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Numerical studies of non-local hyperbolic partial differential equations using collocation methods

Khalid Karam Ali*
Mathematics Department, Faculty of Science, Al-Azhar University,
Nasr City (11884), Cairo, Egypt.
E-mail: khalidkaram2012@yahoo.com.

Kamal Raslan Raslan
Mathematics Department, Faculty of Science, Al-Azhar University,
Nasr City (11884), Cairo, Egypt.
E-mail: kamal_raslan@yahoo.com

Adel Rashad Hadhoud
Mathematics Department, Faculty of Science,
Menoufia University, Shebein El-Koom, Egypt.
E-mail: adelhadhoud_2005@yahoo.com

Abstract The non-local hyperbolic partial differential equations have many applications in sciences and engineering. A collocation finite element approach based on exponential cubic B-spline and quintic B-spline are presented for the numerical solution of the wave equation subject to nonlocal boundary condition. Von Neumann stability analysis is used to analyze the proposed methods. The efficiency, accuracy and stability of the methods are assessed by applying it to the test problem. The results are found to be in good agreement with known solutions and with existing collocation schemes in literature.

Keywords. Collocation methods, Exponential cubic B-spline, Quintic B-spline, Finite difference, Wave equation.

2010 Mathematics Subject Classification.

1. Introduction

Many physical phenomena are modeled by non-classical hyperbolic boundary value problems with nonlocal boundary conditions. In place of the classical specification of boundary data, we impose a nonlocal boundary condition. Partial differential equations with non-local boundary conditions have received much attention in the last twenty years.

In this paper we will consider a non-classic hyperbolic equation [3, 24].

We consider the following problem of this family of equations:

$$u_{tt} - \mu\, u_{xx} = f(x,\, t), \qquad a \le x \le b, \quad 0 \le t \le T, \tag{1.1}$$

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∗ Corresponding author.

with the initial condition

$$u(x,0) = g(x), \qquad a \le x \le b, \tag{1.2}$$

and the boundary conditions

$$u(a,t) = p_1(t), \qquad 0 \le t \le T, \tag{1.3}$$

$$u(b,t) = p_2(t), \qquad 0 \le t \le T, \tag{1.4}$$

$$u_x(a,t) = h(t), \qquad 0 \le t \le T, \tag{1.5}$$

and the non-local boundary condition

$$\int_a^b u(x,t)\,dx = m(t), \tag{1.6}$$

where the functions $f(x,\, t)$, $g(x)$, $p_1(t)$, $p_2(t)$, $h(t)$ and $m(t)$ and the parameter μ are known. The exact solutions and numerical solutions of linear and nonlinear partial differential equations and nonlinear systems of partial differential equations are very important in applied science, for example, exact solutions of coupled GEWE and the space-time fractional RLW and MRLW equations they were obtained by Raslan et al. [20, 23], the KDV equation solved by Raslan et al.[18, 22], the Hirota-Satsuma coupled KDV equation studied by Raslan et al. [13]. Also, the Hirota equation has been solving by Raslan et al. [17, 19]. The generalized long wave equation system has been solved by El- Danaf et al.[6, 7]. The coupled-BBM system has been solved by Raslan et al. [12]-[14]. Coupled Burgers' equations has been studied by Ali et al.[9] and by Raslan et al. [15, 16].

There are many studies of our equation such as, Ang solved the problem using a scheme based on an integro-differential equation and local interpolating functions [1]. Then, B-spline functions were found to be an efficient method for solving wave equation such as, Dehghan et al [4], Khury et al. [10], Goh et al. [8], Caglar et al. [2] and Zin et al.[11]. Also, M. Dehghan and A. Shokri used a meshless method [5].

In this paper, we apply the exponential cubic B-spline and Quintic B-spline methods for computing an approximate solution to (1.1)-(1.6).

This paper is organized as follows. Section 2 is description of the collocation methods for solving equations (1.1)-(1.6). In section 3, some numerical results are presented to demonstrate the efficiency of the methods. Some concluding remarks are presented in section 4.

