



Application of cubic B-spline collocation method for reaction diffusion Fisher's equation

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Abstract In this study, an effective collocation method based on cubic B-spline has been implemented to get the numerical solutions for the non-linear Fisher's equation. After separating this scheme with this method, the stability of the method was proven. To check the efficiency and accuracy of the proposed method, some numerical problems have been considered. The numerical results are found in good agreement with the exact solutions.

Keywords. B-Spline, Collocation, Fisher's equation.

2010 Mathematics Subject Classification. 35Q35,35Q92.

1. INTRODUCTION

An always better understanding of nature is mixed with math problems. So, we should look for a way to answer mathematical problems with the least possible errors. In this case, partial differential equations (PDEs) are one of the most fundamental issues that appear in many natural phenomena. Therefore, finding a way for solving numerical partial differential equations with fewer errors, help us to understand natural phenomena. Fisher's equation is one of the well-known equations appearing in some natural phenomena and it is usually viewed as a population growth model. Many numerical approaches have been used to solve this equation. Cattani & Kudreyko [1] used the wavelet-Galerkin approach to find the numerical solution of Fisher's equation. Verma [15] used the Lie symmetry method for analytic and the numerical study of the non-linear diffusion equations of Fisher's types. Trigonometric B-splines [18] have been used by Zahra. Dag and Ersoy [2] considered cubic B-spline algorithm for solving Fisher equation. Wang and Jiao [17] used a fully discrete pseudo spectral scheme for Fisher's equation and generalized it with Hermite interpolation. In this study, we have paid attention to the Fisher equation and introduce and apply

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a method for numerical solution. In section 2, we introduce the Fisher equation and its application in other sciences, particularly in fluid mechanics and biology and in the context of biological invasion and also the applications of this equation checked out. In section 3, we bring some numerical methods for the Fisher equation and prove stability of the method. In section 4, we have explained the effectiveness of the method by using some numerical examples and analysis of the obtained errors.

2. REACTION-DIFFUSION EQUATION

Reaction-diffusion (RD) equations arise naturally in systems consisting of many interacting components, (e.g., chemical reactions) and are widely used to describe pattern-formation phenomena in a variety of biological, chemical, and physical systems. The principal ingredients of all these models are the equation of the form

$$\partial_t u = v\Delta^2 u + \zeta(u) \quad (2.1)$$

where $u = u(x, t)$ is a vector of concentration variables, $\zeta(u)$ describes a local reaction kinetics and the Laplace operator Δ^2 acts on the vector u componentwise. v denotes a diagonal diffusion coefficient matrix. Note that we suppose the system (2.1) is isotropic and uniform, so v is represented by a scalar matrix, independent on coordinates.

Case A In this equation, if $\zeta(u) = u(1 - u)$, we have Fisher's equation which is written as follow:

$$u_t = vu_{xx} + ku(1 - u) \quad (2.2)$$

This is a non-linear parabolic equation that first used by Fisher [4] to introduce virile gene in an infinite long habitat. v is the diffusion coefficient, k is a positive multiplication factor, t is time and x is distance. Fishers equation is used for studying biological invasion in which way, we can study migration and population habits of a variety of biological species. [11]

Case B If $\zeta(u) = u(1 - u)(u - \beta)$, we have switching waves, which are written as follows

$$u_t = vu_{xx} + u(1 - u)(u - \beta) \quad (2.3)$$

that β is a number in $(0, 1)$.

2.1. Physical interpretation of equations. The one-dimensional displacement reactions can be described by the following equation

$$u_t + vu_x = 0. \quad (2.4)$$

In this reaction, the diffusion process does not occur and u is a dependent variable and v is the velocity of displacement of a fluid stream. For example, if an oil drop is placed in a stream of water, the oil drop will be displaced without any spread. Its shape and density do not change and the oil drop only moves with the speed of water [6]. Figure 1 describes this phenomenon. But in the reaction-diffusion process, the displacement and diffusion occur simultaneously. For example, if a pollutant or a drop of ink is added to the water, the concentration of the pollutants will be reduced (diffuse) as the



