent and dependent variables of

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the system with the property that they transform solutions of the system to other solutions. The determination of symmetry group of geometric object can be regarded as a special case of the general equivalence problem. Indeed, provided it lies in the admissible class of changes of variables, a symmetry is merely a self-equivalence of the object. Thus, for instance, the solution of the equivalence problem for differential equations will include a determination of all symmetries of a given differential equation. Two equivalent objects have isomorphic symmetry group, indeed, conjugating any symmetry of the first object by the equivalence transformation produces a symmetry of the second. Thus, one means of recognizing equivalent objects is by inspecting their symmetry group: If the two symmetry groups are not equivalent, e.g., they have different dimensions, or different structures, then the two objects cannot be equivalent. Of course, having isomorphic symmetry groups is no guarantee that the two objects are equivalent; nevertheless, in many highly symmetric cases, including linearization problems, the existence of a suitable symmetry group is both necessary and sufficient for the equivalence of two objects.

One of the most important application of symmetry's method is the reducing systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. There is a lot of papers in the literature for this process and one can find the classical reduction method in [13, 16, 17, 18, 33].

The fractional calculus can be used to the modeling of the processes from various fields of physics and engineering, also these calculations have so many applications in many branches of sciences besides mathematics and physics such as economics, biology, viscoelasticity (for example, see [4, 5, 7, 19, 20, 21, 24, 25, 26, 36, 37]). Therefore, by considering applications, solving equations in the fractional differential range is very important. From the past to the present, there are different fractional derivatives, for example, the Riemann-Liouville, modified Riemann-Liouville, Caputo, Caputo-Fabrizio and etc. Some of the fractional calculus are based on space-time fractional derivatives and the other includes space-fractional derivatives (fractional Laplacian) and the rest of them, are obtained by replacing time-fractional order α by integer order n. The third type of equations is more general than the other types of FDEs and PDEs.

A method for solving ODEs and PDEs via Lie symmetries are established by Sophus Lie (1842-1899) and for FDEs by Gazizov et al. [11]. Towards the end of the nineteenth century, Sophus Lie introduced the notion of Lie group in order to study the solutions of ODEs. He showed the following main property: the order of an ODE can be reduced by one if it is invariant under one-parameter Lie group of point transformations. This observation unified and extended the available integration techniques. Lie devoted the remainder of his mathematical career to developing these continuous groups that have now an impact on many areas of mathematically based sciences. The applications of Lie groups to differential systems were mainly established by Lie and Emmy Noether, and then advocated by Elie Cartan.

It is necessary to mention that Gazizov et al. investigated the Lie symmetry method for an equation with Riemann-Liouville and Caputo derivative. An important advantage of Lie symmetry method as it discussed above, is to find an equivalent equation called reduced equation. This reduced equation obtained from similarity variables constructed from symmetry operators of the equation. These equations give an exact solution for the primary equation called similarity or group-invariant solution [2, 15, 34, 38]. It is noteworthy that the advantage of Lie symmetry method is that, it is free of the kind of a given system. So, this makes the Lie symmetry method as one of the best candidate for solving a system of differential equations.

The Fokker-Planck equations are the worthful tools to manage fluctuations in a large range of dynamic systems and have been widely used in various field of sciences such as physics, mathematical finance, chemistry, and etc. Specially in physics the Fokker-Planck equations are used to modeling of the complex dynamics, for instance, quantum mechanics, astrophysics, statistical physics, and are applied to the protein folding biophysics problem [30]. In financial mathematics, the Fokker-Planck equations are applied for example to explain the behavior of returns for foreign exchange markets in different time scales [12, 31].

Exact and numerical solutions of these equations are presented in a lot of papers. The applied methods for solving of these equations are so different. For example, in [14], numerical solution of the stationary and transient form of the Fokker-Planck equation are obtained by standard sequential finite element method (FEM) using C^0-shape function and Crank-Nicolson time integration scheme. Lie symmetry analysis is another method in order to find solutions of Fokker-Planck equations [6]. In fact this analysis classifies the solution format of the Fokker-Planck equation by the Lie algebra of symmetries.

The time-fractional Fokker-Plank equation for Ornstein-Uhlenbeck process

$$\mathcal{D}_t^{\beta} p - \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} - \alpha p - \alpha x \frac{\partial p}{\partial x} = 0, \tag{1.1}$$

is the main purpose of the present work, where β is the fractional order with $0 < \beta < 1$. Symmetry analysis in the case of $\beta = 1$ is furnished by E. Dastranj and S. R. Hejazi [6].

