Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 6, No. 2, 2018, pp. 141-156

Numerical quasilinearization scheme for the integral equation form of the Blasius equation

Esmaeil Najafi
Department of Mathematics, Faculty of Science,
Urmia University, Urmia 51818-57561, Iran.
E-mail: e.najafi@urmia.ac.ir

Abstract The method of quasilinearization is an effective tool to solve nonlinear equations when some conditions on the nonlinear term of the problem are satisfied. When the conditions hold, applying this technique gives two sequences of coupled linear equations and the solutions of these linear equations are quadratically convergent to the solution of the nonlinear problem. In this article, using some transformations, the well-known Blasius equation which is a nonlinear third order boundary value problem, is converted to a nonlinear Volterra integral equation satisfying the conditions of the quasilinearization scheme. By applying the quasilinearization, the solutions of the obtained linear integral equations are approximated by the collocation method. Employing the inverse of the transformation gives the approximation solution of the Blasius equation. Error analysis is performed and comparison of results with the other methods shows the priority of the proposed method.

Keywords. Quasilinearization technique, Volterra integral equations, Blasius equation, collocation method.
2010 Mathematics Subject Classification. 76N20, 45D05, 65L60.

1. Introduction

The idea of the method of quasilinearization which is developed by Bellman and Kalaba [5, 6] provides existence results for a wide variety of nonlinear problems and when is combined with the technique of lower and upper solutions, generalized quasilinearization technique is derived that yields pointwise lower and upper estimates. The lower and upper estimates are the solutions of the corresponding linear problems that converge quadratically to the solution of the given nonlinear problem. This fruitful idea goes back to Chaplygin [8, 21].
The method of generalized quasilinearization is an effective tool to obtain lower or upper bounds for the solutions of nonlinear equations and then has been noticed in many areas of sciences that their arisen problems are nonlinear. In [11] Devi et al. employed this method to the Caputo fractional differential equations. Koleva [19] used quasilinearization in solving high order nonlinear differential equations. Also this method has been developed for solving reaction diffusion equations [27, 39]. In [40] Zhu et al. presented two iterative methods for solving the Falkner-Skan equation

Received: 5 June 2017 ; Accepted: 10 March 2018.

based on the quasilinearization method. This powerful method has attracted the attention of a number of researchers [4, 13, 18, 20, 22, 23, 29]. In the area of integral equations consider the following nonlinear equation:

$$u(t) = h(t) + \int_0^t k(t,s,u(s))ds. \tag{1.1}$$

The method of generalized quasilinearization is developed in [30] to solve Eq. (1.1). When k is nonincreasing in u and satisfies a Lipschitz condition, this technique offers coupled monotonic sequences of linear iterations

$$\begin{aligned} v_p(t) = h(t) + \int_0^t \Big(& k(t,s,w_{p-1}(s)) \\ & + k_u(t,s,w_{p-1}(s))\big(w_p(s) - w_{p-1}(s)\big)\Big)ds, \\ w_p(t) = h(t) + \int_0^t \Big(& k(t,s,v_{p-1}(s)) \\ & + k_u(t,s,w_{p-1}(s))\big(v_p(s) - v_{p-1}(s)\big)\Big)ds, \end{aligned}$$

uniformly and quadratically convergent to the unique solution of (1.1) where $v_0(t)$ and $w_0(t)$ are coupled lower and upper solutions of (1.1) defined in following. It is possible to employ numerical methods to solve these two coupled linear integral equations in a piecewise continuous polynomials space and to combine it with the iterative schemes (where with respect to their linearity and quadratically convergent is more rapid in convergence) to approximate the unique solution of the nonlinear integral equation (1.1), under some conditions on k as mentioned before.

To introduce the background of the Blasius equation, this problem arises when considering the two-dimensional steady flow of an incompressible fluid with a free-stream velocity, U_∞, over a semi-infinite flat plate which is parallel to the fluid flow. Since the viscous effects are limited to a thin layer near the surface, across which the velocity changes from zero at the wall to the main-stream velocity U_∞ at the edge of this layer, such a flow is called boundary layer flow [31, 33]. A theoretical analysis of the Blasius boundary layer problem was given by Weyl [38] for the first time. Many researchers in resent years have tried to solve this problem numerically. Gauss-Laguerre quadrature [34], Converting the Blasius problem to a BVP and then applying shooting method [26], ADM and ADMPade [15, 36], variational iterative method [37], Homotopy analysis method [24] and parameter iteration method [25] are of numerical methods that are applied on the Blasius equation. In this study by converting the Blasius problem to nonlinear integral equation (1.1), we give a new computational method for solving this problem. Arisen integral equation satisfies in the assumptions required to the method of quasilinearization. Idea of generalized quasilinearization provides this computational way to the Blasius problem and because of linearizaion property of this method, we avoid to face with nonlinear algebraic systems because we solve a series of linear integral equations in place of nonlinear one and it is possible to approximate the solution of Blasius equation in a large interval rather than the previous works.

The structure of this paper is as follows: In section 2 we convert the Blasius equation to a nonlinear Volterra integral equation. Section 3 contains a general framework for the idea of generalized quasilinearization used to solve the nonlinear integral equations. Section 4 shows using of step-by-step collocation method in a piecewise polynomials space to approximate the solution of the coupled linear integral equations. In section 5, discretization to a linear algebraic system and the convergence of the method are discussed. In section 6 the numerical solution of the Blasius equation is given.