FIGURE 1. displacement reactions

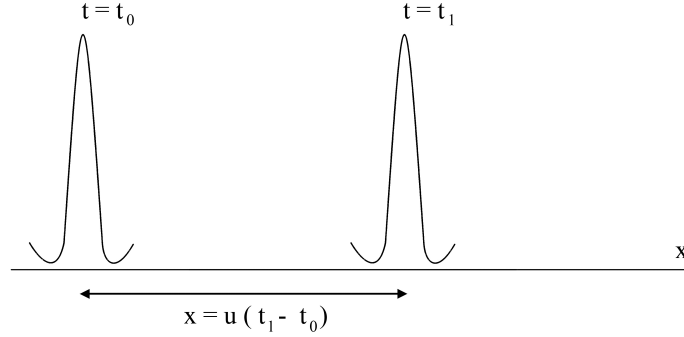
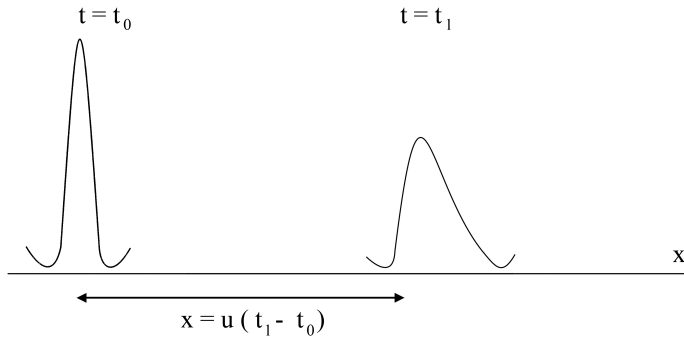


FIGURE 2. reaction-diffusion



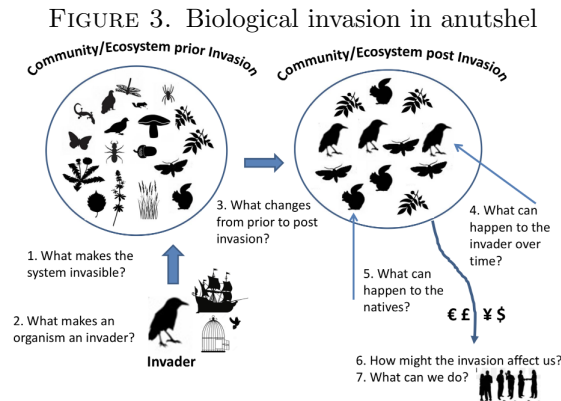
current moves away from the source. The phenomenon of reaction-diffusion, without source or by considering source as a natural reaction, is widely used in industry and engineering sciences. The reaction-diffusion equation in one dimension is presented as follows

$$u_t + vu_x = \rho u_{xx}, \quad (2.5)$$

where ρ is the diffusion coefficient. The Fishers equation is a reaction-diffusion equation. Figure 2 shows the reaction-diffusion process in a symbolic way.

2.2. Biological invasion. The term biological invasion is a common name for a variety of phenomena related to the introduction and spread of alien or exotic species, i.e., a species that has not been present in a given ecosystem before it is brought in. Biological invasion usually has dramatic consequences for the native ecological community. Invasion of alien species often results in virtual eradication of some native species, and now it is considered as one of the main reasons for biodiversity loss all over the world. Figure 3 can be a simple show for this phenomenon.





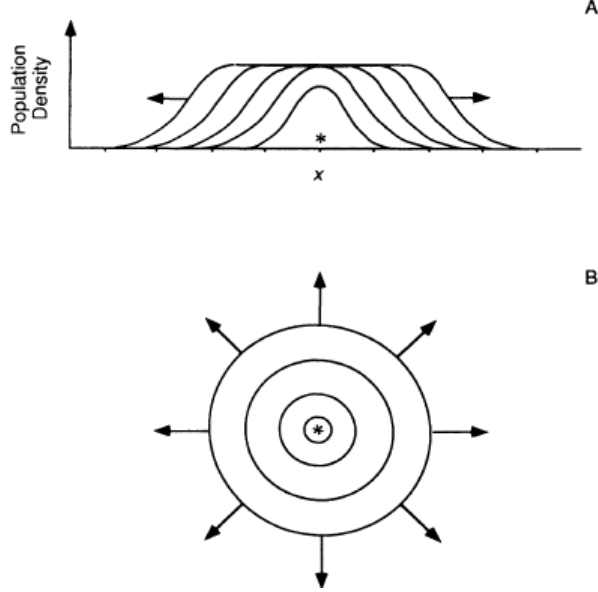
Biological invasion is one of the most challenging and important issues in contemporary ecology. It often causes considered the damage to agriculture or to aquaculture, in the case of marine ecosystems and thus it may result in pivotal economic losses as well. The impact of various biological and environmental factors, patterns, and rates of species spread has been under intensive study for a few decades. New effective tools and approaches have developed, important work has been done and considerable progress has been made towards a better understanding of this phenomenon.

