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# An efficient numerical method for a nonlinear system of singular perturbation problems with different magnitudes of a single parameter

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**Abstract**

Aim of this article is to develop a fitted mesh method for a nonlinear system of two coupled singular perturbation problems of reaction-diffusion type with different magnitudes of a single parameter such that the method preserves the nature of nonlinearity and it works well for all values of the perturbation parameter. The fitted mesh method is a combination of finite difference operators and a Shishkin mesh. The newly developed numerical method is proved to be uniform convergence of order $O(N_t^{-1} \ln N_t)$ where $N_t$ is the discretization parameter. Numerical experiments presented support the theoretical results. Test problems included helps to understand the singular perturbation nature of the nonlinear system under consideration.

**Keywords.** Nonlinear system of singular perturbation problems, boundary layers, fitted mesh method, Shishkin mesh, the continuation method.
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Uncorrected P

## 1. Introduction

A differential equation whose highest order derivative term is multiplied by a small positive parameter is known as a Singular Perturbation Problem (SPP). Many of the SPPs arising in nature are exhibiting nonlinear characteristics, for example, in two-time scale system [16] and catalytic reaction theory [8]. For linear system of SPPs several non-classical numerical methods are available in literature whereas for Nonlinear System of SPPs (NSSPPs) only a few methods are available.

An asymptotic expansion is constructed for different types of NSSPPs in the articles [2–6]. Article [11] deals with an asymptotic method for a NSSPPs of reaction diffusion type and for the same system a numerical method is developed in [17]. A finite difference scheme of Hermite type combined with standard central difference scheme in a special way on a piecewise-uniform Shishkin mesh is developed in [7] for a NSSPPs of reaction diffusion type with same perturbation parameter. A numerical method for aforementioned nonlinear system with different perturbation parameters is developed in [10]. A numerical method for a NSSPPs consist of two equations in which one equation with only first derivative term and other with only second derivative term is constructed in [1]. For a convection diffusion type NSSPPs, a computational method is constructed in [14]. Numerical treatment for aforementioned system with multiscale nature is reported in [15]. A numerical algorithm for a NSSPPs consist of two equations in which one equation is of reaction diffusion type and other is of convection diffusion type is reported in [13].

Most of the SPPs of practical importance are nonlinear in nature whose nonlinear terms are involving the perturbation parameters also [2, 5]. The present article deals with a NSSPPs of reaction diffusion type with different magnitudes of a single parameter $\varepsilon$ in which the nonlinear terms are also involving the parameter $\varepsilon$. A parameter independent fitted mesh method is developed in this article such that no artificial condition is imposed either on the parameter $\varepsilon$ or on the system under consideration.

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The following norms for any vector function $\vec{z}$ on $[0,1]$ are introduced: $\| \vec{z}(t) \| = \max_i |z_i(t)|$ and $\| \vec{z} \| = \sup_{t\in[0,1]} \| \vec{z}(t) \|$ . For any mesh function $\vec{Z}$ on $\overline{D}^{N_t} = \{t_j\}_{j=0}^{N_t}$ the following discrete norms are introduced: $\| \vec{Z}(t_j) \| = \max_i |Z_i(t_j)|$ and $\| \vec{Z} \| = \max_{t_j\in\overline{D}^{N_t}} \| \vec{Z}(t_j) \|$ .

Throughout the article $C$ denotes a generic positive constant which is independent of $t, \varepsilon$ and $N_t$ (discretization parameter).

## 2. The nonlinear system

The NSSPPs considered in this article is

$$\begin{aligned} -\varepsilon^2 \frac{d^2u(t)}{dt^2} + f(t,\, \varepsilon,\, u(t),\, v(t)) &= 0, \\ -\varepsilon \frac{d^2v(t)}{dt^2} + g(t,\, \varepsilon,\, u(t),\, v(t)) &= 0,\ t \in D = (0,1), \end{aligned} \tag{2.1}$$

$$\text{with } u(0) = \theta_{01},\ \ v(0) = \theta_{02}, u(1) = \theta_{11} \text{ and } v(1) = \theta_{12}. \tag{2.2}$$

Here $\theta_{01}, \theta_{02}, \theta_{11}$ and $\theta_{12}$ are known constants and $0 < \varepsilon < 1$. For all $t \in \overline{D} = [0,1]$, $f,\, g \in C^3(\overline{D} \times \mathbb{R}^2)$ such that $\frac{\partial f}{\partial u} > 0$, $\frac{\partial f}{\partial v} \le 0$, $\frac{\partial g}{\partial u} \le 0$, $\frac{\partial g}{\partial v} > 0$. Further,

$$\min_{t\in\overline{D}} \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} \right) \ge \alpha > 0. \tag{2.3}$$

Implicit function theorem along with condition (2.3) guarantees the existence of a unique solution $(u(t),\ v(t))$ of the system (2.1)-(2.2) such that $u(t), v(t) \in C^3(\overline{D})$ [12]. Both the components $u(t)$ and $v(t)$ of the system (2.1)-(2.2) exhibit boundary layers near both the boundaries $t = 0$ and $t = 1$ [9]. The first component $u(t)$ exhibits boundary layers of width $\varepsilon$ whereas the second component $v(t)$ exhibits boundary layers of width $\sqrt{\varepsilon}$.

For convenience, (2.1)-(2.2) is rewritten as

$$\begin{aligned} -\varepsilon^2 \frac{d^2z_1(t)}{dt^2} + f_1(t,\, \varepsilon,\, z_1(t),\, z_2(t)) &= 0, \\ -\varepsilon \frac{d^2z_2(t)}{dt^2} + f_2(t,\, \varepsilon,\, z_1(t),\, z_2(t)) &= 0,\ t \in D \end{aligned} \tag{2.4}$$

with

$$z_1(0) = \theta_{01},\ \ z_2(0) = \theta_{02}, z_1(1) = \theta_{11} \text{ and } z_2(1) = \theta_{12}. \tag{2.5}$$

Note that $z_1 \equiv u$ and $z_2 \equiv v$. In operator form (2.4)-(2.5) is written as

$$\vec{\mathbb{T}}\vec{z}(t) = -E\,\vec{z}''(t) + \vec{f}(t,\, \varepsilon,\, \vec{z}(t)) = \vec{0},\ t \in D, \tag{2.6}$$

with

$$\vec{z}(0) = \vec{\theta}_0 \text{ and } \vec{z}(1) = \vec{\theta}_1, \tag{2.7}$$

where $\vec{\theta}_0 = (\theta_{01},\, \theta_{02})^T$, $\vec{\theta}_1 = (\theta_{11},\, \theta_{12})^T$ and $E = \mathrm{diag}(\varepsilon^2,\, \varepsilon)$. For all $t \in \overline{D}$, $\vec{z}(t) = (z_1(t),\, z_2(t))^T$, $\vec{f}(t,\, \varepsilon,\, \vec{z}(t)) = (f_1(t,\, \varepsilon,\, \vec{z}(t)),\, f_2(t,\, \varepsilon,\, \vec{z}(t)))^T$. A reduced problem of (2.6)-(2.7) is defined to be

$$\vec{f}(t,\, 0,\, \vec{z}_0(t)) = \vec{0},\ t \in \overline{D}. \tag{2.8}$$

From (2.8), for any $t \in \overline{D}$,

$$|z_{0_k}^{(i)}(t)| \le C \text{ for } k = 1, 2,\ 0 \le i \le 4. \tag{2.9}$$

Uncorrected Proof

## 3. Analytical results

Let $\vec{\mathbb{T}}'$ be a linear operator which is given by

$$\vec{\mathbb{T}}'\vec{\psi}(t) = -E\,\vec{\psi}''(t) + A(t,\varepsilon)\vec{\psi}(t)$$

where $A(t,\varepsilon) = \begin{bmatrix} a_{11}(t,\varepsilon) & a_{12}(t,\varepsilon) \\ a_{21}(t,\varepsilon) & a_{22}(t,\varepsilon) \end{bmatrix}$ with $a_{ij}(t,\varepsilon) \le 0$, for $i,j = 1,2$, $i \ne j$ and

$$\min_{t\in\overline{D}} \left(a_{11}(t,\varepsilon) + a_{12}(t,\varepsilon),\ a_{21}(t,\varepsilon) + a_{22}(t,\varepsilon)\right) > 0. \tag{3.1}$$

**Theorem 3.1.** *Let $\vec{\phi}(t) = (\phi_1(t),\, \phi_2(t))^T$ be a function such that $\vec{\phi}(0) \ge \vec{0}$, $\vec{\phi}(1) \ge \vec{0}$ and $\vec{\mathbb{T}}'\vec{\phi} \ge \vec{0}$ on $D$ then $\vec{\phi} \ge \vec{0}$ on $\overline{D}$.*

