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On the solving of matrix equation of Sylvester type

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Abstract Solutions of two problems related to the matrix equation of Sylvester type is given. In the first problem, the procedures for linear matrix inequalities are used to construct the solution of this equation. In the second problem, when a matrix is given which is not a solution of this equation, it is required to find such solution of the original equation, which most accurately approximates the given matrix. For this, an algorithm for constructing a general solution of the Sylvester matrix equation is used. The effectiveness of the proposed approaches is illustrated on the examples.

Keywords. Matrix equation, Linear matrix inequalities (LMI), Matrix Sylvester equation, Sylvester type matrix equation, Complex matrices.

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1. Introduction

In various problems of motion control an important place is occupied by questions connected to the development of algorithms for solving matrix equations (see, for example, [7], and references therein). Here it can be noted that the algorithms for constructing the solution of the Sylvester matrix equations continue to attract the attention of researchers [1-3, 6, 9]. Thus, in [9] two problems are considered for constructing a solution of a matrix equation of the Sylvester type:

$$AXD + CX^TD = E. \tag{1.1}$$

In (1.1) the sought matrix X has the dimension $m \times n$, the superscript "T" hereinafter means transposition. In the first problem, we need to find the solution of (1.1). The second problem is as follows. Let X_f be the given matrix, which is not a solution

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of (1.1). It is necessary to find a matrix X^*, belonging to the set S_r of solutions of (1.1), which minimizes the following norm

$$\|X^* - X_f\|_F = \min_{X \in S_r} \|X - X_f\|_F . \tag{1.2}$$

Hereinafter $\|\cdot\|_F$ means the Frobenius norm (Euclidean or spherical norm [8]).

Below the algorithms for solving these problems will be considered. Thus, to solve the first problem, an algorithm based on the use of procedures of linear matrix inequalities (LMI [4]) will be considered. For solving the second problem the approach described in [1] will be used.

2. General relations of LMI [4]

As noted in [4] (relations (2.3), (2.4)), the matrix inequality:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{2.1}$$

where the matrices $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, $S(x)$ linearly depend on x, is equivalent to the following matrix inequalities:

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \tag{2.2}$$

Consider the LMI:

$$\begin{bmatrix} Z & T \\ T^T & I \end{bmatrix} > 0, Z = Z^T, \tag{2.3}$$

which, according to (2.1), (2.2), can be written in the form

$$Z > TT^T. \tag{2.4}$$

Hereinafter I is the unit matrix of the corresponding size.
The relations (2.1) - (2.3) allow to consider the following standard LMI problem on eigenvalues (relations (2.9) (§2.2.2 [4])), namely, the problem of minimizing of a linear function cx(for example, $cx = tr(Z)$, where $tr(Z)$ is the trace of the matrix Z) under the conditions (2.3). To solve this problem, the mincx.m procedure of the MATLAB package [5] can be used.

3. The solution of equation (1.1)

We use the above relations for solving the first problem. Thus, it is necessary to find a matrix X, satisfying equation (1.1). Let

$$T = AXB + CX^TD - E$$

in (2.3). Using the procedure mincx.m of the MATLAB package, we minimize $tr(Z)$ in inequality (2.4). For a sufficiently small value $tr(Z) \cong 0$, it can be assumed that $T \cong 0$, consequently, the corresponding value of X is a solution of (1.1).
Let us illustrate the effectiveness of such algorithm for solving equation (1.1) by the following example.
Example 1. Suppose that in (1.1)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 8 & 9 \end{bmatrix}, E = \begin{bmatrix} 410 & 539 \\ 646 & 849 \\ 886 & 1163 \end{bmatrix}.$$

With these initial data, the exact solution of (1.1) will be the matrix

$$X_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Using the algorithm described above, we find the matrix X, to which correspond the following discrepancies:

$$n_c = \left\| AXB + CX^T D - E \right\|_F = 9.86 \cdot 10^{-13},$$

$$n_x = \|X - X_c\|_F = 1.48 \cdot 10^{-11}.$$

Thus, the given estimates of the accuracy of the obtained solution of equation (1.1) show the effectiveness of using the LMI procedures in this problems. However, to solve the second problem considered in [9], it is expedient to use an approach based on the procedure of the Kronecker product. As noted in [9], equation (1.1) can be represented as a system of linear algebraic equations:

$$H\, vec(X) = vec(E). \tag{3.1}$$

In (3.1) $H = B^T \otimes A + (D^T \otimes C)P(m, n)$,
where the symbol $\otimes$ denotes the Kronecker product ($A \otimes B = (a_{ij}B)$. see. [8]), and

$$vec(X) = (x_{11}, x_{12}, \dots, x_{m1}, x_{12}, x_{22}, \dots, x_{m2}, \dots, x_{mn})^T \in R^{mn}.$$

The matrix $P(m, n)$ is defined as follows:

$$P(m,n) = \begin{bmatrix} P_{11}^T & P_{12}^T & \cdots & P_{1n}^T \\ P_{21}^T & P_{22}^T & \cdots & P_{2n}^T \\ \vdots & \vdots & \vdots & \vdots \\ P_{m1}^T & P_{m2}^T & \cdots & P_{mn}^T \end{bmatrix} \in R^{mn \times mn} \tag{3.2}$$

In (3.2) the matrices P_{ij} of dimension $m \times n$, have the following structure: the element in the position (i, j) is 1, all others are zeros. Denoting $z = vec(X)$, $b = vec(E)$, we rewrite system (3.1) as follows

$$H\,z = b. \tag{3.3}$$

To construct the general solution of (3.3), one can use the approach of [1].