Although a lot of results in regarding biological invasion were obtained through field studies and analysis of field observations, recent advances could hardly be possible without extensive use of mathematics, in particular, mathematical modeling. The reason for this has its roots in the very nature of the problem. A regular study based on manipulated field experiments is very difficult due to the virtual impossibility of reproducing the environmental and initial conditions. Laboratory experiments are often not effective due to the inconsistency of spatial scales. In these simulations, mathematical modeling takes to some extent. The role is normally played by an experimental study in other natural sciences. It should also be mentioned that the issue of biological invasion has been an inspiration, for example, some classical works by Fisher (1937) and Kolmogorov et al. (1937), this subject has fascinated ever since and eventually became one of the cornerstones for contemporary nonlinear science. [10] Reaction-diffusion (RD) invasion models display more significant behavior when population growth is not exponential but instead is regulated by density-dependent mortality. RD invasion models produce traveling waves of invaders that spread out from their place at a constant speed and form. Figure 4 shows this subject. Populations spread across a region that was not previously occupied.

Many ecological phenomena may be modelled using apparently partial differential equations (PDEs) involving space and possibly time [5, 12]. Traveling waves are a customary feature of many RD models. A common and classic RD model of ecological import is the Fisher model [7] which is described by equation (2.2). Fishers equation is also used in auto catalytic chemical reactions [3]. So it seems the progress in understanding nature has always been tightly related to progress in mathematics.



FIGURE 4. Instance of travelling waves



3. NUMERICAL APPROACH

In this section, we use the collocation method with cubic B-spline to find an approximate solution $\tilde{U}(x, t)$ to the exact solution $u(x, t)$ in the form

$$\tilde{U}(x, t) = \sum_{i=-1}^{N+1} c_i(t) B_i(x) \quad (3.1)$$

where $c_i(t)$ are time-dependent quantities to be determined of boundary conditions and collocation form of the differential equations and $B_i(x)$ are the cubic B-spline basis functions:

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}) \\ (x - x_{i-2})^3 - 4(x - x_{i-1})^3 & x \in [x_{i-1}, x_i) \\ (x_{i+2} - x)^3 - 4(x_{i+1} - x)^3 & x \in [x_i, x_{i+1}) \\ (x_{i+2} - x)^3 & x \in [x_{i+1}, x_{i+2}) \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

The values of $B_i(x)$ and its derivatives in knot points are shown in Table 1

C	M
D	E

TABLE 1. B-spline and its derivatives values

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B'_i(x)$	0	$-\frac{3}{h}$	0	$\frac{3}{h}$	0
$B''_i(x)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

3.1. **B-Spline collocation method.** According to the [11], we have these approximations:

$$u(x_i, t) = \tilde{U}(x_i, t), \quad 0 \leq i \leq N \tag{3.3}$$

$$u'(x_i, t) = \tilde{U}'(x_i, t) + O(h^4), \quad 0 \leq i \leq N \tag{3.4}$$

and

$$\begin{aligned} u''(x_0, t) &= \frac{14\tilde{U}''(x_0, t) - 5\tilde{U}''(x_1, t) + 4\tilde{U}''(x_2, t) - \tilde{U}''(x_3, t)}{12} + O(h^4) & i = 0 \\ u''(x_i, t) &= \frac{\tilde{U}''(x_{i-1}, t) + 10\tilde{U}''(x_i, t) + \tilde{U}''(x_{i+1}, t)}{12} + O(h^4) & 1 \leq i \leq N \\ u''(x_N, t) &= \frac{14\tilde{U}''(x_N, t) - 5\tilde{U}''(x_{N-1}, t) + 4\tilde{U}''(x_{N-2}, t) - \tilde{U}''(x_{N-3}, t)}{12} + O(h^4) & i = N \end{aligned} \tag{3.5}$$

