Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-33
DOI:10.22034/cmde.2026.69379.3417

An accurate spectral collocation matrix method: Utilizing generalized Chebyshev–Gegenbauer polynomials of the seventh and eighth kinds for fractional differential equations

Yasser A. Amer[1], **Kamal R. Raslan**[2], **Mohammed A. Abd EL Salam**[2,3,*] **and Aziza A. Mahdi**[1]

[1]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt.

[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt.

[3]Basic and Applied Science Department, College of Engineering and Technology, Arab Academy for Science, Technology and Maritime Transport, Smart Village Campus, Giza 12577, Egypt.

Abstract

Fractional differential equations (FDEs) have attracted considerable attention due to their ability to accurately model memory-dependent and nonlocal phenomena arising in various scientific and engineering applications. Motivated by the need for accurate and computationally efficient numerical techniques, this paper introduces a novel spectral collocation method for numerically solving fractional differential equations using generalized Chebyshev polynomials of the seventh and eighth kinds. These new classes of orthogonal polynomials are constructed through a Gegenbauer-type generating formulation, providing enhanced symmetry and flexibility in function approximation. By deriving operational matrices for both the Caputo fractional derivative and the ordinary derivative corresponding to the proposed basis functions, the original fractional problem is transformed into a system of algebraic equations that can be solved efficiently. The proposed method is applied to several benchmark linear and nonlinear examples to demonstrate its accuracy and spectral convergence. Numerical results show that the proposed approach outperforms classical Chebyshev- and Legendre-based methods in terms of error minimization and numerical stability. In addition, a convergence and approximation analysis of the proposed polynomial bases is successfully established. The obtained results confirm that the proposed framework provides a highly accurate, stable, and computationally efficient numerical tool for solving a broad class of fractional differential equations. This framework opens the door to more generalized polynomial constructions in the context of fractional calculus and offers a powerful alternative for modeling nonlocal and memory-dependent systems.

Keywords. Fractional differential equation, Generalized Chebyshev–Gegenbauer polynomials, Seventh and eighth Chebyshev polynomials, Spectral collocation method, Convergence and approximation.

2010 Mathematics Subject Classification. 26A33, 65L05, 65D15.

Uncorrected Proof

1. Introduction

Fractional differential equations (FDEs) provide a rigorous mathematical framework to model processes exhibiting memory and nonlocal behavior, with widespread applications in anomalous diffusion, viscoelastic materials, control theory and biological transport phenomena [14, 32, 41]. The theoretical development of fractional calculus and canonical examples of fractional models are well documented in modern surveys and monographs; recent computational research has focused on bridging analytical theory and efficient numerical schemes for applied fractional models [2, 3].

Spectral methods constitute a particularly powerful family of numerical techniques for FDEs because their global basis functions often yield spectral convergence for sufficiently solutions. The classical orthogonal families Chebyshev of first and second kinds, Legendre and Jacobi-Gegenbauer polynomials are well understood in this context and serve as the basic for many high-accuracy collocation, Galerkin and tau schemes [36, 45].

Received: 29 September 2025; Accepted: 15 November 2026.
∗ Corresponding author. Email: mohamedsalam1985@yahoo.com, mohamed_salam@azhar.edu.eg.

In the fractional setting, operational-matrix approaches representing fractional integrals and derivatives as algebraic matrices acting on basis-coefficient vectors are widely used to reduce fractional operators to matrix equations, thereby simplifying discretizations and substantially accelerating computations for both linear and nonlinear problems. Recent methodological advances include improved Caputo-sense operational matrices and their application in optimal control, delay systems, and physics-informed neural networks (PINNs) that incorporate fractional operators [21, 30, 42]. Furthermore, wavelet Galerkin techniques have recently been developed for fractional delay differential equations, providing accurate and stable numerical approximations for this important class of fractional models [22].

While most spectral studies for FDEs have concentrated on the classical polynomial bases, recent years have witnessed the introduction and development of higher-kind Chebyshev families fifth to eighth kinds and related Gegenbauer-type constructions. These generalized Chebyshev polynomials possess distinctive orthogonality weights and symmetry properties that can be advantageous in function approximation, in handling boundary or singular behavior, and in producing structured system matrices that influence conditioning and solver performance [18, 26, 42, 46].

In particular, seventh- and eighth-kind Chebyshev polynomials have been proposed and analyzed for fractional problems: seventh-kind polynomials have been used in spectral tau approaches for time-fractional diffusion equations with accompanying expansion formulae and error discussions [17], while eighth-kind polynomials have been employed in collocation algorithms for nonlinear time-fractional generalized Kawahara models with demonstrated convergence properties [42].

Beyond these specific polynomial families, the literature has reported several complementary advances that motivate a careful comparative study: new derivative operational matrices for the Caputo sense that improve sparsity or accuracy, hybrid or wavelet–polynomial operational matrices for multi-order and multi-term FDEs, and variable-order spectral discretizations using Jacobi-type bases that address time-varying fractional orders and initial singularities . These advances expand the computational toolbox and underscore the importance of assessing alternative orthogonal bases within a unified computational framework [6, 34, 35].

Despite progress, a systematic comparative study that develops Caputo's operational matrices for shifted seventh- and eighth-kind Chebyshev bases, integrates them into a single spectral collocation framework, and rigorously compares their behavior (accuracy, spectral convergence, conditioning, and stability) against classical Chebyshev and Legendre schemes across representative linear and nonlinear benchmark problems is still lacking. The present paper fills that gap by deriving compact operational-matrix formulae for shifted seventh- and eighth-kind Chebyshev polynomials, implementing a collocation algorithm that converts Caputo's FDEs into algebraic systems, and presenting a comprehensive numerical comparison highlighting when and why higher-kind bases provide practical advantages. The scope of this paper extends beyond its numerical results; it also provides a thorough investigation of the convergence and approximation of the proposed polynomias.

The paper is organized as follows. Section 2 collects definitions and notation for Caputo derivatives, Section 3 discuss the generalized gegenbaur polynomials with seventh and eighth kind, Section 4 develops the Caputo operational matrices for the proposed bases. Section 6 describes the collocation formulation and implementation details, Section 7 presents numerical benchmarks linear and nonlinear and comparative performance results.

Uncorrected proof

2. Preliminaries and definitions

In this section, we present the essential mathematical concepts required for the development of our numerical method. We begin with basic definitions from fractional calculus, then provide a brief review of Gegenbauer polynomials, followed by the construction of the generalized Chebyshev polynomials of the seventh and eighth kinds.

Definition 2.1. Let $\lambda > 0$, and let $\eta = \lceil \lambda \rceil \in \mathbb{N}$. Assume that the function $\mathcal{U}(r)$ belongs to the space of continuous functions $\eta[r_0, r]$. Then, the left-sided Caputo fractional derivative of order λ starting from r_0 is defined by [19, 27]:

$$ {}^{C}D^{\lambda}_{r_0+}\mathcal{U}(r) := \frac{1}{\Gamma(\eta-\lambda)} \int_{r_0}^{r} \frac{\mathcal{U}^{(\eta)}(s)}{(r-s)^{\lambda-\eta+1}}\, ds, \quad r > r_0. \tag{2.1} $$

Definition 2.2. Let $\mathcal{M}_\mu(r) = r^\mu$, where $\mu \in \mathbb{R}$. The Caputo fractional derivative of the monomial function is given by [29]:

$$ {}^{C}D^{\lambda}_{r_0+}\mathcal{M}_\mu(r) = \begin{cases} 0, & \text{if } \mu < \eta, \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\lambda)}\, r^{\mu-\lambda}, & \text{if } \mu \geq \eta, \end{cases} \tag{2.2} $$

where $\eta = \lceil \lambda \rceil$.

Definition 2.3. If $\mathcal{U}(r) = C \in \mathbb{R}$ is a constant function, then its Caputo derivative vanishes:

$$ {}^{C}D^{\lambda}_{r_0+}C = 0, \quad \forall \lambda > 0. \tag{2.3} $$

3. General form of Jacobi and Gegenbauer-type polynomials

The Jacobi polynomials, denoted by $\mathcal{P}_i^{(\mu,\eta)}(r)$, form a classical family of orthogonal polynomials on the interval $[-1, 1]$, characterized by two real parameters μ, η where $\mu, \eta > -1$. They are solutions of the second-order differential equation [23, 37, 40, 44]:

$$ (1-r)^\mu (1+r)^\eta \frac{d}{dr}\left[(1-r)^\mu (1+r)^\eta \frac{d}{dr}\mathcal{P}_i^{(\mu,\eta)}(r)\right] + i(i+\mu+\eta+1)\mathcal{P}_i^{(\mu,\eta)}(r) = 0. \tag{3.1} $$

An explicit formula for these polynomials is provided by the Rodrigues representation:

$$ \mathcal{P}_i^{(\mu,\eta)}(r) = \frac{(-1)^i}{2^i i!}(1-r)^{-\mu}(1+r)^{-\eta}\frac{d^i}{dr^i}\left[(1-r)^{\mu+i}(1+r)^{\eta+i}\right]. \tag{3.2} $$

Jacobi polynomials are orthogonal with respect to the weight function $w^{(\mu,\eta)}(r) = (1-r)^\mu(1+r)^\eta$ and satisfy the orthogonality condition:

$$ \int_{-1}^{1} \mathcal{P}_j^{(\mu,\eta)}(r)\mathcal{P}_i^{(\mu,\eta)}(r)w^{(\mu,\eta)}(r)\, dr = 0, \quad \text{for } j \neq n. \tag{3.3} $$

These polynomials are symmetric under reflection:

$$ \mathcal{P}_i^{(\mu,\eta)}(-r) = (-1)^i \mathcal{P}_i^{(\mu,\eta)}(r), \tag{3.4} $$

and they satisfy the recurrence relation, is given by:

$$ \mathcal{P}_{i+1}^{(\mu,\eta)}(r) = \frac{2i+\mu+\eta+1}{i+1}\left[r\mathcal{P}_i^{(\mu,\eta)}(r) - \frac{\mu^2-\eta^2}{(2i+\mu+\eta)(2i+\mu+\eta+2)}\mathcal{P}_{i-1}^{(\mu,\eta)}(r)\right]. \tag{3.5} $$

Jacobi polynomials also unify and generalize several well-known families of orthogonal polynomials:
Legendre polynomials arise when both parameters of the Jacobi polynomials vanish, that is:

$$ \mathcal{L}_i(r) = \mathcal{P}_i^{(0,0)}(r). \tag{3.6} $$

Chebyshev polynomials consist of two families:

First kind:

$$ \mathcal{T}_i(r) = \frac{i!}{\left(\frac{1}{2}\right)_i}\mathcal{P}_i^{(-\frac{1}{2},-\frac{1}{2})}(r), \tag{3.7} $$

where $(a)_i$ denotes the Pochhammer symbol [5].

Second kind:

$$ U_i(r) = \frac{(i+1)!}{\left(\frac{3}{2}\right)_i}\mathcal{P}_i^{(\frac{1}{2},\frac{1}{2})}(r). \tag{3.8} $$

Uncorrected Proof

Gegenbauer polynomials (ultra-spherical polynomials) are a symmetric subclass of Jacobi polynomials and are defined by [20, 43]:

$$\mathfrak{C}_i^{(\mu)}(r) = \frac{(2\mu)_i}{\left(\mu+\frac{1}{2}\right)_i}\,\mathcal{P}_i^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(r). \tag{3.9}$$

The Pochhammer symbol $(a)_i$, defined as:

$$(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)}.$$

These representations demonstrate how Jacobi polynomials encompass several fundamental families of orthogonal polynomials as special cases.

3.1. **Construction of Generalized Gegenbauer-Type Polynomials.** This subsection introduces the generalized orthogonal polynomials $\mathfrak{C}_i^{(\mu,\eta)}(r)$, which extend the classical Gegenbauer polynomials and include the Chebyshev polynomials of the seventh and eighth kinds as special cases. Constructed using symmetric Jacobi polynomials with an additional algebraic factor, this formulation provides enhanced control over symmetry and endpoint behavior.
As, $i \in \mathbb{N}_0$, $r \in [-1,1]$, and consider real parameters $\mu > 0$ and $\eta > -\frac{1}{2}$, the Gegenbauer-type polynomial of degree i define as [16]:

$$\mathfrak{C}_i^{(\mu)}(r) \;=\; \frac{(2\mu)_i}{\left(\mu+\frac{1}{2}\right)_i}\,\mathcal{P}_i^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(r). \tag{3.10}$$

where $\mathcal{P}_i^{(\mu,\eta)}(r)$ denotes the Jacobi polynomial of degree i.