2. Conversion of the Blasius Equation to an Integral Equation

Boundary layer flow of Newtonian fluids can be studied by means of the Navier-Stokes equations, which are derived from the laws of conservation of mass and of momentum. For a fluid flow with a velocity U_∞ over a semi-infinite flat plate, the Navier-Stokes equations and their boundary conditions become

$$\begin{cases} u_x + v_y = 0, \\ uu_x + vu_y = \nu u_{yy}, \\ u(x,0) = v(x,0) = 0,\ u(x,\infty) = U_\infty, \end{cases}$$

that are known as Prandtl boundary-layer equations [12]. Here u and v are the velocity components in the x and y directions, respectively, and ν is the kinematic viscosity. The goal in this problem of the flat plate is the shear at the plate, $u_y(x,0)$. Knowing this value, we can calculate the viscous drag on the plate. Blasius [14] in 1908 by introducing a stream function ψ such that $u = \psi_y$ and $v = -\psi_x$ and a similarity transformation

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}},\ f(\eta) = \frac{\psi}{\sqrt{\nu x U_\infty}},$$

converted the Prandtl boundary-layer equations to a boundary value problem (Blasius equation)

$$\begin{aligned} &2f'''(\eta) + f(\eta)f''(\eta) = 0, \\ &f(0) = f'(0) = 0,\ f'(\infty) = 1. \end{aligned}$$

Because of some difficulties in solving this boundary value problem, using transformation groups in [28, 2] defined by relations $\eta = (F'(\infty))^{\frac{1}{2}}t$ and $f = (F'(\infty))^{\frac{-1}{2}}F$ its equivalent initial value problem

$$2F'''(t) + F(t)F''(t) = 0, \tag{2.1}$$
$$F(0) = F'(0) = 0,\ F''(0) = 1. \tag{2.2}$$

is considered. Then, the value of the shear at the plate is derived by the relation

$$u_y(x,0) = \sqrt{\frac{U_\infty^3}{\nu x}} f''(0) = \sqrt{\frac{U_\infty^3}{\nu x}} F'(\infty)^{\frac{-3}{2}},$$

which shows that the second derivative of $f(\eta)$ at zero plays an important role in solving the problem. In the initial value problem (2.1)-(2.2) by letting $F''(t) = e^{-u(t)}$

we have

$$F'''(t) = -u'(t)e^{-u(t)},$$
$$F(t) = \int_0^t (t-s)e^{-u(s)}ds,$$

and by substituting in equation (2.1), this problem is converted to the following nonlinear second kind Volterra integral equation

$$u(t) = \frac{1}{4}\int_0^t (t-s)^2 e^{-u(s)}ds, \tag{2.3}$$

and to compute the value of the shear at the plate we need the following value

$$F'(\infty) = \int_0^\infty e^{-u(s)}ds,$$

to be computed. Integral equation (2.3) is a kind of integral equation (1.1) and to solve it we employ the generalized quasilinearization technique which yields two iterative schemes of linear Volterra integral equations. The solutions of these linear equations can easily be approximated using step-by-step collocation method which gives an approximation for $u(t)$, the solution of the equation (2.3). In the next two sections we explain these two steps.

3. Volterra Integral Inequalities and Generalized Quasilinearization

For $T \in \mathbb{R}$ and $T > 0$ let $J = [0, T]$ and $D = \{(t, s) \in J \times J : s \le t\}$. Consider the nonlinear integral equation

$$u(t) = h(t) + \int_0^t k(t, s, u(s))ds, \tag{3.1}$$

where $h \in C[J, \mathbb{R}]$ and $k \in C[D \times \mathbb{R}, \mathbb{R}]$.

Definition 3.1. A function $v \in C[J, \mathbb{R}]$ is called a lower solution of (3.1) on J if

$$v(t) \le h(t) + \int_0^t k(t, s, v(s))ds, \ \ t \in J,$$

and an upper solution, if the reversed inequality holds. If

$$w(t) \ge h(t) + \int_0^t k(t, s, v(s))ds, \ \ v(t) \le h(t) + \int_0^t k(t, s, w(s))ds, \ \ t \in J,$$

then v and w are said to be coupled lower and upper solutions of (2.1) on J.

There are interesting consequences about coupled lower and upper solutions of (3.1) where the following two theorems in [30] state them.

Theorem 3.2. *Assume that $h \in C[J, \mathbb{R}]$, $k \in C[D \times \mathbb{R}, \mathbb{R}]$ and $v_0(t), w_0(t) \in C[J, \mathbb{R}]$ are coupled lower and upper solutions of (3.1) and $k(t, s, u)$ is nonincreasing in u for each fixed pair $(t, s) \in D$ and satisfies one-sided Lipschitz condition*

$$k(t, s, \alpha) - k(t, s, \beta) \ge -L(\alpha - \beta), \ \ \alpha \le \beta, \ \ L > 0.$$

Then $v_0(0) \le w_0(0)$ *implies*

$$v_0(t) \le w_0(t),\ t \in J.$$

Let

$$\Omega = \{(t,s,u) \in D \times \mathbb{R}\ ;\ v_0(t) \le u \le w_0(t),\ t \in J\},$$

then if Theorem 3.2 holds and $k \in C[\Omega, \mathbb{R}]$, it is shown in [30] that there exists a unique solution $u(t)$ of (3.1) such that

$$v_0(t) \le u(t) \le w_0(t),\ t \in J.$$

By letting $\|u\| = \max_{t\in J} |u(t)|$ and defining two iterative schemes as coupled system of linear integral equations

$$\begin{aligned} v_p(t) = h(t) + \int_0^t \Big(& k(t,s,w_{p-1}(s)) \\ & + k_u(t,s,w_{p-1}(s))\big(w_p(s) - w_{p-1}(s)\big)\Big) ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} w_p(t) = h(t) + \int_0^t \Big(& k(t,s,v_{p-1}(s)) \\ & + k_u(t,s,w_{p-1}(s))\big(v_p(s) - v_{p-1}(s)\big)\Big) ds, \end{aligned} \tag{3.3}$$

for $p = 1, 2, \dots,$ and $v_0(t), w_0(t) \in C[J, \mathbb{R}]$, coupled lower and upper solutions of (2.1), the following theorem shows the quadratically convergence of two coupled sequences $\{v_p(t)\}$ and $\{w_p(t)\}$ derived from (3.2) and (3.3) to the unique solution of equation (3.1). It is needed to use vectorial inequalities which are understood to mean that the same inequalities hold between their corresponding components. In the whole of the work we refer to $\|\cdot\|$ as the maximum norm of the functions or matrices.