*Proof.* Let $\tilde{i}$ and $\tilde{t}$ be such that $\phi_{\tilde{i}}(\tilde{t}) = \min\limits_{i,\,t} \phi_i(t)$ and suppose that $\phi_{\tilde{i}}(\tilde{t}) < 0$. Then $\tilde{t} \notin \{0,1\}$ and $\phi''_{\tilde{i}}(\tilde{t}) \ge 0$. Let $\tilde{t} \in D$ and consider

$$\left(\vec{\mathbb{T}}'\vec{\phi}\right)_{\tilde{i}}(\tilde{t}) = \begin{cases} -\varepsilon^2\phi_1''(\tilde{t}) + a_{11}(\tilde{t},\varepsilon)\phi_1(\tilde{t}) + a_{12}(\tilde{t},\varepsilon)\phi_2(\tilde{t}), & \text{if } \tilde{i} = 1, \\ -\varepsilon\,\phi_2''(\tilde{t}) + a_{21}(\tilde{t},\varepsilon)\phi_1(\tilde{t}) + a_{22}(\tilde{t},\varepsilon)\phi_2(\tilde{t}), & \text{if } \tilde{i} = 2. \end{cases}$$

Thus,

$$(\vec{\mathbb{T}}'\vec{\phi})_{\tilde{i}}(\tilde{t}) \le \begin{cases} -\varepsilon^2\,\phi_1''(\tilde{t}) + \left(a_{11}(\tilde{t},\varepsilon) + a_{12}(\tilde{t},\varepsilon)\right)\phi_1(\tilde{t}), & \text{if } \tilde{i} = 1, \\ -\varepsilon\,\phi_2''(\tilde{t}) + \left(a_{21}(\tilde{t},\varepsilon) + a_{22}(\tilde{t},\varepsilon)\right)\phi_2(\tilde{t}), & \text{if } \tilde{i} = 2. \end{cases}$$

Using (3.1), $(\vec{\mathbb{T}}'\vec{\phi})_{\tilde{i}}(\tilde{t}) < 0$ for $\tilde{t} \in D$, a contradiction. Hence $\vec{\phi} \ge \vec{0}$ on $\overline{D}$. □

Decompose the solution $\vec{z}$ of (2.6)-(2.7) into $\vec{z}_s$ and $\vec{z}_l$ such that $\vec{z} = \vec{z}_s + \vec{z}_l$ where

$$-E\,\vec{z}_s''(t) + \vec{f}(t,\,\varepsilon,\,\vec{z}_s(t)) = 0,\ t \in D, \tag{3.2}$$

with $\vec{z}_s(0)$ and $\vec{z}_s(1)$ are suitably chosen and

$$-E\,\vec{z}_l''(t) + \vec{f}(t,\,\varepsilon,\,\vec{z}_s(t) + \vec{z}_l(t)) - \vec{f}(t,\,\varepsilon,\,\vec{z}_s(t)) = 0, \qquad t \in D, \tag{3.3}$$

with

$$\vec{z}_l(0) = \vec{z}(0) - \vec{z}_s(0),\ \ \vec{z}_l(1) = \vec{z}(1) - \vec{z}_s(1). \tag{3.4}$$

### 3.1. Bounds on $\vec{z}_s(t)$ and its derivatives.

**Theorem 3.2.** *For all $t \in \overline{D}$*

$$|z_{s_i}^{(j)}(t)| \ \le\ C,\ \ i = 1,2,\ j = 0,1,\ \ |z_{s_1}''(t)| \le C\,\varepsilon^{-1/2},\ \ |z_{s_1}^{(3)}(t)| \le C\,\varepsilon^{-1},\ \ |z_{s_2}^{(k)}(t)| \le C,\ k = 2,3. \tag{3.5}$$

*Proof.* From (3.2) and (2.8),

$$-\varepsilon^2 z_{s_1}''(t) + c_{11}(t,\varepsilon)z_{s_1}(t) + c_{12}(t,\varepsilon)z_{s_2}(t) = c_{11}(t,0)z_{0_1}(t) + c_{12}(t,0)z_{0_2}(t), \tag{3.6}$$

$$-\varepsilon\, z_{s_2}''(t) + c_{21}(t,\varepsilon)z_{s_1}(t) + c_{22}(t,\varepsilon)z_{s_2}(t) = c_{21}(t,0)z_{0_1}(t) + c_{22}(t,0)z_{0_2}(t), \tag{3.7}$$

where $c_{ij}(t,\varepsilon) = \dfrac{\partial f_i(t,\varepsilon,\vec{\beta}(t))}{\partial u_j}$ are intermediate values.

Equations (3.6), (3.7) can be written as

$$\vec{\mathbb{T}}'\vec{z}_s(t) = -E\,\vec{z}_s''(t) + C(t,\,\varepsilon)\vec{z}_s(t) = C(t,\,0)\vec{z}_0(t) = \vec{y}(t) \tag{3.8}$$

with

$$\vec{z}_s(0), \vec{z}_s(1),\ \text{are suitably chosen,} \tag{3.9}$$

Uncorrected Proof

where $C(t,\varepsilon) = \begin{bmatrix} c_{11}(t,\varepsilon) & c_{12}(t,\varepsilon) \\ c_{21}(t,\varepsilon) & c_{22}(t,\varepsilon) \end{bmatrix}$.

For convenience, $\vec{z}_s(t)$ is decomposed as

$$\vec{z}_s(t) = \vec{z}_{s_0}(t) + \varepsilon^{1/2}\,\vec{z}_{s_1}(t) + \varepsilon\,\vec{z}_{s_2}(t), \tag{3.10}$$

where $\vec{z}_{s_0}(t)$ is the solution of

$$\begin{aligned} c_{11}(t,\varepsilon)z_{s_{01}}(t) + c_{12}(t,\varepsilon)z_{s_{02}}(t) &= y_1(t), \\ c_{21}(t,\varepsilon)z_{s_{01}}(t) + c_{22}(t,\varepsilon)z_{s_{02}}(t) &= y_2(t),\ t \in \overline{D}, \end{aligned} \tag{3.11}$$

$\vec{z}_{s_1}(t)$ is the solution of

$$\begin{aligned} c_{11}(t,\varepsilon)z_{s_{11}}(t) + c_{12}(t,\varepsilon)z_{s_{12}}(t) &= \varepsilon^{3/2}z''_{s_{01}}(t), \\ c_{21}(t,\varepsilon)z_{s_{11}}(t) + c_{22}(t,\varepsilon)z_{s_{12}}(t) &= \varepsilon^{1/2}z''_{s_{02}}(t),\ t \in \overline{D} \end{aligned} \tag{3.12}$$

and $\vec{z}_{s_2}(t)$ is the solution of

$$-\varepsilon^2 z''_{s_{21}}(t) + c_{11}(t,\varepsilon)z_{s_{21}}(t) + c_{12}(t,\varepsilon)z_{s_{22}}(t) = \varepsilon^{3/2}z''_{s_{11}}(t), \tag{3.13}$$

$$-\varepsilon\, z''_{s_{22}}(t) + c_{21}(t,\varepsilon)z_{s_{21}}(t) + c_{22}(t,\varepsilon)z_{s_{22}}(t) = \varepsilon^{1/2}z''_{s_{12}}(t),\ t \in D, \tag{3.14}$$

$$z_{s_{21}}(0) = 0,\ \ z_{s_{22}}(0) = 0,\ \ z_{s_{21}}(1) = 0, z_{s_{22}}(1) = 0. \tag{3.15}$$

From (3.11) and (3.12), for $i = 1, 2$ and for all $t \in \overline{D}$,

$$|z^{(k)}_{s_{0i}}(t)| \le C,\ 0 \le k \le 5,\ |z^{(j)}_{s_{1i}}(t)| \le C,\ 0 \le j \le 3. \tag{3.16}$$

Theorem 3.1 and (3.16) with (3.13)-(3.15) gives, for $i = 1, 2$ and for all $t \in \overline{D}$,

$$|z_{s_{2i}}(t)| \le C\,\varepsilon^{1/2}. \tag{3.17}$$

Mean value theorem and (3.17) gives, for all $t \in \overline{D}$, $i = 1, 2$,

$$|z'_{s_{2i}}(t)| \le C. \tag{3.18}$$

From (3.13), for all $t \in \overline{D}$,

$$|z''_{s_{21}}(t)| \le C\,\varepsilon^{-3/2}. \tag{3.19}$$

From (3.14), for all $t \in \overline{D}$,

$$|z''_{s_{22}}(t)| \le C\,\varepsilon^{-1/2}. \tag{3.20}$$

Differentiating (3.13) with respect to $t$ once, for all $t \in \overline{D}$,

$$|z^{(3)}_{s_{21}}(t)| \le C\,\varepsilon^{-2}. \tag{3.21}$$

Differentiating (3.14) with respect to $t$ once, for all $t \in \overline{D}$,

$$|z^{(3)}_{s_{22}}(t)| \le C\,\varepsilon^{-1}. \tag{3.22}$$

Finally, the bounds for $z_{s_1}(t), z_{s_2}(t)$ and their derivatives follow from the bounds for $\vec{z}_{s_0}(t)$, $\vec{z}_{s_1}(t)$, $\vec{z}_{s_2}(t)$ and their derivatives. □

**Remark 3.3.** Using (3.15) with (3.10) componentwise, $z_{s_1}(0) = z_{s_{01}}(0) + \varepsilon^{1/2}z_{s_{11}}(0)$, $z_{s_2}(0) = z_{s_{02}}(0) + \varepsilon^{1/2}z_{s_{12}}(0)$, $z_{s_1}(1) = z_{s_{01}}(1) + \varepsilon^{1/2}z_{s_{11}}(1)$ and $z_{s_2}(1) = z_{s_{02}}(1) + \varepsilon^{1/2}z_{s_{12}}(1)$.