4. Algorithm for constructing the general solution (1.1) [1]

Thus, the problem of constructing a general solution of equation (1.1) reduces to the problem of constructing a general solution of the linear algebraic equation (3.3).

Consequently, the condition for the existence of a solution of (1.1) can be formulated as follows. For the existence of the solution (1.1), the matrices H and $[H \;\; b]$ must have the same rank [8] (to calculate the rank of the matrix one can use the rank.m procedure).

Let us perform the singular decomposition of the matrix H (procedure svd.m):

$$H = USV^T. \tag{4.1}$$

In (4.1) U, V are orthogonal matrices, S is a diagonal matrix, the first r (r is the rang of matrix H) elements of diagonal of which are not equal to zero. Let consider the matrix $U^T H = SV^T$. In connection with the above structure of the matrix S, only the first r rows of the matrix $U^T H$ will be non-zero. Denote by A_g the matrix formed from the first r rows of the matrix $U^T H$. Multiplying the left and right sides of equation (3.3) by the matrix U^T and leaving only the first r rows in both parts, we rewrite (3.3) as follows:

$$A_g z = b_u. \tag{4.2}$$

Here the vector b_u is formed from the first r components of the vector $U^T b$.

Note that appearing in (4.2) matrix A_g is the matrix of full rang. Therefore, to determine the general solution (3.3), we can use the relations [1]:

$$z = A_g^T \left(A_g A_g^T\right)^{-1} b_u + N\xi,$$

$$N = \left(I - A_g^T \left(A_g A_g^T\right)^{-1} A_g\right). \tag{4.3}$$

Here the first term on the right-hand side defines a particular solution of (3.3) having a minimal norm, ξ is a vector of free parameters, which determines the general solution of (3.3).

Let produce, similar to (4.1), the singular expansion of the matrix N:

$$N = U_n S_n V_n^T.$$

Let the first q diagonal elements of the matrix S_n are not equal to zero. Consequently, the matrix $NV_n = U_n S_n$ will have only the first q columns nonzero. We denote the matrix consisting of the first q columns of the matrix NV_n as N_q (defining the zero subspace of the matrix A_g). We will rewrite relation (4.3) as follows:

$$z = A_g^T \left(A_g A_g^T\right)^{-1} b_u + N_q \xi_q, \tag{4.4}$$

where the dimension of the free parameters vector ξ_q is q.

Defining the vector z according to (4.4) and using the reshape.m procedure, it is possible to construct a matrix X, using the vector z, which defines the general solution of (1.1).

Let us illustrate the procedure for constructing of the general solution of (1.1) by the following example.

Example 2. Suppose that in (1.1)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 4 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 50 & 92 \\ 138 & 260 \\ 150 & 272 \end{bmatrix}.$$

Corresponding to these initial data, the matrix $P_{(m,n)}$, according to (3.2), will have the form:

$$P_{(m,n)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix H, appearing in (3.1) has the following form:

$$H = \begin{bmatrix} 2 & 5 & 1 & 2 & 6 & 0 \\ 5 & 13 & 2 & 6 & 18 & 0 \\ 7 & 16 & 5 & 5 & 15 & 0 \\ 2 & 6 & 0 & 5 & 14 & 1 \\ 6 & 18 & 0 & 14 & 40 & 2 \\ 8 & 24 & 0 & 13 & 34 & 5 \end{bmatrix}.$$

Its rank is 4. The matrices A_g,$A_g^T\left(A_g A_g^T\right)^{-1}$appearing in (4.2), (4.3) and the vector b_u have the form:

$$A_g = \begin{bmatrix} -12.9143 & -36.4667 & -2.2762 & -21.2417 & -59.2648 & -4.4604 \\ -3.8814 & -7.3687 & -4.2753 & 1.8256 & 4.8419 & 0.6350 \\ -0.3851 & 1.3498 & -2.5052 & 0.6525 & -1.1156 & 3.0731 \\ -0.0883 & 0.2489 & -0.5139 & -0.1737 & -0.0137 & -0.5074 \end{bmatrix},$$

$$A_g^T\left(A_g A_g^T\right)^{-1} = \begin{bmatrix} -0.0024 & -0.0338 & -0.0199 & -0.1421 \\ -0.0066 & -0.0642 & 0.0697 & 0.4004 \\ -0.0004 & -0.0372 & -0.1294 & -0.8266 \\ -0.0039 & 0.0159 & 0.0337 & -0.2794 \\ -0.0108 & 0.0422 & -0.0576 & -0.0221 \\ -0.0008 & 0.0055 & 0.1587 & -0.8162 \end{bmatrix},$$

$$b_u = \begin{bmatrix} -440.0004 \\ -20.5351 \\ 6.4192 \\ -5.3579 \end{bmatrix}.$$

