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Numerical solution of time-dependent Foam Drainage Equation (FDE)

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Abstract Reduced Differental Transform Method (RDTM), which is one of the useful and effective numerical method, is applied to solve nonlinear time-dependent Foam Drainage Equation (FDE) with different initial conditions. We compare our method with the famous Adomian Decomposition and Laplace Decomposition Methods. The obtained results demonstrated that RDTM is a powerful tool for solving nonlinear partial differential equations (PDEs), it can be applied very easily and it has less computational work than other existing methods like Adomian decomposition and Laplace decomposition. Additionally, effectiveness and precision of RDTM solutions are shown in tables and graphically.

Keywords. Foam Drainage Equation, Laplace Decomposition Method, Adomian Decomposition Method, Reduced Differential Transform Method.

2010 Mathematics Subject Classification. 35G25, 65M99, 68W25, 68W30.

1. Introduction

Most of the natural events, such as chemical, physical, biological, can be modeled by nonlinear differential equations. Besides exact solutions, we need their approximate solutions in terms of applicability. Therefore, a lot of approximate, numerical and analytical methods are developed and applied for nonlinear models [3–5, 7, 8, 17, 18, 20, 29, 34].

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The process of Foam drainage, which is a natural event, is described in [27], [28] and also shown in Figure 1. Because of the importance, many technological and industrial applications have been developed for foams, which include cleansing, water purification, mineral extraction as mentioned in [27], [28].

More than ten years ago, given studies by Verbist and Weaire described the main features of both free drainage, where liquid drains out of a foam due to gravity and forced drainage, where liquid is introduced to the top of a column of foam [9–11,14,15]. In the second state, a solitary wave of constant velocity is generated when liquid is added at a constant rate [9–11, 13–15]. So forced foam drainage may be the best prototype for certain general phenomena described by nonlinear differential equations, particularly the type of solitary wave [14,15]. The model is developed by Verbist and Weaire to idealize the network of Plateau borders as a set of N identical pipes of cross section A, which is a function of position and time as in show [12–15] and below

$$\frac{\partial A(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(A(x,t)^2-\frac{\sqrt{A(x,t)}}{2}\frac{\partial A(x,t)}{\partial x}\right)=0, \tag{1.1}$$

and exact solution of (1.1) as shown [14,15]

$$A(x,t)=\begin{cases} c\tanh(\sqrt{c}(x-ct))^2, & x\leq ct,\\ 0, & x\geq ct,\end{cases} \tag{1.2}$$

where x and t are location and time respectively, c is the velocity of the wave front. If we substitute $A(x,t)=u(x,t)^2$ and rearrange the eq (1.1), then it can be written with initial condition as follow

$$\begin{aligned} &u(x,t)_t+2u(x,t)^2u(x,t)_x-(u(x,t)_x)^2-\frac{1}{2}u(x,t)_{xx}u(x,t)=0\\ &u(x,0)=g(x). \end{aligned} \tag{1.3}$$

In this context, we consider the famous time-dependent nonlinear forced channel-dominated foam drainage equation (FDE)(1.1) to solve as numerical with reduced differential transform method (RDTM). RDTM, which has an alternative approach of problems, is presented to overcome the demerit of complex calculation, discretization, linearization or small perturbations of well-known numerical and analytical methods such as ADM, DTM, HPM etc. And also the main advantage of this method is that it allows an analytical approximation, in many cases an exact solution, with rapidly

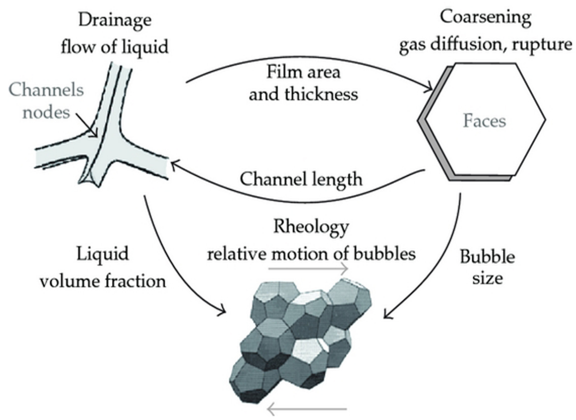

FIGURE 1. Schematic of the interdependence of drainage, coarsening and rheology of foams [28].

convergent sequence [30–33]. Some of the novel research articles related to RDTM can be shown in ref. [1, 2, 6, 21–24, 26].