Definition 3.1. The generalized Gegenbauer-type polynomial $\mathfrak{C}_i^{(\mu,\eta)}(r)$ can be expressed in a unified closed form as:

$$\mathfrak{C}_i^{(\mu,\eta)}(r) = c_i^{(\mu,\eta)} \cdot r^{i-2\lfloor\frac{n}{2}\rfloor} \cdot \mathcal{P}_{\lfloor\frac{i}{2}\rfloor}^{\left(\mu-\frac{1}{2},\,\eta-\frac{1}{2}+i-2\lfloor\frac{i}{2}\rfloor\right)}\left(2r^2-1\right), \tag{3.11}$$

and the normalization coefficient $c_i^{(\mu,\eta)}$ is given by:

$$c_i^{(\mu,\eta)} = \frac{(\mu+\eta)_{\lfloor\frac{i+1}{2}\rfloor}}{\left(\frac{5}{2}\right)_{\lfloor\frac{i+1}{2}\rfloor}}, \tag{3.12}$$

where, the floor function is denoted by $\lfloor\frac{i}{2}\rfloor$, this represents the greatest integer less than or equal to $\frac{i}{2}$.
The polynomials $\mathfrak{C}_i^{(\mu,\eta)}(r)$ satisfy the orthogonality relation:

$$\int_{-1}^{1} w(r)\,\mathfrak{C}_i^{(\mu,\eta)}(r)\,\mathfrak{C}_j^{(\mu,\eta)}(r)\,dr = \begin{cases} 0, & i \neq j, \\ \sigma_i^{(\mu,\eta)}, & i = j, \end{cases} \tag{3.13}$$

where, $w(r) = (1-r^2)^{\mu-1/2}(r)^{2\eta}$, and the normalization constant $\sigma_i^{(\lambda,\mu)}$ is given by:

$$\sigma_i^{(\mu,\eta)} = \frac{\left(\Gamma\left(\eta+\frac{1}{2}\right)\right)^2}{(i+\mu+\eta)\left(\Gamma(\mu+\eta)\right)^2} \times \begin{cases} \dfrac{\Gamma\left(\mu+\frac{i+1}{2}\right)\,\Gamma\left(\frac{i}{2}+\mu+\eta\right)}{\left(\frac{i}{2}\right)!\,\Gamma\left(\frac{i+1}{2}+\eta\right)}, & \text{if } i \text{ even}, \\ \dfrac{\Gamma\left(\frac{i}{2}+\mu\right)\,\Gamma\left(\frac{i}{2}+\mu+\eta+\frac{1}{2}\right)}{\left(\frac{i-1}{2}\right)!\,\Gamma(i)}, & \text{if } i \text{ odd}. \end{cases} \tag{3.14}$$

As Jacobi polynomials unify several well-known families of orthogonal polynomials the generalized Gegenbauer polynomials also do.
By choosing appropriate values of the parameters μ and η, the generalized Gegenbauer polynomials $\mathfrak{C}_i^{(\mu,\eta)}(r)$ reduce

Uncorrected Proof

to several known families. The fifth- $X_i(r)$ and sixth- $Y_i(r)$ kinds Chebyshev polynomials are featured special cases of the $\mathfrak{C}_i^{(\mu,\eta)}(r)$, such that:

$$X_i(r) = \mathfrak{C}_i^{(0,1)}(r), \qquad Y_i(r) = \mathfrak{C}_i^{(1,1)}(r)$$

Most recently, Abd elhameed and Yousri with research group [2] introduced the seventh-$\mathscr{Z}_i(r)$ and eighth- $E_i(r)$ kinds Chebyshev polynomials as follows. The seventh-kind Chebyshev polynomials correspond to the case $\mu = 0$, $\eta = 2$, leading to the form:

$$\mathscr{Z}_i(r) = \mathfrak{C}_i^{(0,2)}(r) = \frac{(2)_{\lfloor \frac{i+1}{2} \rfloor} \cdot r^{i-2\lfloor i/2 \rfloor} \cdot \mathcal{P}_{\lfloor i/2 \rfloor}^{(\frac{-1}{2}, \frac{3}{2}+i-2\lfloor i/2 \rfloor)}(2r^2-1)}{\left(\frac{5}{2}\right)_{\lfloor \frac{i+1}{2} \rfloor}}, \tag{3.15}$$

with the associated weight function $w_Z^*(r) = \frac{r^4 \cdot}{\sqrt{1-r^2}}$.

From the orthogonality condition of the generalized polynomials $\mathfrak{C}_i^{(\mu,\eta)}(r)$ in (3.13), obtain an orthogonal system $\mathscr{Z}_i(r)$ on the interval $[-1, 1]$ with respect to the weight function $w_Z^*(r) = \frac{r^4 \cdot}{\sqrt{1-r^2}}$. The polynomials $\mathscr{Z}_i(r)$ satisfy

$$\int_{-1}^{1} w_Z^*(r)\, \mathscr{Z}_i(r)\, \mathscr{Z}_j(r)\, dr = \begin{cases} \rho_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and the normalization constant ρ_i is explicitly given by:

$$\rho_i = \frac{9\pi}{8} \begin{cases} \dfrac{1}{(i+1)(i+3)}, & \text{if } i \text{ is even}, \\ \dfrac{(i+1)(i+3)}{i(i+2)^2(i+4)}, & \text{if } i \text{ is odd}. \end{cases} \tag{3.16}$$

The eighth-kind Chebyshev polynomials correspond to $\mu = 1$, $\eta = 2$, given:

$$E_i(r) = \mathfrak{C}_i^{(1,2)}(r) = \frac{(3)_{\lfloor \frac{i+1}{2} \rfloor} \cdot r^{i-2\lfloor i/2 \rfloor} \cdot \mathcal{P}_{\lfloor i/2 \rfloor}^{(\frac{1}{2}, \frac{3}{2}+i-2\lfloor i/2 \rfloor)}(2r^2-1)}{\left(\frac{5}{2}\right)_{\lfloor \frac{i+1}{2} \rfloor}}, \tag{3.17}$$

with weight function $w_E^*(r) = (r)^4 \cdot \sqrt{1-r^2}$. These constructions extend the classical Chebyshev framework and maintain orthogonality over the interval $[-1, 1]$.

In addition, the orthogonality relation for the polynomial sequence $E_i(r)$ is given by:

$$\int_{-1}^{1} r^4 \sqrt{1-r^2}\, E_i(r)\, E_j(r)\, dr = \begin{cases} h_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}, \tag{3.18}$$

where, the normalization constant h_i is defined by:

$$h_i = \frac{9\pi}{128} \begin{cases} \dfrac{(i+2)(i+4)}{(i+3)^2}, & \text{if } i \text{ is even}, \\ \dfrac{(i+1)(i+5)}{(i+2)(i+4)}, & \text{if } i \text{ is odd}. \end{cases} \tag{3.19}$$

Lemma 3.2. *Let $i \in \mathbb{N}_0$ be a non-negative integer. Then the polynomials $\mathscr{Z}_{2i}(r)$ and $\mathscr{Z}_{2i+1}(r)$, defined in terms of the generalized Gegenbauer polynomials, admit the following explicit representations:*

$$\mathscr{Z}_{2j}(r) = \sum_{r=0}^{j} (-1)^s \cdot \frac{(2j-s+1)!}{(j-s)!\, s! \cdot \left(\frac{5}{2}\right)_{j-s}} r^{2j-2s}, \tag{3.20}$$

Uncorrected Proof

$$\mathscr{Z}_{2j+1}(x) = \sum_{s=0}^{j} (-1)^s \cdot \frac{(2j-s+2)!}{(j-s)!\, s! \cdot \left(\frac{5}{2}\right)_{j-s+1}} r^{2j-2s+1}. \tag{3.21}$$

Proof. Begin with the generalized orthogonal polynomial defined in (3.15):

$$\mathfrak{C}_i^{(\mu,\eta)}(r) = c_i^{(\mu,\eta)} \cdot r^{i-2\left\lfloor \frac{i}{2} \right\rfloor} \cdot \mathcal{P}_{\left\lfloor \frac{i}{2} \right\rfloor}^{\left(\mu-\frac{1}{2}, \eta-\frac{1}{2}+i-2\left\lfloor \frac{i}{2} \right\rfloor\right)} \left(2r^2-1\right),$$

where, $\mathcal{P}_i^{(\mu,\eta)}(r)$ denotes the Jacobi polynomial, and $c_i^{(\mu,\eta)}$ defined in (3.12).
To derive the expressions for $\mathscr{Z}_{2i}(r)$ and $\mathscr{Z}_{2i+1}(r)$, and choose the parameters $\mu = 0$, $\eta = 2$, the polynomial $\mathscr{Z}_i(r)$ in terms of the generalized Gegenbauer form.
For even values, when $i = 2j$, the general expression simplifies to

$$\mathfrak{C}_i^{(0,2)}(r) = \mathscr{Z}_{2j}(r) = \frac{(2)_{\left\lfloor j+\frac{1}{2} \right\rfloor}}{\left(\frac{5}{2}\right)_{\left\lfloor j+\frac{1}{2} \right\rfloor}} \cdot \mathcal{P}_j^{\left(-\frac{1}{2}, \frac{3}{2}\right)} \left(2r^2-1\right), \tag{3.22}$$

where, $\left\lfloor j+\frac{1}{2} \right\rfloor = j, \quad$ for all integers $j \in \mathbb{N}_0$.

$$\mathfrak{C}_i^{(0,2)}(r) = \mathscr{Z}_{2j}(r) = \frac{(2)_j}{\left(\frac{5}{2}\right)_j} \cdot \mathcal{P}_j^{\left(-\frac{1}{2}, \frac{3}{2}\right)} \left(2r^2-1\right).$$

On the other hand, for odd indices with $i = 2j+1$, the corresponding expression becomes:

$$\mathscr{Z}_{2j+1}(r) = \frac{(j+2)!}{\left(\frac{5}{2}\right)_{j+1}} \cdot r \cdot \mathcal{P}_j^{\left(-\frac{1}{2}, \frac{5}{2}\right)} (2r^2-1).$$

Using Jacobi polynomials in terms of the hypergeometric function as [24]:

$$\mathcal{P}_i^{(\mu,\eta)}(r) = \frac{(-1)^i(\eta+1)_i}{i!} \cdot {}_2F_1\left(-i, i+\mu+\eta+1; \eta+1; \frac{1+r}{2}\right),$$

To overcome this, an alternative representation of the function $\mathscr{Z}_{2j}(r)$ is utilized, expressed in terms of the hypergeometric function as:

$$\mathscr{Z}_{2j}(r) = (-1)^j \cdot (j+1) \cdot {}_2F_1\left(-j, j+2; \tfrac{5}{2}; r^2\right).$$

This expression can be expanded using the series form of the hypergeometric function [15]:

$${}_2F_1\left(-j, j+2; \tfrac{5}{2}; r^2\right) = \sum_{k=0}^{j} \frac{(-j)_s (j+2)_s}{\left(\frac{5}{2}\right)_s\, s!} (r^2)^s.$$

Substituting these into the hypergeometric series, obtains:

$$\mathscr{Z}_{2j}(r) = (-1)^j \cdot (j+1) \sum_{s=0}^{j} (-1)^s \cdot \frac{j!}{(j-s)!\, s!} \cdot \frac{(j+s+1)!}{(j+1)!} \cdot \frac{1}{\left(\frac{5}{2}\right)_s} \cdot r^{2s}.$$

To simplify further, the change of index $j \mapsto j-s$ is applied, this yields a more symmetric form:

$$\mathscr{Z}_{2j}(r) = \sum_{s=0}^{j} (-1)^s \cdot \frac{(2j-s+1)!}{(j-s)!\, s! \cdot \left(\frac{2}{5}\right)_{j-s}} r^{2j-2s}. \tag{3.23}$$

Similarly, for odd case $i = 2j+1$,

$$\mathscr{Z}_{2j+1}(r) = \frac{(j+2)!}{\left(\frac{5}{2}\right)_{j+1}} \cdot r \cdot \mathcal{P}_j^{\left(-\frac{1}{2}, \frac{5}{2}\right)} (2r^2-1),$$

Uncorrected Proof

We obtain:

$$\mathscr{Z}_{2j+1}(r) = \sum_{s=0}^{j} (-1)^s \cdot \frac{(2j-s+2)!}{(j-s)!\, s! \cdot \left(\frac{5}{2}\right)_{j-s+1}} \cdot r^{2j-2s+1}. \tag{3.24}$$

This completes the proof.

□

Lemma 3.3. *The eighth-kind Chebyshev polynomials are given by:*

$$E_{2j}(r) = \sum_{s=0}^{j} (-1)^s \cdot \frac{(3)_{2j-s}}{(j-s)!\, \left(\frac{5}{2}\right)_{j-s}\, s!}\, r^{2j-2s}, \tag{3.25}$$

$$E_{2j+1}(r) = \sum_{s=0}^{j} (-1)^s \cdot \frac{(3)_{1+2j-s}}{(j-s)!\, \left(\frac{5}{2}\right)_{1+j-s}\, s!}\, r^{2j-2s+1}. \tag{3.26}$$

Proof. From the general formula (3.17):

$$E_i(r) = \mathfrak{C}_i^{(2,1)}(r) = \frac{(3)_{\lfloor (i+1)/2 \rfloor} \cdot r^{i-2\lfloor i/2 \rfloor} \cdot \mathcal{P}_{\lfloor i/2 \rfloor}^{(1/2,\, 3/2+i-2\lfloor i/2 \rfloor)}(2r^2-1)}{\left(\frac{5}{2}\right)_{\lfloor (i+1)/2 \rfloor}}.$$

Using the hypergeometric form gives the desired expressions.

□

Corollary 3.4. *By combining the even and odd representations into a unified framework, one obtains the closed-form expression:*

$$\mathscr{Z}_j(r) = \sum_{s=0}^{\lfloor j/2 \rfloor} (-1)^s \frac{(j-s+1)!}{(\lfloor j/2 \rfloor - s)!\, s!\, \left(\frac{5}{2}\right)_{\lfloor \frac{j+1}{2} \rfloor - s}} r^{\,j-2s}, \qquad j \in \mathbb{N}_0. \tag{3.27}$$

Also, the coefficients of the eighth–kind Chebyshev polynomial can be expressed in:

$$E_j(r) = \sum_{s=0}^{\lfloor j/2 \rfloor} (-1)^s \frac{(3)_{j-s}}{(\lfloor j/2 \rfloor - r)!\, r!\, \left(\frac{5}{2}\right)_{\lfloor \frac{j+1}{2} \rfloor - s}} r^{\,j-2s}. \tag{3.28}$$

Proof. The proof directly given by the mathematical induction, by the help of relations (3.23)–(3.26).

□

3.2. **Shifted Expansions and Orthogonality of Seventh and Eighth-Kind Chebyshev Polynomials.** This subsection presents explicit formulas for the shifted seventh-and eighth-kind Chebyshev polynomials and establishes their orthogonality with respect to transformed weight functions. The results provide the basis for later developments in operational matrices and approximation methods.