by using (3.1) in (3.5) we get

$$\begin{aligned} u''(x_0, t) &= \frac{14 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_0) - 5 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_1) + 4 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_2) - \sum_{i=-1}^{N+1} c_i(t) B''_i(x_3)}{12} \\ &+ O(h^4) \quad i = 0 \\ u''(x_i, t) &= \frac{\sum_{i=-1}^{N+1} c_i(t) B''_i(x_{i-1}) + 10 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_i) + \sum_{i=-1}^{N+1} c_i(t) B''_i(x_{i+1})}{12} \\ &+ O(h^4) \quad 1 \leq i \leq N \\ u''(x_N, t) &= \frac{14 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_N) - 5 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_{N-1}) + 4 \sum_{i=-1}^{N+1} c_i(t) B''_i(x_{N-2}) - \sum_{i=-1}^{N+1} c_i(t) B''_i(x_{N-3})}{12} \\ &+ O(h^4) \quad i = N \end{aligned} \tag{3.6}$$

Given that B-Spline functions for each node point x_i only in $i - 1$, i and $i + 1$ have value; According to table 1 and (3.6) we have:

$$\begin{aligned} u''(x_0, t) &= \frac{14c_{-1} - 33c_0 + 28c_1 - 14c_2 + 6c_3 - c_4}{2h^2} \\ u''(x_i, t) &= \frac{c_{i-2} + 8c_{i-1} - 18c_i + 8c_{i+1} + c_{i+2}}{2h^2} \\ u''(x_N, t) &= \frac{14c_{N+1} - 33c_N + 28c_{N-1} - 14c_{N-2} + 6c_{N-3} - c_{N-4}}{2h^2} \end{aligned} \tag{3.7}$$

After applying these approximations, discriminate equation (2.2) by Crank Nicolson scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} = v \frac{u^{n+1} - u^n}{2} + k \frac{(u(1-u))^n + (u(1-u))^{n+1}}{2} \tag{3.8}$$



$$b = \begin{bmatrix} r_1 g_1(t_n) - m_1 g_1(t_{n+1}) \\ z_1 g_1(t_n) + x_2 g_1(t_{n+1}) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ z_1 g_2(t_n) + x_2 g_2(t_{n+1}) \\ b_6 g_2(t_n) - a_6 g_2(t_{n+1}) \end{bmatrix},$$

and

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \end{bmatrix}.$$

Now, we can solve this $N + 1$ in $N + 1$ system in each level. But we need the first level C^0 , that obtain using B-spline approximation of the initial condition [8].

3.2. Stability of the scheme. Consider equation (3.10) using the approximations (3.1) and the assumptions $u^n = \gamma$, $1 - \frac{k\Delta t}{2} + k\Delta t\gamma = \phi_1$, $1 + \frac{k\Delta t}{2} = \phi_2$, and $\frac{v\Delta t}{h^2} = \lambda$. We have

$$\begin{aligned} & -\frac{\lambda}{4}c_{i-2}^{n+1} + (\phi_1 - 2\lambda)c_{i-1}^{n+1} + \left(\frac{9\lambda}{2} + 4\phi_1\right)c_i^{n+1} + (\phi_1 - 2\lambda)c_{i+1}^{n+1} - \frac{\lambda}{4}c_{i+2}^{n+1} = \\ & \frac{\lambda}{4}c_{i-2}^n + (\phi_2 + 2\lambda)c_{i-1}^n + \left(4\phi_2 - \frac{9\lambda}{2}\right)c_i^n + (\phi_2 + 2\lambda)c_{i+1}^n + \frac{\lambda}{4}c_{i+2}^n. \end{aligned} \quad (3.17)$$

