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Space-time radial basis function collocation method for one-dimensional advection-diffusion problem

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Abstract The parabolic partial differential equation arises in many application of technologies. In this paper, we propose an approximate method for solution of the heat and advection-diffusion equations using Laguerre-Gaussians radial basis functions (LG-RBFs). The results of numerical experiments are compared with the other radial basis functions and the results of other schemes to confirm the validity of the presented method.

Keywords. Radial basis functions, Heat and advection-diffusion equations, Laguerre-Gaussians functions.

2010 Mathematics Subject Classification. 65M99, 35K20.

1. Introduction

1.1. **Introduction of the problem.** Parabolic partial differential equations have a wide range of applications for mathematical modelling of many phenomena. Therefore, recently much attention has been paid in the literature to the analysis of accurate methods for the numerical solution of time-dependent partial differential equations. Consider the one-dimensional advection-diffusion equation

$$u_t(x,t) + \beta u_x(x,t) = \alpha u_{xx}(x,t), \quad 0 < x < L,\ 0 < t \le T, \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{1.2}$$

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and the boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad 0 < t \leq T, \tag{1.3}$$

where β is an arbitrary constant which shows the speed of convection and the diffusion coefficient, i.e. α is a positive constant. We assume that $f(x), g_0(t)$ and $g_1(t)$ are suitably given functions. Eq. (1.1) has been used to describe heat transfer in a draining film [25], thermal pollution in river systems [6], the dispersion of dissolved material in estuaries and coastal seas [22], contaminant dispersion in shallow lakes [42], long-range transport of pollutants in the atmosphere [53], water transfer in soils [41], dispersion of dissolved salts in groundwater [19] and flow in porous media [32]. Much efforts has been put into developing accurate numerical methods [28, 29] for the solution of (1.1). Mohebbi [37] proposed a class of new finite difference schemes for solving the one-dimensional heat and advection-diffusion equations. Dehghan [7] presented weighted finite difference techniques for solving this problem. In [48], a high-order compact boundary value method was employed for solution of the heat equations. In the present paper, a numerical scheme will be developed and compared for solving this equation.

1.2. **Introduction of the Radial Basis Function.** During the recent decades, many numerical methods have been designed for solving various types of problems [1, 2, 21, 23, 24]. Recently, the new advanced computational schemes called meshless methods have been widely employed to solve partial differential equations [4, 5, 49]). The meshless methods based on the radial basis functions are very powerful computational schemes to deal with high-dimensional problems or mathematical models with irregular domain. They are classified into two main categories: strong-form methods such as radial basis collocation schemes (Kansa's method) [15, 16, 26, 27, 35, 40] and weak-form methods such as radial point interpolation scheme [3, 8, 36, 44-47]. Recently, authors have developed and well-used some meshless methods for solving various types of problems [8-14].

To our best learning, researchers have introduced the space-time meshless formulation. Li and Mao published work on the space-time approach using RBFs [34]. They used the global collocation scheme using the Multiquadric function. Netuzhylov developed the space-time meshfree collocation scheme based upon the Interpolating Moving Least Squares (IMLS) method and used it to solve coupled problems with moving boundaries [38]. Young et al. [52] have applied time-dependent fundamental solutions to solve homogeneous diffusion equations. Their proposed method can be considered a space-time collocation scheme as it is free from time discretization.

Some well-known RBFs are listed in Table 1. The kind of RBFs, we will be mostly interested in, are the Gaussians $\phi(r) = e^{-\varepsilon^2 r^2}$. Other families of radial basis functions are the Laguerre-Gaussians. The definition of Laguerre-Gaussians functions family comes from the generalized Laguerre polynomials of degree n and order $s/2$ [39]. Laguerre-Gaussians are infinitely smooth, oscillatory functions and strictly positive definite. Specific examples are listed in Table 2. A numerical method using Laguerre-Gaussians functions was proposed for solving the one-dimensional heat equation subject to initial-boundary conditions in [30].

Table 1. Some well-known functions that generate RBFs.