Theorem 3.5. *Let* $\{\mathscr{Z}_j(r)\}_{j\geq 0}$ *and* $\{E_j(r)\}_{j\geq 0}$ *be the seventh and eighth-kind Chebyshev polynomials, respectively, given by*

$$\mathscr{Z}_j(r) = \sum_{s=0}^{\lfloor j/2 \rfloor} a_{j,s}^{(\mathscr{Z})}\, r^{\,j-2s}, \qquad E_j(r) = \sum_{s=0}^{\lfloor j/2 \rfloor} a_{j,s}^{(E)}\, r^{\,j-2s}, \tag{3.29}$$

where,

$$a_{j,s}^{(\mathscr{Z})} = (-1)^s \frac{(j-s+1)!}{(\lfloor j/2 \rfloor - s)!\, s!\, \left(\frac{5}{2}\right)_{\lfloor (j+1)/2 \rfloor - s}}, \qquad a_{j,s}^{(E)} = (-1)^s \frac{(3)_{j-s}}{(\lfloor j/2 \rfloor - s)!\, s!\, \left(\frac{5}{2}\right)_{\lfloor (j+1)/2 \rfloor - s}}. \tag{3.30}$$

Uncorrected Proof

Then, the shifted polynomials define on the interval $[0,\beta]$, $\beta>0$, *admit the explicit representations as:*

$$\mathscr{Z}_j^*(r)=\sum_{s=0}^{\lfloor j/2\rfloor}\sum_{t=0}^{j-2s}B_{j;s,t}^{(Z)}\,r^t,\qquad E_j^*(r)=\sum_{s=0}^{\lfloor j/2\rfloor}\sum_{t=0}^{j-2s}B_{j;s,t}^{(E)}\,r^t, \tag{3.31}$$

where,

$$B_{j;s,t}^{(\mathscr{Z})}:=a_{j,s}^{(\mathscr{Z})}\binom{j-2s}{t}2^t(-\beta)^{j-2s-t},\qquad B_{j;s,t}^{(E)}:=a_{j,s}^{(E)}\binom{j-2s}{t}2^t(-\beta)^{j-2s-t}. \tag{3.32}$$

Proof. Starting from the expansions of $\mathscr{Z}_j(r)$ and $E_j(r)$, For a fixed parameter $\beta>0$, define the shifted families by:

$$\mathscr{Z}_j^*(r):=\mathscr{Z}_j(2r-\beta),\qquad E_j^*(r)=E_j(2r-\beta).$$

For each monomial $(2r-\beta)^{j-2s}$, the binomial theorem gives

$$(2r-\beta)^{j-2s}=\sum_{t=0}^{j-2s}\binom{j-2s}{t}2^t(-\beta)^{j-2s-t}r^t.$$

Substituting this into the original expansions produces:

$$\mathscr{Z}_j^*(r)=\sum_{s=0}^{\lfloor j/2\rfloor}a_{j,s}^{(\mathscr{Z})}\sum_{t=0}^{j-2s}\binom{j-2s}{t}2^t(-\beta)^{j-2s-t}r^t, \tag{3.33}$$

$$E_j^*(r)=\sum_{s=0}^{\lfloor j/2\rfloor}a_{j,s}^{(E)}\sum_{t=0}^{j-2s}\binom{j-2s}{t}2^t(-\beta)^{j-2s-t}r^t. \tag{3.34}$$

Recognizing $B_{j;s,t}^{(\mathscr{Z})}$ and $B_{j;s,t}^{(E)}$ as stated completes the proof.

□

Remark 3.6. Let $\mathscr{Z}_j(r)$ and $E_j(r)$ denote the seventh- and eighth-kind Chebyshev polynomials in the variable r, orthogonal on an interval $\mathcal{I}\subset\mathbb{R}$ with respect to weight functions $w_Z(r)$ and $w_E(r)$, respectively:

$$\int_{\mathcal{I}}\mathscr{Z}_i(r)\,\mathscr{Z}_j(r)\,w_{\mathscr{Z}}(r)\,dr=\rho_i\,\delta_{i,j},\qquad\int_{\mathcal{I}}E_i(r)\,E_j(r)\,w_E(r)\,dr=h_i\,\delta_{i,j}.$$

For the shifted families $\mathscr{Z}_j^*(r)=\mathscr{Z}_j(2r-\beta)$ and $E_j^*(r)=E_j(2r-\beta)$ defined on $r\in[0,\beta]$ the corresponding shifted weights are

$$w_{\mathscr{Z}}^*(r):=w_{\mathscr{Z}}(2r-\beta),\qquad w_E^*(r):=w_E(2r-\beta),$$

The orthogonality relations for the modified basis functions are given by:

$$\int_0^\beta\mathscr{Z}_i^*(r)\,\mathscr{Z}_j^*(r)\,w_{\mathscr{Z}}^*(r)\,dr=\begin{cases}\widehat{\rho}_i, & i=j,\\ 0, & i\neq j,\end{cases}\quad\text{with}\quad\widehat{\rho}_i=2\rho_i,$$

and

$$\int_0^\beta E_i^*(r)\,E_j^*(r)\,w_E^*(r)\,dr=\begin{cases}\widehat{h}_i, & i=j,\\ 0, & i\neq j,\end{cases}\quad\text{with}\quad\widehat{h}_i=\tfrac{1}{4}h_i.$$

These orthogonality relations ensure that the basis functions form a complete orthogonal system, which is fundamental for the spectral expansion method and the uniqueness of the series representation for any function in the appropriate Hilbert space.

Uncorrected Proof

4. Operational matrices of differentiation for the generalized polynomial basis

In this section, we develop the operational matrices of differentiation corresponding to a shifted polynomial basis $\{\mathscr{P}_j^*(r)\}_{j=0}^m$ defined on the interval $[0,\beta]$ such that: $\mathscr{P}_j^*(r) = \mathscr{Z}_j^*(r)$ or $\mathscr{P}_j^*(r) = E_j^*(r)$ according to (3.33) and (3.34). By exploiting the established relationship between this polynomial basis and the standard monomial basis, the differentiation operator is expressed as a similarity transformation of the differentiation matrix. This formulation facilitates the efficient and systematic computation of derivatives for various polynomial families, including the seventh and eighth kinds.

Proposition 4.1. *Let $\{\mathscr{P}_j^*(r)\}_{j=0}^m$ be a generalized shifted polynomial basis defined on the interval $\mathcal{I}^* = [0,1]$, where each polynomial admits an expansion in the standard monomial basis $\{r^t\}_{t=0}^m$ as:*

$$\mathscr{P}_j^*(r) = \sum_{s=0}^{\lfloor j/2 \rfloor} a_{j,s} \sum_{t=0}^{j-2s} \binom{j-2s}{t} 2^t \,(-1)^{\,j-2s-t}\, r^t, \tag{4.1}$$

with coefficients $a_{j,s} = a_{j,s}^{(Z)}$ in the seventh polynomial case, also, $a_{j,s} = a_{j,s}^{(E)}$, in the eighth polynomial case of the relations (3.30). and $B_{j,t,s} = B_{j,t,s}^{(Z)}$, $B_{j,t,s}^{(E)}$, then, they form a lower triangular matrix.

$$B_m = [B_{j,s,t}]_{j=0}^m = a_{j,s} \sum_{t=0}^{j-2s} \binom{j-2s}{t} 2^t \,(-1)^{\,j-2s-t} \in \mathbb{R}^{(m+1)\times(m+1)}. \tag{4.2}$$

In addition, the differentiation operator admits the matrix representation:

$$\frac{d}{dr}\mathscr{P}_j^*(r) = B_m \cdot M \cdot B_m^{-1} \cdot \mathscr{P}^*(r),$$

$$D^{(1)} = B_m \cdot M \cdot B_m^{-1}.$$

where,

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m & 0 \end{pmatrix}, \qquad B_m = \begin{pmatrix} b_{0,0} & 0 & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & 0 & \cdots & 0 \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,0} & b_{m,1} & b_{m,2} & b_{m,3} & \cdots & b_{m,m} \end{pmatrix}.$$

In addition, $D^{(1)}$ of size $(m+1)\times(m+1)$ is the differentiation matrix corresponding to the monomial basis.

Proof. Define the vector of generalized basis polynomials as:

$$\mathscr{P}_j^*(r) = [\mathscr{P}_0^*(r), \mathscr{P}_1^*(r), \ldots, \mathscr{P}_m^*(r)]^T,$$

and the standard monomial basis vector

$$\mathscr{R}(r) = \left[1, r, r^2, \ldots, r^m\right]^T.$$

The differentiation operator acts on the monomial basis as follows:

$$\frac{d}{dr}\mathscr{R}(r) = M \cdot \mathscr{R}(r),$$

Uncorrected Proof

where, the classical differentiation matrix M in $\mathbb{R}^{(m+1)\times(m+1)}$ is given by:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m & 0 \end{pmatrix}.$$

Differentiating $\mathscr{P}_j^*(r)$ yields:

$$\frac{d}{dr}\mathscr{P}_j^*(r) = \frac{d}{dr}\cdot\big(B_m\cdot\mathscr{R}(r)\big) = B_m\cdot\frac{d}{dr}\cdot\mathscr{R}(r) = B_m\cdot M\cdot\mathscr{R}(r),$$

where,

$$B_m = \begin{pmatrix} b_{0,0} & 0 & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & 0 & \cdots & 0 \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,0} & b_{m,1} & b_{m,2} & b_{m,3} & \cdots & b_{m,m} \end{pmatrix}.$$

Since, B_m is invertible (such that $b_{ii}\neq 0$), we have

$$\mathscr{R}(r) = B_m^{-1}\cdot\mathscr{P}_j^*(r).$$

By construction, these satisfy the linear relation:

$$\frac{d}{dr}\mathscr{P}_j^*(r) = B_m\cdot M\cdot B_m^{-1}\mathscr{P}^*(r).$$

Hence, the differentiation operator on the generalized basis is represented by the matrix,

$$D^{(1)} = B_m\cdot M\cdot B_m^{-1}.$$

□

In particular, for the seventh kind $\mathscr{Z}_j^*(r)$ and eighth kind $E_j^*(r)$ polynomial families, the corresponding coefficient matrices $B_n^{(Z)}$ and $B_n^{(E)}$ yield operational differentiation matrices:

$$D^{(\mathscr{Z})} = B_m^{(\mathscr{Z})}\cdot M\cdot\left(B_m^{(\mathscr{Z})}\right)^{-1},\quad D^{(E)} = B_m^{(E)}\cdot M\cdot\left(B_m^{(E)}\right)^{-1},$$

for example, if one takes $m=5$: The coefficient matrices $B_m^{(\mathscr{Z})}$ and $B_m^{(E)}$ for the seventh and eighth kind polynomials with $\beta=1$ are:

$$B_5^{(\mathscr{Z})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{5} & \frac{8}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & -\frac{48}{5} & \frac{48}{5} & 0 & 0 & 0 \\ -\frac{12}{35} & \frac{408}{35} & -\frac{1152}{35} & \frac{768}{35} & 0 & 0 \\ \frac{9}{35} & -\frac{576}{35} & \frac{4416}{35} & -\frac{1536}{7} & \frac{768}{7} & 0 \\ -\frac{8}{35} & \frac{656}{35} & -\frac{1408}{7} & \frac{4352}{7} & -\frac{5120}{7} & \frac{2048}{7} \end{pmatrix},$$

Uncorrected Proof

$$B_5^{(E)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6}{5} & \frac{12}{5} & 0 & 0 & 0 & 0 \\ \frac{9}{5} & -\frac{96}{5} & \frac{96}{5} & 0 & 0 & 0 \\ -\frac{72}{35} & \frac{1104}{35} & -\frac{576}{7} & \frac{384}{7} & 0 & 0 \\ \frac{18}{7} & -\frac{480}{7} & \frac{2784}{7} & -\frac{4608}{7} & \frac{2304}{7} & 0 \\ -\frac{20}{7} & \frac{680}{7} & -\frac{5504}{7} & \frac{15616}{7} & -2560 & 1024 \end{pmatrix}.$$

Remark 4.2. Assume that:

$$\mathscr{P}_j^*(r) = [\mathscr{P}_0^*(r), \mathscr{P}_1^*(r), \ldots, \mathscr{P}_m^*(r)]^T,$$

denotes the vector of shifted orthogonal polynomials defined over the interval $\mathcal{I} = [0, 1]$.
The m-th derivative of $\mathscr{P}_j^*(r)$ can be expressed in the operational matrix form as:

$$\frac{d^m \mathscr{P}_j^*(r)}{dx^m} = D^{(m)}\, \mathscr{P}_j^*(r), \quad m = 1, 2, 3, \ldots,$$

where,

$$D^{(m)} = \left(D^{(1)}\right)^m.$$

The first-order derivative matrix $D^{(1)}$ is obtained via the similarity transformation

$$D^{(1)} = B_m \cdot M \cdot B_m^{-1}.$$

This leads to the general expression:

$$D^{(m)} = B_m \cdot (M)^m \cdot B_m^{-1}, \tag{4.3}$$

where, M and B_m are as defined previously.