Further, we find that the matrix N_q appearing in (4.4) has the form

$$N_q = \begin{bmatrix} -0.9045 & -0.0000 \\ 0.3015 & 0.0000 \\ 0.3015 & -0.0000 \\ 0.0000 & -0.9045 \\ 0.0000 & 0.3015 \\ 0.0000 & 0.3015 \end{bmatrix} \tag{4.5}$$

and, correspondingly, the dimension of the free parameters vector q is equal to 2. To the minimal solution z_0 of the system (4.2), namely, to the solution at the zero value of the vector ξ_q in (4.4):

$$z_0 = A_g^T\left(A_g A_g^T\right)^{-1} b_u = [2.3636\ \ 2.5455\ \ 4.5455\ \ 3.0909\ \ 3.6364\ \ 5.6364]^T, \tag{4.6}$$

there corresponds the following solution of equation (1.1):

$$X_0 = \begin{bmatrix} 2.3636 & 3.0909 \\ 2.5455 & 3.6364 \\ 4.5455 & 5.6364 \end{bmatrix}. \quad (4.7)$$

If in (4.4) we select the following value of the vector ξ_q:

$$\xi_q = \begin{bmatrix} 1.507 \\ 1.206 \end{bmatrix},$$

then the matrix X will have the form

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}. \quad (4.8)$$

We note that both of matrices (4.7), (4.8) are solutions of equation (1.1).

5. The second problem of [9]

Thus, as noted in the Introduction, the second problem of [9] is as follows. Let a matrix X_f be given that does not belong to the of solution set of equation (1.1). It is necessary to find a matrix X^*, belonging to the set of solutions of equation (1.1), which most accurately approximates the matrix X_f (see the relation (1.2)).
Taking into account that for a matrix of dimension $n \times m$ (see §4.48 [6]):

$$\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2$$

it can be argued that

$$\|A\|_F^2 = \|vec(A)\|_2^2 .$$

Thus, the problem is reduced to approximation of the vector $vec(X_f)$, by the vectors which are defined by (4.4), i.e. to the corresponding selection of the vector of free parameters. In other words, it is necessary, choosing the elements of the vector ξ_q, to minimize the discrepancy of the following linear system:

$$N_q \xi_q = b_q, \quad (5.1)$$

$$b_q = vec(X_f) - A_g^T \left(A_g A_g^T \right)^{-1} b_u.$$

The corresponding solution of system (5.1) has the form (see §15.43 [8]):

$$\xi_q = (N_q^T N_q)^{-1} N_q^T b_q. \quad (5.2)$$

Note that the procedure of solving (5.2) is realized by the procedure "\" of the MATLAB package.
Thus, the expression for the matrix X^*, which most accurately approximates the matrix X_f has the form

$$vec(X^*) = A_g^T \left(A_g A_g^T\right)^{-1} b_u + N_q \xi_q = z_0 + N_q \xi_q, \tag{5.3}$$

where the vector ξ_q is defined by (5.2).
Let us illustrate the algorithm described above for solving the second problem of [9] in the following example.
Example 3. In this example, the original data coincides with the original data of Example 2. It is assumed that

$$X_f = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

This matrix does not satisfy equation (1.1). According to the results of Example 2, the matrix N_q is determined by the expression (4.5). Taking into account (5.1), we have, according to (4.7), the following expression for the vector b_q:

$$\mathrm{b_q} = -[\,1.3636 \quad 1.5455 \quad 3.5455 \quad 2.0909 \quad 2.6364 \quad 4.6364]^{\mathrm{T}}.$$

Then according to (5.2) we obtain the value for the vector ξ_q:

$$\xi_q = \begin{bmatrix} -0.3015 \\ -0.3015 \end{bmatrix}.$$

Using (5.3) and taking into account (5.2), we have:

$$\mathrm{vec(X^*)} = [2.6364 \quad 2.4545 \quad 4.4545 \quad 3.3636 \quad 3.5455 \quad 5.5455]^{\mathrm{T}},$$

$$X^* = \begin{bmatrix} 2.6364 & 3.3636 \\ 2.4545 & 3.5455 \\ 4.4545 & 5.5455 \end{bmatrix}.$$

Note that the matrix X^* satisfies equation (1.1) with accuracy of order 10^{-13}.

6. Conclusion

Solutions of the problems considered in [9], connected to the solution of the Sylvester type matrix equation, is given. In the first problem to construct the solution of this equation the procedures for linear matrix inequalities are used [4, 5]. In the second problem is used the algorithm [1] for construct a general solution of the Sylvester matrix equation. The effectiveness of the proposed approaches is illustrated by examples.

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