Name of Radial Basis Function	Definition
Multiquadric(MQ)	$\phi(r)=\sqrt{\varepsilon^2+r^2}$
Inverse Quadratic(IQ)	$\phi(r)=\frac{1}{(\varepsilon^2+r^2)}$
Inverse Multiquadric(IMQ)	$\phi(r)=\frac{1}{\sqrt{\varepsilon^2+r^2}}$
Gaussian(GA)	$\phi(r)=e^{-\varepsilon^2 r^2}$
Thin Plate Splines(TPS)	$\phi(r)=r^2 log(r)$

In RBF theory, for a fixed basis function ϕ and shape parameter ε, good approxi-

Table 2. Laguerre-Gaussians radial functions.

s	n=1	n=2
1	$\phi(r)=(\frac{3}{2}-(\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\phi(r)=(\frac{15}{8}-\frac{5}{2}(\varepsilon r)^2+\frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$
2	$\phi(r)=(2-(\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\phi(r)=(3-3(\varepsilon r)^2+\frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$
3	$\phi(r)=(\frac{5}{2}-(\varepsilon r)^2)e^{(-\varepsilon r)^2}$	$\phi(r)=(\frac{35}{8}-\frac{7}{2}(\varepsilon r)^2+\frac{1}{2}(\varepsilon r)^4)e^{(-\varepsilon r)^2}$

mation quality requires small

$$h_X=\sup_{\chi}\min_{\chi_i}\|\chi-\chi_i\|_2,\quad \chi=(\mathbf{x},t)$$

and fine stability needs large

$$q_X=1/2\min_{1\le i,j\le N}\|\chi_j-\chi_i\|_2,\quad \chi=(\mathbf{x},t).$$

However one cannot minimize h_X and maximize q_X at the same time which is referred to as uncertainty relation in [43]. In the all RBFs consider here, the small shape parameter ε decreases (h_X and subsequently) the error of approximation solution and increases (q_X and subsequently) the condition number, and contrariwise. However many researchers have attempted to develop algorithms for choosing optimal values of the shape parameter but the optimal choice of the shape parameter is still an open question and it is most often selected by brute force. For example, Franke [18] suggested $\varepsilon^2=1.25D/\sqrt{N}$ in MQ basis, where D is the diameter of the smallest circle containing all data points and N is the number of data points. Hardy [20] recommended the use of $\varepsilon^2=0.815d$ where $d=(1/N)\sum_{i=1}^{N}d_i$ and d_i is the distance from the data point x_i to its nearest neighbor. Recently, Fornberg developed a Contour-Padé algorithm that is capable of stably computing the RBF approximation for all $\varepsilon>0$ [17].

2. Radial basis function

2.1. **Definition of RBF.** Let $\mathbb{R}^+=\{x\in\mathbb{R},x\ge 0\}$, $\|.\|_2$ denotes the Euclidean norm and $\phi:\mathbb{R}^+\to\mathbb{R}$ be a continuous function with $\phi(0)\ge 0$. A radial basis function on $\mathbb{R}^d$ is a function of the form:

$$\phi(\|\mathbf{x}-\mathbf{x}_i\|),$$

which depends only on the distance between $\mathbf{x} \in \mathbb{R}^d$ and a fixed point $\mathbf{x}_i \in \mathbb{R}^d$. So that the radial basis function ϕ_i is radially symmetric about the center $\mathbf{x}_i$. Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_N \in \Omega \subset \mathbb{R}^d$ be a given set of scattered data. Let r be the Euclidean distance between a fixed point $\mathbf{x}_i \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$, i.e. $\|\mathbf{x} - \mathbf{x}_i\|_2$.
The standard RBFs are categorized into two major classes [14, 31]:

- Class 1. Infinitely smooth RBFs: These basis functions are infinitely differentiable and heavily depend on the shape parameter ε (such as multiquadric (MQ), Gaussian (GA), inverse multiquadric (IMQ) and inverse quadric (IQ)).
- Class 2. Infinitely smooth (except at centers) RBFs: The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in Class 1 (such as Thin plate Spline (TPS)).

We have the following theorem about the convergence of RBFs interpolation.

Theorem 2.1. *Assume $\{\mathbf{x}_i\}_{i=1}^N$ are N nodes in $\Omega \subset \mathbb{R}^d$ which is convex, let:*

$$h = \max_{\mathbf{x}\in\Omega} \min_{1\leq i\leq N} \|\mathbf{x} - \mathbf{x}_i\|_2,$$

when $\hat{\phi}(\eta) < c(1+ \mid \eta \mid)^{-2l+d}$, for any y satisfing $\int (\hat{y}(\eta))^2/\hat{\phi}(\eta) d\eta < \infty$, we have:

$$\|y_N^{(\alpha)} - y^{(\alpha)}\| < ch^{l-\alpha},$$

where ϕ is RBFs and the constant c depends on the RBFs, $\hat{\phi}$ and $\hat{y}$ are supposed to be the Fourier transforms of ϕ and y respectively, $y^{(\alpha)}$ denotes the αth derivative of y, y_N is the RBFs approximation of y, d is space dimension, l and α are nonnegative integers.