The paper is organized as follows. In section 2 some properties of the Riemann-Liouville and the Caputo derivative are presented. In section 3 the Lie symmetries of the time-fractional PDE for the Ornstein-Uhlenbeck process are constructed. Finally, section 4 includes the classical basis polynomials like Legendre, Laguerre, and Chebyshev polynomials, which are widely used to deal with many problems of dynamic systems. Since solutions of fractional differential equations can contain some fractional power terms, classical polynomials are not a reasonable suggestion for solving these problems [10]. Consequently, we have to consider suitable basis polynomials in the proposed numerical technique. In this section, we consider Müntz-Legendre polynomials (MLPs), which are a family of generalized orthogonal polynomials. These polynomials have been introduced and discussed in [3].

2. Preliminaries Of Fractional calculus And The Lie Groups Method

Some basis concepts should be mentioned before starting the main sections.

2.1. **Definition and Properties.** Let us recall some usual properties of Riemann-Liouville and Caputo fractional time derivatives of order β:

Definition 2.1. The left-hand-sided of the Riemann-Liouville derivative (${}_a\mathcal{D}_t^\beta$) and the right-hand-sided of the Riemann-Liouville derivative (${}_t\mathcal{D}_b^\beta$) are defined as following condition,

$$ {}_a\mathcal{D}_t^\beta p(x,t) = \frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial t^n}\int_a^t \frac{p(x,s)}{(t-s)^{\beta+1-n}}ds, \quad 0<n-1<\beta<n, \tag{2.1}$$

$$ {}_t\mathcal{D}_b^\beta p(x,t) = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{\partial^n}{\partial t^n}\int_t^b \frac{p(x,s)}{(s-t)^{\beta+1-n}}ds, \quad 0<n-1<\beta<n, \tag{2.2}$$

$$ {}_a\mathcal{D}_t^\beta p(x,t) = {}_t\mathcal{D}_b^\beta p(x,t) = \frac{\partial^n p}{\partial t^n}, \quad \beta = n\in\mathbb{N}. \tag{2.3}$$

Definition 2.2. The left-hand-sided of the Caputo fractional derivative (${}_a^C\mathcal{D}_t^\beta$) and the right-hand-sided of the Caputo fractional derivative (${}_t^C\mathcal{D}_t^\beta$) are:

$$ {}_a^C\mathcal{D}_t^\beta p(x,t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \frac{1}{(t-s)^{\beta+1-n}}\frac{\partial^n p(x,s)}{\partial s^n}ds, \quad n-1<\beta<n, \tag{2.4}$$

$$ {}_t^C\mathcal{D}_b^\beta p(x,t) = \frac{(-1)^n}{\Gamma(n-\beta)}\int_t^b \frac{1}{(t-s)^{\beta+1-n}}\frac{\partial^n p(x,s)}{\partial s^n}ds, \quad n-1<\beta<n, \tag{2.5}$$

$$ {}_a^C\mathcal{D}_t^\beta p(x,t) = {}_t^C\mathcal{D}_b^\beta p(x,t) = \frac{\partial^n p}{\partial t^n}, \qquad \beta = n\in\mathbb{N}. \tag{2.6}$$

Before giving some properties of these derivatives, a blanket hypothesis is considered, that is all considered derivatives are the left type in the sequel. A number of properties are expressed as follows:

• The Riemann-Liouville and Caputo derivatives are non-local due to operator integral in their definitions.

$$ \bullet \mathcal{D}_t^\beta t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)}t^{\gamma-\beta}, \tag{2.7}$$

$$ \bullet \mathcal{D}_t^\beta f(x,t) = 0 \Longrightarrow f(x,t) = \sum_{j=1}^m c_j t^{\beta-j}. \tag{2.8}$$

$$ \bullet L\{{}_0\mathcal{D}_x^\beta f(x);s\} = s^\beta F(s) - \sum_{k=0}^{n-1} s^k\mathcal{D}^{\beta-k-1}f(0+), \tag{2.9}$$

$$ \bullet L\{{}_0^C\mathcal{D}_t^\beta f(x);s\} = s^\beta F(s) - \sum_{k=0}^{n-1} s^{\beta-k-1}f^{(k)}(0+), \qquad n-1<\beta\le n. \tag{2.10}$$

where $L\{{}_0\mathcal{D}_x^\beta f(x);s\}$ and $L\{{}_0^C\mathcal{D}_x^\beta f(x);s\}$ are the Laplace transform of Riemann-Liouville and Caputo derivatives respectively.