Remark 4.3. Assume that: ${}^{C}D_{r_0+}^{\lambda}$ denotes the Caputo's fractional derivative operator of order $\lceil \lambda \rceil > 0$. Consider the standard monomial basis vector of degree m:

$$\mathscr{R}_m(r) = \left[1, r, r^2, \ldots, r^m\right]^T,$$

and an arbitrary polynomial basis

$$\mathscr{P}^*(r) = \left[\mathscr{P}_0^*(r), \mathscr{P}_1^*(r), \ldots, \mathscr{P}_m^*(r)\right]^T,$$

related by a matrix $B_m \in \mathbb{R}^{(m+1)\times(m+1)}$ as

$$\mathscr{P}^*(r) = B_m \cdot \mathscr{R}_m(r).$$

Then, the fractional Caputo derivative applied to the basis $\mathscr{P}_j^*(r)$ can be represented as

$${}^{C}D_{r_0+}^{\lambda} \mathscr{P}^*(r) = \mathcal{D}^{(\lambda)} \mathscr{P}^*(r),$$

where, the fractional operational matrix $\mathcal{D}^{(\lambda)} \in \mathbb{R}^{(m+1)\times(m+1)}$ is given by the similarity transformation:

$$\mathcal{D}^{(\lambda)} = B_m \cdot A^{(\lambda)}(r) \cdot B_m^{-1}. \tag{4.4}$$

So,

$$\mathcal{D}^{(\lambda)}_{(m+1)\times(m+1)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\lceil\lambda\rceil,0} & d_{\lceil\lambda\rceil,1} & d_{\lceil\lambda\rceil,\lceil\lambda\rceil} & \cdots & d_{\lceil\lambda\rceil,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{m,0} & d_{m,1} & d_{m,2} & \cdots & d_{m,m} \end{pmatrix}, \tag{4.5}$$

Uncorrected Proof

where, the matrix is lower–triangular in the sense that $d_{i,j} = 0$ for all $j > i$, and the first $\lceil\lambda\rceil - 1$ rows are identically zero.

Proof. By the definition of the Caputo fractional derivative, Definition 2.1, the monomial basis satisfies

$$ {}^{C}D^{\lambda}_{r_0+}\mathscr{R}_m(r) = A^{(\lambda)}(r)\cdot\mathscr{R}_m(r). $$

Also, the diagonal matrix $A^{(\lambda)}(r) \in \mathbb{R}^{(m+1)\times(m+1)}$ is:

$$ A^{(\lambda)}(r) = \begin{bmatrix} a_0^{(\lambda)}(r) & 0 & 0 & \cdots & 0 \\ 0 & a_1^{(\lambda)}(r) & 0 & \cdots & 0 \\ 0 & 0 & a_2^{(\lambda)}(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_m^{(\lambda)}(r) \end{bmatrix}, $$

where,

$$ a_k^{(\lambda)}(r) = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}\, r^{k-\lambda}, & k \geq \lceil\lambda\rceil, \\ 0, & k < \lceil\lambda\rceil. \end{cases} $$

Using the linear transformation between bases,

$$ \mathscr{P}^*_j(r) = B_m\cdot\mathscr{R}(r), $$

and the invertibility of B_m, it follows that:

$$ {}^{C}D^{\lambda}_{r_0+}\mathscr{P}^*_j(r) = {}^{C}D^{\lambda}_{r_0+}\cdot(B_m\cdot\mathscr{R}(r)) = B_m\cdot{}^{C}D^{\lambda}_{r_0+}\cdot\mathscr{R}(r) = B_m\cdot A^{(\lambda)}(r)\cdot\mathscr{R}(r). $$

Substituting $\mathscr{R}(r) = B_m^{-1}\mathscr{P}^*_m(r)$ yields

$$ {}^{C}D^{\lambda}_{r_0+}\mathscr{P}^*_j(r) = B_m\cdot A^{(\lambda)}(r)\cdot B_m^{-1}\cdot\mathscr{P}^*_j(r), \tag{4.6} $$

which completes the proof. □

5. Approximation and convergence analysis

This section is devoted to the study of the approximation properties of the proposed scheme, together with its convergence behavior and associated error bounds. The objective is to demonstrate that the numerical solution converges to the exact one as the number of basis functions increases, and that the approximation error can be rigorously estimated.

In practical computations, the function $\mathcal{U}(r)$ is approximated by considering only the first $(m+1)$ terms of the shifted Gegenbauer polynomial basis. Thus, one may express:

$$ \mathcal{U}(r) \approx \mathcal{U}_m(r) = \sum_{j=0}^{m} C_j\,\mathscr{P}^*_j(r) = C^T\,\mathscr{P}^*(r), \tag{5.1} $$

$$ C^T = [C_0, C_1, \ldots, C_m], \qquad \mathscr{P}^*_j(r) = [\mathscr{P}^*_0(r), \mathscr{P}^*_1(r), \ldots, \mathscr{P}^*_m(r)]^T. \tag{5.2} $$

Here, $\{\mathscr{P}^*_j(r)\}_{j=0}^{m}$ denotes the shifted Gegenbauer polynomial basis (seventh- or eighth-kinds Chebyshev bases, as listed before) corresponding to the parameter r, while the coefficients C_j are the expansion coefficients of $\mathcal{U}(r)$ in terms of this basis.

Uncorrected Proof

Lemma 5.1. *Let $\{\mathscr{P}_j^*(r)\}_{j=0}^m$ be a set of shifted polynomials defined on $[0,1]$, associated with a weight function $w^*(r)$. Then, the left-sided Caputo fractional derivative of order $0<\lambda<1$ of $\mathscr{P}_i^*(r)$ can be represented approximately as:*

$$ {}^{\mathrm{C}}D_{0+}^{\lambda}\mathscr{P}_i^*(r) \;\approx\; \sum_{j=0}^{i}\psi_\lambda(i,j)\,\mathscr{P}_j^*(r), \tag{5.3}$$

where, the expansion coefficients are defined by

$$\psi_\lambda(i,j)=\sum_{s=0}^{i}B_{i,s,t}\,\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\,\eta_{s,\lambda}^{(j)}, \tag{5.4}$$

and $\eta_{s,\lambda}^{(j)}$ are given by

$$\eta_{s,\lambda}^{(j)}=\frac{1}{N_j}\int_0^1 r^{s-\lambda}\,w^*(r)\,\mathscr{P}_j^*(r)\,dr \quad ,\text{where} \quad N_j=\int_0^1\left(\mathscr{P}_j^*(r)\right)^2 w^*(r)\,dr, \tag{5.5}$$

where,

$$N_j=\hat{\rho}_j \text{ or } \hat{h}_j,$$

Proof. Each polynomial can be expanded in monomial form as

$$\mathscr{P}_i^*(r)=\sum_{s=0}^{i}B_{i,s}\,r^s.$$

Applying the Caputo derivative of order λ to each monomial gives

$${}^{\mathrm{C}}D_{0+}^{\lambda}\mathscr{P}_i^*(r)=\sum_{s=0}^{i}B_{i,s}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\,r^s=\sum_{s=0}^{i}B_{i,s,t}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\,r^{s-\lambda}.$$

Since, $r^{s-\lambda}$ generally cannot be exactly represented by the finite set of polynomials $\{\mathscr{P}_j^*(r)\}_{j=0}^m$, we represent it approximately as a linear combination of these basis polynomials as follows:

$$r^{s-\lambda}\approx\sum_{j=0}^{m}\eta_{s,\lambda}^{(j)}\,\mathscr{P}_j^*(r),$$

where, the coefficients $\eta_{s,\lambda}^{(j)}$ are defined in (5.5), $0\le s\le\lfloor j/2\rfloor$. Substituting this into the derivative and collecting terms yields

$${}^{\mathrm{C}}D_{0+}^{\lambda}\mathscr{P}_i^*(r)\approx\sum_{j=0}^{m}\left(\sum_{s=0}^{\lfloor j/2\rfloor}B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\,\eta_{s,\lambda}^{(j)}\right)\mathscr{P}_j^*(r),$$

which proves the lemma.

□

Corollary 5.2. *Let $\mathscr{P}_{\boldsymbol{j}}^*(r)=[\mathscr{P}_0^*(r),\mathscr{P}_1^*(r),\ldots,\mathscr{P}_m^*(r)]^T$ be the vector of shifted Gegenbaur polynomials. The left-sided Caputo fractional derivative of order $0<\lambda<1$ can be expressed in matrix form as*

$${}^{\mathrm{C}}D_{0+}^{\lambda}\mathscr{P}_j^*(r)\approx\hat{\mathcal{D}}^{(\lambda)}\,\mathscr{P}_j^*(r), \tag{5.6}$$

Uncorrected Proof

where $\hat{\mathcal{D}}^{(\lambda)} \in \mathbb{R}^{(m+1)\times(m+1)}$ *is the operational matrix of the Caputo derivative, with entries defined by*

$$\hat{\mathcal{D}}^{(\lambda)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{s=\lceil\lambda\rceil}^{\lceil\lambda\rceil} \psi_\lambda(s,0) & \sum_{s=\lceil\lambda\rceil}^{\lceil\lambda\rceil} \psi_\lambda(s,1) & \cdots & \sum_{s=\lceil\lambda\rceil}^{\lceil\lambda\rceil} \psi_\lambda(s,m) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=\lceil\lambda\rceil}^{i} \psi_\lambda(s,0) & \sum_{s=\lceil\lambda\rceil}^{i} \psi_\lambda(s,1) & \cdots & \sum_{s=\lceil\lambda\rceil}^{i} \psi_\lambda(s,m) \end{pmatrix}, \tag{5.7}$$

for $i = \lceil\lambda\rceil, \ldots, m$, *so that:* $\psi_\lambda(s,j)$ *is defined in Eq. (5.4). For* $i = 0, 1, \ldots, \lceil\lambda\rceil - 1$, *all rows are zero:*

$${}^{\mathrm{C}}D^{\lambda}_{0+}\mathscr{P}^*_i(r) = 0.$$

Proof. From Lemma (5.1), the Caputo derivative of each basis polynomial can be approximated as

$${}^{\mathrm{C}}D^{\lambda}_{0+}\mathscr{P}^*_i(r) \approx \sum_{s=0}^{\lfloor i/2\rfloor} \sum_{j=0}^{m} B_{i,s,t} \frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)} \eta^{(j)}_{s,\lambda} \mathscr{P}^*_j(r).$$

By the definition of $\psi_\lambda(s,j)$ in Eq.(5.4), this can be rewritten as

$${}^{\mathrm{C}}D^{\lambda}_{0+}\mathscr{P}^*_i(r) \approx \sum_{j=0}^{m} \left(\sum_{s=0}^{\lfloor j/2\rfloor} \psi_\lambda(s,j) \right) \mathscr{P}^*_j(r).$$

Stacking these relations for all $i = 0, \ldots, m$ forms the nonzero portion of the matrix $\hat{\mathcal{D}}^{(\lambda)}$. For $i < \lfloor j/2\rfloor$, the Caputo derivative vanishes, yielding zero rows. Combining these observations establishes Eq. (5.6).

□

Remark 5.3. The two differential matrices, $\mathcal{D}^{(\lambda)}$ and $\hat{\mathcal{D}}^{(\lambda)}$ for the same order λ are equivalent.

Theorem 5.4. *(see[33]): Let* V *be a Hilbert space, and let* $\bar{V} \subset V$ *be a subspace with finite dimensional, such that:* $\dim(\bar{V}) < \infty$. *Suppose that*

$$\bar{V} = \operatorname{span}\{v_1, v_2, \ldots, v_m\}.$$

For any $v \in V$, *let* $\bar{v} \in \bar{V}$ *denote its unique best approximation in* $\bar{V}$. *Then the approximation error satisfies*

$$\|v - \bar{v}\|^2_{L^2} = \frac{\operatorname{Gram}(v, v_1, \ldots, v_m)}{\operatorname{Gram}(v_1, \ldots, v_m)},$$

where, the Gram determinant is defined as:

$$\operatorname{Gram}(v_1, v_2, \ldots, v_m) = \begin{vmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_m\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \cdots & \langle v_2, v_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_m, v_1\rangle & \langle v_m, v_2\rangle & \cdots & \langle v_m, v_m\rangle \end{vmatrix}.$$

Lemma 5.5. *Let* $\mathcal{U}(r) \in L^2[0,1]$. *Its best approximation in the subspace*

$$\mathcal{V}_n = \operatorname{span}\{\mathscr{P}^*_0(r), \mathscr{P}^*_1(r), \ldots, \mathscr{P}^*_m(r)\}$$

is expressed as

$$\mathcal{U}(r) \approx \mathcal{U}_m(r) = \sum_{i=0}^{m} C_i \mathscr{P}^*_i(r) = C^T \mathscr{P}^*(r),$$

where

$$\mathscr{P}^*(r) = [\mathscr{P}^*_0(r), \mathscr{P}^*_1(r), \ldots, \mathscr{P}^*_m(r)]^T.$$

Uncorrected Proof

Furthermore,

$$\lim_{n\to\infty} \|\mathcal{U}(r) - \mathcal{U}_m(r)\|_{L^2} = 0.$$

Define the error vector associated with the Caputo operational matrix as

$$\mathscr{E}^{(\lambda)} = {}^{\mathrm{C}}D^{\lambda}_{0+}\mathscr{P}^*_j(r) - D^{(\lambda)}\mathscr{P}^*_j(r),$$

with

$$\mathscr{E}^{(\lambda)} = \left[e^{(\lambda)}_0, e^{(\lambda)}_1, \ldots, e^{(\lambda)}_n\right].$$

Then, each error component satisfies

$$\|e^{(\lambda)}_i\|_{L^2} \le \sum_{s=0}^{i} B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)} \cdot \frac{\mathrm{Gram}(r^{\,s-\lambda}, \mathscr{P}^*_0(r), \ldots, \mathscr{P}^*_m(r))}{\mathrm{Gram}(\mathscr{P}^*_0(r), \ldots, \mathscr{P}^*_m(r))}. \tag{5.8}$$

Proof. Consider the subspace

$$\mathcal{V}_m = \mathrm{span}\{\mathscr{P}^*_0(r), \mathscr{P}^*_1(r), \ldots, \mathscr{P}^*_m(r)\} \subset L^2[0,1].$$

The function $r^{s-\lambda}$ admits an optimal approximation in $\mathcal{V}_n$ of the form

$$r^{s-\lambda} \approx \sum_{j=0}^{m} C^{(\lambda)}_j\, \mathscr{P}^*_j(r).$$

By Theorem 5.4, the corresponding error is

$$\left\| r^{s-\lambda} - \sum_{j=0}^{m} C^{(\lambda)}_j\, \mathscr{P}^*_j(r)\right\|^2_{L^2} = \frac{\mathrm{Gram}(r^{s-\lambda}, \mathscr{P}^*_0(r), \ldots, \mathscr{P}^*_m(r))}{\mathrm{Gram}(\mathscr{P}^*_0(r), \ldots, \mathscr{P}^*_m(r))}. \tag{5.9}$$