Uncorrected Proof

3.2. **Bounds on $\vec{z}_l(t)$ and its derivatives.** Using (3.3),

$$-\varepsilon^2 z_{l_1}''(t) + d_{11}(t,\varepsilon) z_{l_1}(t) + d_{12}(t,\varepsilon) z_{l_2}(t) = 0, \tag{3.23}$$

$$-\varepsilon\, z_{l_2}''(t) + d_{21}(t,\varepsilon) z_{l_1}(t) + d_{22}(t,\varepsilon) z_{l_2}(t) = 0, \tag{3.24}$$

where $d_{ij}(t,\varepsilon) = \dfrac{\partial f_i(t,\varepsilon,\vec{\zeta}(t))}{\partial u_j}$ are intermediate values.

Equations (3.23), (3.24) can be written as

$$\vec{\mathbb{T}}'\vec{z}_l(t) = -E\,\vec{z}_l''(t) + D(t,\varepsilon)\vec{z}_l(t) = 0,\ t \in D, \tag{3.25}$$

with

$$\vec{z}_l(0) = \vec{z}(0) - \vec{z}_s(0),\ \ \vec{z}_l(1) = \vec{z}(1) - \vec{z}_s(1), \tag{3.26}$$

where $D(t,\varepsilon) = \begin{bmatrix} d_{11}(t,\varepsilon) & d_{12}(t,\varepsilon) \\ d_{21}(t,\varepsilon) & d_{22}(t,\varepsilon) \end{bmatrix}$.

For all $t \in \overline{D}$, let $\mathbb{B}_\varepsilon(t) = e^{-\sqrt{\alpha}\,t/\sqrt{\varepsilon}} + e^{-\sqrt{\alpha}(1-t)/\sqrt{\varepsilon}}$.

**Theorem 3.4.** *For all $t \in \overline{D}$,*

$$|z_{l_i}(t)| \le C\,\mathbb{B}_\varepsilon(t),\ |z_{l_i}'(t)| \le C\,\varepsilon^{-1/2}\mathbb{B}_\varepsilon(t),\ i = 1,2,\ |z_{l_1}''(t)| \le C\,\varepsilon^{-2}\mathbb{B}_\varepsilon(t),$$
$$|z_{l_1}^{(3)}(t)| \le C\,\varepsilon^{-5/2}\mathbb{B}_\varepsilon(t),\ |z_{l_2}''(t)| \le C\,\varepsilon^{-1}\mathbb{B}_\varepsilon(t),\ |z_{l_2}^{(3)}(t)| \le C\,\varepsilon^{-3/2}\mathbb{B}_\varepsilon(t).$$

*Proof.* Let $\vec{\psi}^{\pm}(t) = C\,\mathbb{B}_\varepsilon(t)\,\vec{e} \pm \vec{z}_l(t)$ where $\vec{e} = (1,1)^T$. Then, for a proper choice of $C$,

$$\vec{\psi}^{\pm}(0) = C\,\left(1 + e^{-\sqrt{\alpha}/\sqrt{\varepsilon}}\right)\vec{e} \pm \vec{z}_l(0) \ge \vec{0},$$
$$\vec{\psi}^{\pm}(1) = C\,\left(1 + e^{-\sqrt{\alpha}/\sqrt{\varepsilon}}\right)\vec{e} \pm \vec{z}_l(1) \ge \vec{0},$$

and for $t \in D$,

$$\begin{aligned}
\vec{\mathbb{T}}'\vec{\psi}^{\pm}(t) &= \begin{cases} C\left(-\varepsilon^2\dfrac{\alpha}{\varepsilon} + d_{11}(t,\varepsilon) + d_{12}(t,\varepsilon)\right)\mathbb{B}_\varepsilon(t) \pm \mathbb{T}_1' z_{l_1}(t), \\ C\left(-\varepsilon\dfrac{\alpha}{\varepsilon} + d_{21}(t,\varepsilon) + d_{22}(t,\varepsilon)\right)\mathbb{B}_\varepsilon(t) \pm \mathbb{T}_2' z_{l_2}(t), \end{cases} \\
&= \begin{cases} C\,(-\alpha\,\varepsilon + (d_{11}(t,\varepsilon) + d_{12}(t,\varepsilon)))\,\mathbb{B}_\varepsilon(t), \\ C\,(-\alpha + (d_{21}(t,\varepsilon) + d_{22}(t,\varepsilon)))\,\mathbb{B}_\varepsilon(t), \end{cases} \\
&\ge \begin{cases} C\,(-\alpha\,\varepsilon + \alpha)\,\mathbb{B}_\varepsilon(t), \\ C\,(-\alpha + \alpha)\,\mathbb{B}_\varepsilon(t), \end{cases} \\
&\ge \vec{0}.
\end{aligned}$$

Thus $\vec{\mathbb{T}}'$ satisfies Theorem 3.1 with $\vec{\psi}^{\pm}(t)$. Hence, $\vec{\psi}^{\pm}(t) \ge \vec{0}$ on $\overline{D}$ which gives for all $t \in \overline{D}$, $i = 1,2$,

$$|z_{l_i}(t)| \le C\,\mathbb{B}_\varepsilon(t). \tag{3.27}$$

The mean value theorem and (3.27) gives, for all $t \in \overline{D}$, $i = 1,2$,

$$|z_{l_i}'(t)| \le C\,\varepsilon^{-1/2}\mathbb{B}_\varepsilon(t). \tag{3.28}$$

Using (3.27) with (3.23), for all $t \in \overline{D}$,

$$|z_{l_1}''(t)| \le C\,\varepsilon^{-2}\mathbb{B}_\varepsilon(t). \tag{3.29}$$

Uncorrected Proof

Using (3.27) with (3.24), for all $t \in \overline{D}$,

$$|z''_{l_2}(t)| \leq C\,\varepsilon^{-1}\mathbb{B}_\varepsilon(t). \tag{3.30}$$

Differentiating (3.23) with respect to $t$ once and then using (3.27) and (3.28), for all $t \in \overline{D}$,

$$|z^{(3)}_{l_1}(t)| \leq C\,\varepsilon^{-5/2}\mathbb{B}_\varepsilon(t). \tag{3.31}$$

Differentiating (3.24) with respect to $t$ once and then using (3.27) and (3.28), for all $t \in \overline{D}$,

$$|z^{(3)}_{l_2}(t)| \leq C\,\varepsilon^{-3/2}\mathbb{B}_\varepsilon(t). \tag{3.32}$$

□

## 4. Mesh and numerical scheme

4.1. **The mesh.** A Shishkin mesh is constructed on $\overline{D}$ as follows. Divide $\overline{D}$ into 3 sub-intervals $[0, \lambda_\varepsilon)$, $[\lambda_\varepsilon, 1-\lambda_\varepsilon]$ and $(1-\lambda_\varepsilon, 1]$ such that $\overline{D} = [0, \lambda_\varepsilon) \cup [\lambda_\varepsilon, 1-\lambda_\varepsilon] \cup (1-\lambda_\varepsilon, 1]$ where

$$\lambda_\varepsilon = \min\left\{\frac{1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N_t\right\}.$$

$\frac{N_t}{4}$, $\frac{N_t}{2}$ and $\frac{N_t}{4}$ mesh points are placed uniformly on $[0, \lambda_\varepsilon)$, $[\lambda_\varepsilon, 1-\lambda_\varepsilon]$ and $(1-\lambda_\varepsilon, 1]$, respectively. Let $h_1$ denote the step size in $[0, \lambda_\varepsilon)$ and $(1-\lambda_\varepsilon, 1]$ whereas $h_2$ denote the step size in $[\lambda_\varepsilon, 1-\lambda_\varepsilon]$. Then $h_1 = \dfrac{4\,\lambda_\varepsilon}{N_t}$ and $h_2 = \dfrac{2(1-2\lambda_\varepsilon)}{N_t}$. Thus,

$$t_j = \begin{cases} jh_1, & \text{for } 0 \leq j \leq \dfrac{N_t}{4} - 1, \\ \lambda_\varepsilon + \left(j - \dfrac{N_t}{4}\right)h_2, & \text{for } \dfrac{N_t}{4} \leq j \leq \dfrac{3N_t}{4}, \\ 1 - \lambda_\varepsilon + jh_1, & \text{for } \dfrac{3N_t}{4} + 1 \leq j \leq N_t. \end{cases}$$