2.2. **Lie group analysis of time-fractional PDE (1.1).** The results of Lie symmetry method for FDEs are the same as PDEs almost every where, for further details, readers are referred to [11]. The symmetry group of the Eq. (1.1) will be formed by the vector field or the infinitesimal operator of the form

$$X = \tau(x,t,p)\frac{\partial}{\partial t} + \xi(x,t,p)\frac{\partial}{\partial x} + \eta(x,t,p)\frac{\partial}{\partial p}. \tag{2.11}$$

Thus, the Eq. (1.1) admits (2.11) as an infinitesimal generator of a Lie point symmetry if

$$X^{(\beta,2)}\left(\mathcal{D}_t^\beta p - \frac{1}{2}\sigma^2 p_{xx} - \alpha p - \alpha x p_x\right) = 0, \tag{2.12}$$

where $X^{(\beta,2)}$ denotes the second prolongation of operator 2.11 and in the meantime satisfies the following equation:

$$X^{(\beta,2)} = X + \eta_t^\beta \frac{\partial}{\partial p_t^\beta} + \eta^x \frac{\partial}{\partial p_x} + \eta^{xx}\frac{\partial}{\partial p_{xx}}. \tag{2.13}$$

Regarding the symmetry condition (2.13), the prolongation's coefficients η^x, η^{xx} and η_t^β that must satisfy the following invariance identity:

$$\eta_t^\beta - \frac{1}{2}\sigma^2\eta^{xx} - \alpha\eta - \alpha\xi p_x - \alpha x\eta^x = 0, \tag{2.14}$$

where η^x, η^{xx} and η_t^β obtained by

$$\begin{aligned}
\eta_t^\beta &= \mathcal{D}_t^\beta(\eta - \tau p_t - \xi p_x) + \tau\mathcal{D}_t^\beta(u_t) + \xi\mathcal{D}_t^\beta(u_x), \qquad (2.15)\\
\eta^x &= D_x(\eta - \tau u_t - \xi u_x) + \tau u_{tx} + \xi u_{xx},\\
\eta^{xx} &= D_{xx}(\eta - \tau u_t - \xi u_x) + \tau u_{txx} + \xi u_{xxx}.
\end{aligned}$$

The expanded forms of the above equations are written by using the total derivative operator D_x, the generalized chain rule, and the generalized Leibnitz rule, for example:

$$\begin{aligned}
\eta_t^\beta &= \sum_{n=1}^{\infty}\left[\binom{\beta}{n}\partial_t^n\eta_u - \binom{\beta}{n+1}D_t^{n+1}(\tau)\right]\partial_t^{\beta-n}u + \partial_t^\beta\eta - u\partial_t^\beta\eta_u\\
&\quad - \sum_{n=1}^{\infty}\binom{\beta}{n}D_t^n(\xi)\partial_t^{\beta-n}(u_x) + [\eta_u - \beta D_t(\tau)]\,\partial_t^\beta u + \mu, \qquad (2.16)
\end{aligned}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\binom{\beta}{n}\binom{\beta}{m}\frac{t^{n-\beta}U_k}{k!\Gamma(n+1-\beta)}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k},$$

and

$$U_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}g^k(t)\partial_t^\beta(g^{n-k}(t)).$$

Substituting the values of Eqs. (**??**) into (2.14), together with Eq. (1.1), the determining system for invariance condition (2.14) is derived. The solution of this system gives

$$\begin{aligned}
&\tau = C_2\beta\sigma^2 x - C_4\sigma^2,\\
&\xi = 2C_2\sigma^2 t + C_1\sigma^2,\\
&\eta = C_2\beta\sigma^2 p - C_2\alpha\beta p x^2 + C_4\alpha p x + C_5 p - C_2\sigma^2 p
\end{aligned} \tag{2.17}$$

as the coefficients of the symmetry operator (2.11) where $C_i(i = 1, 2, 3)$ are arbitrary constants. Thus, in terms of the Lie symmetry analysis method, Eq. (1.1) admits the following generators:

$$\begin{aligned}
&X_1 = \beta\sigma^2 x\frac{\partial}{\partial x} + 2\sigma^2 t\frac{\partial}{\partial t} + \left(\beta p\sigma^2 - p\beta\alpha x^2 - \sigma^2 p\right)\frac{\partial}{\partial p},\\
&X_2 = -\sigma^2\frac{\partial}{\partial x} + p\alpha x\frac{\partial}{\partial p},\\
&X_3 = p\frac{\partial}{\partial p}.
\end{aligned} \tag{2.18}$$

As regards the Eq. (1.1) has Riemann-Liouville derivative, so this equation doesn't admit $\frac{\partial}{\partial t}$ as a geometric vector field [32].