Proof. A complete proof is given by authors [50, 51]. □

2.2. **Function approximation.** Let $X = L^2([0, L] \times [0, T])$ and

$$\{\phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t), ..., \phi_{1M}(x,t), ..., \phi_{N0}(x,t), ..., \phi_{NM}(x,t)\} \subset X$$

be the set of RBFs and

$$H = span\{\phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t), ..., \phi_{1M}(x,t), ..., \phi_{N0}(x,t), ..., \phi_{NM}(x,t)\},$$

suppose that h be an arbitrary element in X. Since H is a finite dimensional vector space, h has the unique best approximation out of H as $h_{NM} \in H$, that is [33]:

$$\forall g \in H, \|h - h_{NM}\|_2 \leq \|h - g\|_2.$$

Since $h_{NM} \in H$, there exist unique coefficients $c_{00}, ..., c_{0M}, c_{10}, ..., c_{1M}, ..., c_{N0}, ..., c_{NM}$ such that:

$$h \simeq h_{NM} = \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x,t) = C^T\Phi_{NM}(x,t) = \Phi_{NM}^T(x,t)C,$$

where C and $\Phi_{NM}(x,t)$ are vectors with the form:

$$C = [c_{00}, ..., c_{0M}, c_{10}, ..., c_{1M}, ..., c_{N0}, ..., c_{NM}]^T, \tag{2.1}$$

$$\Phi_{NM}(x,t) = [\phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t)..., \phi_{NM}(x,t)]^T. \tag{2.2}$$

3. The solution of the problem via radial basis functions

In This section we present a numerical scheme to solve the one-dimensional advection-diffusion equations using the collocation method and Laguerre-Gaussian radial basis functions. Radial basis function methods are known as a mesh-less scheme for solving partial differential equations numerically. Other methods such as finite-difference methods are known as an efficient class of techniqes for solving PDEs, but there are some problems in using these methods. These schemes are efficient especially for solving problems with arbitrary geometry. But finding a body-fitted mesh is time-consuming and hard to use. Also, it is difficult to obtain results with high order of accuracy.

In the rest of this section we discuss the application of the radial basis functions for solving parabolic partial differential equation.

Let

$$u_t(x,t) + \beta u_x(x,t) = \alpha u_{xx}(x,t),\ (x,t) \in (0,L) \times (0,T], \tag{3.1}$$

with the following initial and boundary conditions:

$$u(x,t) = f(x),\ (x,t) \in (0,L) \times \{0\}, \tag{3.2}$$

$$u(x,t) = g_0(t),\ (x,t) \in \{0\} \times (0,T], \tag{3.3}$$

$$u(x,t) = g_1(t),\ (x,t) \in \{L\} \times (0,T]. \tag{3.4}$$

Let

$$\Xi = \{(x_i,t_j) | x_i = L\frac{i}{N}, t_j = T\frac{j}{M}, i = 0,1,\cdots,N, j = 0,1,\cdots,M\}. \tag{3.5}$$

Using a RBFs method, the solution of the problem is considered as

$$\widetilde{u}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x,t), \tag{3.6}$$

where c_{ij} are unknown which remain to be determined and $\phi_{ij}(x,t)$ is the Laguerre-Gaussians, i.e. $\phi_{ij}(x,t) = (2-\varepsilon^2((x-x_i)^2+(t-t_j)^2))e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}$. Now by the collocation approach we impose the approximate solution $\widetilde{u}$ to satisfy the differential equation and the initial and boundary conditions at $(x_i,t_j), i = 0,1,...,N, j = 0,1,...,M$. So, we have

$$\widetilde{u}_t(x_i,t_j) + \beta\widetilde{u}_x(x_i,t_j) = \alpha\widetilde{u}_{xx}(x_i,t_j),\ (x_i,t_j) \in (0,L) \times (0,T], \tag{3.7}$$

$$\widetilde{u}(x_i,t_j) = f(x_i),\ (x_i,t_j) \in (0,L) \times \{0\}, \tag{3.8}$$

$$\widetilde{u}(x_i,t_j) = g_0(t_j),\ (x_i,t_j) \in \{0\} \times (0,T], \tag{3.9}$$

$$\widetilde{u}(x_i,t_j) = g_1(t_j),\ (x_i,t_j) \in \{L\} \times (0,T], \tag{3.10}$$