4.2. **The numerical scheme.** The discrete system corresponding to (2.6)-(2.7) is defined to be

$$\vec{\mathbb{T}}^{N_t}\vec{Z}(t_j) = -E\,\delta^2\vec{Z}(t_j) + \vec{f}(t_j, \varepsilon, \vec{Z}(t_j)) = \vec{0},\ t_j \in D^{N_t}, \tag{4.1}$$

with

$$\vec{Z}(t_j) = \vec{z}(t_j) \text{ for } t_j \in D_B^{N_t}, \tag{4.2}$$

where $D^{N_t} = \{t_j\}_{j=1}^{N_t-1}$ and $D_B^{N_t} = \{t_0, t_{N_t}\}$. Then $\overline{D}^{N_t} = \{t_j\}_{j=0}^{N_t}$.
Here, $\delta^2 F(t_j) = \dfrac{(D^+ - D^-)F(t_j)}{\overline{h}_j}$, $D^+F(t_j) = \dfrac{F(t_{j+1}) - F(t_j)}{h_{j+1}}$, $D^-F(t_j) = \dfrac{F(t_j) - F(t_{j-1})}{h_j}$,
$h_j = t_j - t_{j-1}$, $\overline{h}_j = \dfrac{h_{j+1} + h_j}{2}$, $\overline{h}_0 = \dfrac{h_1}{2}$ and $\overline{h}_{N_t} = \dfrac{h_{N_t}}{2}$.

## 5. Uniform convergence of the numerical method

Let $\vec{\varphi}_1(t_j) = (\varphi_{11}(t_j), \varphi_{12}(t_j))^T$ and $\vec{\varphi}_2(t_j) = (\varphi_{21}(t_j), \varphi_{22}(t_j))^T$ be two mesh functions such that $\vec{\varphi}_1(t_0) = \vec{\varphi}_2(t_0)$ and $\vec{\varphi}_1(t_{N_t}) = \vec{\varphi}_1(t_{N_t})$. Now for $t_j \in D^{N_t}$,

$$\begin{aligned}(\vec{\mathbb{T}}^{N_t}\vec{\varphi}_1 - \vec{\mathbb{T}}^{N_t}\vec{\varphi}_2)(t_j) &= \begin{cases} (\mathbb{T}_1^{N_t}\varphi_{11} - \mathbb{T}_1^{N_t}\varphi_{21})(t_j), \\ (\mathbb{T}_2^{N_t}\varphi_{12} - \mathbb{T}_2^{N_t}\varphi_{22})(t_j), \end{cases} \\ &= \begin{cases} -\varepsilon^2\delta^2(\varphi_{11} - \varphi_{21})(t_j) + f_1(t_j, \varepsilon, \vec{\varphi}_1(t_j)) - f_1(t_j, \varepsilon, \vec{\varphi}_2(t_j)), \\ -\varepsilon\,\delta^2(\varphi_{12} - \varphi_{22})(t_j) + f_2(t_j, \varepsilon, \vec{\varphi}_1(t_j)) - f_2(t_j, \varepsilon, \vec{\varphi}_2(t_j)), \end{cases}\end{aligned}$$

Uncorrected Proof

$$
\begin{aligned}
&= \begin{cases} -\varepsilon^2\delta^2(\varphi_{11}-\varphi_{21})(t_j) + e_{11}(t_j,\varepsilon)(\varphi_{11}-\varphi_{21})(t_j) + e_{12}(t_j,\varepsilon)(\varphi_{12}-\varphi_{22})(t_j), \\ -\varepsilon\,\delta^2(\varphi_{12}-\varphi_{22})(t_j) + e_{21}(t_j,\varepsilon)(\varphi_{11}-\varphi_{21})(t_j) + e_{22}(t_j,\varepsilon)(\varphi_{12}-\varphi_{22})(t_j), \end{cases} \\
&= \begin{cases} (\mathbb{T}_1^{N_t})'(\varphi_{11}-\varphi_{21})(t_j), \\ (\mathbb{T}_2^{N_t})'(\varphi_{12}-\varphi_{22})(t_j), \end{cases} \\
&= (\vec{\mathbb{T}}^{N_t})'(\vec{\varphi}_1-\vec{\varphi}_2)(t_j), \qquad (5.1)
\end{aligned}
$$

where $e_{ik}(t_j,\varepsilon) = \dfrac{\partial f_i(t_j,\,\varepsilon,\,\vec{\beta}(t_j))}{\partial u_k}$ are intermediate values and $(\vec{\mathbb{T}}^{N_t})'$ is the Frechet derivative of $\vec{\mathbb{T}}^{N_t}$.

**Theorem 5.1.** *For any function $\vec{\Phi}(t_j) = (\Phi_1(t_j),\,\Phi_2(t_j))^T$, the inequalities $\vec{\Phi}(t_0) \geq \vec{0}$, $\vec{\Phi}(t_{N_t}) \geq \vec{0}$ and $(\vec{\mathbb{T}}^{N_t})'\vec{\Phi} \geq \vec{0}$ on $D^N$ imply that $\vec{\Phi} \geq \vec{0}$ on $\overline{D}^{N_t}$.*

*Proof.* Let $\hat{i}$ and $\hat{j}$ be such that $\Phi_{\hat{i}}(t_{\hat{j}}) = \min\limits_{i,\,j}\Phi_i(t_j)$ and suppose $\Phi_{\hat{i}}(t_{\hat{j}}) < 0$. Then $t_{\hat{j}} \notin D_B^{N_t}$. Let $t_{\hat{j}} \in D^{N_t}$. Consider,

$$
\left((\vec{\mathbb{T}}^{N_t})'\vec{\Phi}\right)_{\hat{i}}(t_{\hat{j}}) = \begin{cases} -\varepsilon^2\delta^2\Phi_1(t_{\hat{j}}) + e_{11}(t_{\hat{j}},\varepsilon)\Phi_1(t_{\hat{j}}) + e_{12}(t_{\hat{j}},\varepsilon)\Phi_2(t_{\hat{j}}), \text{ if } \hat{i}=1, \\ -\varepsilon\,\delta^2\Phi_2(t_{\hat{j}}) + e_{21}(t_{\hat{j}},\varepsilon)\Phi_1(t_{\hat{j}}) + e_{22}(t_{\hat{j}},\varepsilon)\Phi_2(t_{\hat{j}}), \text{ if } \hat{i}=2. \end{cases}
$$

Thus,

$$
((\vec{\mathbb{T}}^{N_t})'\vec{\Phi})_{\hat{i}}(t_{\hat{j}}) \leq \begin{cases} -\varepsilon^2\delta^2\Phi_1(t_{\hat{j}}) + (e_{11}(t_{\hat{j}},\varepsilon) + e_{12}(t_{\hat{j}},\varepsilon))\Phi_1(t_{\hat{j}}), \text{ if } \hat{i}=1, \\ -\varepsilon\,\delta^2\Phi_2(t_{\hat{j}}) + (e_{21}(t_{\hat{j}},\varepsilon) + e_{22}(t_{\hat{j}},\varepsilon))\Phi_2(t_{\hat{j}}), \text{ if } \hat{i}=2 \end{cases}
$$

which gives $((\vec{\mathbb{T}}^{N_t})'\vec{\Phi})_{\hat{i}}(t_{\hat{j}}) < 0$, for $t_{\hat{j}} \in D^{N_t}$, a contradiction. Hence $\vec{\Phi} \geq \vec{0}$ on $\overline{D}^{N_t}$. □

**Theorem 5.2.** *If $\vec{\Phi}(t_j)$ is any function on $\overline{D}^{N_t}$ then for any $t_j \in \overline{D}^{N_t}$,*

$$
||\vec{\Phi}(t_j)|| \leq ||\vec{\Phi}(t_0)|| + ||\vec{\Phi}(t_{N_t})|| + \tfrac{1}{\alpha}||(\vec{\mathbb{T}}^{N_t})'\vec{\Phi}(t_j)||_{D^{N_t}}.
$$

*Proof.* Let $t_j \in \overline{D}^{N_t}$ and $M = ||\vec{\Phi}(t_0)|| + ||\vec{\Phi}(t_{N_t})|| + \frac{1}{\alpha}||(\vec{\mathbb{T}}^{N_t})'\vec{\Phi}(t_j)||_{D^{N_t}}$. Consider

$$
\vec{\Psi}^{\pm}(t_j) = M\vec{e} \pm \vec{\Phi}(t_j).
$$

Then, $\vec{\Psi}^{\pm}(t_0) \geq \vec{0}$ and $\vec{\Psi}^{\pm}(t_{N_t}) \geq \vec{0}$. Further, for $t_j \in D^{N_t}$

$$
(\vec{\mathbb{T}}^{N_t})'\vec{\Psi}(t_j) = \begin{cases} M(e_{11}(t_j,\varepsilon) + e_{12}(t_j,\varepsilon)) \pm (\mathbb{T}_1^{N_t})'\Phi_1(t_j), \\ M(e_{21}(t_j,\varepsilon) + e_{22}(t_j,\varepsilon)) \pm (\mathbb{T}_2^{N_t})'\Phi_2(t_j). \end{cases}
$$