Theorem 3.3. *Suppose that*
(H_1) $v_0, w_0 \in C[J, \mathbb{R}]$, $v_0(t) \le w_0(t)$ *on* J, *are coupled lower and upper solutions of (2.1) on* J *respectively.*
(H_2) $k \in C^2[\Omega, \mathbb{R}]$, $k_u(t,s,u) \le 0$ *and the sign of* $k_{uu}(t,s,u)$ *does not change on* J *for* $(t,s,u) \in \Omega$.
Then the two coupled iterative schemes (3.2) and (3.3) define a nondecreasing sequence $\{v_p(t)\}$ *and a nonincreasing sequence* $\{w_p(t)\}$ *in* $C[J,\mathbb{R}]$ *such that* $v_p \longrightarrow u$ *and* $w_p \longrightarrow u$ *uniformly on* J, *and the following quadratic convergent estimate holds:*

$$\|r_p(t)\| \le \Big(TQ + \frac{T^2}{2} PQ \exp(PT)\Big) \|r_{p-1}(t)\|^2,$$

where

$$\|r_p(t)\|^i = \begin{bmatrix} \|u(t) - v_p(t)\|^i \\ \|w_p(t) - u(t)\|^i \end{bmatrix},\ i = 1, 2,$$

$$Q = \begin{bmatrix} 0 & M_1 \\ 2M_1 & M_1 \end{bmatrix},\ P = \begin{bmatrix} 0 & M_2 \\ M_2 & 0 \end{bmatrix},$$

and $M_1 = \max_\Omega k_{uu}$, $M_2 = \max_\Omega k_u$. *Also these two sequences satisfy the relation*

$$v_0 \leq v_1 \leq \cdots \leq v_p \leq w_p \leq \cdots \leq w_1 \leq w_0.$$

These two theorems constitute the basis of the generalized quasilinearization for integral equations. It is easy to see that the integral equation (2.3) satisfies in the assumptions of Theorems 3.2 and 3.3, then the method of generalized quasilinearization can be employed to this equation. We will need the following lemma in [10].

Lemma 3.4. *Suppose* E *is a matrix such that* $\|E\| < 1$. *Then* $(I - E)$ *is nonsingular and*

$$\|(I - E)^{-1}\| \leq (1 - \|E\|)^{-1}.$$

4. Collocation Method in the Piecewise Polynomials Space

Suppose $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ is a partition on J and let $h_n = (t_{n+1} - t_n)$, $n = 0, \cdots, N-1$, and $h = \max_n h_n$, then the above partition is denoted by J_h.

Definition 4.1. Suppose that J_h is a given partition on J. The piecewise polynomials space $\mathbb{S}_\mu^{(d)}(J_h)$ with $\mu \geq 0$, $-1 \leq d \leq \mu$ is defined by

$$\mathbb{S}_\mu^{(d)}(J_h) = \{q(t) \in C^d[J, \mathbb{R}^2] : q|_{\sigma_n} \in \pi_\mu \ \times \pi_\mu \ ; 0 \leq n \leq N-1\}.$$

Here $\sigma_n = (t_n, t_{n+1}]$ and π_μ denotes the space of polynomials of degree not exceeding μ and it is easy to see that $\mathbb{S}_\mu^{(d)}(J_h)$ is a real vector space and its dimension is given by

$$dim(\mathbb{S}_\mu^{(d)}(J_h)) = 2(N(\mu - d) + d + 1).$$

Now, consider the coupled system of linear integral equations (3.2) and (3.3). They may be shown in the form of

$$v_p(t) = H_p(t) + \int_0^t k_p(t,s) w_p(s) ds, \tag{4.1}$$

$$w_p(t) = G_p(t) + \int_0^t k_p(t,s) v_p(s) ds, \tag{4.2}$$

where

$$H_p(t) = h(t) + \int_0^t \Big(k(t,s,w_{p-1}(s)) - k_u(t,s,w_{p-1}(s)) w_{p-1}(s) \Big) ds, \tag{4.3}$$

$$G_p(t) = h(t) + \int_0^t \Big(k(t,s,v_{p-1}(s)) - k_u(t,s,w_{p-1}(s)) v_{p-1}(s) \Big) ds, \tag{4.4}$$

$$k_p(t,s) = k_u(t,s,w_{p-1}(s)), \ (t,s) \in D. \tag{4.5}$$

By letting

$$x_p(t) = \begin{bmatrix} v_p(t) \\ w_p(t) \end{bmatrix}, \ y_p(t) = \begin{bmatrix} H_p(t) \\ G_p(t) \end{bmatrix}, \ f_p(t,s) = \begin{bmatrix} 0 & k_p(t,s) \\ k_p(t,s) & 0 \end{bmatrix},$$

the coupled equations (4.1) and (4.2) is formulated in the following form of the system of equations:

$$x_p(t) = y_p(t) + \int_0^t f_p(t,s)x_p(s)ds. \tag{4.6}$$

The solution of this coupled system will be approximated by collocation method in the piecewise polynomials space

$$\mathbb{S}^{(-1)}_{m-1}(J_h) = \{q(t) \in C^{-1}[J, \mathbb{R}^2] : q|_{\sigma_n} \in \pi_{m-1} \ \times \pi_{m-1}; 0 \le n \le N-1\},$$

corresponding to the choices $\mu = m-1$ and $d = -1$ in Definition 4.1 which the polynomials are allowed to have finite jumps in the partition points. The collocation solution is denoted by

$$\hat{x}_p(t) = \left[\begin{array}{c} \hat{v}_p(t) \\ \hat{w}_p(t) \end{array} \right] \in \mathbb{S}^{(-1)}_{m-1}(J_h),\ p = 1, 2, \dots,$$

and defined by the system of collocation equations

$$\hat{x}_p(t) = y_p(t) + \int_0^t f_p(t,s)\hat{x}_p(s)ds,\ t \in X_h,\ p = 1, 2, \dots, \tag{4.7}$$

where X_h contains the collocation points

$$X_h = \{t_n + c_i h_n :\ 0 \le c_1 \le \dots \le c_m \le 1;\ 0 \le n \le N-1\}, \tag{4.8}$$

and is determined by the points of the partition J_h and the given collocation parameters $\{c_i\} \in [0, 1]$.