For the Caputo derivative of $\mathscr{P}^*_i(\tau)$, we have

$$\|e^{(\lambda)}_i\|_{L^2} = \left\| {}^{\mathrm{C}}D^{\lambda}_{0+}\mathscr{P}^*_i(r) - \sum_{j=0}^{m} \psi_{\lambda}(i,j)\mathscr{P}^*_j(r)\right\|_{L^2}.$$

Using the expansion of the fractional derivative yields

$$\|e^{(\lambda)}_i\|_{L^2} = \left\| \sum_{s=0}^{\lfloor i/2\rfloor} B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\, r^{\,s-\lambda} - \sum_{s=0}^{\lfloor i/2\rfloor}\sum_{j=0}^{m} B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\, \eta^{(j)}_{s,\lambda}\mathscr{P}^*_j(r)\right\|_{L^2}.$$

Finally, applying the triangle inequality and substituting (5.9) yields the bound

$$\|e^{(\lambda)}_i\|_{L^2} \le \sum_{s=0}^{\lfloor i/2\rfloor} B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)}\left\| r^{\,s-\lambda} - \sum_{j=0}^{m} \eta^{(j)}_{s,\lambda}\mathscr{P}^*_j(r)\right\|_{L^2}.$$

Substituting from Eq.(5.8), obtain

$$\|e^{(\lambda)}_i\|_{L^2} \le \sum_{s=0}^{\lfloor i/2\rfloor} B_{i,s}\frac{\Gamma(s+1)}{\Gamma(s-\lambda+1)} \cdot \frac{\mathrm{Gram}\left(r^{\,s-\varphi}, \mathscr{P}^*_0(\tau), \ldots, \mathscr{P}^*_m(r)\right)}{\mathrm{Gram}\left(\mathscr{P}^*_0(r), \ldots, \mathscr{P}^*_m(\tau)\right)}.$$

which completes the proof. □

Uncorrected Proof

Theorem 5.6. *Assume that* ${}^{\mathrm{C}}D_{0+}^{i\lambda}\mathcal{U}(r)$ *is continuous on the interval* $[0,1]$*, for* $i=0,1,\ldots,m$*, and that:*

$$ {}^{\mathrm{C}}D_{0+}^{i\lambda}\mathcal{U}(\xi) \le \tilde{M}, \quad \text{for some real constant } \tilde{M},\ 0<\xi\le r.$$

Let $\mathcal{V}_m = \mathrm{Span}\{\mathscr{P}_0^*(r),\mathscr{P}_1^*(r),\ldots,\mathscr{P}_m^*(r)\}$ *be an* m*-dimensional subspace of* $L^2[0,1]$*, and let* $\mathcal{U}_m(r)$ *denote the best approximation to* $\mathcal{U}(r)$ *from* $\mathcal{V}_m$*. Then the error bound is given by:*

$$\|\mathcal{U}(r)-\mathcal{U}_m(r)\|^2 \le \frac{\tilde{M}\,\zeta^*_{m,\lambda}}{\Gamma(m\lambda+1)},$$

where,

$$\zeta^*_{m,\lambda}=\begin{cases}\sqrt{\dfrac{\pi\left(9+4m\lambda(1+2m\lambda)\big(6+m\lambda(5+2m\lambda)\big)\right)\Gamma\left(\frac{1}{2}+2m\lambda\right)}{2\,\Gamma(5+2m\lambda)}}, & \text{for the seventh kind case,}\\[2ex] \sqrt{\dfrac{\pi\left(45+4m\lambda(1+2m\lambda)\big(12+m\lambda(5+2m\lambda)\big)\right)\Gamma\left(\frac{3}{2}+2m\lambda\right)}{\Gamma(7+2m\lambda)}}, & \text{for the eighth kind case.}\end{cases}$$

which holds for all $r\in[0,1]$*.*

Proof. By the generalized Taylor expansion of $\mathcal{U}(r)$ on $[0,1]$ (see [11, 33]), we have

$$\mathcal{U}(r) = \sum_{i=0}^{m-1}\frac{r^{i\lambda}}{\Gamma(i\lambda+1)}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(0) + \frac{r^{m\lambda}}{\Gamma(m\varphi+1)}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(\xi),\quad \xi\in[0,r].$$

Thus,

$$\left|\mathcal{U}(r)-\sum_{i=0}^{m-1}\frac{r^{i\lambda}}{\Gamma(i\lambda+1)}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(0)\right| \le \frac{r^{m\lambda}}{\Gamma(m\lambda+1)}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(\xi) \le \frac{\tilde{M}\,r^{m\lambda}}{\Gamma(m\lambda+1)}.$$

Since $\mathcal{U}_m(r)$ is the best approximation of $\mathcal{U}(r)$ in $\mathcal{V}_m$, we obtain

$$\|\mathcal{U}(r)-\mathcal{U}_m(r)\|^2 \le \left\|\mathcal{U}(r)-\sum_{i=0}^{m-1}\frac{r^{i\lambda}}{\Gamma(i\lambda+1)}\,{}^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(0)\right\|^2.$$

Consequently,

$$\int_0^1 |\mathcal{U}(r)-\mathcal{U}_m(r)|^2\,w^*(r)\,dr \le \frac{\tilde{M}^2}{\Gamma(m\lambda+1)^2}\int_0^1 r^{2m\lambda}\,w^*(r)\,dr,$$

Consequently, the error bound can be expressed as follows, where $w^*(r)$ denotes the weight function corresponding to the shifted Gegenbauer polynomials.By evaluating the integral, get:

$$\|\mathcal{U}(r)-\mathcal{U}_m(r)\|^2 \le \frac{\tilde{M}^2\zeta^*_{m,\lambda}}{\Gamma(m\lambda+1)^2}.$$

This concludes the results. □

6. Numerical scheme based on the collocation method

In this section, we develop a robust numerical scheme utilizing the collocation method coupled with shifted Gegenbauer polynomial expansions to solve both linear and nonlinear fractional differential equations (FDEs) [25, 28, 31].

Uncorrected Proof

6.1. **Preliminaries.** Consider a general fractional differential equation defined on the interval $[0,1]$. The solution $\mathcal{U}(r)$ is approximated using shifted Gegenbauer polynomials $\mathscr{P}_j^*(r)$, which form an orthogonal basis with respect to the weight function $w^*(r)$ and, assume $\{\mathscr{P}_j^*(r)\}_{j=0}^m$ be the shifted Gegenbauer polynomials defined on $[0,1]$. Define the basis vector

$$\mathscr{P}^*(r) = \big[\,\mathscr{P}_0^*(r)\ \mathscr{P}_1^*(r),\ \ldots,\ \mathscr{P}_m^*(r)\big].$$

For any fractional order λ, there exist operational matrices with respect to the Gegenbauer basis:

$$^{C}D_{0+}^{\lambda}\mathscr{P}^*(r) = \mathcal{D}^{(\lambda)}\mathscr{P}^*(r), \tag{6.1}$$

where, $\mathcal{D}^{(\lambda)} \in \mathbb{R}^{(m+1)\times(m+1)}$ are precomputed (4.5). In addition, the solution is approximated in terms of the Gegenbauer basis as:

$$\mathcal{U}(r) \approx \mathcal{U}_m(r) = C^T\mathscr{P}^*(r). \tag{6.2}$$

6.2. **Equidistant Collocation Method.** To determine the coefficients, we employ the *equidistant collocation method*, where the collocation points are chosen as:

$$r_k = \frac{k}{m}, \quad k = 0,1,\ldots,m.$$

At each collocation point r_k, the differential equation is enforced exactly, leading to a system of algebraic equations.

6.3. **Linear Fractional Differential Equations.** Consider the general linear fractional differential equation [12] on the interval $[0,1]$:

$$\sum_{h=0}^{N} a_h(r)\,{}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r) = f(r). \tag{6.3}$$

With boundary conditions:

$$\mathcal{U}^{(q)}(r_q) = \xi_q, \qquad q = 0,1,\ldots,Q-1. \tag{6.4}$$

Such that: $Q = Max\lceil\lambda_h\rceil$, turning to the matrix collocation approach, we now express the proposed equation and its condition in matrix form. The diagonal matrix representation for $a_h(r)$:

$$a_h = \begin{pmatrix} a_h(r_0) & 0 & \cdots & 0 \\ 0 & a_h(r_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_h(r_m) \end{pmatrix},$$

$$\hat{\mathcal{U}} = [{}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r_0) \quad {}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r_1) \quad \ldots \quad {}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r_m)]^T,$$

where, $a_h(r_k)$ represents the value of the function $a_h(r)$ at the discrete points r_k for $k = 0,1,\ldots,m$.
Approximate $\mathcal{U}(r)$ by a truncated expansion of shifted Gegenbauer polynomials $\{\mathscr{P}_j^*(r)\}_{j=0}^m$:

$$\mathcal{U}_m(r) = \sum_{j=0}^{m} C_j\mathscr{P}_j^*(r) = C^\top\mathscr{P}^*(r), \tag{6.5}$$

with unknown coefficients $C = [C_0, C_1, \ldots, C_N]^\top$.
The fractional and integer derivatives of $\mathcal{U}_m(r)$ can be expressed compactly using the operational matrices $\mathcal{D}^{(\lambda_h)}$ associated with the basis:

$$^{C}D_{0+}^{\lambda_h}\mathcal{U}_m(r) = C^\top\mathcal{D}^{(\lambda_h)}\mathscr{P}^*(r), \tag{6.6}$$

Define the residual function by substituting the approximations into the original equation:

$$R(r;C) = \sum_{h=0}^{N} a_h(r)\,{}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r) - f(r), \tag{6.7}$$

Uncorrected Proof

$$R(r_k;C)=\sum_{h=0}^{N} a_h(r_k)\,{}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r_k)-f(r_k), \tag{6.8}$$

in the matrix form:

$$R(r_k;C)=a_h\,{}^{C}D_{0+}^{(\lambda_h)}\,\hat{\mathcal{U}}-F, \tag{6.9}$$

or,

$$R(r_k;C)=a_h\,{}^{C}D_{0+}^{(\lambda_h)}\,C^T\cdot\mathscr{P}^*(r_k)-F, \tag{6.10}$$

substitute from (4.6) becomes:

$$R(r_k;C)=a_h\,C^T(B_m\cdot A^{(\lambda)}(r_k)\cdot B_m^{-1})\cdot\mathscr{P}^*(r_k)-F, \tag{6.11}$$

and,

$$\mathscr{P}^*(r_k)=\begin{pmatrix}\mathscr{P}_0^*(r_0) & \mathscr{P}_1^*(r_0) & \dots & \mathscr{P}_m^*(r_0)\\ \mathscr{P}_0^*(r_1) & \mathscr{P}_1^*(r_1) & \dots & \mathscr{P}_m^*(r_1)\\ \vdots & \vdots & \ddots & \vdots\\ \mathscr{P}_0^*(r_m) & \mathscr{P}_1^*(r_m) & \dots & \mathscr{P}_m^*(r_m)\end{pmatrix}, \tag{6.12}$$

to determine C, impose the collocation conditions at $m+1$ distinct collocation points $\{r_k\}_{k=0}^{m}$:

$$R(r_k;C)=0,\quad k=0,1,\dots,m.$$

This yields the linear algebraic system:

$$AC=F=\begin{bmatrix}f(r_0)\\ f(r_1)\\ f(r_2)\\ \vdots\\ f(r_m)\end{bmatrix}. \tag{6.13}$$

An additional equations, generated from the condition (6.4), is integrated into the system (6.13). A suitable solver is then used to solve the resulting system for coefficients column C. By substituting these coefficients into equation (6.5), the approximate solution is obtained. Finally, the generated equations from the condition may take the following form:

$$\mathcal{U}^{(q)}(r_q)=C^TB_m\cdot(M)^q\cdot B_m^{-1}\mathscr{P}^*(r_q)=[\xi_h],\quad q=0,1,2,\dots,(Q-1),$$

6.4. **Nonlinear Fractional Differential Equations.** Consider the nonlinear problem:

$$\mathcal{N}\big(r,\mathcal{U}(r),\mathcal{U}^{(p)}(r),{}^{C}D^{\lambda}\mathcal{U}(r)\big)=f(r). \tag{6.14}$$

For instance [1, 8, 9], a representative form can be written as

$$\sum_{l=0}^{N}\sum_{h=0}^{N_1}Q_{l,h}(r)\,\big(\mathcal{U}(r)\big)^l\,{}^{C}D_{0+}^{(\lambda_h)}\mathcal{U}(r)+\sum_{p=1}^{N_2}\sum_{q=0}^{N_3}P_{p,q}(r)\,D^{(p)}\mathcal{U}(r)\,{}^{C}D_{0+}^{(\phi_q)}\mathcal{U}(r)=f(r). \tag{6.15}$$

$$\mathcal{U}^{(h)}(r_h)=\xi_h,\quad h=0,1,2,\dots,(Q-1), \tag{6.16}$$

Such that: $Q=Max\big[N,\lceil\lambda_h\rceil,N_2,\lceil\phi_q\rceil\big]$. Similar to the linear case, the functions $\mathcal{U}(r),\mathcal{U}^{(h)}(r),\mathcal{U}^{(\lambda_h)}(r)$, and $\mathcal{U}^{(\phi_q)}(r)$ are approximated using shifted Gegenbaur polynomials via proposed collocation-based schemes. The residual can be expressed as:

$$R_m(r_k;C)=\mathcal{N}\big(r_k,\ C^T\mathscr{P}^*(r_k),C^\top\mathcal{D}^{(\lambda)}\mathscr{P}^*(r_k),\ C^T\mathcal{D}^{(\lambda)}\mathscr{P}^*(r_k)\big)-f(r_k). \tag{6.17}$$