Table 3. Computational results for Example 1.

x	Method [37]		Present method		
	$P_e = 1000$	$P_e = 10,000$	LG-RBF	MQ-RBF	IMQ-RBF
0.25	1.4e-06	1.0e-06	2.9e-09	1.2e-05	3.2e-06
0.50	1.7e-06	4.9e-07	5.8e-09	2.2e-05	6.3e-06
0.75	1.9e-06	6.9e-06	5.8e-09	4.0e-05	1.1e-05
1.00	9.7e-07	2.6e-05	1.8e-08	5.9e-05	1.6e-05
1.25	1.1e-07	7.0e-05	1.3e-08	8.3e-05	1.8e-05
1.50	2.0e-07	1.6e-04	3.7e-07	9.9e-05	1.5e-05
1.75	2.9e-07	2.7e-04	1.2e-07	8.9e-05	2.7e-06

which results a linear system of equations. Solving the resulted system, the unknown values $c_{ij}, i = 0, 1, ..., N, j = 0, 1, ..., M$ can be found. Similarly, we approximate the solution for MQ and IMQ basis functions.

4. Numerical examples

In this section we give some computational results of numerical experiments with the method based on the preceding sections, to support our theoretical discussion. In the process of computation, all the symbolic and numerical computations are performed by using Maple. The readers can see the efficiency of the proposed method from the provided figures and tables in the following examples.

Example 1. Consider Eqs. (1.1)-(1.3) with L = 2, T = 1 and

$$f(x) = \sin(x),\ g_0(t) = e^{-\alpha t}\sin(-\beta t),\ g_1(t) = e^{-\alpha t}\sin(1-\beta t), \tag{4.1}$$

which has the exact solution

$$u(x,t) = e^{-\alpha t}\sin(x - \beta t). \tag{4.2}$$

For this problem we put $\beta = 1$. In Table 3 we give the absolute errors with $dx = dt = 0.0714$ for LG-RBFs with $\varepsilon = 0.5$, and for MQ and IMQ basis functions with $\varepsilon = 5.3$ at final time $T = 1$. To compare our result we give the absolute errors for the Compact finite difference scheme [37]. Analytical and numerical solutions for $0 \le t \le 1$ and $T = 1$ are given in Figure 1.

Example 2. Consider the heat equation

$$u_t(x,t) = \frac{1}{\pi^2}u_{xx}(x,t), \tag{4.3}$$

with $L = 1, T = 1$ and

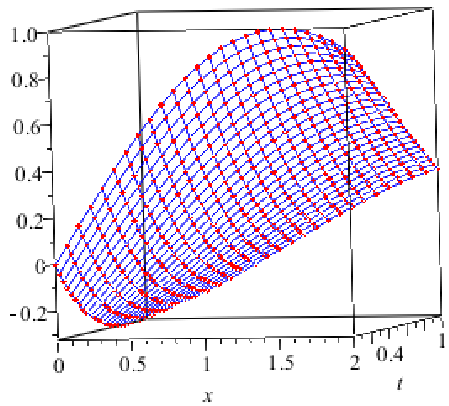
$$f(x) = \sin(\pi x),\ g_0(t) = 0,\ g_1(t) = 0, \tag{4.4}$$

which has the exact solution

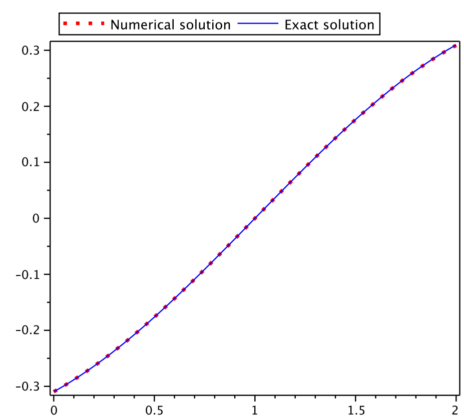
$$u(x,t) = e^{-t}\sin(\pi x). \tag{4.5}$$

In Table 4 we give the absolute errors for LG-RBFs with $dx = dt = 0.1$ and $\varepsilon = 0.5$, and with $dx = dt = 0.0667$, $dx = dt = 0.05$ and $\varepsilon = 0.4$ at final time $T = 1$. In Table

FIGURE 1. Analytical (line) and estimated (point) solutions with $dx = dt = 0.0714$ and $\varepsilon = 0.5$ for (a) $0 \leq t \leq 1$ and (b) $T = 1$ from Example 1.