5. Discretization and Error Analysis

In this section the Lagrange polynomials based on collocation parameters $\{c_i\}$ are used to discretize the system (4.6) to a linear algebraic system of equations. When this polynomials are chosen as basis functions in each subinterval σ_n for the space $\mathbb{S}^{(-1)}_{m-1}(J_h)$, a comfort computational form for the collocation equations system (4.7) is obtained. Lagrange polynomials in σ_n are defined as

$$L_j(z) = \prod_{k \ne j}^{m} \frac{z - c_k}{c_j - c_k},\ z \in [0, 1],\ j = 1, 2, \dots, m, \tag{5.1}$$

where belong to π_{m-1}. Also set

$$X^p_{n,j} = \hat{x}_p(t_n + c_j h_n) = \left[\begin{array}{c} \hat{v}_p(t_n + c_j h_n) \\ \hat{w}_p(t_n + c_j h_n) \end{array} \right] = \left[\begin{array}{c} V^p_{n,j} \\ W^p_{n,j} \end{array} \right],\ j = 1, 2, \dots, m. \tag{5.2}$$

The collocation solution $\hat{x}_p(t) \in \mathbb{S}^{(-1)}_{m-1}(J_h)$ when is restricted to the subinterval σ_n is as

$$\hat{x}_p(t) = \hat{x}_p(t_n + z h_n) = \sum_{j=1}^{m} L_j(z) X^p_{n,j} = \left[\begin{array}{c} \sum_{j=1}^m L_j(z) V^p_{n,j} \\ \\ \sum_{j=1}^m L_j(z) W^p_{n,j} \end{array} \right]. \tag{5.3}$$

Now, by letting $t = t_{n,i} = t_n + c_i h_n$, the collocation equations system (4.7) has the following form

$$\hat{x}_p(t_{n,i}) = y_p(t_{n,i}) + \int_0^{t_n} f_p(t_{n,i}, s)\hat{x}_p(s)ds \\ + h_n \int_0^{c_i} f_p(t_{n,i}, t_n + sh_n)\hat{x}_p(t_n + sh_n)ds,$$

and by employing (5.2) and (5.3), we have

$$X^p_{n,i} = y_p(t_{n,i}) + F^n_p(t_{n,i}) \\ + h_n \sum_{j=1}^{m} \Big(\int_0^{c_i} f_p(t_{n,i}, t_n + sh_n) L_j(s)ds \Big) X^p_{n,j}, \tag{5.4}$$

for $i = 1, 2, \ldots, m$, where

$$F^n_p(t) = \int_0^{t_n} f_p(t, s)\hat{x}_p(s)ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 f_p(t, t_\ell + sh_\ell)\hat{x}_p(t_\ell + sh_\ell)ds. \tag{5.5}$$

$F^n_p(t)$ is known with respect to the collocation solution on $[0, t_n]$ and by applying representation (5.3) and setting $t = t_{n,i}$ in (5.5) has the form

$$F^n_p(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \Big(\int_0^1 f_p(t_{n,i}, t_\ell + sh_\ell) L_j(s)ds \Big) X^p_{\ell,j}.$$

By letting

$$\Phi^{\ell,p,n}_{i,j} = \int_0^1 f_p(t_{n,i}, t_\ell + sh_\ell) L_j(s)ds \\ = \begin{bmatrix} 0 & \int_0^1 k_p(t_{n,i}, t_\ell + sh_\ell) L_j(s)ds \\ \int_0^1 k_p(t_{n,i}, t_\ell + sh_\ell) L_j(s)ds & 0 \end{bmatrix},$$

and defining $2m \times 2m$ matrix

$$B^{\ell,n}_p = \begin{bmatrix} \Phi^{\ell,p,n}_{1,1} & \cdots & \Phi^{\ell,p,n}_{1,m} \\ \vdots & \ddots & \vdots \\ \Phi^{\ell,p,n}_{m,1} & \cdots & \Phi^{\ell,p,n}_{mm} \end{bmatrix}, \quad 0 \le \ell < n \le N - 1,$$

the term (5.5) is reduced to the $2m \times 1$ matrix

$$\mathbb{F}^n_p = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} B^{\ell,n}_p X^\ell_p, \quad i = 1, 2, \ldots, m,$$

$$X^\ell_p = \begin{bmatrix} X^p_{\ell,1} \\ \vdots \\ X^p_{\ell,m} \end{bmatrix}.$$

Now, by denoting

$$y_p = \begin{bmatrix} y_p(t_{n,1}) \\ \vdots \\ y_p(t_{n,m})] \end{bmatrix} = \Big[H_p(t_{n,1}), G_p(t_{n,1}), \cdots, H_p(t_{n,m}), G_p(t_{n,m})\Big]^T,$$

$$\begin{aligned}\Psi_{i,j}^{p,n} &= \int_0^{c_i} f_p(t_{n,i}, t_n + sh_n)L_j(s)ds \\ &= \begin{bmatrix} 0 & \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)L_j(s)ds \\ \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)L_j(s)ds & 0 \end{bmatrix},\end{aligned}$$

and putting the $2m \times 2m$ matrix

$$B_p^n = \begin{bmatrix} \Psi_{1,1}^{p,n} & \cdots & \Psi_{1,m}^{p,n} \\ \vdots & \ddots & \vdots \\ \Psi_{m,1}^{p,n} & \cdots & \Psi_{mm}^{p,n} \end{bmatrix}, \quad 0 \leq n \leq N-1,$$

the collocation equation (4.7) is transformed to the following algebraic system of linear equations

$$(I_{2m} - h_n B_p^n)X_p^n = y_p + \mathbb{F}_p^n, \ 0 \leq n \leq N-1, \ p = 1, 2, \ldots. \tag{5.6}$$

Here I_{2m} denotes the $2m \times 2m$ identity matrix. The existence and uniqueness of the solution for the system (5.6) or the collocation solution for (4.7) in $\mathbb{S}_{m-1}^{(-1)}(J_h)$ is considered in the following theorem.