Thus, $(\vec{\mathbb{T}}^{N_t})'\vec{\Psi}^{\pm} \geq \vec{0}$ on $D^{N_t}$. Hence using Theorem 5.1, $\vec{\Psi}^{\pm} \geq \vec{0}$ on $\overline{D}^{N_t}$. □

Since $\vec{\varphi}_1 = \vec{\varphi}_2$ on $D_B^{N_t}$, using Theorem 5.2 with (5.1), for $t_j \in \overline{D}^{N_t}$,

$$
\begin{aligned}
||(\vec{\varphi}_1 - \vec{\varphi}_2)(t_j)|| &\leq C\,||(\vec{\mathbb{T}}^{N_t})'(\vec{\varphi}_1 - \vec{\varphi}_2)(t_j)||_{D^{N_t}} \\
&= C\,||\vec{\mathbb{T}}^{N_t}\vec{\varphi}_1(t_j) - \vec{\mathbb{T}}^{N_t}\vec{\varphi}_2(t_j)||_{D^{N_t}}. \qquad (5.2)
\end{aligned}
$$

Similar to the continuous case, $\vec{Z}$ is divided into $\vec{Z}_S$ and $\vec{Z}_L$ such that $\vec{Z} = \vec{Z}_S + \vec{Z}_L$ where, for $t_j \in D^{N_t}$

$$
\begin{aligned}
&-\varepsilon^2\delta^2 Z_{S_1}(t_j) + c_{11}(t_j,\,\varepsilon)Z_{S_1}(t_j) + c_{12}(t_j,\,\varepsilon)Z_{S_2}(t_j) = y_1(t_j), \\
&-\varepsilon\,\delta^2 Z_{S_2}(t_j) + c_{21}(t_j,\,\varepsilon)Z_{S_1}(t_j) + c_{22}(t_j,\,\varepsilon)Z_{S_2}(t_j) = y_2(t_j), \qquad (5.3)
\end{aligned}
$$

Uncorrected Proof

with

$$\vec{Z}_S(t_0) = \vec{z}_s(t_0),\ \vec{Z}_S(t_{N_t}) = \vec{z}_s(t_{N_t}) \tag{5.4}$$

and for $t_j \in D^{N_t}$

$$\begin{aligned} &-\varepsilon^2\delta^2 Z_{L_1}(t_j) + d_{11}(t_j,\varepsilon)Z_{L_1}(t_j) + d_{12}(t_j,\varepsilon)Z_{L_2}(t_j) = 0,\\ &-\varepsilon\,\delta^2 Z_{L_2}(t_j) + d_{21}(t_j,\varepsilon)Z_{L_1}(t_j) + d_{22}(t_j,\varepsilon)Z_{L_2}(t_j) = 0, \end{aligned} \tag{5.5}$$

with

$$\vec{Z}_L(t_0) = \vec{z}_l(t_0),\ \vec{Z}_L(t_{N_t}) = \vec{z}_l(t_{N_t}). \tag{5.6}$$

Systems (5.3)-(5.4) and (5.5)-(5.6) can be written as

$$(\vec{\mathbb{T}}^{N_t})'\vec{Z}_S(t_j) = -E\,\delta^2\vec{Z}_S(t_j) + \vec{C}(t_j,\varepsilon)\vec{Z}_S(t_j) = \vec{y}(t_j), \tag{5.7}$$

with

$$\vec{Z}_S(t_0) = \vec{z}_s(t_0),\ \vec{Z}_S(t_{N_t}) = \vec{z}_s(t_{N_t}), \tag{5.8}$$

and

$$(\vec{\mathbb{T}}^{N_t})'\vec{Z}_L(t_j) = -E\,\delta^2\vec{Z}_L(t_j) + \vec{D}(t_j,\varepsilon)\vec{Z}_L(t_j) = \vec{0}, \tag{5.9}$$

with

$$\vec{Z}_L(t_0) = \vec{z}_l(t_0),\ \vec{Z}_L(t_{N_t}) = \vec{z}_l(t_{N_t}). \tag{5.10}$$

### 5.1. The error $\vec{Z}_S - \vec{z}_s$.

**Theorem 5.3.** *Let $\vec{z}_s$ be the solution of (3.8)-(3.9) and $\vec{Z}_S$ be that of (5.7)-(5.8). Then for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,*

$$|(\vec{Z}_S - \vec{z}_s)_i(t_j)| \ \le\ C\,N_t^{-1}.$$

*Proof.* Let $t_j \in \overline{D}^{N_t}$. Since $\vec{Z}_S = \vec{z}_s$ on $D_B^{N_t}$, from (5.2), for $i = 1, 2$,

$$|(\vec{Z}_S - \vec{z}_s)_i(t_j)| \ \le\ C\left|\left((\vec{\mathbb{T}}^{N_t})'(\vec{Z}_S - \vec{z}_s)\right)_i(t_j)\right|_{D^{N_t}}. \tag{5.11}$$

For $t_j \in D^{N_t}$, using the truncation error estimates in [**?** ],

$$\left|\left((\vec{\mathbb{T}}^{N_t})'(\vec{Z}_S - \vec{z}_s)\right)_i(t_j)\right| \le \begin{cases} C\,N_t^{-1}\varepsilon^2\,|z_{s_1}|_3, & \text{if } i = 1\\ C\,N_t^{-1}\varepsilon\,|z_{s_2}|_3, & \text{if } i = 2. \end{cases} \tag{5.12}$$

Using (3.5) with (5.12), for $t_j \in D^{N_t}$ and $i = 1, 2$,

$$\left|\left((\vec{\mathbb{T}}^{N_t})'(\vec{Z}_S - \vec{z}_s)\right)_i(t_j)\right| \le C\,N_t^{-1}. \tag{5.13}$$

From (5.11) and (5.13), for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$|(\vec{Z}_S - \vec{z}_s)_i(t_j)| \ \le\ C\,N_t^{-1}.$$

□

Uncorrected Proof

5.2. **The error $\vec{Z}_L - \vec{z}_l$.**

**Theorem 5.4.** *Let $\vec{z}_l$ be the solution of (3.25)-(3.26) and $\vec{Z}_L$ be that of (5.9)-(5.10). Then for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,*

$$|(\vec{Z}_L - \vec{z}_l)_i(t_j)| \;\le\; C\, N_t^{-1} \ln N_t. \tag{5.14}$$

*Proof.* Let $t_j \in \overline{D}^{N_t}$. Since $\vec{Z}_L = \vec{z}_l$ on $D_B^{N_t}$, from (5.2), for $i = 1, 2$,

$$|(\vec{Z}_L - \vec{z}_l\,)_i(t_j)| \;\le\; C\, \left| \Big( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \Big)_i (t_j) \right|_{D^{N_t}}. \tag{5.15}$$

The proof depends on whether $\lambda_\varepsilon = \frac{1}{4}$ or $\lambda_\varepsilon = \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N_t$. If $\lambda_\varepsilon = \frac{1}{4}$ then $\varepsilon^{-1/2} \le C \ln N_t$. For this choice of $\lambda_\varepsilon$, $t_{j+1} - t_{j-1} = 2N_t^{-1}$. And for $t_j \in D^{N_t}$,

$$\left| \Big( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \Big)_i (t_j) \right| \le \begin{cases} C\, N_t^{-1} \varepsilon^2\, |z_{l_1}|_3, & \text{if } i = 1 \\ C\, N_t^{-1} \varepsilon\, |z_{l_2}|_3, & \text{if } i = 2. \end{cases} \tag{5.16}$$

Using (3.31) and (3.32) with (5.16), for $t_j \in D^{N_t}$ and $i = 1, 2$,

$$\left| \Big( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \Big)_i (t_j) \right| \le C\, N_t^{-1} \varepsilon^{-1/2}. \tag{5.17}$$

Since $\varepsilon^{-1/2} \le C \ln N_t$, (5.17) becomes, for $t_j \in D^{N_t}$ and $i = 1, 2$,

$$\left| \Big( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \Big)_i (t_j) \right| \le C\, N_t^{-1} \ln N_t. \tag{5.18}$$

From (5.15) and (5.18), for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$|(\vec{Z}_L - \vec{z}_l\,)_i(t_j)| \;\le\; C\, N_t^{-1} \ln N_t. \tag{5.19}$$