2. Description of collocation B-spline methods

In this section, we discuss collocation B-spline methods for solving numerically the one-dimensional hyperbolic equation (1.1). To construct numerical solution, consider nodal points (x_j, t_n) defined in the region $[a,b] \times [0,T]$ where

$$a = x_0 < x_1 < ... < x_N = b, \qquad h = x_{j+1} - x_j = \frac{b-a}{N},\ j = 0,1,...,N.$$

$$0 = t_0 < t_1 < ... < t_n < ... < T, \qquad t_n \ = n\Delta t, \qquad n = 0,1,.... \ .$$

2.1. **Description of exponential cubic B-spline method.**

Our approach for one-dimensional hyperbolic equation using collocation method with exponential cubic B-spline is to seek an approximate solution as

$$U_N(x,t) = \sum_{j=-1}^{N+1} c_j(t)\ B_j(x), \tag{2.1}$$

where $c_j(t)$ are to be determined for the approximated solutions $U_N(x,t)$, to the exact solutions $u(x,t)$ at the point (x_j, t_n) whilst $B_j(x)$ are exponential cubic B-spline basis functions defined by

$$B_j(x) = \begin{cases} b_2\left((x_{j-2}-x) - \frac{1}{p}\left(\sinh\left(p\left(x_{j-2}-x\right)\right)\right)\right), & x_{j-2} \le x \le x_{j-1}\,, \\ a_1 \ + \ b_1\left(x_j - x\right) \ + c_1 \ \exp\ \left(p\left(x_j - x\right)\right) & \\ +d_1 \exp\ \left(-p\left(x_j - x\right)\right) & x_{j-1} \le x \le x_j, \\ a_1 \ + \ b_1\left(x - x_j\right) \ + c_1 \ \exp\ \left(p\left(x - x_j\right)\right) & \\ +d_1 \exp\ \left(-p\left(x - x_j\right)\right), & x_j \le x \le x_{j+1}\,, \\ b_2\left((x - x_{j+2}) - \frac{1}{p}\left(\sinh\left(p\ \left(x - x_{j+2}\right)\right)\right)\right), & x_{j+1} \le x \le x_{j+2}\ , \\ 0 & otherwise\,, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} a_1 &= \frac{p\,h\,c}{p\,h\,c - s}, \qquad b_1 = \frac{p}{2}\left[\frac{c\,(c-1)+s^2}{(p\,h\,c-s)\,(1-c)}\right], \\ b_2 &= \frac{p}{2\,(p\,h\,c-s)},\ \ c_1 = \frac{1}{4}\left[\frac{\exp\left(-p\,h\right)(1-c)+s\left(\exp\left(-p\,h\right)-1\right)}{(p\,h\,c-s)\,(1-c)}\right], \\ d_1 &= \frac{1}{4}\left[\frac{\exp\left(p\,h\right)(1-c)+s\left(\exp\left(p\,h\right)-1\right)}{(p\,h\,c-s)\,(1-c)}\right], \\ s &= \sinh\left(p\,h\right),\ \ c = coh\ \left(p\,h\right),\quad h = \frac{b-a}{N}. \end{aligned}$$

Using approximate function (2.1) and exponential cubic B-spline functions (2.2), the approximate values $U(x)$ and their derivatives up to second order are determined in terms of the time parameters $c_j(t)$ as

$$\begin{aligned} U_{\rm j} &= U(x_j) = m_1 c_{j-1} + c_j + m_1 c_{j+1}, \\ U'_j &= U'(x_j) = m_2 c_{j-1} - m_2 c_{j+1}, \\ U''_j &= U''(x_j) = m_3 c_{j-1} - 2m_3 c_j + m_3 c_{j+1}, \end{aligned} \tag{2.3}$$

where $m_1 = \frac{(s - p\,h)}{2\,(p\,h\,c-s)}$, $m_2 = \frac{p\,(1-c)}{2\,(p\,h\,c-s)}, m_3 = \frac{p^2 s}{2\,(p\,h\,c-s)}$.

To apply the proposed method, we can rewrite (1.1) as

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \mu\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,\ t) = 0,$$

we take the approximations$u(x,t) = U_j^n$and$v(x,t) = V_j^n$, then from famous Cranck-Nicolson scheme and forward finite difference approximation for the derivative t. We get

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} - \mu \left[\frac{{U_{xx}}_j^{n+1} + {U_{xx}}_j^n}{2} \right] - f(x_j,\, t_n) = 0, \tag{2.4}$$

where $k = \Delta t$ is the time step.