3. Similarity reductionss

In this section, some discussions on similarity reductions of the Eq. (1.1) are given. The method of obtaining the group invariant solution on FDEs is the same as PDEs.

Case1. For the generator X_2 readers are referred to [12], Habibi et al. in their study reduced FPDE to another form.

Case2. For the generator X_1, the similarity variable and similarity solution will be found by solving the associated characteristic equation

$$\frac{dx}{\beta\sigma^2 x} = \frac{dt}{2t\sigma^2} = \frac{dp}{\beta\sigma^2 p - p\beta\alpha x^2 - \sigma^2 p}. \tag{3.1}$$

Integration from (3.1) gives the invariants

$$\frac{t}{\sqrt[\beta]{x^2}} \quad \text{and} \quad \frac{x^{1-\frac{1}{\beta}} e^{-\frac{\alpha x^2}{2\sigma^2}}}{p(x,z)}. \tag{3.2}$$

Consequently, we have

$$p(x,z) = \frac{x^{1-\frac{1}{\beta}} e^{-\frac{\alpha x^2}{2\sigma^2}}}{g(z)}. \tag{3.3}$$

Substituting $p(x,z)$ and derivatives with respect to x into (1.1), one can obtain the reduced ODE form of Eq. (1.1) such as following:

$$
\begin{aligned}
-\frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial x^2} - \alpha p - \alpha x\frac{\partial p}{\partial x} = {} & \frac{\sigma^2 x^{-1-\frac{1}{\beta}}}{2g(z)e^{\frac{\alpha x^2}{2\sigma^2}}}\left(\frac{1}{\beta}\right)\left(1-\frac{1}{\beta}\right) \\
& - \frac{\alpha^2 x^{3-\frac{1}{\beta}}}{2\sigma^2 g(z)e^{\frac{\alpha x^2}{2\sigma^2}}} + \frac{\alpha x^{1-\frac{1}{\beta}}}{2g(z)e^{\frac{\alpha x^2}{2\sigma^2}}} \\
& + \frac{\sigma^2 g''(z)x^{1-\frac{1}{\beta}}}{2g^2(z)e^{\frac{\alpha x^2}{2\sigma^2}}}z'^2 + \frac{\sigma^2 g'(z)x^{1-\frac{1}{\beta}}}{2g^2(z)e^{\frac{\alpha x^2}{2\sigma^2}}}z'' \\
& - \frac{\sigma^2 g'^2(z)x^{1-\frac{1}{\beta}}}{g^3(z)e^{\frac{\alpha x^2}{2\sigma^2}}}z'^2 + \frac{\sigma^2 g'(z)x^{-\frac{1}{\beta}}}{g^2(z)e^{\frac{\alpha x^2}{2\sigma^2}}} \\
& \times z'\left(1-\frac{1}{\beta}\right) - \frac{\alpha x^{1-\frac{1}{\beta}}}{g(z)e^{\frac{\alpha x^2}{2\sigma^2}}} \\
& + \frac{\alpha^2 x^{3-\frac{1}{\beta}}}{\sigma^2 g(z)e^{\frac{\alpha x^2}{2\sigma^2}}},
\end{aligned} \tag{3.4}
$$

where

$$
g(z) = \frac{x^{1-\frac{1}{\beta}}e^{-\frac{\alpha x^2}{2\sigma^2}}}{p(x,z)},
$$

satisfies the above equation and z', z'' are derivatives respect to x and $g'(z)$, $g''(z)$ are derivatives respect to z. Thus, the fractional part of Eq. (1.1) is described by:

$$
\begin{aligned}
\mathcal{D}_t^\beta p(t,x) &= \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t \frac{p(\tau,x)d\tau}{(t-\tau)^\beta} \\
&= \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t \frac{x^{1-\frac{1}{\beta}}e^{-\frac{\alpha x^2}{2\sigma^2}}d\tau}{g\left(\frac{\tau}{\sqrt[\beta]{x^2}}\right)(t-\tau)^\beta}.
\end{aligned} \tag{3.5}
$$