(a)



(b)

5 maximum errors obtained for LG-RBF are presented. Also we give maximum errors for MQ and IMQ basis functions with $dx = dt = 0.1$ and $\varepsilon = 5.6$, and with $dx = dt = 0.05$ and $\varepsilon = 4.6$. We compared our method together with high-order compact

boundary value method [48] and compact finite difference scheme [37]. Analytical and numerical solutions for $0 \leq t \leq 1$ and $T = 1$ are given in Figure 2.

5. Conclusion

A RBF-based numerical method was proposed for solving the one-dimensional heat and advection-diffusion equations. The Laguerre-Gaussians radial basis functions (LG-RBFs) on interval $x \in [0, L]$ and $t \in [0, T]$ were employed. The method was based upon reducing the system into a set of algebraic equations. This algorithm proposed in the current paper was tested for MQ and IMQ functions on several examples from the literature. The obtained results showed that this approach using LG-RBFs can solve the problem effectively.

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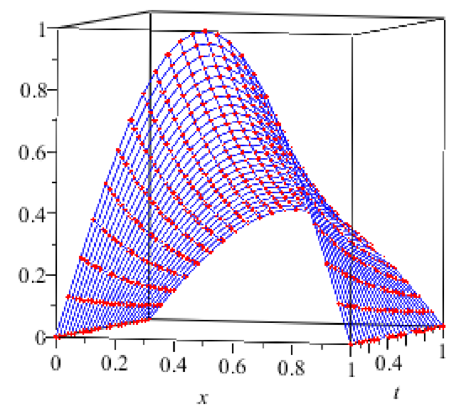
Table 4. Computational results for Example 2.

x	Present method with		
	$M = N = 10, \varepsilon = 0.5$	$M = N = 15, \varepsilon = 0.4$	$M = N = 20, \varepsilon = 0.4$
0.1	1.4239e-07	3.8969e-09	1.5686e-10
0.2	2.7072e-07	1.0942e-08	8.6829e-11
0.3	3.7207e-07	1.3111e-08	8.8855e-11
0.4	4.3692e-07	1.0763e-08	1.6711e-10
0.5	4.5931e-07	1.0829e-08	1.0856e-10
0.6	4.3732e-07	2.6629e-09	2.8712e-10
0.7	3.7285e-07	1.2411e-08	9.8855e-11
0.8	2.7173e-07	6.1578e-09	7.8830e-11
0.9	1.4323e-07	1.4603e-08	2.1134e-11

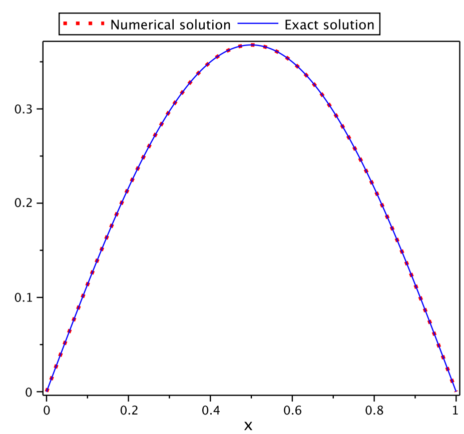
Table 5. Maximum errors obtained for Example 2.

$M = N$	CBVM [48]	method [37]	Present method with		
			LG-RBF	MQ-RBF	IMQ-RBF
10	1.5e-05	1.5e-05	4.6e-07	3.6e-08	9.9e-07
20	9.5e-07	9.4e-07	2.9e-10	1.3e-09	2.7e-09

FIGURE 2. Analytical (line) and estimated (point) solutions with $dx = dt = 0.1$ and $\varepsilon = 0.5$ for (a) $0 \leq t \leq 1$ and (b) $T = 1$ from Example 2.



(a)



(b)

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