On substituting the approximate solution for U and its derivatives from Eq. (2.3) at the knots in Eq. (2.4) yields the following difference equation with the variables $c_j(t)$.

$$\begin{aligned} &A_1 c_{j-1}^{n+1} + A_2 c_j^{n+1} + A_1 c_{j+1}^{n+1} = \\ &\quad A_3 c_{j-1}^n + A_4 c_j^n + A_3 c_{j+1}^n - m_1 c_{j-1}^{n-1} - c_j^{n-1} - m_1 c_{j+1}^{n-1} + k^2 f(x_j,\, t_n), \end{aligned} \tag{2.5}$$

where $A_1 = m_1 - \dfrac{k^2\mu}{2} m_3, \qquad A_2 = 1 + k^2\mu\, m_3, \qquad A_3 = 2m_1 + \dfrac{k^2\mu}{2} m_3,$
$A_4 = 2 - k^2 \mu\, m_3$.

It is clear that the system (2.5) consists of $(N+1)$ linear equations in the $(N+3)$ unknown $(c_{-1}^n, c_0^n, ..., c_N^n, c_{N+1}^n)^T$. Hence, the following two additional equation from the boundary conditions given in (1.3) and (1.6) are needed for calculation.

(1)

$$m_1 c_{-1}^n + c_0^n + m_1 c_1^n = p_1(t_n), \tag{2.6}$$

(2)

$$m_2 c_{N-1}^n - m_2 c_{N+1}^n - m_2 c_{-1}^n + m_2 c_1^n = m''(t_n) - \int_a^b f(x, t_n)\, dx. \tag{2.7}$$

Then the system is tridiagonal matrix system of dimension $(N+3) \times (N+3)$ that can be solved by any algorithm.

The above system requires two initial time levels at $t = 0$, and $t = \Delta t = k$, we can evaluate the initial conditions at $t = 0$ and $t = k$ as

At level time $n = 0 \quad (t = 0)$

$$U_N(x_j, 0) = \sum_{j=-1}^{N+1} c_j^0\, B_j(x) = m_1 c_{j-1}^0 + c_j^0 + m_1 c_{j+1}^0 = g(x), j = 0, 1, ..., N. \tag{2.8}$$

At level time $n = 1 \quad (t = k)$

$$U_N(x_j, k) = \sum_{j=-1}^{N+1} c_j^1\, B_j(x) = m_1 c_{j-1}^1 + c_j^1 + m_1 c_{j+1}^1 = g(x), j = 0, 1, ..., N. \tag{2.9}$$

It is clear that the systems (2.8) and (2.9) consists of $(N+1)$ linear equations in the $(N+3)$ unknowns $(c_{-1}^0, c_0^0, ..., c_N^0, c_{N+1}^0)^T, (c_{-1}^1, c_0^1, ..., c_N^1, c_{N+1}^1)^T$.

Hence, to solve these systems we can be using (2.6) and (2.7) at two time levels at $n = 0$, and $n = k$. Then the system is tridiagonal matrix system of dimension $(N+3) \times (N+3)$ that can be solved by any algorithm.

TABLE 1. Numerical errors at the grid points for various mesh sizes with $k = 0.01$ at $t = 5$.

x	$h = 0.1$, $p = 3.456 \times 10^{-5}$ Exponential cubic B-spline	$h = 0.02$ $p = 3.456 \times 10^{-5}$ Exponential cubic B-spline	$h = 0.01$ $p = 3.456 \times 10^{-5}$ Exponential cubic B-spline	$h = 0.02$ Quintic B-spline	$h = 0.01$ Quintic B-spline
0.1	1.97090 E-3	3.83207 E-5	2.14171 E-5	1.64398 E-4	5.73333 E-5
0.2	2.88999 E-3	5.97271 E-5	2.86301 E-5	2.71918 E-4	9.57796 E-5
0.3	2.71061 E-3	5.83163 E-5	2.53084 E-5	2.70807 E-4	9.63597 E-5
0.4	1.61636 E-3	3.55555 E-5	1.45997 E-5	1.65951 E-4	5.93989 E-5
0.5	3.1214 E-16	5.71213E-16	4.16733E-16	1.99285E-16	3.54061E-15
0.6	1.61636 E-3	3.55555 E-5	1.45997 E-5	1.65951 E-4	5.93989 E-5
0.7	2.71061 E-3	5.83163 E-5	2.53084 E-5	2.70807 E-4	9.63597 E-5
0.8	2.88999 E-3	5.97271 E-5	2.86301 E-5	2.71918 E-4	9.57796 E-5
0.9	1.97090 E-3	3.83207 E-5	2.14171 E-5	1.64398 E-4	5.73333 E-5