Theorem 5.1. *If $y_p(t)$ and $f_p(t,s)$ in the Volterra integral equation (3.6) are continuous on their domains J and D respectively, then there exists an $\bar{h} > 0$ such that for any partition J_h with partition diameter h, $0 < h < \bar{h}$, the linear algebraic system (4.7) has a unique solution X_p^n for $0 \leq n \leq N-1$ and $p = 1, 2, \ldots$.*

Proof. The continuity of $y_p(t)$ and $f_p(t,s)$ is obvious. Since the components of $f_p(t,s)$ are continuous and its domain is compact, then the components of the matrix B_p^n for $0 \leq n \leq N-1$ and $p = 1, 2, \cdots$, are all bounded. These implies if h_n's are chosen sufficiently small, the inequality $h_n\|B_p^n\| < 1$ holds and by Lemma 3.4 the inverse of the matrix $(I_m - h_n B_p^n)$ exists. In other words, there is a $\bar{h} > 0$ so that for any partition J_h with $h = \max\{h_n;\ 0 \leq n \leq N-1\} < \bar{h}$ the matrix $(I_m - h_n B_p^n)$ has a uniformly bounded inverse. Then the collocation equation (4.7) has a unique solution. □

We can obtain a bound for $\|B_p^n\|$ and an estimation for $\bar{h}$. Consider the components of the matrix B_p^n. Since

$$k_p(t,s) = k_u(t, s, w_{p-1}(s)) \leq 0, \ (t,s) \in D,$$

then the kernel $k_p(t,s)$ has constant sign on J and the mean-value theorem of integral calculus gives

$$\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)L_j(s)ds = L_j(\xi)\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)ds, \tag{5.7}$$

for some $\xi \in [0, c_i]$. By attention to the continuity of $k_p(t,s)$ on D, the phrase $|\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)ds|$ has a finite bound as $K = \max |k_u(t,s,u)|$ on Ω and by letting $L = \max_j \|L_j(s)\|$ estimation $\bar{h} = \frac{1}{LK}$ is obtained.
When the unknown vector X_p^n is computed from (5.7), the collocation solution for $t = t_n + zh_n \in \bar{\sigma}_n = [t_n, t_{n+1}]$ is given by

$$\hat{x}_p(t) = y_p(t) + F_p^n(t) + h_n \sum_{j=1}^{m} \Big(\int_0^{c_i} f_p(t, t_n + sh_n) L_j(s) ds \Big) X_{n,j}^p.$$

The following theorem shows that the solution of system of collocation equations (4.7) is convergent to the solution of system of linear integral equations (4.6). The proof of Theorem when the approximated function is 1-dimensional, is showed in [7]. But its proof with some changes and definitions can be extended for 2-dimensional vector functions, thereby for the following theorem.

Theorem 5.2. *Suppose that in (4.6), $f_p \in C^i[D, \mathbb{R}^2]$ and $y_p \in C^i[J, \mathbb{R}^2]$, where $1 \le i \le m$, and $\hat{x}_p \in \mathbb{S}_{m-1}^{(-1)}(J_h)$ is the collocation solution of (4.6) with $h \in (0, \bar{h})$, defined by (4.7). If $x_p(t)$ is the exact solution of (4.6), then*

$$\|x_p - \hat{x}_p\| = \begin{bmatrix} \|v_p - \hat{v}_p\| \\ \|w_p - \hat{w}_p\| \end{bmatrix} \le C \|x_p^{(i)}\| h^i = C \begin{bmatrix} \|v_p^{(i)}\| \\ \|w_p^{(i)}\| \end{bmatrix} h^i,$$

holds on J, for any collocation points X_h. The constant C depends on the parameters $\{c_i\}$ but not on h.

The above argument provides an approximation solution $\hat{x}_p(t)$ to the unique solution of the system of linear integral equations (4.6) in the space $\mathbb{S}_{m-1}^{(-1)}(J_h)$ and the iterative schemes (3.2) and (3.3) produce two sequences $\{v_p(t)\}$ and $\{w_p(t)\}$ that are quadratically convergent to the unique solution of nonlinear integral equation (3.1). The inequality

$$\begin{bmatrix} \|u - \hat{v}_p\| \\ \|\hat{w}_p - u\| \end{bmatrix} \le \begin{bmatrix} \|u - v_p\| \\ \|w_p - u\| \end{bmatrix} + \begin{bmatrix} \|v_p - \hat{v}_p\| \\ \|w_p - \hat{w}_p\| \end{bmatrix}, \tag{5.8}$$

and Theorems 3.3 and 5.2 show that the sequences of the collocation solutions $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ is convergent to the unique solution $u(t)$ of nonlinear integral equation (3.1). In relation (5.8) the first term in the right hand side is quadratically convergent and the second term is of order $O(h^i), 1 \le i \le m$.

6. Numerical Solution of the Blasius Equation

In this section the presented method is applied to approximate the solution of the Blasius problem. Consider the converted Blasius problem in integral equation form

$$u(t) = \frac{1}{4} \int_0^t (t-s)^2 e^{-u(s)} ds, \ 0 \le t \le T. \tag{6.1}$$

The kernel $k = \frac{1}{4}(t-s)^2 e^{-u(s)}$ is nonincreasing and convex with respect to the u on $D \times \mathbb{R}$. It is easy to see that $v_0(t) = 0$ and $w_0(t) = \frac{t^3}{3}$ are coupled lower and upper solutions of (6.1) on $[0, T], T > 0$. This shows that the exact solution is between $v_0(t)$

Table 1. Differences between the upper and lower solutions, $|\hat{w}_p(t_i) - \hat{v}_p(t_i)|$.