2. Reduced Differential Transform Method (RDTM)

Let, suppose that $u(x,t)$ can be represented two variable function as a product of two single variable functions $f(x)$ and $g(t)$ to show following manner [30–33]

$$u(x,t) = f(x)g(t). \tag{2.1}$$

From the similar meaning of definition of Differential Transform Method and its properties, we can write the transforming form of RDTM [30–33]

$$u(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{2.2}$$

where $U_k(x)$ is called t dimensional spectrum function of $u(x,t)$. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}. \tag{2.3}$$

Thus, from (2.3), it can be written the inverse transform of a sequence $U_k(x)_{k=0}^{\infty}$

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{2.4}$$

then combining (2.3) and (2.4), we obtain the RDTM solution as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k. \tag{2.5}$$

If we consider the expressions (2.3), (2.4) and (2.5), it's clearly shown that the concept of the reduced differential transform is derived from the power series expansion. So, we give a table which included fundamental transformation properties of RDTM in Table 1. The proofs of Table 1 and the basic definitions of reduced differential transform method can be found in [32]. For illustration of the proposed method, we write the Foam Drainage Equation (1.3) in the standard operator form [30–33]

$$L(u(x,t)) + Nu(x,t) = g(x,t) \tag{2.6}$$

with initial condition

$$u(x,0) = f(x), \tag{2.7}$$

where $L = \frac{\partial}{\partial t}$ is a linear operator, $Nu(x,t)$ is a nonlinear terms and $g(x,t)$ inhomogeneous term. According to the RDTM and Table 1, we can construct the following iteration formula [30–33]

$$(k+1)U_{k+1}(x) = G_k(x) - NU_k(x). \tag{2.8}$$

Here, $U_k(x)$, $G_k(x)$ and $NU_k(x)$ are the transformations of the functions $L(u(x,t))$, $g(x,t)$ and $Nu(x,t)$ respectively. From the initial condition, we write

$$U_0(x) = f(x) \tag{2.9}$$

Substituting (2.9) into (2.8) and by straightforward iterative calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $U_k(x)_{k=0}^{n}$ gives the approximation solution as

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k, \tag{2.10}$$

where n is order of approximate solution. Therefore, the exact solution of the problem is given by [30–33]

$$u(x,t) = \lim_{x \to \infty} \tilde{u}_n(x,t). \tag{2.11}$$

Table 1. Basic transformations of RDTM for some functions.

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$, α constant
$w(x,t) = x^m t^n$	$x^m \delta(k-n)$, $\delta(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)(k+2)\ldots(k+r)U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{d}{dx}U_k(x)$
$w(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$	$W_k(x) = \frac{d^2}{dx^2}U_k(x)$

3. Implementation of presented method

Problem 1: The one-dimensional homogeneous forced time-dependent foam drainage equation following

$$u(x,t)_t + 2u(x,t)^2 u(x,t)_x - (u(x,t)_x)^2 - \frac{1}{2}u(x,t)_{xx}u(x,t) = 0, \tag{3.1}$$

with initial condition

$$u(x,0) = -\sqrt{c}\tanh\left(\sqrt{c}x\right), \tag{3.2}$$

where c is the velocity of wave front as in [9, 11]. Let, $U_k(x)$ denotes transformation form of the function $u(x,t)$. Then, by using the basic properties of the reduced differential transformation as shown in Table 1, we can write the transformed form of equation (3.1) as

$$\begin{aligned}(k+1)U_{k+1}(x) = &-2\sum_{r=0}^{k}\sum_{s=0}^{k-r}U_r(x)U_s(x)\frac{d}{dx}U_{k-r-s}(x) \\ &+\sum_{r=0}^{k}\frac{d}{dx}U_r(x)\frac{d}{dx}U_{k-r}(x) + \frac{1}{2}\sum_{r=0}^{k}U_r(x)\frac{d^2}{dx^2}U_{k-r}(x)\end{aligned} \tag{3.3}$$

and using the initial condition (3.2), we get the reduced transform form

$$U_0(x) = -\sqrt{c}\tanh\left(\sqrt{c}x\right). \tag{3.4}$$

Now, put (3.4) into place (3.3), from hence we have the $U_k(x)$ values following

$$\begin{aligned} U_1(x) &= \frac{c^2}{cosh^2\left(\sqrt{c}x\right)}, \\ U_2(x) &= \frac{c^{\frac{7}{2}} sinh\left(\sqrt{c}x\right)}{cosh^3\left(\sqrt{c}x\right)}, \\ U_3(x) &= \frac{1}{3}\frac{c^5\left(2cosh^2\left(\sqrt{c}x\right)-3\right)}{cosh^4\left(\sqrt{c}x\right)}, \\ U_4(x) &= \frac{1}{3}\frac{c^{\frac{13}{2}} sinh\left(\sqrt{c}x\right)\left(cosh^2\left(\sqrt{c}x\right)-3\right)}{cosh^5\left(\sqrt{c}x\right)}, \cdots \end{aligned} \tag{3.5}$$