Let us assume that $y = \frac{\tau}{\sqrt[\beta]{x^2}}$, $\left(d\tau = \sqrt[\beta]{x^2}dy\right)$ and in the meantime, according to our previous notation that, $t = z\sqrt[\beta]{x^2}$, $\left(\frac{\partial}{\partial t} = \frac{1}{\sqrt[\beta]{x^2}}\frac{\partial}{\partial z}\right)$, so

$$
\begin{aligned}
\mathcal{D}_t^\beta p(t,x) &= \frac{x^{-1-\frac{1}{\beta}}e^{-\frac{\alpha x^2}{2\sigma^2}}}{\Gamma(1-\beta)}\frac{\partial}{\partial z}\int_0^z \frac{dy}{g(z)(z-y)^\beta} \\
&= x^{-1-\frac{1}{\beta}}e^{-\frac{\alpha x^2}{2\sigma^2}}\mathcal{D}_z^\beta\left(\frac{1}{g(z)}\right).
\end{aligned} \tag{3.6}
$$

Considering z as a new independent variable, x as prior independent variable and g as the new dependent variable Eq. (1.1) converts to:

$$\begin{aligned} x^{-1-\frac{1}{\beta}}\mathcal{D}_z^{\beta}\left(\frac{1}{g(z)}\right) &= \frac{\alpha}{2g(z)}x^{1-\frac{1}{\beta}} - \frac{\sigma^2 g''(z)}{2g^2(z)}z'^2 x^{1-\frac{1}{\beta}} \\ &- \frac{\sigma^2 g'(z)}{g^2(z)}z'(1-\frac{1}{\beta})x^{-\frac{1}{\beta}} \\ &- \frac{\sigma^2}{2g(z)}\left(\frac{1}{\beta}\right)\left(1-\frac{1}{\beta}\right)x^{-1-\frac{1}{\beta}} \\ &- \frac{\sigma^2 g'(z)}{g^2(z)}z'' x^{1-\frac{1}{\beta}} - \frac{\alpha^2}{2\sigma^2 g(z)}x^{3-\frac{1}{\beta}}. \end{aligned} \tag{3.7}$$

We will use the symbol $h(z)$ to denote $\frac{1}{g(z)}$. By using this placement, we have the following derivatives:

$$\begin{aligned} h(z) &= \frac{1}{g(z)}, \\ h'(z)z' &= -\frac{g'(z)z'}{g^2(z)}, \\ h''(z)z'^2 + h'(z)z'' &= \frac{-g''(z)z'^2 g^2(z) - g'(z)z'' g^2(z) + 2g(z)g'^2(z)z'^2}{2g^4(z)}. \end{aligned} \tag{3.8}$$

By applying the above derivatives Eq. (1.1) transforms to,

$$\begin{aligned} \mathcal{D}_z^{\beta}(h(z)) &= h(z)\left[-\frac{(\beta-1)\sigma^2}{2\beta^2} - \frac{\alpha^2 x^4}{2\sigma^2} + \frac{\alpha x^2}{2}\right] + h''(z)\left(\frac{z'\sigma^2 x^2}{2}\right) \\ &+ h'(z)\left[\frac{z''\sigma^2 x^2}{2} + \frac{z'(\beta-1)\sigma^2 x}{\alpha}\right]. \end{aligned} \tag{3.9}$$

Now, a solution for (3.9) with new symmetries is investigated in the sequel. The use of Lie algorithm method again, concludes that

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = h\frac{\partial}{\partial h}. \tag{3.10}$$

are two symmetries for Eq. (3.9).

The symmetry $Y_1 + Y_2$ yields the following associated Lagrange's equation

$$\frac{dx}{1} = \frac{dz}{0} = \frac{dh}{h}. \tag{3.11}$$

By solving the above statement, the similarity variables z and $\frac{e^x}{h(z)}$ are obtained. Therefore a solution for Eq. (3.9) is written by,

$$h(z) = \frac{e^x}{k(z)}. \tag{3.12}$$

According to Eq. (3.7), $h(z) = \frac{1}{g(z)}$ so $g(z) = \frac{k(z)}{e^x}$. On the other hand $p(x,z) = \frac{x^{1-\frac{1}{\beta}}e^{-\frac{\alpha x^2}{2\sigma^2}}}{g(z)}$ is a solution for Eq. (1.1), then another solution for Eq. (1.1) is considered by:

$$p(x,z) = \frac{x^{1-\frac{1}{\beta}} e^{x-\frac{\alpha x^2}{2\sigma^2}}}{k(z)}. \tag{3.13}$$