	$m=3$, $N=20$			$m=4$, $N=20$		
t_i	$p=2$	$p=3$	$p=4$	$p=2$	$p=3$	$p=4$
0	0	0	0	0	0	0
2	3.7331 E -15	6.9388 E -18	6.9388 E -18	2.0122 E -16	8.6736 E -19	8.6736 E -19
4	1.0626 E -05	4.2188 E -15	5.5511 E -17	2.7791 E -10	2.7755 E -17	6.9388 E -18
6	4.0348 E -03	7.2240 E -10	4.4408 E -16	1.1200 E -06	3.3306 E -16	5.5511 E -17
8	2.9709 E -02	1.5106 E -08	1.7763 E -15	6.1259 E -05	6.3948 E -14	2.2204 E -16
10	8.2840 E -02	5.3608 E -08	7.1054 E -15	9.6275 E -04	7.2581 E -12	4.4408 E -16
12	1.6353 E -01	1.1660 E -07	1.4210 E -14	5.7787 E -03	5.1154 E -10	8.8817 E -15
14	2.7178 E -01	2.0410 E -07	2.8421 E -14	1.5131 E -02	4.1997 E -09	3.5527 E -15
16	4.0759 E -01	3.1610 E -07	1.4210 E -14	3.2229 E -02	1.3526 E -08	7.1054 E -15
18	5.7096 E -01	4.9056 E -07	2.8421 E -14	5.6216 E -02	2.8961 E -08	3.5528 E -15
20	7.6189 E -01	7.2667 E -07	3.2467 E -14	1.0045 E -01	9.0184 E -08	8.5427 E -15

and $w_0(t)$ and with respect to the Theorem 3.3 the solutions of the coupled iterative schemes

$$\begin{aligned} v_p(t) =& \frac{1}{4}\int_0^t (t-s)^2\big(1+w_{p-1}(s)\big)e^{-w_{p-1}(s)}ds \\ &-\frac{1}{4}\int_0^t (t-s)^2 e^{-w_{p-1}(s)} w_p(s)ds, \end{aligned} \tag{6.2}$$

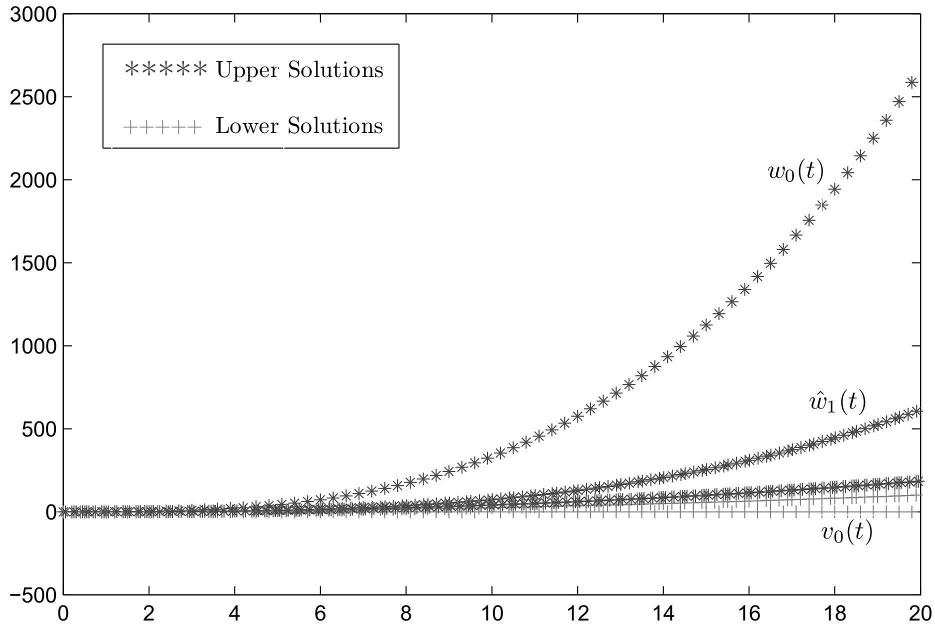
$$\begin{aligned} w_p(t) =& \frac{1}{4}\int_0^t (t-s)^2\big(e^{-v_{p-1}(s)}+v_{p-1}(s)e^{-w_{p-1}(s)}\big)ds \\ &-\frac{1}{4}\int_0^t (t-s)^2 e^{-w_{p-1}(s)} v_p(s)ds, \end{aligned} \tag{6.3}$$

for $p=1,2,\ldots,$ are convergent to the exact solution of (6.1). To employ the given numerical procedure for these two coupled linear integral equations, subintervals and the collocation parameters are chosen such that

$$\begin{aligned} &h = h_n = \frac{T}{N},\ n=0,1,\ldots,N-1, \\ &c_i = \frac{i-1}{m-1},\ i=1,2,\ldots,m. \end{aligned}$$

By getting $m=3,4$, $N=40$ and $T=20$ we have approximated the solutions of the linear equations (6.2) and (6.3) for $p=2,3,4$ and the absolute values of the differences between $\hat{v}_p(t)$ and $\hat{w}_p(t)$ are computed and shown in Table 1 and Figure 1. As mentioned before, the value of $f''(0)$ plays an important role in solving the problem. An accurate numerical solution of the Blasius equation is performed by Howarth [17], who calculated the initial slope $f''(0) = 0.332057$. Asaithambi [3] found this number correct to nine decimal positions as $f''(0) = 0.332057336$. By homotopy method, He [16] achieved 0.3095 in the first iteration step and in the second iteration step 0.3296. By Adomians decomposition method Abbasbandy [1] acquired $f''(0) = 0.333329$, also Tajvidi et al. [35] obtained $f''(0) = 0.33209$. Our relation for computation of its value

FIGURE 1. The convergence of the sequences $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ to the exact solution.



is

$$f''(0) = \Big(F'(\infty)\Big)^{\frac{-3}{2}} = \Big(\int_0^{\infty} e^{-u(s)} ds\Big)^{\frac{-3}{2}},$$

and for approximation, first we let $u(t) \simeq \varphi_p(t) = \frac{\hat{v}_p(t)+\hat{w}_p(t)}{2}$ and then compute

$$f''(0) \simeq \Big(\int_0^{20} e^{-\varphi_p(s)} ds\Big)^{\frac{-3}{2}}.$$

In this manner we obtain results shown in Table 2 where are more accurate in comparison with the other results. Using the relations

$$\begin{aligned}
&F(t) = \int_0^t (t-s)e^{-u(s)} ds \simeq F_p(t) = \int_0^t (t-s)e^{-\varphi_p(s)} ds,\\
&F'(t) = \int_0^t e^{-u(s)} ds \simeq F_p'(t) = \int_0^t e^{-\varphi_p(s)} ds,\\
&F''(t) = e^{-u(t)} \simeq F_p''(t) = e^{-\varphi_p(s)},\\
&\eta = (F'(\infty))^{\frac{1}{2}} t, \;\; f = (F'(\infty))^{\frac{-1}{2}} F,
\end{aligned}$$

we estimate the values of the $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ for $m = 4$, $N = 40$, $p = 5$ in a large interval $T = 20$ which are shown in the Tables 3 and 4. Also a comparison with

Table 2. Approximations of $f''(0)$ obtained by the presented method and the methods Parand [32] and Liao [24].