Assume $c_j^n = A\delta^n e^{ij\theta h}$ where $i = \sqrt{-1}$

$$\begin{aligned} & -\frac{\lambda}{4}A\delta^{n+1}e^{i(j-2)\theta h} + (\phi_1 - 2\lambda)A\delta^{n+1}e^{i(j-1)\theta h} + \left(\frac{9\lambda}{2} + 4\phi_1\right)A\delta^{n+1}e^{ij\theta h} + \\ & (\phi_1 - 2\lambda)A\delta^{n+1}e^{i(j+1)\theta h} - \frac{\lambda}{4}A\delta^{n+1}e^{i(j+2)\theta h} = \frac{\lambda}{4}A\delta^n e^{i(j-2)\theta h} + \\ & (\phi_2 + 2\lambda)A\delta^n e^{i(j-1)\theta h} + \left(4\phi_2 - \frac{9\lambda}{2}\right)A\delta^n e^{ij\theta h} + (\phi_2 + 2\lambda)A\delta^n e^{i(j+1)\theta h} \\ & + \frac{\lambda}{4}A\delta^n e^{i(j+2)\theta h}. \end{aligned} \quad (3.18)$$

After simplify common factors from the sides of the equation we have

$$\begin{aligned} & \delta\left(-\frac{\lambda}{4}e^{i(-2)\theta h} + (\phi_1 - 2\lambda)e^{i(-1)\theta h} + \left(\frac{9\lambda}{2} + 4\phi_1\right) + (\phi_1 - 2\lambda)e^{i\theta h} - \frac{\lambda}{4}e^{i(2)\theta h}\right) = \\ & \frac{\lambda}{4}e^{i(-2)\theta h} + (\phi_2 + 2\lambda)e^{i(-1)\theta h} + \left(4\phi_2 - \frac{9\lambda}{2}\right) + (\phi_2 + 2\lambda)e^{i\theta h} + \frac{\lambda}{4}e^{i(2)\theta h}. \end{aligned} \quad (3.19)$$



So

$$\begin{aligned} \delta &= \frac{\frac{\lambda}{4}e^{i(-2)\theta h} + (\phi_2 + 2\lambda)e^{i(-1)\theta h} + (4\phi_2 - \frac{9\lambda}{2}) + (\phi_2 + 2\lambda)e^{i\theta h} + \frac{\lambda}{4}e^{i(2)\theta h}}{-\frac{\lambda}{4}e^{i(-2)\theta h} + (\phi_1 - 2\lambda)e^{i(-1)\theta h} + (\frac{9\lambda}{2} + 4\phi_1) + (\phi_1 - 2\lambda)e^{i\theta h} - \frac{\lambda}{4}e^{i(2)\theta h}} \\ &\Rightarrow \delta = \frac{\frac{\lambda}{2}\cos 2\theta h + 2(\phi_2 + 2\lambda)\cos\theta h + (4\phi_2 - \frac{9\lambda}{2})}{\frac{\lambda}{2}\cos 2\theta h + 2(\phi_1 - 2\lambda)\cos\theta h + (4\phi_2 + \frac{9\lambda}{2})} \end{aligned} \tag{3.20}$$

or $\delta = \frac{p}{q}$.

For stability of the scheme, we should have $|\delta| < 1$ so

$$\left| \frac{p}{q} \right| < 1 \Rightarrow \begin{cases} \frac{p}{q} < 1 & \Rightarrow q - p > 0 \\ \frac{p}{q} > -1 & \Rightarrow q + p > 0 \end{cases} \tag{3.21}$$

For

$$\begin{aligned} p + q &= 2(\phi_1 + \phi_2) + 4(\phi_1 + \phi_2) \\ M &= \phi_1 + \phi_2 = 2 + k\Delta t\gamma > 0 \\ &\Rightarrow p + q = 2M\cos\theta h + 4M \end{aligned}$$

In the worst case $\cos\theta h = -1$ and $p + q = 2M > 0$.