4. Numerical simulation

In this section, we will propose a numerical solution for the equation

$$^{C}\mathcal{D}_t^{\beta} p = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} + \alpha p + \alpha x \frac{\partial p}{\partial x}, \tag{4.1}$$

with the initial conditions as follows

$$p(0,t) = \phi_1(t), \tag{4.2}$$

$$\frac{\partial p}{\partial x}(0,t) = \phi_2(t), \tag{4.3}$$

$$p(x,0) = \psi(x), \tag{4.4}$$

where $x, t \in [0,T]$. Since that is not the focus of this work, we briefly describe an efficient numerical method to perform it here. The readers are recommended [22, 23, 27, 28, 39] for another useful numerical methods about our purpose.

4.1. **Müntz–Legendre polynomials.** In this part of paper, we consider a special case of MLPs discussed in [10] that are defined on the interval $[0,T]$ by:

$$L_{n,\beta}(t) := \sum_{k=0}^{n} C_{n,k} \left(\frac{t}{T}\right)^{k\beta}, \; C_{n,k} = \frac{(-1)^{n+k}}{\beta^n k!(n-k)!} \prod_{v=0}^{n-1} ((k+v)\beta + 1). \tag{4.5}$$

Also, they are orthogonal with respect to inner product on the interval $[0,T]$ with the weight function $\omega(x) = 1$, i.e.

$$(L_{n,\beta}, L_{m,\beta}) = \int_0^T L_{n,\beta}(t) L_{m,\beta}(t) dt = \frac{T}{2n\beta + 1}\delta_{nm}. \tag{4.6}$$

Any function $f(t) \in L^2[0,T]$ can be approximated in terms of MLPs as

$$f(t) = \sum_{n=0}^{\infty} c_n L_{n,\beta}(t), \;\; c_n = \frac{(2n\beta+1)}{T} \int_0^T f(t) L_{n,\beta}(t) dt.$$

Theorem 4.1. *Let $^{C}D_t^{\beta i} f(t) \in C(0,T]$ be bounded on $[0,T]$ for $i = 0, \cdots, N+1$ and $0 < \beta \leq 1$. Then function $f(t)$ can be approximated by $N+1$ terms of MLPs such that*

$$\|f(t) - \sum_{n=0}^{N} c_n L_{n,\beta}(t)\|_\infty \leq \frac{k_1}{\Gamma((N+1)\beta + 1)},$$

where $k_1 = \|^{C}\mathcal{D}_t^{(N+1)\beta} f(t)\|$.

Proof. See [9]. □

We also recall important property of MLPs. The fractional derivative of MLPs is defined as

$$ {}^{C}\mathcal{D}_t^{\beta} L_{n,\beta}(t) = \sum_{k=\lceil \beta \rceil}^{n} D_{n,k} t^{(k-1)\beta}, \; D_{n,k} = \frac{\Gamma(k\beta+1)}{T^{k\beta}\Gamma(k\beta+1-\beta)} C_{n,k}. $$

Proposition 4.2. *For $\beta = 1$, MLPs (4.5) are shifted Legendre polynomials on the interval $[0,T]$. An evaluation of MLPs in the form (4.5) can be problematic in finite arithmetic, especially when n is a large number and x is close to 1. For this reason we use the following theorem which is proved in* [10].

Theorem 4.3. *Let $\beta > 0$ be a real number and $J_{T,n}^{(0,-1+1/\beta)}(t)$ be the Jacobi polynomial with parameters $a = 0, b = -1 + \frac{1}{\beta}$ on the interval $[0,T]$; then for MLPs we have*

$$ L_{n,\beta}(t) = J_{T,n}^{(0,-1+1/\beta)} \left[2\left(\frac{t}{T}\right)^{\beta} - 1 \right]. \tag{4.7} $$

So, in view of (4.5) and (4.6), the MLPs $L_{n,\beta}(t)$ can be obtained by means of the three-term recursion formula. We should remark that, using the analytical form of MLPs can be written as [10]

$$ L_{n,\beta}(t) = \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma(n+\frac{1}{\beta})\Gamma(n+k+\frac{1}{\beta})}{\Gamma(k+\frac{1}{\beta})\Gamma(n+\frac{1}{\beta})(n-k)!k!} \left(\frac{t}{T}\right)^{k\beta}. \tag{4.8} $$

4.2. **Direct collocation.** In this part, we will approximate the solution of the problem as follows

$$ p(x,t) \simeq p_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} p_{mn} L_{m,\beta}(x) L_{n,\beta}(t), $$

where u_{mn}, $m = n = 0, 1, \cdots, N$ are unknown coefficients and should be determined by our presented method and $L_{m,\beta}(x)$ and $L_{n,\beta}(t)$ are the MLPs of order m and n, respectively.