Uncorrected Proof

Applying the relations established in (4.6) and (6.2), the residual becomes:

$$R_m(r_k;C) = \sum_{l=0}^{N}\sum_{h=0}^{N_1} Q_{l,h}(r_k)\left(C^T\mathscr{P}^*(r_k)\right)^l C^T D_{0+}^{(\lambda_h)}\mathscr{P}^*(r_k) + \sum_{p=1}^{N_2}\sum_{q=0}^{N_3} P_{p,q}(r_k)C^T D^{(p)}\mathscr{P}^*(r_k)\, C^T\, D_{0+}^{(\phi_q)}\mathscr{P}^*(r_k) - f(r_k).$$

An equivalent form incorporating the transformation matrix B_m is

$$R_m(r_k;C) = \sum_{l=0}^{N}\sum_{h=0}^{N_1} Q^{l,h}\left(C^T\mathscr{P}^*(r_k)\right)^l C^T B_m\mathcal{D}^{(\lambda_h)}(B_m)^{-1}\mathscr{P}^*(r_k)_{(}r_k) + \sum_{p=1}^{N_2}\sum_{q=0}^{N_3} P^{p,q}C^T B_m\mathcal{D}^{(p)}(B_m)^{-1}\mathscr{P}^*(r_k)\times C^T B_m\mathcal{D}^{(\phi_q)}(B_m)^{-1}\mathscr{P}^*(r_k) - f(r_k). \tag{6.18}$$

Here,

$$Q^{l,h} = \begin{pmatrix} Q_{l,h}(r_0) & 0 & \cdots & 0 \\ 0 & Q_{l,h}(r_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{l,h}(r_m) \end{pmatrix}, \quad P^{p,q} = \begin{pmatrix} P_{p,q}(r_0) & 0 & \cdots & 0 \\ 0 & P_{p,q}(r_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{p,q}(r_m) \end{pmatrix}, \quad F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

In such a way that $\mathscr{P}^*(r_k)$ is defined before in (6.12), furthermore, the nonlinear constraints (6.16) is represented in matrix form as:

$$\mathcal{U}^{(h)}(r_h) = C^T B_m\cdot(M)^h\cdot B_m^{-1}\mathscr{P}^*(r_h) = [\xi_h], \quad h = 0,1,2,\ldots,(Q-1),$$

as explained before in the linear case, the system (6.18) together with the produced columns of the conditions, they result in a set of nonlinear algebraic equations that are solved iteratively for C.

Enforce:

$$R_m(r_k;C) = 0, \qquad k = 0,\ldots,m.$$

This procedure leads to the following nonlinear algebraic system:

$$R(r_k;C) = \begin{bmatrix} R_m(r_0;C) \\ R_m(r_1;C) \\ \vdots \\ R_m(r_m;C) \end{bmatrix}.$$

Here, $R_m(r_k;C)$ denotes the residual function evaluated at the k-th collocation point r_k for the approximate solution parameterized by the unknown coefficient vector $C = [C_0, C_1, \ldots, C_m]^T$. The collocation points $\{r_k\}_{k=0}^m$ are chosen to be equally spaced in the domain of the problem, and the system $R(r_m;C) = 0$ must be solved to determine the coefficients C.

7. Test examples and applications

In this section, we assess the proposed spectral collocation method through illustrative test problems involving both linear and nonlinear FDEs. All computations are performed with double precision, with collocation points taken as Gegenbauer polynomials shifted on $[0,1]$, and the operational matrices of previous sections used for ordinary and fractional derivative approximations.

Uncorrected Proof

7.1. **Collocation formulation.** We approximate the solution $\mathcal{U}(r)$ by the truncated expansion:

$$\mathcal{U}_m(r) \approx \sum_{j=0}^{m} C_j \mathscr{P}_j^*(r) = C^\top \mathscr{P}^*(r).$$

The coefficient's vector C is determined by enforcing the residual at the chosen collocation points together with the boundary constraints. For the seventh- and eighth-kind shifted Gegenbauer families, the interior collocation points are taken as equidistance points though the interval $[0, 1]$, used in both basis polynomials: $\mathscr{Z}_m^*(r)$, $E_m^*(r)$. Together with the boundary conditions these provide a system of $m+1$ equations for the $m+1$ unknown coefficients.
Let $B_m^{(\mathscr{Z})}$ and $B_m^{(E)}$ denote the connection matrices mapping the monomial basis

$$\mathscr{R}_m(r) = [1, r, \ldots, r^m]^T,$$

to the shifted Gegenbauer bases:

$$\mathscr{P}^*(r) = B_m^{(\mathscr{Z})}\, \mathscr{R}_m(r), \qquad \mathscr{P}^*(r) = B_m^{(E)}\, \mathscr{R}_m(r).$$

If $A^{(\lambda)}$ denotes the truncated monomial Caputo matrix, the corresponding operational matrices for the two families are obtained via the similarity transformations:

$$\mathcal{D}^{(\lambda;\mathscr{Z})} = B_m^{(\mathscr{Z})}\, A^{(\lambda)} \left(B_m^{(\mathscr{Z})}\right)^{-1}, \qquad \mathcal{D}^{(\lambda;E)} = B_m^{(E)}\, A^{(\lambda)} \left(B_m^{(E)}\right)^{-1}.$$

These matrices generate the fractional derivatives in the shifted Gegenbauer representations.

Example 7.1. Consider the nonlinear Bratu–type problem

$$^{\mathrm{C}}D_{0+}^{\lambda}\mathcal{U}(r) + \beta\, e^{\mathcal{U}(r)} = 0, \tag{7.1}$$

Bratu-type problem (7.1) is a fractional version or a generalization of the classic ordinary model, that allows for a more flexible model of physical systems with memory or non-local behavior. Such that a classic ordinary model gives variety of physical phenomena, such as a spontaneous combustion model or a suspension bridge's deflection under a uniform load, whose exact solution is given by:

$$\mathcal{U}(r) = -2\ln\left[\frac{\cosh\left(\frac{\alpha}{4}(2r-1)\right)}{\cosh\left(\frac{\alpha}{4}\right)}\right],$$

where, the parameter α satisfies the equation

$$\alpha = \sqrt{2\beta}\,\cosh(\alpha).$$

The equation is typically solved with Dirichlet boundary conditions, such as:

$$\mathcal{U}(0) = \mathcal{U}(1) = 0.$$

The residual for the approximation

$$\mathcal{U}_m(r) = C^T \mathscr{P}^*(r),$$

is

$$R_m(r_k; C) \;=\; C^T D^{(\lambda)} \mathscr{P}^*(r_k) \;+\; \beta\, e^{\left(C^T \mathscr{P}^*(r_k)\right)}. \tag{7.2}$$

Specializing to each family, replace $D^{(\lambda)}$ and $\mathscr{P}^*(r)$:

$$R_m^{(\mathscr{Z})}(r_k; C) \approx C^T \mathcal{D}^{(\lambda;\mathscr{Z})} \mathscr{P}^*(r_k) + \beta\, e^{C^T \mathscr{P}^*(r_k)} = C^T B_m^{(\mathscr{Z})}\, A^{(\lambda;\mathscr{Z})} \left(B_m^{(\mathscr{Z})}\right)^{-1} \mathscr{P}^*(r_k) + \beta\, e^{C^T \mathscr{P}^*(r_k)},$$

$$R_m^{(E)}(r_k; C) \approx C^T \mathcal{D}^{(\lambda;E)} \mathscr{P}^*(r_k) + \beta\, e^{C^T \mathscr{P}^*(r_k)} = C^T B_m^{(E)}\, A^{(\lambda;E)} \left(B_m^{(E)}\right)^{-1} \mathscr{P}^*(r_k) + \beta\, e^{C^T \mathscr{P}^*(r_k)}.$$

also, the residual of the clasical model is being as:

$$R_m^{(\star)}(r_k; C) = C^T B_m^{(\star)} M^2 (B_m^{(\star)})^{-1} \mathscr{P}^*(r_k) + \beta\, e^{C^T \mathscr{P}^*(r_k)}.$$

Uncorrected Proof

Enforce the residual to vanish at the interior collocation points and impose boundary conditions:

$$R_m^{(\star)}(r_k^{(\star)}; C) = 0, \quad k = 1, \ldots, m-1, \qquad \mathcal{U}_m(0) = 0, \quad \mathcal{U}_m(1) = 0,$$

or

$$R_m^{(\star)}(r_k^{(\star)}; C) = 0, \quad k = 1, \ldots, m-1, \qquad \mathcal{C}^T \mathscr{P}^*(0) = [0], \quad \mathcal{C}^T \mathscr{P}^*(1) = [0],$$

where $\star \in \{\mathscr{Z}, E\}$ indicates the family. This yields a nonlinear system of $m+1$ equations for C After solving for Coefficients, compute pointwise values and errors at $\lambda = 2$, $m = 16$.

In the context of seventh and eighth-kind Gegenbauer polynomials, we adopt the natural convention of all operational matrices are ordered according to polynomial degree $0, 1, \ldots, 5$. For $m = 5$, the explicit form of the change-of-basis matrix $B_5^{(\mathscr{Z})}$ associated with the seventh kind is given by

$$B_5^{(\mathscr{Z})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{5} & \frac{8}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & -\frac{48}{5} & \frac{48}{5} & 0 & 0 & 0 \\ -\frac{12}{35} & \frac{408}{35} & -\frac{1152}{35} & \frac{768}{35} & 0 & 0 \\ \frac{9}{35} & -\frac{576}{35} & \frac{4416}{35} & -\frac{1536}{7} & \frac{768}{7} & 0 \\ -\frac{8}{35} & \frac{656}{35} & -\frac{1408}{7} & \frac{4352}{7} & -\frac{5120}{7} & \frac{2048}{7} \end{pmatrix}.$$

By contrast, the eighth-kind change-of-basis matrix takes the exact rational form

$$B_5^{(E)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6}{5} & \frac{12}{5} & 0 & 0 & 0 & 0 \\ \frac{9}{5} & -\frac{96}{5} & \frac{96}{5} & 0 & 0 & 0 \\ -\frac{72}{35} & \frac{1104}{35} & -\frac{576}{7} & \frac{384}{7} & 0 & 0 \\ \frac{18}{7} & -\frac{480}{7} & \frac{2784}{7} & -\frac{4608}{7} & \frac{2304}{7} & 0 \\ -\frac{20}{7} & \frac{680}{7} & -\frac{5504}{7} & \frac{15616}{7} & -2560 & 1024 \end{pmatrix}.$$

In both cases the differential operator, expressed in the ordered monomial basis, reduces to the standard nil-potent super-sub-diagonal matrix. For $m = 5$ this operator is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix},$$

while, the second moment matrix, denoted M^2, is sparse with entries concentrated on the lower band:

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \end{pmatrix}.$$

Uncorrected Proof

Applying the similarity transform, the seventh-kind basis yields a transformed moment matrix of the form

$$B_5^{(\mathscr{Z})} \cdot M \cdot B_5^{(\mathscr{Z})} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{96}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{576}{7} & 0 & 0 & 0 & 0 \\ \frac{6912}{35} & 0 & \frac{960}{7} & 0 & 0 & 0 \\ 0 & \frac{2720}{7} & 0 & \frac{800}{3} & 0 & 0 \end{pmatrix},$$

whereas for the eighth kind the corresponding product is

$$B_5^{(E)} \cdot M^2 \cdot B_5^{(E)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{192}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{960}{7} & 0 & 0 & 0 & 0 \\ \frac{2976}{7} & 0 & \frac{1440}{7} & 0 & 0 & 0 \\ 0 & \frac{14080}{21} & 0 & \frac{1120}{3} & 0 & 0 \end{pmatrix}.$$

where, the differential matrices $A^{(\lambda;\star)}$ at $m = 5$, with differntt order take the following form

$$A^{(1.99;\mathscr{Z})} = A^{(1.99;E)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2.01141}{x^{1.99}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5.97449}{x^{1.99}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{11.8895}{x^{1.99}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{19.7501}{x^{1.99}} \end{pmatrix},$$

and

$$A^{(1.88;\mathscr{Z})} = A^{(1.88;E)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2.11956}{x^{1.88}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5.6774}{x^{1.88}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{10.7121}{x^{1.88}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{17.1668}{x^{1.88}} \end{pmatrix}.$$

The results in Table 1 confirm the very high accuracy of the proposed collocation schemes at $\beta = 1$. For $m = 16$, the absolute errors of both the seventh- and eighth-kind approximations remain on the order of 10^{-7} throughout the domain, indicating near machine-level precision. By contrast, the reference values from [10] at $m = 20$ are several orders of magnitude larger between 10^{-4} and 10^{-5}, demonstrating a clear superiority of the present approach insolving the classical model of (7.1). Moreover, the strong consistency between the seven and eighth-kind approximations indicates that both approaches perform equally well in representing the exact solution. Table 2 shows that the proposed collocation schemes achieve errors several orders of magnitude smaller than those reported in [10] and [7], particularly for $\beta = 1.00$ and $\beta = 2.00$, confirming their superior accuracy at $\lambda = 2$. Figure 1 illustrates that the approximation remains close to the exact solution, with the absolute error decreasing as r increases through the interval. Figure 2 Shows the comparative analysis of logarithmic absolute errors for different β parameter values. Error magnitudes increase with higher β values, suggesting optimal parameter ranges for numerical stability in the seventh and eighth-kind Chebyshev Gegenbaur polynomial. Besides, Table 3 compares the exact and approximate solutions for the seventh and Eighth-kind Chebyshev polynomials, evaluating three different values of the order $\lambda = 2$, 1.99, and 1.88. The comparative assessment provides significant insights into the performance and accuracy of the

Uncorrected Proof



TABLE 1. Comparison of the exact solution and collocation approximations of the present method with the absolute error For Example 7.1.