TABLE 2. Compare absolute errors of our schemes with absolute errors of [8] and [11] at $k = 0.01$ and $t = 5$.

x	$h = 0.01$ $p = 3.456 \times 10^{-5}$ Exponential cubic B-spline	$h = 0.01$ Quintic B-spline	$h = 0.01$[8] cubic B-spline	$h = 0.01$[11]
0.1	2.14171 E-5	5.73333 E-5	7.97 E-5	7.39 E-5
0.2	2.86301 E-5	9.57796 E-5	1.21 E-4	1.12 E-4
0.3	2.53084 E-5	9.63597 E-5	1.15 E-4	1.07 E-4
0.4	1.45997 E-5	5.93989 E-5	6.88 E-5	6.40 E-5
0.5	4.16733E-16	3.54061E-15	2.03E-13	5.05E-15
0.6	1.45997 E-5	5.93989 E-5	6.88 E-5	6.40 E-5
0.7	2.53084 E-5	9.63597 E-5	1.15 E-4	1.07 E-4
0.8	2.86301 E-5	9.57796 E-5	1.21 E-4	1.12 E-4
0.9	2.14171 E-5	5.73333 E-5	7.97 E-5	7.39 E-5

2.1.1. *Stability analysis of the method.*

Our stability analysis will be based on the Von-Neumann concept in which the growth factor of a typical Fourier mode defined as

$$c_j^n = \zeta^n \exp(ij\phi), \tag{2.10}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where, $\phi = k\,h$, k is the mode number, $i=\sqrt{-1}$ and g is the amplification factor of the schemes. We will apply the stability of the exponential cubic scheme (2.5).Thus, $f(x,t)$ in (1.1) is assumed to be 0.

Substituting (2.10) into the difference (2.5), we get

$$(2A_1\cos\ (\phi)+A_2)\ g^2-(2A_3\cos\ (\phi)+A_4)\ g+(2m_1\cos\ (\phi)+1)=0\,, \tag{2.11}$$

Let
$A=(2A_1\cos\ (\phi)+A_2)$, $B=(2A_3\cos\ (\phi)+A_4)$, $C=(2m_1\cos\ (\phi)+1)$,

Then, Eq. (2.11) becomes

$$A\,g^2-B\ g+C=0\,, \tag{2.12}$$

Applying the Routh–Hurwitz criterion on Eq. (2.12), the necessary and sufficient condition for Eq. (2.5) to be stable as follows:

Using the transformation $g=\dfrac{1+\nu}{1-\nu}$ and simplifying, Eq. (2.12) takes the form [25]

$$(A+B+C)\,\nu^2+2\,(A-C)\ \ \nu+(A-B+C)\ =0.$$

The necessary and sufficient condition for $|g|\le 1$ is that

$$(A+B+C)\ge 0,\quad (A-C)\ge 0,\quad (A-B+C)\ge 0.$$

It can be easily proved that

$$\begin{aligned}&(A+B+C)=8m_1\ \cos\ (\phi)+4\ge 0,\\&(A-C)=k^2\mu\,m_3\,(1-\cos\ (\phi))\ge 0,\\&(A-B+C)=2k^2\mu\,m_3\,(1-\cos\ (\phi))\ge 0.\end{aligned}$$

It is evident that the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on h and, but h should be chosen in such a way that the accuracy of the scheme is not degraded