It should be noted that in some research works [1, 8], the authors used an operational matrix of fractional differentiation to solve the problems of this type. However, in this part, we compute the Caputo fractional derivative of order β by some suitable commands in mathematical softwares such as `Maple` or `Matlab`. The fractional derivative of order β can be determined using the following command in `Maple` software

$$ {}^{C}\mathcal{D}_t^{\beta} f(t) = \texttt{fracdiff}(f(t), t, \beta). $$

Using this command the fractional and second derivative which we need in our approach could be computed. Since there exist $(N+1)^2$ unknown coefficients u_{mn}, $m = n = 0, 1, \cdots, N$ we should construct system of $(N+1)^2$ algebraic equations. Suppose that x_i and t_i are the $N+1$ roots of Chebyshev polynomials $T_{N+1}(x)$ on the interval $[0,T]$ which are defined as

$$ t_i = \tfrac{T}{2} - \tfrac{T}{2}\cos\left[\tfrac{(2i-1)\pi}{2(N+1)}\right], \quad i = 1, 2, \ldots, N+1, $$
$$ x_i = \tfrac{T}{2} - \tfrac{T}{2}\cos\left[\tfrac{(2i-1)\pi}{2(N+1)}\right], \quad i = 1, 2, \ldots, N+1. $$

Now, we discretize equation (4.1) using the points t_i, x_i as

$$ {}^{C}\mathcal{D}_t^{\beta} p_N(x_i,t_i) = \frac{1}{2}\sigma^2 \frac{\partial^2 p_N}{\partial x^2}(x_i,t_i) + \alpha p_N(x_i,t_i) + \alpha x_i \frac{\partial p_N}{\partial x}(x_i,t_i), $$

and the initial conditions (4.2)-(4.4) as follows

$$ \begin{aligned} p(0,t_i) &= \phi_1(t_i), \\ \frac{\partial p}{\partial x}(0,t_i) &= \phi_2(t_i), \\ p(x_i,0) &= \psi(x_i). \end{aligned} $$

In this case, the considered equations are collocated and then transformed into the associated systems of $(N+1)^2$ algebraic equations and $(N+1)^2$ unknowns which can be solved through an iterative method in Maple software by `fsolve` command. In the following, we consider a numerical example and check the accuracy of our proposed numerical approach by replacing the $p_N(x,y), \mathcal{D}_t^{\beta} p_N$, $\frac{\partial^2 p_N}{\partial x^2}$, p_N and $\frac{\partial p_N}{\partial x}$ in equations (4.1)-(4.4). Then (4.1) can be satisfied approximately. In other words, we define the absolute error as

$$ E(x,t) = \left| {}^{C}\mathcal{D}_t^{\beta} p - \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} - \alpha p - \alpha x \frac{\partial p}{\partial x} \right| \to 0. \tag{4.9} $$

Example 4.4. Let $\beta = 0.9$, $\sigma = 0.2$, $T = 1$ and $\alpha = -0.02$ in equation (4.1). Thus we have

$$ {}^{C}\mathcal{D}_t^{0.9} p = 0.02 \left(\frac{\partial^2 p}{\partial x^2} - p - x \frac{\partial p}{\partial x} \right), \tag{4.10} $$

with initial conditions as follows

$$ p(0,t) = 1, \tag{4.11} $$

$$ \frac{\partial p}{\partial x}(0,t) = 0, \tag{4.12} $$

$$ p(x,0) = e^{\frac{1}{2}x^2}. \tag{4.13} $$

Diagrams of the solution using the suggested numerical method are shown in Figure 1. The absolute error $E(x,t)$ is also depicted in Figure 2. In Table 2 we determined the CPU time for different values of N. The rate of convergence of the Müntz-Legendre polynomials are discussed in several papers such as [29, 35]. It worth mentioning that in these papers the superiority of Müntz-Legendre basis in compared with the classical polynomials is shown by several examples. The Table 1 shows the rate of convergence of the method with increasing the number of N.