For

$$\begin{aligned} q - p &= -\lambda\cos 2\theta h + 9\lambda + 4(\phi_1 - \phi_2) + \cos\theta h(2\phi_1 - 8\lambda - 2\phi_2) \\ &= -\lambda(2\cos^2\theta h - 1) + 2(\phi_1 - \phi_2 - 4\lambda)\cos\theta h + 9\lambda + 4(\phi_1 - \phi_2) \\ &= -2\lambda\cos^2\theta h + 2(-k\Delta t + k\Delta t\gamma - 4\lambda)\cos\theta h + 10\lambda + 4(-k\Delta t + k\Delta t\gamma) \end{aligned}$$

In the worst case $\cos\theta h = 1$ and $q - p = 6k\Delta t(\gamma - 1) > 0$. So the scheme is unconditionally stable.

4. NUMERICAL RESULTS

Now, we demonstrate the effectiveness of this method by several numerical examples and for calculating error, we use norm 2 as below

$$L_2 = \sqrt{\sum_0^N |u_j - \tilde{U}_j|^2}. \tag{4.1}$$

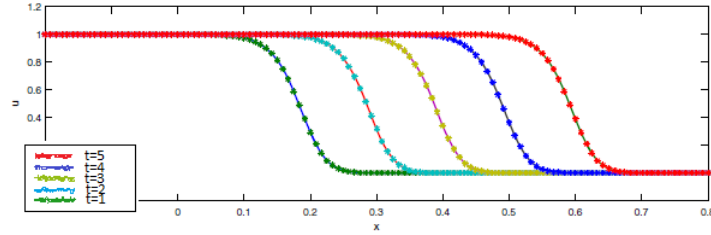
Example 1. We solve Fisher's equation (2.2) with the following exact solutions obtained by Wang [16] as:

$$u(x, t) = \frac{1}{(1 + e^{\sqrt{\frac{k}{6}}x - \frac{5k}{6}t})}. \tag{4.2}$$

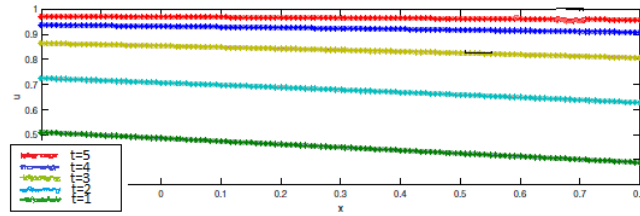
In figure 5 (a), results are displayed for $t = 0.001$ to $t = 0.003$ and in figure 5 (b), results are displayed for $t = 1$ to $t = 5$; in both case $N = 120$. According to figure 5 (a) and figure 5 (b) numerical results obtained from the method approximate exact solution very well and characters are matching and the results are satisfactory.



FIGURE 5. (a) Numerical solutions $t = 0.001$ to $t = 0.003$ (b) Numerical solutions $t = 1$ to $t = 5$

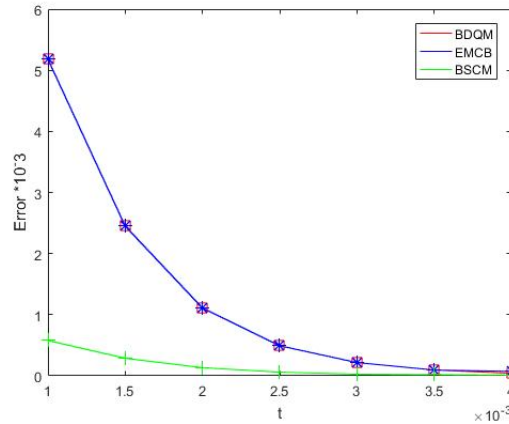


(a)



(b)

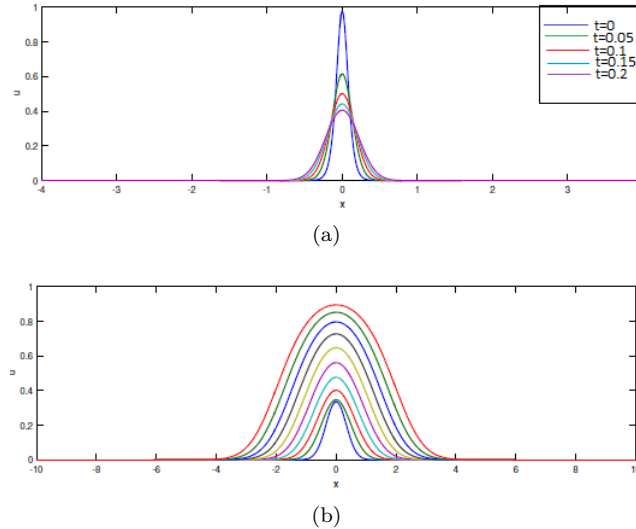
FIGURE 6. Comparison of errors between cubic trigonometric B-spline differential quadrature method (BDQM), Extended modified cubic B-spline method (EMCB) and B-spline collocation method(BSCM) for the Example 1