	$m=16$					$m=20$
r	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	Error$_{\mathscr{Z}}$	Error$_E$	[10]
0.0	0.00000	5.095×10^{-18}	7.873×10^{-18}	5.095×10^{-18}	7.873×10^{-18}	–
0.1	0.0498468	0.0498465	0.0498465	2.59204×10^{-7}	2.59204×10^{-7}	2.01×10^{-4}
0.2	0.0891899	0.0891894	0.0891894	4.60783×10^{-7}	4.60783×10^{-7}	2.83×10^{-4}
0.3	0.1176090	0.1176080	0.1176080	6.00241×10^{-7}	6.00241×10^{-7}	2.95×10^{-4}
0.4	0.1347900	0.1347890	0.1347890	6.74235×10^{-7}	6.74235×10^{-7}	2.56×10^{-4}
0.5	0.1405390	0.1405380	0.1405380	6.80797×10^{-7}	6.80797×10^{-7}	1.96×10^{-4}
0.6	0.1347900	0.1347890	0.1347890	6.19517×10^{-7}	6.19517×10^{-7}	1.37×10^{-4}
0.7	0.1176090	0.1176080	0.1176080	4.92099×10^{-7}	4.92099×10^{-7}	9.16×10^{-5}
0.8	0.0891897	0.0891894	0.0891894	3.06009×10^{-7}	3.06009×10^{-7}	5.71×10^{-5}
0.9	0.0498466	0.0498465	0.0498465	9.43141×10^{-8}	9.43141×10^{-8}	2.83×10^{-5}
1.0	4.57901×10^{-18}	0.0000000	0.0000000	4.57901×10^{-18}	3.58690×10^{-18}	—

TABLE 2. MAE of the seventh- and eighth-kind collocation approximations For Example 7.1.

β	max | Error |$_Z$	max | Error |$_E$	[10]	[7]
1.00	6.80797×10^{-7}	6.80797×10^{-7}	2.95×10^{-4}	8.62×10^{-6}
2.00	4.20608×10^{-4}	4.20608×10^{-4}	1.32×10^{-3}	1.28×10^{-5}
3.51	9.92010×10^{-1}	2.67850×10^{-1}	–	5.46×10^{-4}

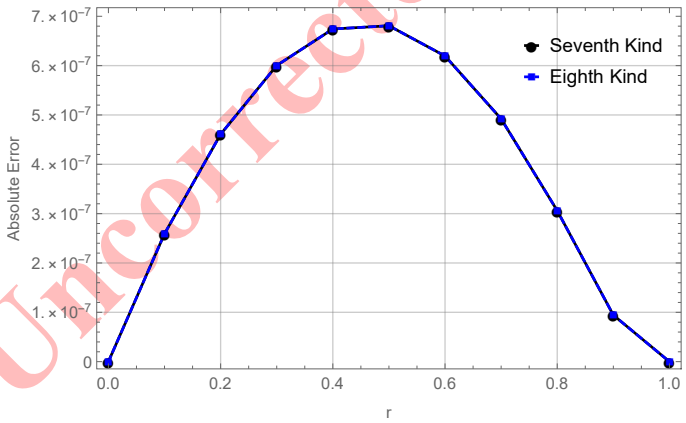


FIGURE 1. The comparison of absolute error for both 7th and 8th kinds through interval for Example 7.1 at $m=16$.

employed numerical approximation technique. Figure 3 illustrates the comparison of the accuracy of the seventh- and eighth-kind Chebyshev polynomial methods For Example 7.1, when the parameter λ is perturbed from its exact value of 2, highlighting the effective of their numerical solutions to variations in λ.

Example 7.2. Consider the standard Lane–Emden equation [13, 39]:

$$ {}^{C}D_{0+}^{\lambda}\mathcal{U}(r)+\frac{2}{r}\,{}^{C}D_{r_0+}^{\lambda-1}\mathcal{U}(r)+\mathcal{U}^{n}(r)=0,\qquad 0\le n\le 5,\quad \lambda>1. \tag{7.3}$$

Uncorrected Proof

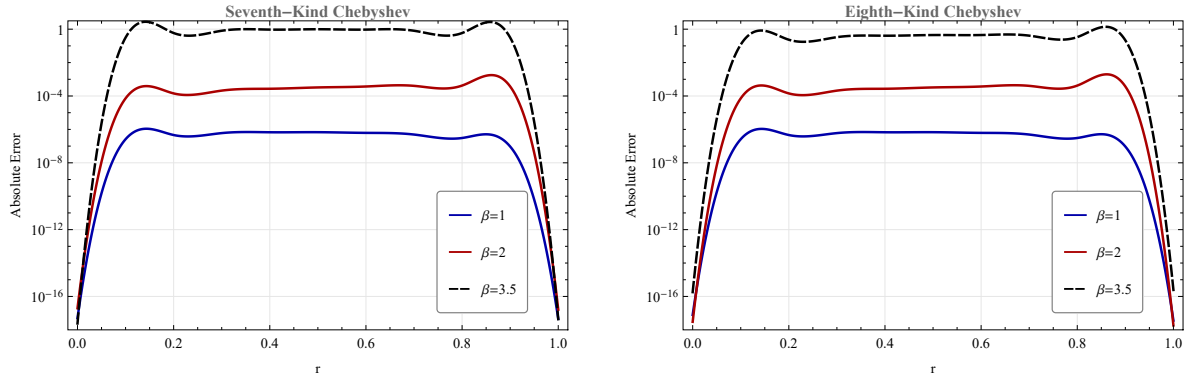


FIGURE 2. Log Error For Example 7.1 For $\beta = 1, 2, 3.5$, $m = 16$, 7th (Left) and 8th (Right) kinds.

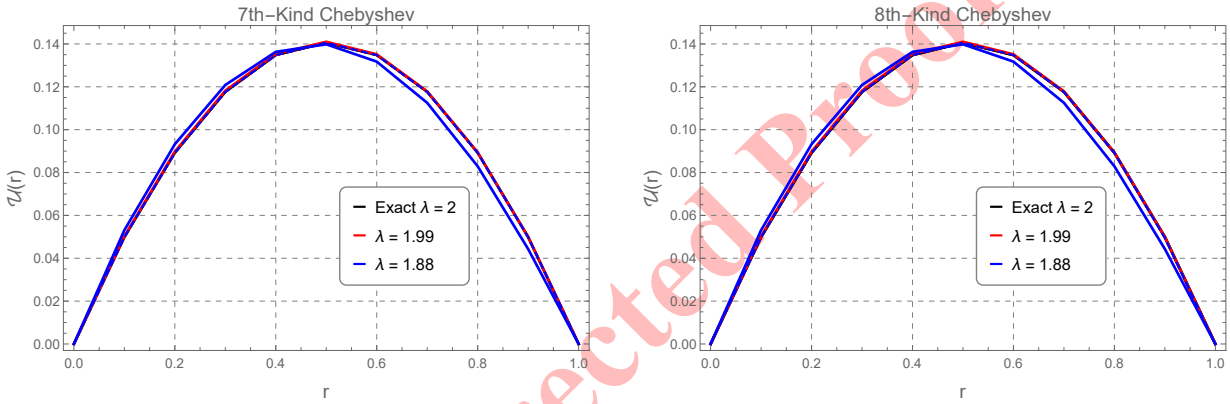


FIGURE 3. Comparison of 7th (Left) and 8th (Right)-kind Chebyshev under order variation λ, for Example 7.1.

TABLE 3. Seventh and Eighth-kind Chebyshev approximations for three choices of λ for Example 7.1.

r	$\lambda = 2$			$\lambda = 1.99$		$\lambda = 1.88$	
	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$
0.0	2.404×10^{-18}	0	0	0	0	2.220×10^{-16}	0
0.1	0.0498608	0.0498465	0.0498465	0.0498465	0.0498465	0.0498468	0.0498468
0.2	0.0892177	0.0891894	0.0891894	0.0891894	0.0891894	0.0891899	0.0891899
0.3	0.1176500	0.1176080	0.1176080	0.1176080	0.1176080	0.1176090	0.1176090
0.4	0.1348450	0.1347890	0.1347890	0.1347890	0.1347890	0.1347900	0.1347900
0.5	0.1406070	0.1405380	0.1405380	0.1405380	0.1405380	0.1405390	0.1405390
0.6	0.1348700	0.1347890	0.1347890	0.1347890	0.1347890	0.1347900	0.1347900
0.7	0.1177000	0.1176080	0.1176080	0.1176080	0.1176080	0.1176090	0.1176090
0.8	0.0892862	0.0891894	0.0891894	0.0891894	0.0891894	0.0891899	0.0891899
0.9	0.0499259	0.0498465	0.0498465	0.0498465	0.0498465	0.0498468	0.0498468
1.0	4.376×10^{-18}	0	0	0	0	2.220×10^{-16}	0

Uncorrected Proof

TABLE 4. Exact, 7th-kind and 8th-kind approximate values at $\lambda = 2$, with absolute errors for Example 7.2, case 1 at $n = 0$, $m = 5$.

r	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	Error$_{\mathscr{Z}}$	Error$_E$
0.0	1.000000	1.000000	1.000000	0	0
0.1	0.998333	0.998333	0.998333	0	0
0.2	0.993333	0.993333	0.993333	0	0
0.3	0.985000	0.985000	0.985000	0	0
0.4	0.973333	0.973333	0.973333	0	1.11022×10^{-16}
0.5	0.958333	0.958333	0.958333	1.11022×10^{-16}	1.11022×10^{-16}
0.6	0.940000	0.940000	0.940000	0	1.11022×10^{-16}
0.7	0.918333	0.918333	0.918333	0	1.11022×10^{-16}
0.8	0.893333	0.893333	0.893333	1.11022×10^{-16}	1.11022×10^{-16}
0.9	0.865000	0.865000	0.865000	0	0
1.0	0.833333	0.833333	0.833333	1.11022×10^{-16}	1.11022×10^{-16}

The classical form of the standard Lane–Emden equation (7.3) is given by replacing λ with 2 is formed to describe the structure of polytropic stars, and $\mathcal{U}(r)$ represents the normalized temperature, and r is the normalized radius, in addition n is the polytropic index. This equation (7.3) is typically solved with the initial conditions:

$$\mathcal{U}(0) = 1, \qquad \mathcal{U}'(0) = 0. \tag{7.4}$$

The fractional version of (7.3), provides a more generalized model that can capture phenomena with non-local or memory effects. For an approximate the solution $\mathcal{U}_m(r)$, the residual function is expressed in operator form as:

$$R_m(r_k; C) \approx C^T \mathcal{D}^{(\lambda)} \mathscr{P}^*(r_k) + \frac{2}{r_k} C^T \mathcal{D}^{(\lambda-1)} \mathscr{P}^*(r_k) + (C^T \mathscr{P}^*(r_k))^n,$$

$$R_m^{(\star)}(r_k; C) = (C^T B_m^\star A^{(\lambda;\star)} \left(B_m^\star\right)^{-1} + C^T B_m^\star A^{(\lambda-1;\star)} \left(B_m^\star\right)^{-1}) \cdot \mathscr{P}^*(r_k)) + (C^T \mathscr{P}^*(r_k)))^n.$$

for the initial conditions:

$$C^T \mathscr{P}^*(0) = [1], \quad C^T B_m^{(\star)} M (B_m^{(\star)})^{-1} \mathscr{P}^*(0) = [0].$$

Case 1. The Lane–Emden equation exhibits distinct analytical and numerical characteristics depending on the index n. We begin with the simplest case, $n = 0$. In this case, the problem reduces to a linear fractional differential equation, and its exact analytical solution is $\mathcal{U}(r) = 1 - \frac{r^2}{3!}$. To evaluate the performance of the proposed spectral schemes, we approximate the solution using the 7th-kind $\mathcal{U}_m^{(\mathscr{Z})}$ and the 8th-kind $\mathcal{U}_m^{(E)}$ Chebyshev polynomial using the present method, the corresponding numerical results obtained. Table 4 demonstrates that both the 7th- and 8th-kind Chebyshev approximations at $n = 0$ reproduce the exact solution with machine precision, yielding absolute errors on the order of 10^{-16} or identically zero using $m = 5$. For $\lambda = 1.8$, $\lambda = 1.9$ and $m = 5$ in Table 5 the comparison of approximate solutions (7th and 8th kinds solution's) with the exact solution (λ=2) are obtained. Figure 4 shows 7th and 8th-kind Chebyshev polynomials achieve excellent numerical accuracy; however, their performance is fundamentally limited by the precision of parameter specification. Both variants show similar changes, indicating that improved numerical schemes cannot compensate for inaccurate parameter values.

Case 2. The Lane–Emden equation (7.3) at $n = 1$. Table 6 presents a comprehensive comparison of numerical solutions obtained using 7th-kind ($\mathcal{U}_m^{(\mathscr{Z})}$) and 8th-kind ($\mathcal{U}_m^{(E)}$) Chebyshev methods with $n = 1$, examining their performance under several λ order variation compared with the exact solution ($\lambda = 2$). Figure 5 provides a visual comparison of how 7th and 8th-kind Chebyshev polynomial approximations respond to order λ changes for $n = 1$, $m = 5$.

Uncorrected Proof

TABLE 5. 7th and 8th-kind Chebyshev approximations at $\lambda = 1.8$ and $\lambda = 1.9$ For Example 7.2. case 1 at $n = 0$, $m = 5$.