5. Concluding remark

In the previous research the Lie group analysis for system of differential equations is extended to fractional system. This extension is applied for solving an important PDE in financial mathematics called time-fractional Fokker-Planck equation of the Ornstein-Uhlenbeck process. Symmetries are found and some reductions are presented by similarity variables extracted from symmetries. As it shown this method is an algorithmic method which could be applied for every kind of system of differential

FIGURE 1. Numerical solution $p_N(x,t)$ with $N = 10$.

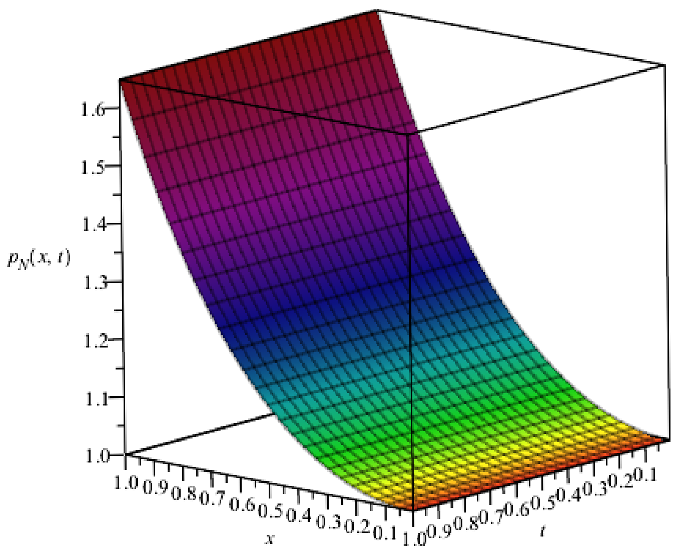

TABLE 1. Some numerical results of $p_N(x, y)$ for different values of N.

(x,t)	$N=4$	$N=8$	$N=10$
(0.1, 0.1)	3.937852E−04	5.247359E−08	1.441447E−10
(0.2, 0.2)	2.891708E−05	3.090832E−09	1.194652E−10
(0.3, 0.3)	1.765937E−04	1.110404E−08	1.543577E−10
(0.4, 0.4)	4.929371E−05	1.981757E−08	7.376778E−11
(0.5, 0.5)	3.818000E−20	1.011751E−19	7.700786E−21
(0.6, 0.6)	2.496016E−05	1.496851E−08	1.136318E−10
(0.7, 0.7)	1.364715E−04	1.497029E−08	1.271956E−10
(0.8, 0.8)	2.601971E−05	1.161986E−08	2.063365E−10
(0.9, 0.9)	3.759005E−04	3.348389E−08	2.014449E−10
(1, 1)	3.694219E−03	6.744763E−07	3.931629E−09

equation. But if the variables and the order of system increase, computations are also increase. Thus, some computational softwares such as `Maple` and `Mathematica` are needed. Finally numerical simulation based on Müntz-Legendre polynomials (MLPs), which are a family of generalized orthogonal polynomials, is given.

FIGURE 2. Absolute error $E(x,t)$ of the presented method for $N = 10$.

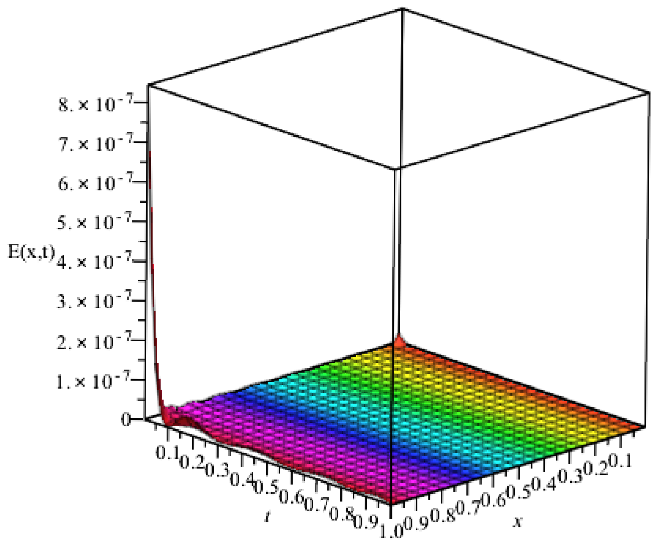

TABLE 2. CPU time for different values of N.

N	$N=4$	$N=8$	$N=10$
CPU time	0.187	2.996	11.263

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