Thus, if we continue this process and also the inverse transformation of the set of $U_k(x)_{k=0}^{\infty}$ values are written

$$\begin{aligned} \sum_{k=0}^{\infty} U_k(x)t^k &= -\sqrt{c}tanh\left(\sqrt{c}x\right) + \frac{c^2}{cosh^2\left(\sqrt{c}x\right)}t \\ &+\frac{c^{\frac{7}{2}} sinh\left(\sqrt{c}x\right)}{cosh^3\left(\sqrt{c}x\right)}t^2 + \frac{1}{3}\frac{c^5\left(2cosh^2\left(\sqrt{c}x\right)-3\right)}{cosh^4\left(\sqrt{c}x\right)}t^3 \\ &+\frac{1}{3}\frac{c^{\frac{13}{2}} sinh\left(\sqrt{c}x\right)\left(cosh^2\left(\sqrt{c}x\right)-3\right)}{cosh^5\left(\sqrt{c}x\right)}t^4 + \cdots \end{aligned} \tag{3.6}$$

Arranging (3.6) and from (2.4) and (2.5), we obtain RDTM solution of (3.1) as

$$\begin{aligned} u(x,t) = \lim_{n\to\infty} u_n(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k = -\sqrt{c}tanh\left(\sqrt{c}x\right) \\ &+c^2tsech_2\left(\sqrt{c}x\right) + c^{\frac{7}{2}}t^2tanh\left(\sqrt{c}x\right)sech^2\left(\sqrt{c}x\right) \\ &-\frac{1}{3}\left(3c^5sech^4\left(\sqrt{c}x\right) - 2c^5sech^2\left(\sqrt{c}x\right)\right)t^3 + \cdots \end{aligned} \tag{3.7}$$

Higher accuracy and efficient convergence of solution (3.7) presented in Figure 2, Figure 3 and Table 2, Table 3 as shown below. Also, from Table 4, it can be say that our presented method is faster than ADM and LDM.

Problem 2: Secondly, in order to test efficiency, accuracy and reliability of RDTM, we consider the nonlinear foam drainage equation (1.3) for different initial value such that

$$\begin{aligned} &u(x,t)_t + 2u(x,t)^2u(x,t)_x - (u(x,t)_x)^2 - \frac{1}{2}u(x,t)_{xx}u(x,t) = 0 \\ &u(x,0) = -\frac{1}{2} + \frac{1}{1+e^x} \qquad . \end{aligned} \tag{3.8}$$

Taking differential transform of (3.8), we obtain the same transformed form in (3.3) which has a different initial condition as follows

$$U_0(x) = -\frac{1}{2} + \frac{1}{1+e^x}. \tag{3.9}$$

Then, substituting (3.9) into (3.3), we obtain the following $U_k(x)$ values,

$$\begin{aligned} U_1(x) &= \frac{1}{4}\frac{e^x}{(1+e^x)^2}, U_2(x) = \frac{1}{32}\frac{e^x(-1+e^x)}{(1+e^x)^3}, \\ U_3(x) &= -\frac{1}{384}\frac{e^x(-1+4e^x-e^{2x})}{(1+e^x)^4}, \\ U_4(x) &= \frac{1}{6144}\frac{e^x(e^{3x}-1+11e^x-11e^{2x})}{(1+e^x)^5}, \\ U_5(x) &= \frac{1}{122880}\frac{e^x(26e^{3x}-1+26e^x-66e^{2x}-e^{4x})}{(1+e^x)^6} \end{aligned} \tag{3.10}$$

Therefore, if we go on this process and the inverse transformation of the set of $U_k(x)_{k=0}^{\infty}$ values gives RDTM solution as

$$\begin{aligned} u(x,t) = \lim_{n\to\infty} u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k &= -\frac{1}{2} + \frac{1}{1+e^x} \\ &+\frac{1}{4}\frac{e^x}{(1+e^x)^2}t + \frac{1}{32}\frac{e^x(-1+e^x)}{(1+e^x)^3}t^2 \\ &-\frac{1}{384}\frac{e^x(-1+4e^x-e^{2x})}{(1+e^x)^4}t^3 + \cdots \end{aligned} \tag{3.11}$$

Table 2. Comparison of absolute errors for **Problem 1** by solving RDTM,ADM and LDM with 15 terms at $c = 1$.