2.2. **Description of quintic B-spline method.**

The quintic B-spline basis functions at knots are given by:

$$B_j(x)=\frac{1}{h^5}\begin{cases}(x-x_{j-3})^5\,, & x_{j-3}\le x\le x_{j-2}\\(x-x_{j-3})^5\ -6\,(x-x_{j-2})^5, & x_{j-2}\le x\le x_{j-1}\\(x-x_{j-3})^5\ -6\,(x-x_{j-2})^5\ +15(\,x-x_{j-1})^5, & x_{j-1}\le x\le x_j\\(-x+x_{j+3})^5\ +6\,(x-x_{j+2})^5\ -15(\,x-x_{j+1})^5\,, & x_j\le x\le x_{j+1}\\(-x+x_{j+3})^5\ +6\,(x-x_{j+2})^5\ , & x_{j+1}\le x\le x_{j+2}\\(-x+x_{j+3})^5, & x_{j+2}\le x\le x_{j+3}\\0 & otherwise\end{cases} \tag{2.13}$$

Expressing $U(x,t)$ by using quintic B-spline functions $B_j(x)$ and the time dependent parameters$c_j(t)$ for $U(x,t)$, the approximate solution can be written as:

$$U_N(x,t)=\sum_{j=-2}^{N+2}c_j(t)\ B_j(x). \tag{2.14}$$

Using approximate function (2.14) and quintic B-spline functions (2.13), the approximate values $U(x)$ and their derivatives up to second order are determined in terms of the time parameters $c_j(t)$, as

$$
\begin{aligned}
U_{\mathrm{j}} &= U(x_j) = c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2},\\
U'_j &= U'(x_j) = \frac{5}{h}(c_{j+2} + 10c_{j+1} - 10c_{j-1} - c_{j-2}),\\
U''_j &= U''(x_j) = \frac{20}{h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}).
\end{aligned}
\tag{2.15}
$$

On substituting the approximate solution for U and its derivatives from Eq. (2.15) at the knots in Eq. (2.4) yields the following difference equation with the variables$c_j(t)$.

$$
\begin{aligned}
&B_1c_{j-2}^{n+1} + B_2c_{j-1}^{n+1} + B_3c_j^{n+1} + B_2c_{j+1}^{n+1} + B_1c_{j+2}^{n+1} = B_4c_{j-2}^n + B_5c_{j-1}^n\\
&+B_6c_j^n + B_5c_{j+1}^n + B_4c_{j+2}^n - c_{j-2}^{n-1} - 26c_{j-1}^{n-1} - 66c_j^{n-1} - 26c_{j+1}^{n-1}\\
&\quad - c_{j+2}^{n-1} + k^2 f(x_j,\, t_n),
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
&B_1 = 1 - \frac{20k^2\mu}{2h^2}, \qquad B_2 = 26 - \frac{20k^2\mu}{h^2}, \qquad B_3 = 66 + \frac{120k^2\mu}{2h^2},\\
&B_4 = 2 + \frac{20k^2\mu}{2h^2},\ B_5 = 52 + \frac{20k^2\mu}{h^2},\ B_6 = 132 - \frac{120k^2\mu}{2h^2}.
\end{aligned}
$$

It is clear that the system (2.16) consists of $(N+1)$ linear equations in the$(N+5)$ unknown $(c_{-2}^n, c_{-1}^n, c_0^n, ..., c_N^n, c_{N+1}^n, c_{N+2}^n)^T$. Hence, the following four additional equation from the boundary conditions given in (1.3), (1.4), (1.5) and (1.6) are needed for calculation.

(1)

$$
c_{-2}^n + 26c_{-1}^n + 66c_0^n + 26c_1^n + c_2^n = p_1(t_n), \tag{2.17}
$$

(2)

$$
c_{N-2}^n + 26c_{N-1}^n + 66c_N^n + 26c_{N+1}^n + c_{N+2}^n = p_2(t_n), \tag{2.18}
$$

(3)

$$
\frac{5}{h}(c_2^n + 10c_1^n - 10c_{-1}^n - c_{-2}^n) = h\left(t_n\right), \tag{2.19}
$$

(4)

$$
\begin{aligned}
&\frac{5}{h}(c_{N+2}^n + 10c_{N+1}^n - 10c_{N-1}^n - c_{N-2}^n) - \frac{5}{h}(c_2^n + 10c_1^n - 10c_{-1}^n - c_{-2}^n)\\
&\qquad = m''(t_n) - \textstyle\int_a^b f(x, t_n)\, dx.
\end{aligned}
\tag{2.20}
$$

Then the system is penta-diagonal matrix system of dimension $(N+5)\times(N+5)$ that can be solved by any algorithm.