In figure 6, the comparison of errors (according to norm 2 in (4.1)) between cubic trigonometric B-spline differential quadrature method (BDQM) [14], extended modified cubic B-spline method(EMCB) [13] and B-spline collocation method (BSCM)



FIGURE 7. (a) Numerical solutions $t = 0$ to $t = 0.2$ and $\Delta t = 0.05$.
 (b) Numerical solutions $t = 0.5$ to $t = 5$ and $\Delta t = 0.5$



displayed. Based on the results in this chart, B-spline collocation method has less error than the extended modified cubic B-spline method and B-spline differential quadrature method.

Example 2. Consider Fisher’s equation (2.2) with $v = 0.1$, $k = 1$ with initial condition as bellow

$$u(x, 0) = \operatorname{sech}^2(10x) \tag{4.3}$$

from [9].

Figure 8(a) shows the results from $t = 0$ to $t = 0.2$ and $\Delta t = 0.05$ and figure 8(b) shows the results from $t = 0.5$ to $t = 5$ and $\Delta t = 0.5$. The results are a good approximation for the exact solution and this shows the effectiveness of the B-spline collocation method for this case.

Example 3. Consider Fisher’s equation (2.3) with $v = 1$, $\beta = 0.5, 1.5$, $\Delta t = 0.0001$, $h = 0.02$ and the following exact solution

$$u(x, t) = \frac{1}{2} + \frac{1}{2}\beta + \left(\frac{1}{2} - \frac{1}{2}\beta\right)\operatorname{tanh}\left(1 + \frac{1}{4}\sqrt{2}(-1 + \beta)(x - \theta t)\right)$$

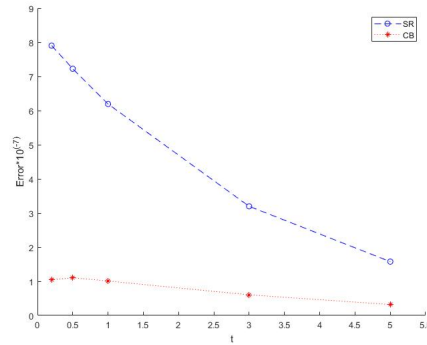
where $\theta = (1 + \beta)/\sqrt{2}$

In Table 2 obtained results with $\beta = 1.5$ are shown and compared with [15]. Also, in Figure 8 error diagram with $\beta = 0.5$ is drawn. The results show the high accuracy of the method.



TABLE 2. L_2 for example 3 with $\beta = 1.5$, $h = 0.02$ and $\Delta t = 0.0001$

t	[15]	peresent
0.02	7.054E-03	9.879E-08
0.5	6.210E-03	1.209E-07
1.0	3.335E-03	1.346E-07
3.0	3.767E-05	6.164E-08
5.0	3.760E-07	1.177E-07

FIGURE 8. Comparison of errors between symmetry reductions method (SR) [15] and cubic B-spline collocation method (CB) for Example 3 with $\beta = 0.5$ 

5. CONCLUSION

In this study, we use a numerical approach for solving the biological invasion in Fisher's equation. Fisher's equation is usually viewed as a population growth model. Biological invasion is one of the issues that can be described with partial differential equations (PDEs). Fisher's equation describes this phenomenon. So a collocation method based on cubic B-spline functions (BSCM) has been used to solve the biological invasion in Fisher's equation. This method was implemented for solving the Fisher equation. The unconditional sustainability of this method was proved using Von Neumann's approach. The numerical results compared with some other studies are acceptable and satisfactory. Also, the numerical results and the actual answer are acceptable to a very reasonable level.



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