Now let $\lambda_\varepsilon = \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N_t$. For this choice of $\lambda_\varepsilon$, the result is established on the sub-intervals $[0, \lambda_\varepsilon)$, $[\lambda_\varepsilon, 1 - \lambda_\varepsilon]$ and $(1 - \lambda_\varepsilon, 1]$ separately. First let either $t_j \in (0, \lambda_\varepsilon)$ or $t_j \in (1 - \lambda_\varepsilon, 1)$. Then $t_{j+1} - t_{j-1} = \frac{8\sqrt{\varepsilon}\, N_t^{-1} \ln N_t}{\sqrt{\alpha}}$. Proceeding as before, for either $t_j \in (0, \lambda_\varepsilon)$ or $t_j \in (1 - \lambda_\varepsilon, 1)$ and $i = 1, 2$,

$$|(\vec{Z}_L - \vec{z}_l\,)_i(t_j)| \;\le\; C\, N_t^{-1} \ln N_t. \tag{5.20}$$

Finally let $t_j \in [\lambda_\varepsilon, 1 - \lambda_\varepsilon]$. In this case

$$\begin{aligned}
\left| \Big( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \Big)_i (t_j) \right| &\le \begin{cases} 2\varepsilon^2 \max\limits_{\zeta \in [t_{j-1}, t_{j+1}]} |z_{l_1}''(\zeta)|, & \text{if } i = 1 \\ 2\varepsilon \max\limits_{\mu \in [t_{j-1}, t_{j+1}]} |z_{l_2}''(\mu)|, & \text{if } i = 2. \end{cases} \\
&\le 2 \max_{\kappa \in [t_{j-1}, t_{j+1}]} \mathbb{B}_\varepsilon(\kappa), i = 1, 2 \\
&= 2 \left( e^{-\sqrt{\alpha} t_{(N_t/4)-1}/\sqrt{\varepsilon}} + e^{-\sqrt{\alpha}/\sqrt{\varepsilon}} e^{\sqrt{\alpha} t_{(3N/4_t)+1}/\sqrt{\varepsilon}} \right), \; i = 1, 2 \\
&= 2 \left( e^{-\sqrt{\alpha}(\lambda_\varepsilon - h_1)/\sqrt{\varepsilon}} + e^{-\sqrt{\alpha}/\sqrt{\varepsilon}} e^{\sqrt{\alpha}((1-\lambda_\varepsilon)+h_1)/\sqrt{\varepsilon}} \right), \; i = 1, 2 \\
&= 2 \left( e^{-\sqrt{\alpha}\lambda_\varepsilon/\sqrt{\varepsilon}} e^{\sqrt{\alpha} h_1/\sqrt{\varepsilon}} + e^{-\sqrt{\alpha}\lambda_\varepsilon/\sqrt{\varepsilon}} e^{\sqrt{\alpha} h_1/\sqrt{\varepsilon}} \right), \; i = 1, 2 \\
&= 4\, e^{-\sqrt{\alpha}\lambda_\varepsilon/\sqrt{\varepsilon}} e^{\sqrt{\alpha} h_1/\sqrt{\varepsilon}}, i = 1, 2 \\
&= 4\, N_t^{-1} \left( N_t^{1/N_t} \right)^4, i = 1, 2 \\
&\le C\, N_t^{-1}, i = 1, 2, \text{ since } x^{1/x} \to 1 \text{ as } x \to \infty.
\end{aligned} \tag{5.21}$$

Uncorrected Proof

Thus, for $t_j \in [\lambda_\varepsilon, 1-\lambda_\varepsilon]$ and for $i = 1, 2$,

$$\left| \left( (\vec{\mathbb{T}}^{N_t})'(\vec{Z}_L - \vec{z}_l\,) \right)_i (t_j) \right| \leq C\, N_t^{-1}. \tag{5.22}$$

From (5.15) and (5.22), for $t_j \in [\lambda_\varepsilon, 1-\lambda_\varepsilon]$ and for $i = 1, 2$,

$$|(\vec{Z}_L - \vec{z}_l\,)_i(t_j)| \;\leq\; C\, N_t^{-1}. \tag{5.23}$$

From (5.19), (5.20) and (5.23), for any choice of $\lambda_\varepsilon$, for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$|(\vec{Z}_L - \vec{z}_l\,)_i(t_j)| \;\leq\; C\, N_t^{-1} \ln N_t.$$

□

### 5.3. The error $\vec{Z} - \vec{z}$.

**Theorem 5.5.** *Let $\vec{z}$ be the solution of (2.6)-(2.7) and $\vec{Z}$ be that of (4.1)-(4.2). Then for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,*

$$|(\vec{Z} - \vec{z}\,)_i(t_j)| \;\leq\; C\, N_t^{-1} \ln N_t.$$

*Proof.* Note that for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$(\vec{Z} - \vec{z}\,)_i(t_j) = (\vec{Z}_S - \vec{z}_s)_i(t_j) + (\vec{Z}_L - \vec{z}_l)_i(t_j).$$

Thus, for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$|(\vec{Z} - \vec{z}\,)_i(t_j)| \leq |(\vec{Z}_S - \vec{z}_s)_i(t_j)| + |(\vec{Z}_L - \vec{z}_l)_i(t_j)|. \tag{5.24}$$

Using Theorem 5.3 and Theorem 5.4 with (5.24), for $t_j \in \overline{D}^{N_t}$ and $i = 1, 2$,

$$|(\vec{Z} - \vec{z}\,)_i(t_j)| \;\leq\; C\, N_t^{-1} \ln N_t.$$

□

## 6. The continuation method

The nonlinear system of ordinary differential equations in (2.6)-(2.7) is modified to an artificial nonlinear system of partial differential equation as given below. For $(t, x) \in D \times (0, X\,]$,

$$\begin{aligned} \frac{\partial z_1(t,x)}{\partial x} - \varepsilon^2 \frac{\partial^2 z_1(t,x)}{\partial t^2} + f_1(t,\, \varepsilon,\, z_1(t,x), z_2(t,x)) \;&=\; 0, \\ \frac{\partial z_2(t,x)}{\partial x} - \varepsilon \frac{\partial^2 z_2(t,x)}{\partial t^2} + f_2(t,\, \varepsilon,\, z_1(t,x), z_2(t,x)) \;&=\; 0, \end{aligned} \tag{6.1}$$

with $\vec{z}(0,x) = \vec{\theta}_0,\;\; \vec{z}(1,x) = \vec{\theta}_1,\; x \geq 0$ and $\vec{z}(t,0) = \vec{z}_{init}(t),\; 0 < t < 1$.

The above system of equations can be written together in vector form as follows.

$$\begin{aligned} &\frac{\partial \vec{z}(t,x)}{\partial x} + E\, \frac{\partial^2 \vec{z}(t,x)}{\partial t^2} + \vec{f}(t,\, \varepsilon,\, \vec{z}(t,x)) = \vec{0}, \\ &\vec{z}(0,x) = \vec{\theta}_0,\;\; \vec{z}(1,x) = \vec{\theta}_1,\; x \geq 0,\;\; \text{and}\;\; \vec{z}(t,0) = \vec{z}_{init}(t),\; 0 < t < 1, \end{aligned} \tag{6.2}$$

where $E$ is the diagonal matrix in (2.6). The continuation method developed for a scalar nonlinear differential equation of reaction-diffusion type in [9] is modified appropriately for the nonlinear system of differential equations mentioned in (2.6); the modified structure is given below which is used to solve (6.2). For $t_j \in D^{N_t}$ and $k = 1, \ldots K$,

$$D_x^- \vec{Z}(t_j, x_k) - E\, \delta_t^2 \vec{Z}(t_j, x_k) + \vec{f}(t_j,\, \varepsilon,\, \vec{Z}(t_j, x_{k-1})) = \vec{0}, \tag{6.3}$$

Uncorrected Proof

$$\begin{aligned} \vec{Z}(t_0, x_k) &= \vec{z}(t_0), \quad \vec{Z}(t_{N_t}, x_k) = \vec{z}(t_{N_t}) \text{ for all } k \text{ and} \\ \vec{Z}(t_j, x_0) &= \vec{z}_{init}(t_j) \text{ for all } t_j \in \overline{D}^{N_t} \end{aligned} \tag{6.4}$$

where $\delta_t^2 S(t_j, x_k) = \frac{(D_t^+ - D_t^-)S(t_j,x_k)}{\overline{h}_j}$, $D_t^+ S(t_j, x_k) = \frac{S(t_{j+1},x_k)-S(t_j,x_k)}{h_{j+1}}$, $D_t^- S(t_j, x_k) = \frac{S(t_j,x_k)-S(t_{j-1},x_k)}{h_j}$, $D_x^- S(t_j, x_k) = \frac{S(t_j,x_k)-S(t_j,x_{k-1})}{h_x}$.