The above system requires two initial time levels at $t = 0$, and $t = \Delta t = k$, we can evaluate the initial conditions at $t = 0$ and $t = k$ as

At level time $n = 0 \quad (t = 0)$

$$\begin{aligned} U_N(x_j, 0) &= \sum_{j=-2}^{N+2} c_j^0 B_j(x) \\ &= c_{j-2}^0 + 26c_{j-1}^0 + 66c_j^0 + 26c_{j+1}^0 + c_{j+2}^0 = g(x), \end{aligned} \tag{2.21}$$

$j = 0, 1, ..., N.$

At level time $n = 1 \quad (t = k)$

$$\begin{aligned} U_N(x_j, k) &= \sum_{j=-2}^{N+2} c_j^1 \; B_j(x) \\ &= \frac{5}{h}\left(-c_{j-2}^1 - 10c_{j-1}^1 + 10c_{j+1}^1 + c_{j+2}^1\right) = g(x), \end{aligned} \tag{2.22}$$

$j = 0, 1, ..., N.$

It is clear that the systems (2.21) and (2.22) consists of $(N + 1)$ linear equations in the $(N + 5)$ unknown $(c_{-2}^0, c_{-1}^0, ..., c_{N+1}^0, c_{N+2}^0)^T$, $(c_{-2}^1, c_{-1}^1, ..., c_{N+1}^1, c_{N+2}^1)^T$. Hence, to solve these systems we can be using (2.17), (2.18), (2.19) and (2.20) at two time levels at $n = 0$, and $n = k$. Then the system is penta-diagonal matrix system of dimension $(N + 5) \times (N + 5)$ that can be solved by any algorithm.

Figure 1. Space-time graph of analytical solution of our problem using quintic B-spline with $h = 0.01$ and $k = 0.01$.

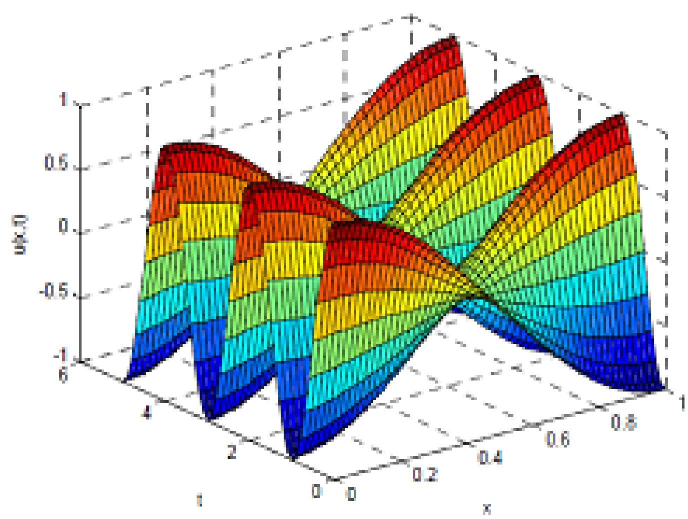


Figure 2. Space-time graph of numerical solution of our problem using quintic B-spline with $h = 0.01$ and $k = 0.01$.