	RDTM		ADM		LDM	
x	$t = 0.2$	$t = 0.7$	$t = 0.2$	$t = 0.7$	$t = 0.2$	$t = 0.7$
0	$1e-15$	$1.145166e-6$	$1e-15$	$1.145166e-6$	$1e-15$	$1.145166e-6$
1	0	$3.2766458e-8$	0	$3.2766458e-8$	0	$3.2766458e-8$
2	$4e-15$	$5.44128e-10$	$4e-15$	$5.44128e-10$	$4e-15$	$5.44128e-10$
3	$4e-15$	$3.559e-12$	$4e-15$	$3.559e-12$	$4e-15$	$3.559e-12$
4	$2e-15$	$1.33e-13$	$2e-15$	$1.33e-13$	$2e-15$	$1.33e-13$
5	0	0	0	0	0	0
6	0	$2e-15$	0	$2e-15$	0	$2e-15$
7	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$
8	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$
9	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$	$2e-15$
10	$3e-15$	$3e-15$	$3e-15$	$3e-15$	$3e-15$	$3e-15$

TABLE 3. Comparison of absolute errors for **Problem 1** by solving RDTM,ADM and LDM with 15 terms at $t = 0.5$.

	RDTM		ADM		LDM	
x	$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$
0	$4.0874e-9$	0.1668503	$4.0874e-9$	0.1668503	$4.0874e-9$	0.1668503
1	$4.0766e-11$	$2.16074e-3$	$4.0766e-11$	$2.16074e-3$	$4.0766e-11$	$2.16074e-3$
2	$2.058e-12$	$2.11298e-6$	$2.058e-12$	$2.11298e-6$	$2.058e-12$	$2.11298e-6$
3	$1.1e-14$	$7.22317e-9$	$1.1e-14$	$7.22317e-9$	$1.1e-14$	$7.22317e-9$
4	0	$6.9e-13$	0	$6.9e-13$	0	$6.9e-13$
5	0	$1.84e-12$	0	$1.84e-12$	0	$1.84e-12$
6	$2e-15$	$9e-14$	$2e-15$	$9e-14$	$2e-15$	$9e-14$
7	$2e-15$	$2e-14$	$2e-15$	$2e-14$	$2e-15$	$2e-14$
8	0	0	0	0	0	0
9	$1e-15$	$1e-14$	$1e-15$	$1e-14$	$1e-15$	$1e-14$
10	$3e-15$	$1e-14$	$3e-15$	$1e-14$	$3e-15$	$1e-14$

TABLE 4. The comparison of computation times which computed with Intel(R) Core (TM) i5-3230M CPU for 2.60 GHz between ADM, LDM and RDTM for equation (3.1) with initial condition (3.2).

Iterations	CPU times of RDTM	CPU times of ADM	CPU times of LDM
10 terms	0.841 sec	0.879 sec	1.874 sec
20 terms	4.822 sec	5.127 sec	9.221 sec
30 terms	14.618 sec	22.049 sec	34.298 sec
40 terms	30.049 sec	115.077 sec	147.342 sec
50 terms	75.106 sec	753.342 sec	835.238 sec

Now, we compare the solution (3.11) with well-known ADM and LDM solutions for eq. (3.8) as Figure 4 and also in Table 5.

4. Conclusion

Foam Drainage Equation is solved numerically by RDTM. Our results compared with ADM-LDM by displaying in figures and tables. The solutions obtained by RDTM shows that it has higher accuracy and efficiency with compare ADM and LDM. At the same time presented method is more quickly than LDM and ADM as seen Table 4. Additionally, we can say that RDTM is very simple and powerful numerical method to solve various nonlinear partial differential equations.

Acknowledgment

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TABLE 5. Comparison of RDTM,ADM and LDM solutions for **Problem 2** with 15 terms at $t = 0.8$.

x	RDTM	ADM	LDM
0	0.04983399731	0.04983399731	0.04983399731
1	−0.1899744810	−0.1899744810	−0.1899744810
2	−0.3581489352	−0.3581489352	−0.3581489352
3	−0.4426758241	−0.4426758241	−0.4426758241
4	−0.4781187290	−0.4781187290	−0.4781187290
5	−0.4918374288	−0.4918374288	−0.4918374288
6	−0.4969815836	−0.4969815836	−0.4969815836
7	−0.4988874639	−0.4988874639	−0.4988874639
8	−0.4995904329	−0.4995904329	−0.4995904329
9	−0.4998492896	−0.4998492896	−0.4998492896
10	−0.4999445515	−0.4999445515	−0.4999445515

FIGURE 2. Error values for RDTM solution of **Problem 1** at $c = 1.0$, (Left) $t = 0.1$, (Middle) $t = 0.5$ and (Right) $t = 1.0$.