Our method		Parand [32]		Liao [24]	
p	$f''(0)$	order	$f''(0)$	order	$f''(0)$
1	0.32813953605	7	0.332061648	5	0.28098
2	0.33204248667	9	0.332058234	10	0.32992
3	0.33205738279	11	0.332057001	15	0.33164
4	0.33205733625	18	0.332057314	20	0.33198
5	0.33205733632	32	0.332057334	–	–
6	0.33205733633	–	–	–	–

Table 3. Approximation of $f(\eta)$ for the presented method with $m = 4$, $N = 40$, $p = 5$ and $T = 20$ and the solutions of Parand [32] and Cortell [9].

η	Our method	Parand [32]	Cortell [9]
0	0.0000000	0.0000000	0.00000
1	0.1655734	0.1655724	0.16557
2	0.6500308	0.6500351	0.65003
3	1.3968230	1.3968223	1.39682
4	2.3057710	2.3057618	2.30576
5	3.2833090	3.2832910	3.28330
6	4.2796670	4.2796435	4.27965
7	5.2792950	5.2792684	5.27927
8	6.2792810	6.2792336	6.27923
9	7.2792900	7.2792358	7.27925
10	8.2793120	–	–
12	10.279320	–	–
14	12.279330	–	–
16	14.279340	–	–
18	16.279350	–	–
20	18.279360	–	–

the other results (Parand [32] and Cortell [9]) has been given. Figure 2 shows the solution of the Blasius problem and its derivatives in the interval $[0, 20]$.

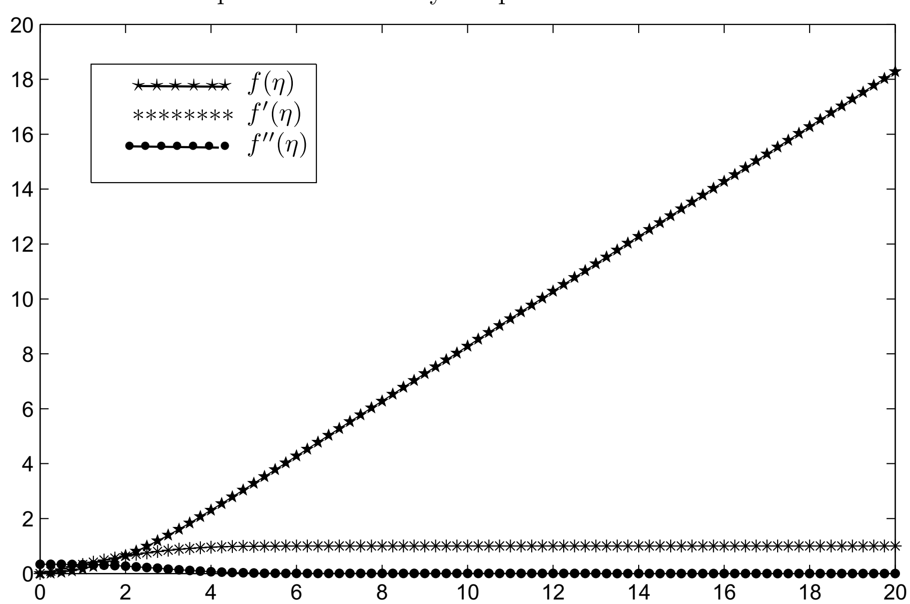
7. Conclusion

In this article, using a transformation we converted the Blasius equation to a nonlinear Volterra integral equation which is satisfied in the conditions of the quasi-linearization technique. We applied this technique and derived two sequences of the coupled linear integral equations with solutions quadratically convergent to the solution of the nonlinear equation. Using collocation method, we approximated these solutions and obtain an approximation solution for the Blasius equation. Comparing

TABLE 4. Approximation of $f'(\eta)$ and $f''(\eta)$ for the presented method with $m = 4$, $N = 40$ and $p = 5$ and the solutions of Parand [32] and Cortell [9].

	Results for $f'(\eta)$			Results for $f''(\eta)$		
η	Our method	Parand [32]	Cortell [9]	Our method	Parand [32]	Cortell [9]
0	0.0000000	0.0000000	0.00000	0.3320573	0.3320571	0.33206
1	0.3297851	0.3297963	0.32978	0.3230071	0.3230136	0.32301
2	0.6297758	0.6297763	0.62977	0.2667516	0.2667557	0.26675
3	0.8460622	0.8460595	0.84605	0.1613603	0.1613637	0.16136
4	0.9555380	0.9555236	0.95552	0.0642341	0.0642411	0.06423
5	0.9915627	0.9915546	0.99155	0.0159068	0.0159134	0.01591
6	0.9989942	0.9989817	0.99898	0.0024020	0.0024016	0.00240
7	0.9999430	0.9999236	0.99993	0.0002202	0.0002234	0.00022
8	0.9999994	1.0000000	1.00000	0.0000124	0.0000100	0.00001
9	1.0000000	1.0000000	1.00000	0.0000004	0.0000000	0.00000
10	1.0000000	—	—	0.0000000	—	—

FIGURE 2. Graph of the approximations of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ for the Blasius equation obtained by the presented method.



the numerical results to the other methods shows the proirity of our scheme which gives a new effective computational way to approximate the solution of the Blasius equation.