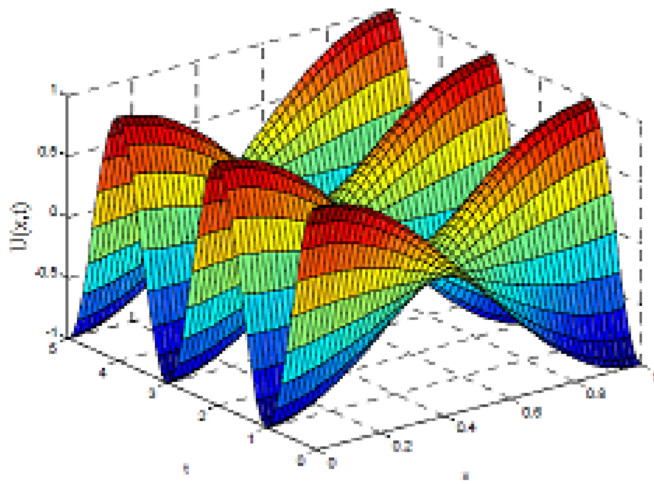
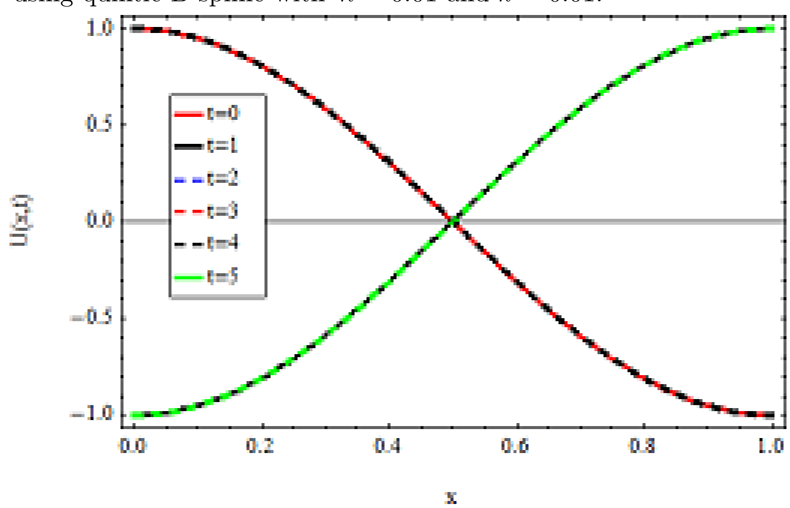


Figure 3. Space-time graph of numerical solution of our problem using quintic B-spline with $h = 0.01$ and $k = 0.01$.



2.2.1. *Stability analysis of the method.*

Our stability analysis will be based on the Von-Neumann concept in which the growth factor of a typical Fourier mode defined as

$$c_j^n = \zeta^n \exp(ij\phi), \tag{2.23}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where, $\phi = k\,h$, k is the mode number, $i = \sqrt{-1}$ and g is the amplification factor of the schemes. We will apply the stability of the quintic scheme (2.16).

Substituting (2.23) into the difference (2.16), we get

$$\begin{aligned}&(2B_1 \cos\ (2\phi) + 2B_2 \cos\ (\phi) + B_3)\ g^2 - (2B_4 \cos\ (2\phi) + \\ &2B_5 \cos\ (\phi) + B_6)\, g + (2 \cos\ (2\phi) + 52 \cos\ (\phi) + 66) = 0\,.\end{aligned} \tag{2.24}$$

Let

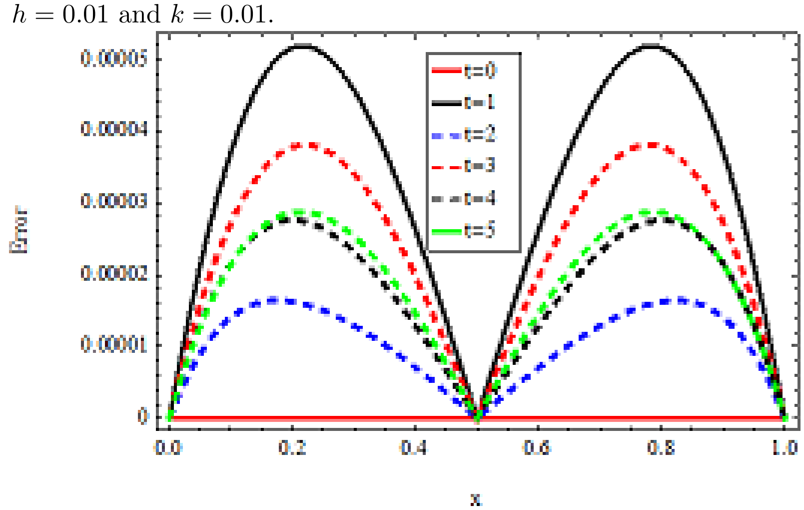
$$\begin{aligned} D &= (2B_1 \cos\ (2\phi) + 2B_2 \cos\ (\phi) + B_3)\ , \\ E &= (2B_4 \cos\ (2\phi) + 2B_5 \cos\ (\phi) + B_6)\ , \\ F &= \quad (2 \cos\ (2\phi) + 52 \cos\ (\phi) + 66)\ . \end{aligned}$$