The initial guess $\vec{z}_{init}(t)$ is taken to be

$$\vec{z}_{init}(t) = \begin{bmatrix} z_1(0) + t(z_1(1) - z_1(0)) \\ z_2(0) + t(z_2(1) - z_2(0)) \end{bmatrix}.$$

The choices of the step size $h_x = x_k - x_{k-1}$ and the number of iterations $K$ are determined as follows. Define,

$$Err(k) = \max_{t_j \in D^{N_t}} \left( \frac{||\vec{Z}(t_j, x_k) - \vec{Z}(t_j, x_{k-1})||}{h_x} \right) \text{ for } k = 1, \ldots, K. \tag{6.5}$$

The step size $h_x$ is chosen sufficiently small so that $Err(k)$ decreases with the increasing $k$. Precisely, we choose $h_x$ such that

$$Err(k) \leq Err(k-1) \text{ for all } k, 1 < k \leq K. \tag{6.6}$$

and $K$ such that

$$Err(K) \leq tol \tag{6.7}$$

where $tol$ is a prescribed small tolerance.

## 7. Computational experiments

Nonlinear system in (2.1)-(2.2) is examined in this section. The continuation method constructed in Section 6 is used to solve (2.1)-(2.2). The error tolerance $'tol'$ is taken to be $10^{-7}$ and the number of iterations $K$ is taken to be 100. Notations $E^{N_t}$, $C_p^{N_t}$ and $p^{N_t}$ denote the parameter-uniform maximum pointwise error, parameter-uniform error constant and parameter-uniform rate of convergence respectively and given by

$$E^{N_t} = \max_{\varepsilon} E_{\varepsilon}^{N_t}, \quad \text{where } E_{\varepsilon}^{N_t} = \| \vec{Z}^{N_t} - \vec{Z}^{2N_t} \|,$$

$$p^{N_t} = \log_2 \frac{E^{N_t}}{E^{2N_t}}, \quad C_p^{N_t} = \frac{E^{N_t} N_t^{p^\star}}{1 - 2^{-p^\star}}, \quad \text{where } p^\star = \min_{N_t} p^{N_t}.$$

**Example 7.1.** Consider

$$\begin{aligned} -\varepsilon^2 \frac{d^2u(t)}{dt^2} + f(t, \varepsilon, u(t), v(t)) &= 0, \\ -\varepsilon \frac{d^2v(t)}{dt^2} + g(t, \varepsilon, u(t), v(t)) &= 0, \ t \in D, \end{aligned}$$

with $u(0) = 0.5$, $u(1) = \sin(\cos(1))$, $v(0) = e^{-0.6}$, $v(1) = \cos(1 + \sin(0.1))$,

$$f(t, \varepsilon, u(t), v(t)) = u^5(t) + (3 + \varepsilon)u(t) - v(t) - 1 - \varepsilon - t/4,$$
$$g(t, \varepsilon, u(t), v(t)) = v^7(t) - (1 - \varepsilon^2)u(t) + (4 + \varepsilon^2)v(t) - 2.$$

For Example 7.1, the values of $E^{N_t}, C_p^{N_t}, p^{N_t}$ are provided in Table 1 and the CPU time (in seconds) for each row is given in the last column. Decomposition of the domain $[0, 1]$ for $N_t = 64$ and $\varepsilon = 2^{-15}$ is portrayed in Figure 1. A graph of the numerical solution for $N_t = 256$ and $\varepsilon = 2^{-10}$ is portrayed in Figure 2. Further, for $N_t = 256$ and $\varepsilon = 2^{-2}, 2^{-6}, 2^{-12}$, the changes in the components $u(t)$ and $v(t)$ are portrayed in Figure 3. The *Log-log plot* for the error in the suggested computational method for Example 7.1 is presented in Figure 6.

Uncorrected Proof

**Example 7.2.** Consider

$$-\varepsilon^2 \frac{d^2u(t)}{dt^2} + f(t, \varepsilon, u(t), v(t)) = 0,$$

$$-\varepsilon \frac{d^2v(t)}{dt^2} + g(t, \varepsilon, u(t), v(t)) = 0, \ t \in D,$$

with $u(0) = 0.5$, $u(1) = \cos(5)$, $v(0) = 0.5$, $v(1) = \sin(5)$,

$$f(t, \varepsilon, u(t), v(t)) = u^3(t) + (3 - \varepsilon)u(t) - v(t) - 1 + \varepsilon^2,$$

$$g(t, \varepsilon, u(t), v(t)) = v^9(t) - (1 - \varepsilon^2)u(t) + (4 - \varepsilon^2)v(t) - t.$$

For Example 7.2, the values of $E^{N_t}, C_p^{N_t}, p^{N_t}$ are provided in Table 2 and the CPU time (in seconds) for each row is given in the last column. A graph of the numerical solution for $N_t = 256$ and $\varepsilon = 2^{-10}$ is portrayed in Figure 4. Further, for $N_t = 256$ and $\varepsilon = 2^{-2}, 2^{-6}, 2^{-12}$, the changes in the components $u(t)$ and $v(t)$ are portrayed in Figure 5. The *Log-log plot* for the error in the suggested computational method for Example 7.1 is presented in Figure 7.

TABLE 1. *Values of* $E^{N_t}, C_p^{N_t}$ & $p^{N_t}$ *for* $\alpha = 1/2$.

| $\varepsilon$ | $N_t$ | | | | | CPU time (in seconds) |
|---|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 | |
| $2^{-3}$ | 1.3606e-03 | 4.5188e-04 | 1.1989e-04 | 3.0690e-05 | 7.7182e-06 | 1.3038 |
| $2^{-6}$ | 4.7219e-03 | 1.8519e-03 | 1.1304e-03 | 9.1937e-04 | 3.3409e-04 | 2.4076 |
| $2^{-9}$ | 1.7789e-03 | 1.6109e-03 | 6.8051e-04 | 9.1776e-04 | 5.3345e-04 | 3.4815 |
| $2^{-12}$ | 1.8309e-03 | 1.6129e-03 | 1.3601e-03 | 9.1315e-04 | 4.6197e-04 | 3.8753 |
| $2^{-15}$ | 1.8507e-03 | 1.6138e-03 | 1.3605e-03 | 9.1326e-04 | 4.6198e-04 | 4.2701 |
| $2^{-18}$ | 1.8578e-03 | 1.6142e-03 | 1.3607e-03 | 9.1332e-04 | 4.6199e-04 | 4.6561 |
| $2^{-21}$ | 1.8603e-03 | 1.6143e-03 | 1.3607e-03 | 9.1334e-04 | 4.6200e-04 | 5.0400 |
| $2^{-24}$ | 1.8612e-03 | 1.6143e-03 | 1.3608e-03 | 9.1334e-04 | 4.6200e-04 | 5.4313 |
| $2^{-27}$ | 1.8616e-03 | 1.6143e-03 | 1.3608e-03 | 9.1335e-04 | 4.6200e-04 | 5.8155 |
| $2^{-30}$ | 1.8617e-03 | 1.6144e-03 | 1.3608e-03 | 9.1335e-04 | 4.6200e-04 | 6.1990 |
| $2^{-33}$ | 1.8617e-03 | 1.6144e-03 | 1.3608e-03 | 9.1335e-04 | 4.6200e-04 | 6.6315 |
| $2^{-36}$ | 1.8617e-03 | 1.6144e-03 | 1.3608e-03 | 9.1335e-04 | 4.6200e-04 | 7.0163 |
| $E^{N_t}$ | 4.7219e-03 | 1.8519e-03 | 1.3608e-03 | 9.1937e-04 | 5.3345e-04 | |
| $p^{N_t}$ | 4.4456e-01 | 4.4456e-01 | 5.6571e-01 | 7.8528e-01 | | |
| $C_p^{N_t}$ | 4.4878e-02 | 2.3953e-02 | 2.3953e-02 | 2.2024e-02 | 1.7391e-02 | |

Uncorrected Proof

## 8. CONCLUSION

In this article a numerical method consist of classical finite difference operators and a piecewise uniform Shishkin mesh has been developed for a nonlinear system of singular perturbation problems with different magnitudes of a single parameter. It has been proved both theoretically and numerically that the developed computational method is robust, layer resolving and also parameter uniform convergent.

From the table, we find that the maximum pointwise error ($E^{N_t}$) decreases when the number of mesh points ($N_t$) increases; on the other hand $E^{N_t}$ stabilizes for each $N_t$ as the perturbation parameter approaches zero. Moreover, from the table, we notice that the proposed computational method is almost first order parameter uniform convergent. This is in agreement with Theorem 5.5.