References

[1] S. Abbasbandy, *A numerical solution of Blasius equation by Adomians decomposition method and comparison with homotopy perturbation method*, Chaos Solitons Fractals, *31* (2007), 257–260.

[2] W. F. Ames, *Nonlinear Ordinary Differential Equations in Runsport Processes*, Academic Press, New York, 1968.

[3] A. Asaithambi, *Solution of the Falkner-Skan equation by recursive evaluation of Taylor coefficients*, J. Comput. Appl. Math., *176* (2005), 203–214.

[4] F. M. Atici and S. Gulsan Topal, *The generalized quasilinearization method and three point boundary value problems on time scales*, Applied Mathematics Letters, *18* (2005), 577-585.

[5] R. Bellman, *Methods of Nonlinear Analysis*, Vol. II, Academic Press, New York, 1973.

[6] R. Bellman and R. Kalaba, *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York, 1965.

[7] H. Brunner, *Collocation methods for Volterra integral and related functional equations*, Cambridge University Press, 2004.

[8] S. A. Chaplygin, *Collected Papers on Mechanics and Mathematics*, Moscow, 1954.

[9] R. Cortell, *Numerical solutions of the classical Blasius flat-plate problem*, Appl. Math. Comput., *170* (2005), 706–710.

[10] B. Datta, *Numerical linear algebra and applications*, Brooks/Cole Publishing Company, 1995.

[11] J. V. Devi, F. A. McRae, and Z. Drici, *Generalized quasilinearization for fractional differential equations*, Computers and Mathematics with Applications, *59* (2010), 1057–1062.

[12] L. Dresner,*Similarity solutions of nonlinear partial differential equations*, Pitman Publishing INC., Massachusetts, 1983.

[13] Z. Dricia, F. A. McRae, and D. J. Vasundhara, *Quasilinearization for functional differential equations with retardation and anticipation*, Nonlinear Analysis, *70* (2009), 1763–1775.

[14] S. Goldstein, *Modern Developments in Fluid Dynamics*, Oxford Univ. Press, London, 1957.

[15] I. Hashim, *Comments on a new algorithm for solving classical Blasius equation*, Appl. Math. Comput., *181* (2006), 1417–1422.

[16] J. He, *A simple perturbation approach to Blasius equation*, Appl. Math. Comput., *114* (2003), 217–224.

[17] L. Howarth, *On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream*, Aero Res. Counc. Lond. R & M, *164* (1935), 16–32.

[18] R. A. Khan, *The generalized quasilinearization technique for a second order differential equation with separated boundary conditions*, Mathematical and Computer Modelling, *43* (2006), 727-742.

[19] M. N. Koleva and L. G. Vulkov, *Two-grid quasilinearization approach to ODEs with applications to model problems in physics and mechanics*, Computer Physics Communications, *181* (2010), 663–670.

[20] V. Lakshmikantham, *Further improvement of generalized quasilinearization*, Nonlinear Analysis, *27* (1996), 315–321.

[21] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I, Academic Press, New York, 1969.

[22] V. Lakshmikantham, S. Leela, and S. Sivasundaram, *Extensions of the method of quasilinearization*, J. Opt. Th. Appl, 1994, 315–321.

[23] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers, Netherlands, 1998.

[24] S. J. Liao, *An explicit, totally analytic approximate solution for Blasius viscous flow problems*, Int. J. Non-Linear Mech., *34* (1999), 759–778.

[25] J. Lin, *A new approximate iteration solution of Blasius equation*, Commun. Nonlinear Sci. Numer. Simul., *4* (1999), 97–106.

[26] C. S. Liua and J. R. Chang, *The Lie-group shooting method for multiple–solutions of Falkner–Skan equation under suction–injection conditions*, Int. J. Nonlinear Mech. *176* (2008), 700–703.

[27] T. G. Melton and A. S. Vatsala, *Improved generalized quasilinearization method and rapid convergence for reaction diffusion equations*, Applied Mathematics and Computation, *203* (2008), 563–572.

[28] T. Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York, 1979.

[29] J. J. Neito, *Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions*, Proc. Amer. Math. Soc., *125* (1997), 2599–2604.

[30] S. G. Pandit, *Quadratically converging iterative schemes for nonlinear Volterra integral equations and an application*, J. AMSA, *10* (1997), 169–178.

[31] T. C. Papanastasiou, G. C. Georgiou, and A. N. Alexandrou, *Viscous Fluid Flow*, CRC Press, Washington, D. C., 2000.

[32] K. Parand and A. Taghavi, *Rational scaled generalized Laguerre function collocation method for solving the Blasius equation*, J. Comput. Appl. Math., *233* (2009), 980–989.

[33] H. Schlichting, *Boundary Layer Theory*, McGRAW-HILL, New York, 1979.

[34] W. Squire, *Application of generalized Gauss–Laguerre quadrature to boundary-layer problems*, Journal of Aero/Space Sci., *3*(26) (1959), 540–541.

[35] T. Tajvidi, M. Razzaghi, and M. Dehghan, *Modified rational Legendre approach to laminar viscous flow over a semi-infinite flat plate*, Chaos Solitons Fractals *35* (2008), 59–66.

[36] L. Wang, *A new algorithm for solving classical Blasius equation*, Appl. Math. Comput., *176* (2006), 700–703.

[37] A. M. Wazwaz, *The variational iteration method for solving two forms of Blasius equation on a half-infinite domain*, Appl. Math. Comput., *188* (2007), 485–491.

[38] H. Weyl, *On the differential equations of the simplest boundary-layer theory problems*, Ann. of Math., *43* (1942), 381–407.

[39] J. Yang and A. S. Vatsala, *Numerical investigation of generalized quasilinearization method for reaction diffusion systems*, Computers and Mathematics with Applications, *50* (2005), 587–598.

[40] S. Zhu, Q. Wu, and X. Cheng, *Numerical solution of the Falkner-Skan equation based on quasilinearization*, Applied Mathematics and Computation, *215* (2009), 2472–2485.