Then, Eq. (2.24) becomes

$$Dg^2 - E\,g + F = 0. \tag{2.25}$$

Applying the Routh–Hurwitz criterion on Eq. (2.25), the necessary and sufficient condition for Eq. (2.16) to be stable as follows:

Figure 4. Absolute error of our problem using quintic B-spline with $h = 0.01$ and $k = 0.01$.



Using the transformation $g = \dfrac{1+\nu}{1-\nu}$ and simplifying, Eq. (2.25) takes the form [25]

$$(D+E+F)\,\nu^2 + 2\,(D-F)\ \ \nu + (D-E+F)\ = 0.$$

The necessary and sufficient condition for $|g| \le 1$ is that

$$(D+E+F) \ge 0, \quad (D-F) \ge 0, \quad (D-E+F) \ge 0.$$

It can be easily proved that

$$\begin{aligned}
(D+E+F) &= 8\ \cos\ (2\phi) + 208\ \cos\ (\phi) + 264 \ge 0,\\
(D-F) &= \frac{20k^2\mu}{2h^2}\,(6 - 2\cos\ (2\phi) - 4\cos\ (\phi)) \ge 0,\\
(D-E+F) &= \frac{40k^2\mu}{2h^2}\,(6 - 2\cos\ (2\phi) - 4\cos\ (\phi)) \ge 0.
\end{aligned}$$

It is clear that the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on h and, but $h$should be chosen in such a way that the accuracy of the scheme is not degraded

3. Numerical Tests and Results of wave equation

In this section, we present numerical example to test validity of our schemes for solving wave equation.

We consider the wave equation is considered as [8]-[11]

$$u_{tt} - \mu\, u_{xx} = 0, \qquad 0 \le x \le 1, \quad 0 \le t \le T,$$

with the initial condition

$$u(x,0) = \cos(\pi\, x), \qquad 0 \le x \le 1,$$

and the boundary conditions

$$\begin{aligned}
&u(0,t) = \cos(\pi\, x), && 0 \le t \le T,\\
&u(1,t) = 0.5\left\{\cos\ \left[\pi\,(\,1+t)\right] + \cos\ \left[\pi\,(\,1-t)\right]\right\}, && 0 \le t \le T,\\
&u_x(0,t) = 0, && 0 \le t \le T,
\end{aligned}$$

and the non-local boundary condition

$$\int_0^1 u(x,t)\, dx = 0.$$

The analytical solution is given by

$$u(x,t) = 0.5\left\{\cos\ \left[\pi\,(\,x+t)\right] + \cos\ \left[\pi\,(\,x-t)\right]\right\}, 0 \le x \le 1, \quad 0 \le t \le T.$$

The space-time plot for this analytical solution and approximate solution obtained with $h = 0.01$ and $k = 0.01$ are shown in Figures 1, 2 and 3, respectively. The accuracy of the present methods is tested by calculating the absolute error of the problem. Figure 4 depicts the absolute error of the problem at different time level with $h = 0.01$ and $k = 0.01$. It can be seen that the errors decrease as time increases. Numerically, the absolute errors of this problem are listed in Table 1 at $t = 5$, the

Table 2 shows that the present methods gives smaller absolute error compare with [8] and [11].

4. Conclusion

In this work, a numerical methods incorporating finite difference approach with exponential cubic B-spline and quantic B-spline had been developed to solve one-dimensional wave equation. B-spline functions had been used to interpolate the solution in x-direction and finite difference approach had been applied to discretize the time derivative. Based on von Neumann stability analysis, this approach was proved to be unconditionally stable.

The problem was tested. It was found that the solutions are approximated very well. Tables 1 and 2 showe the errors obtained from present methods are less than the errors obtained from the methods proposed in literature. Hence, we conclude that this present methods approximate the solutions very well. Hence, we conclude that the quintic B-spline method approximates the solutions very well.

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