Figures given in this present article exhibit the complex boundary layer nature of the solution of the nonlinear system of singular perturbation problems with different magnitudes of a single parameter. From Figure 2 and Figure 4 we find that both the components $u(t)$ and $v(t)$ of the solution exhibit overlapping boundary layers at both the boundaries $t = 0$ and $t = 1$ of the domain of the system and they are smooth away from the boundaries. Further, it is

TABLE 2. *Values of* $E^{N_t}, C_p^{N_t}$ & $p^{N_t}$ *for* $\alpha = 1/2$.

| $\varepsilon$ | $N_t$ | | | | | CPU time (in seconds) |
|---|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 | |
| $2^{-3}$ | 6.3393e-03 | 2.2488e-03 | 5.9408e-04 | 1.5037e-04 | 3.7702e-05 | 1.4363 |
| $2^{-6}$ | 3.2693e-02 | 1.2596e-02 | 6.7263e-03 | 6.1048e-03 | 2.2035e-03 | 2.6081 |
| $2^{-9}$ | 1.3007e-02 | 1.1095e-02 | 4.4213e-03 | 6.3552e-03 | 3.4782e-03 | 3.6940 |
| $2^{-12}$ | 1.3486e-02 | 1.1103e-02 | 9.0322e-03 | 5.9458e-03 | 2.7790e-03 | 4.0881 |
| $2^{-15}$ | 1.3668e-02 | 1.1106e-02 | 9.0331e-03 | 5.9470e-03 | 2.7801e-03 | 4.5428 |
| $2^{-18}$ | 1.3734e-02 | 1.1107e-02 | 9.0334e-03 | 5.9473e-03 | 2.7804e-03 | 4.9414 |
| $2^{-21}$ | 1.3757e-02 | 1.1107e-02 | 9.0335e-03 | 5.9474e-03 | 2.7805e-03 | 5.3414 |
| $2^{-24}$ | 1.3765e-02 | 1.1107e-02 | 9.0335e-03 | 5.9474e-03 | 2.7806e-03 | 5.7373 |
| $2^{-27}$ | 1.3768e-02 | 1.1108e-02 | 9.0335e-03 | 5.9475e-03 | 2.7806e-03 | 6.1250 |
| $2^{-30}$ | 1.3769e-02 | 1.1108e-02 | 9.0336e-03 | 5.9475e-03 | 2.7806e-03 | 6.5139 |
| $2^{-33}$ | 1.3769e-02 | 1.1108e-02 | 9.0336e-03 | 5.9475e-03 | 2.7806e-03 | 6.9389 |
| $2^{-36}$ | 1.3770e-02 | 1.1108e-02 | 9.0336e-03 | 5.9475e-03 | 2.7806e-03 | 7.3243 |
| $E^{N_t}$ | 3.2693e-02 | 1.2596e-02 | 9.0336e-03 | 6.3552e-03 | 3.4782e-03 | |
| $p^{N_t}$ | 4.7959e-01 | 4.7959e-01 | 5.0735e-01 | 8.6961e-01 | | |
| $C_p^{N_t}$ | 3.1337e-01 | 1.6835e-01 | 1.6835e-01 | 1.6514e-01 | 1.2602e-01 | |

Proof

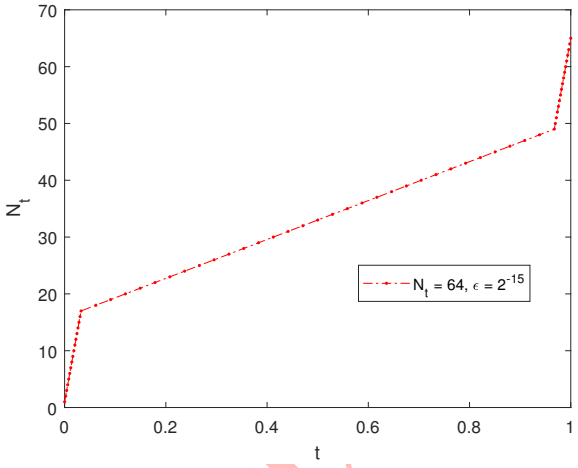



FIGURE 1. Decomposition of $[0, 1]$ for $N_t = 64, \varepsilon = 2^{-15}$.

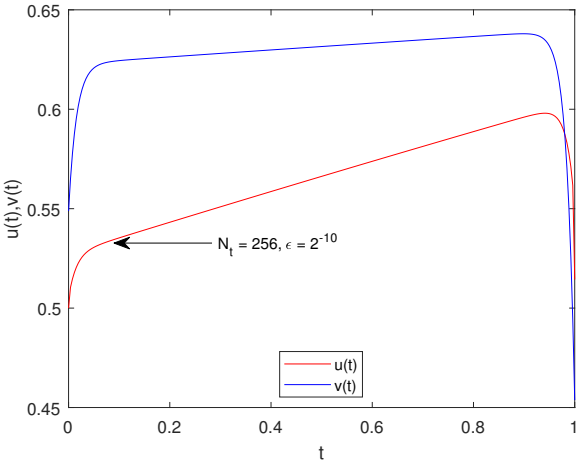



FIGURE 2. Solution profile of Example 7.1.

evident from Figure 3 and Figure 5 that both the components $u(t)$ and $v(t)$ of the solution of the system undergoes a rapid change near both the boundaries of the domain of the system when the parameter $\varepsilon$ approaches zero.

Moreover, the $Log - log\ plot$ of the error in the suggested numerical method for the system in Example 7.1 is presented in Figure 6 and in Figure 7 for Example 7.2. From Figure 6 and Figure 7 we find that the maximum pointwise errors are bounded by $O(N^{-1} \ln N)$ which is proved in Theorem 5.5.

It is to be noted that the analysis provided here can be suitably modified to construct numerical methods for systems with three or more equations, systems with discontinuous source terms and parabolic problems.

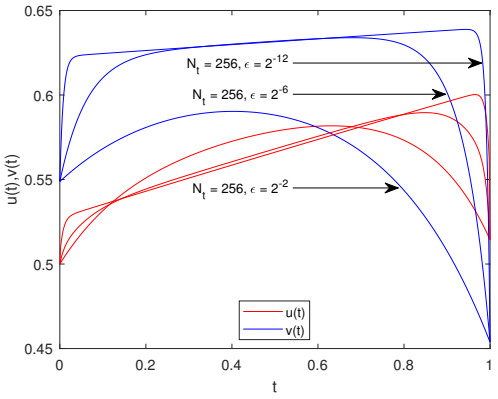



FIGURE 3. Changes in the components of the solution in Example 7.1.

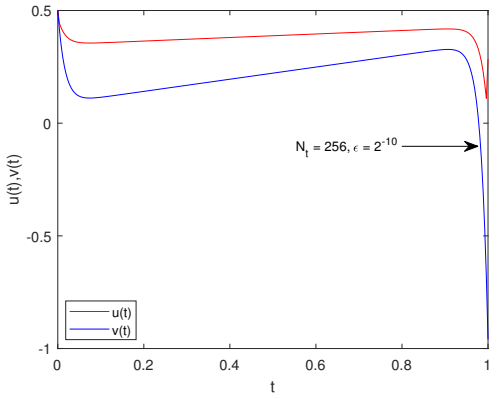



FIGURE 4. Solution profile of Example 7.2.

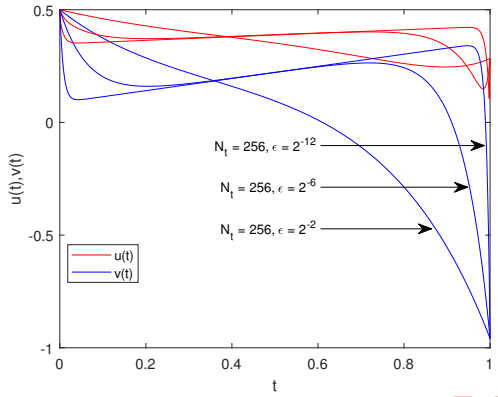



FIGURE 5. Changes in the components of the solution in Example 7.2.

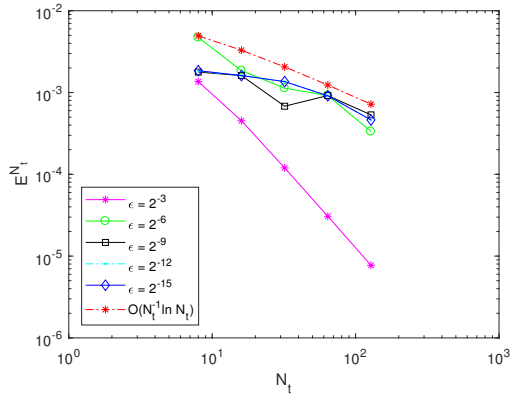



FIGURE 6. *Log-log plot* for the error in Example 7.1.

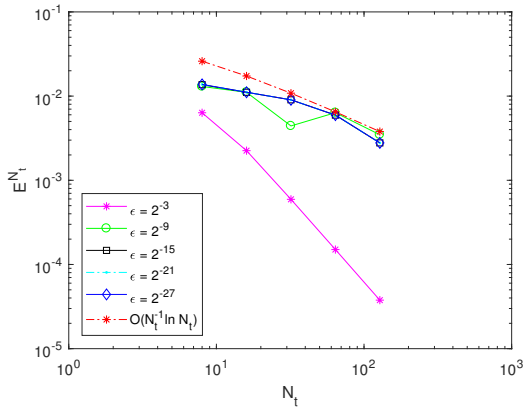



FIGURE 7. *Log-log plot* for the error in Example 7.2.

Uncorrected Proof

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