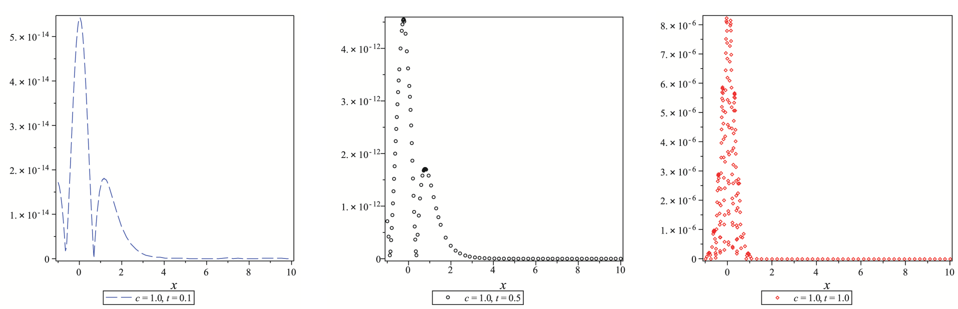

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FIGURE 3. Error values for RDTM solution of **Problem 1** at $t = 0.5$, (Left) $c = 0,5$ and $c = 1.0$, (Right) $c = 2.0$.

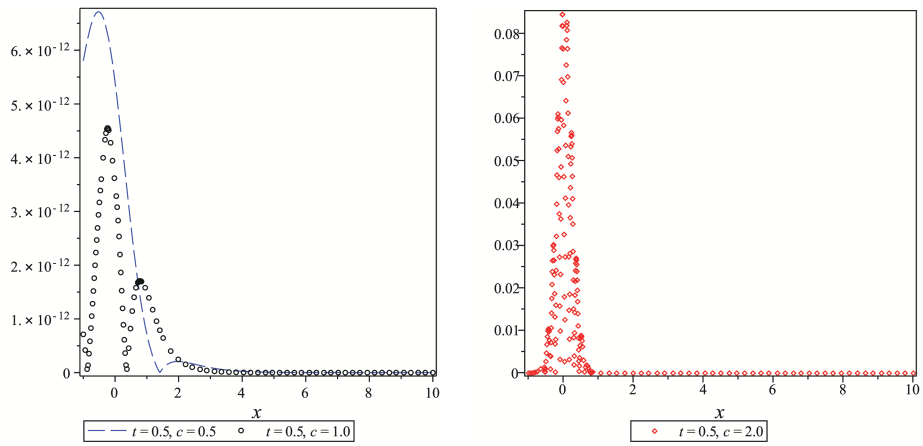
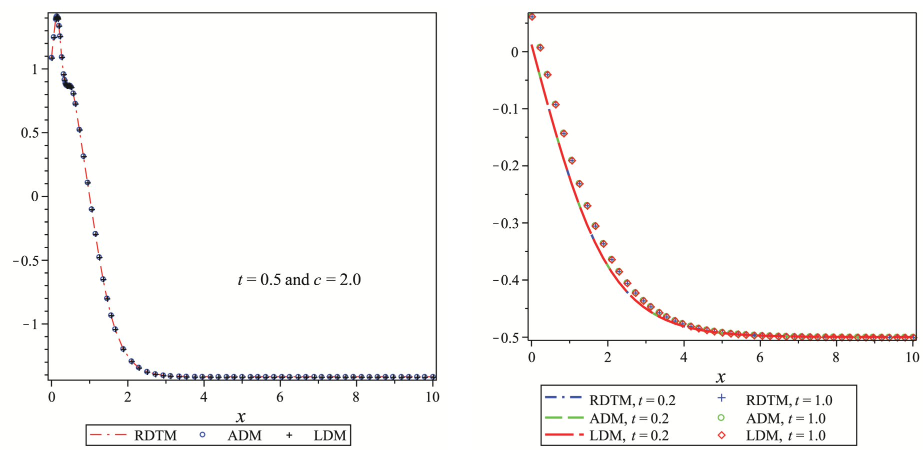


FIGURE 4. Comparison of RDTM, ADM and LDM solutions of (left) **Problem 1** and (Right) **Problem 2**.



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