	$\lambda = 1.8$			$\lambda = 1.9$		
r	Exact	$\mathcal{U}_m^{(\mathscr{Z})}$	$\mathcal{U}_m^{(E)}$	Exact	$\mathcal{U}_m^{(\mathscr{Z})}$	$\mathcal{U}_m^{(E)}$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.997764	0.998333	0.998333	0.998292	0.998333	0.998333
0.2	0.991339	0.993333	0.993333	0.993186	0.993333	0.993333
0.3	0.981070	0.985000	0.985000	0.984705	0.985000	0.985000
0.4	0.967206	0.973333	0.973333	0.972866	0.973333	0.973333
0.5	0.949918	0.958333	0.958333	0.957683	0.958333	0.958333
0.6	0.929318	0.940000	0.940000	0.939164	0.940000	0.940000
0.7	0.905473	0.918333	0.918333	0.917316	0.918333	0.918333
0.8	0.878424	0.893333	0.893333	0.892141	0.893333	0.893333
0.9	0.848205	0.865000	0.865000	0.863645	0.865000	0.865000
1.0	0.814854	0.833333	0.833333	0.831831	0.833333	0.833333

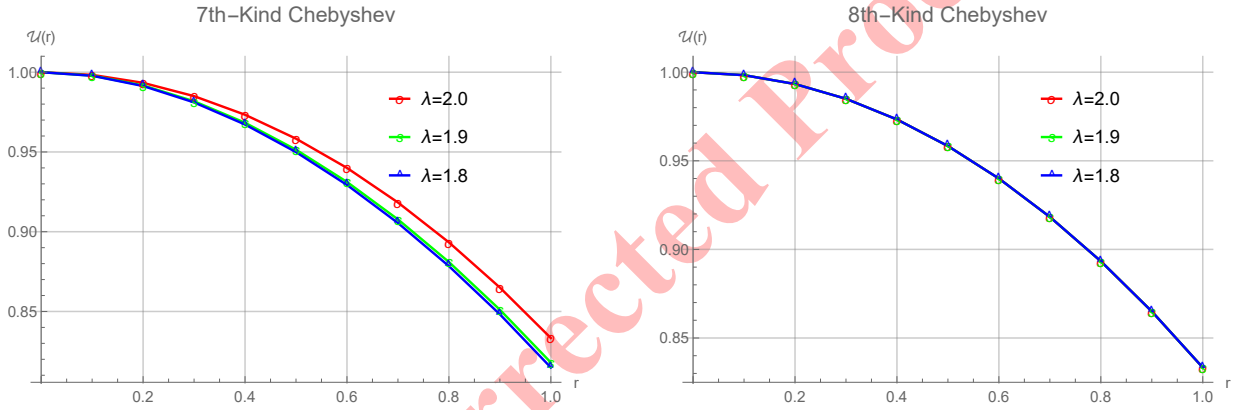


FIGURE 4. Comparison of 7th (Left) and 8th (Right) kind of variant λ For Example 7.2 case 1 at $n = 0$, $m = 5$.

TABLE 6. seventh- and Eighth-kind Chebyshev exact and approximate values for three choices of λ for Example 7.2 case 2 at $n = 1$, $m = 5$.

r	$\lambda = 2$			$\lambda = 1.9$		$\lambda = 1.8$	
	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$
0.1	0.998334	0.998335	0.998335	0.997871	0.997935	0.997284	0.997284
0.2	0.993347	0.993347	0.993347	0.991728	0.991920	0.989733	0.989733
0.3	0.985067	0.985067	0.985068	0.981894	0.982213	0.978092	0.978092
0.4	0.973546	0.973547	0.973547	0.968638	0.969045	0.962924	0.962924
0.5	0.958851	0.958852	0.958852	0.952189	0.952635	0.944652	0.944652
0.6	0.941071	0.941072	0.941072	0.932747	0.933189	0.923589	0.923589
0.7	0.920311	0.920312	0.920312	0.910494	0.910905	0.899975	0.899975
0.8	0.896695	0.896696	0.896696	0.885606	0.885978	0.874014	0.874014
0.9	0.870363	0.870363	0.870363	0.858265	0.858605	0.845906	0.845906
1.0	0.841471	0.841470	0.841471	0.828669	0.828988	0.815885	0.815885

Uncorrected Proof

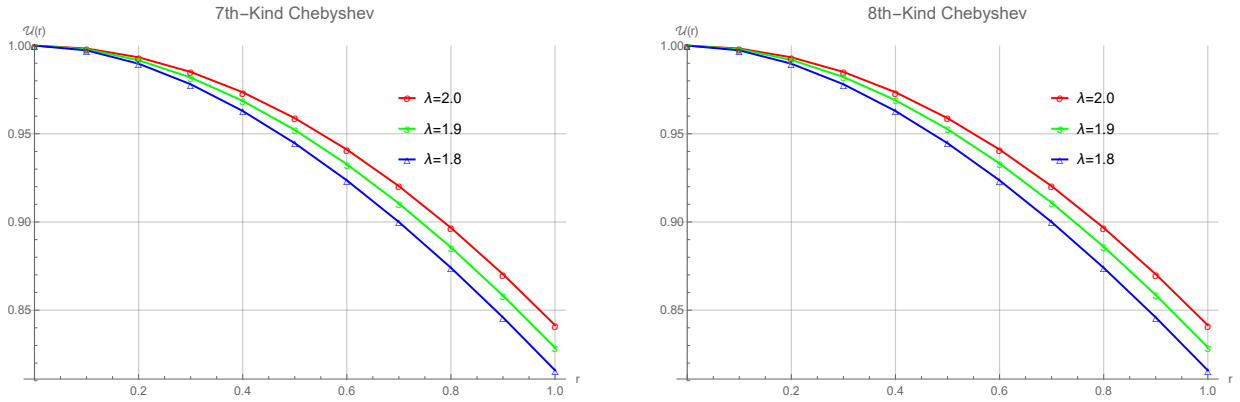


FIGURE 5. 7th-kind (Left) and 8th (Right)-kind Chebyshev approximations for Example 7.2 case 2 at $n = 1$ with different order values λ.

Example 7.3. Consider the nonlinear Riccati equation [38]:

$$ {}^{\mathrm{C}}D^{\lambda}_{r_0+}\mathcal{U}(r) + \big(\mathcal{U}(r)\big)^2 - 2\mathcal{U}(r) = 1, \quad \mathcal{U}(0) = 0, \quad 0 \le r \le 1. \tag{7.5} $$

The exact solution of this problem (7.5) is given when $\lambda = 1$ (the classical version) by:

$$ \mathcal{U}(r) = 1 + \sqrt{2}\,\tanh\Big(\tfrac{1}{2}\log\Big(\tfrac{\sqrt{2}-1}{\sqrt{2}+1}\Big) + \sqrt{2}r\Big). \tag{7.6} $$

Using the shifted Gegenbauer polynomial approximation, the associated residual at the collocation points r_k is then written as:

$$ R^{(\star)}_m(r_k; C) = C^T B^{\star}_m A^{(\lambda;\star)} \left(B^{\star}_m\right)^{-1} \cdot \mathscr{P}^*(r_k) + (C^T \mathscr{P}^*(r_k))^2 - 2(C^T \mathscr{P}^*(r_k)) = [1], \tag{7.7} $$

and for the condition,

$$ C^T \mathscr{P}^*(0) = [0]. $$

The results in Table 7 demonstrate that the approximate solutions closely match the exact values, confirming the accuracy and effectiveness of the Chebyshev spectral method across different polynomial kinds and parameter values. Figure 6 compares 7th- and 8th-kind Chebyshev approximations for $\lambda = 1, 0.98, 0.88$, showing similar trends with increasing differences as λ decreases, highlighting parameter and method reliability. Besides, Figure 7 the approximate solutions (7th, 8th) at different values of m namely $m = 6, 10, 16$ are compared with the exact solution tell that for bigger m the accurate solution obtained, while, Figure 8, the Log10 error comparisons at different values of m ($m = 6, 10, 16$) are given in both 7th and 8th approximations. At the end, Table 8 gives Max absolute error of the seventh- and eighth-kind approximations at $m = 6, 10, 16$.

Example 7.4. Consider the boundary value problem [4]:

$$ {}^{\mathrm{C}}D^{\lambda}_{r_0+}\mathcal{U}(r) + \frac{1}{r}\,{}^{\mathrm{C}}D^{\lambda-1}_{r_0+}\mathcal{U}(r) + \frac{1}{1-r}\mathcal{U}(r) = 4\cos(r) - 5r\sin(r) + \frac{r^3}{1-r}, \quad r \in (0,1), \tag{7.8} $$

subject to the boundary conditions

$$ \mathcal{U}(0) = 0, \qquad \mathcal{U}(1) = \cos(1). $$

The exact solution of this problem (7.8) is given when $\lambda = 2$ by:

$$ \mathcal{U}(r) = r^2\cos(r). $$

Uncorrected Proof

TABLE 7. Seventh and Eighth-kind Chebyshev exact and approximate values for three choices of λ for Example 7.3.

r	$\lambda = 1$			$\lambda = 0.98$		$\lambda = 0.88$	
	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathcal{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathcal{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathcal{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$
0.1	0.110295	0.110295	0.110295	0.153891	0.116417	0.153891	0.153891
0.2	0.241977	0.241977	0.241977	0.324886	0.253830	0.324886	0.324886
0.3	0.395105	0.395105	0.395105	0.514669	0.412551	0.514669	0.514669
0.4	0.567812	0.567812	0.567812	0.718210	0.590353	0.718210	0.718210
0.5	0.756014	0.756014	0.756014	0.926933	0.782517	0.926933	0.926933
0.6	0.953566	0.953566	0.953566	1.131240	0.982254	1.131240	1.131240
0.7	1.152950	1.152950	1.152950	1.322530	1.181620	1.322530	1.322530
0.8	1.346360	1.346360	1.346360	1.494700	1.372770	1.494700	1.494700
0.9	1.526910	1.526910	1.526910	1.644620	1.549130	1.644620	1.644620
1.0	1.689520	1.689520	1.689520	1.771120	1.706040	1.771120	1.771120

TABLE 8. Max error of the seventh- and eighth-kind approximations For Example 7.3. at $m = 6, 10, 16$.

m	max | Error $\|_Z$	max | Error $\|_E$
6	27767×10^{-3}	2.7766×10^{-3}
10	12949×10^{-5}	1.29483×10^{-5}
16	1.20838×10^{-8}	1.20839×10^{-8}

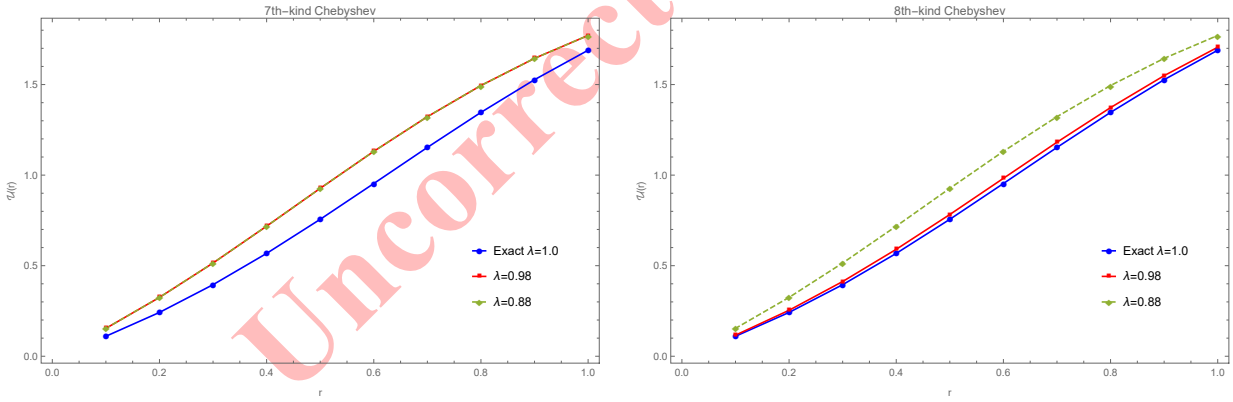


FIGURE 6. 7th (Left) and 8th (Right)-kinds chebyshev approximation at different λ. For Example 7.3.

Table 9 shows that the seventh and eighth-kind Chebyshev methods demonstrate exceptional numerical performance, and efficiency compared to the reference method [47], making them highly promising for high-precision computational applications. Table 10 presented the reliability of the proposed seventh and eighth-kind Chebyshev polynomial approximations at $m = 5$. For all three choices of the order λ, the approximate solutions show close agreement with the exact values, reflecting the accuracy of the method across different λ highlights the efficiency of the approach methods. Figure 9 shows approximations closely follow the exact solution for $\lambda = 2$, with only minor deviations for $\lambda = 1.88$, confirming the accuracy of the method.

Uncorrected Proof

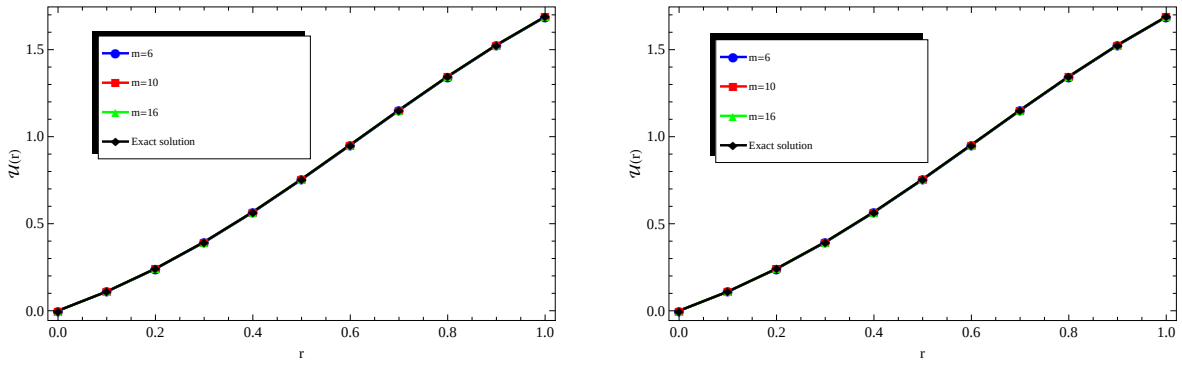


FIGURE 7. The approximate solutions at different values of m compared with the exact solution (7th approximate Left, 8th approximate Right) For Example 7.3.

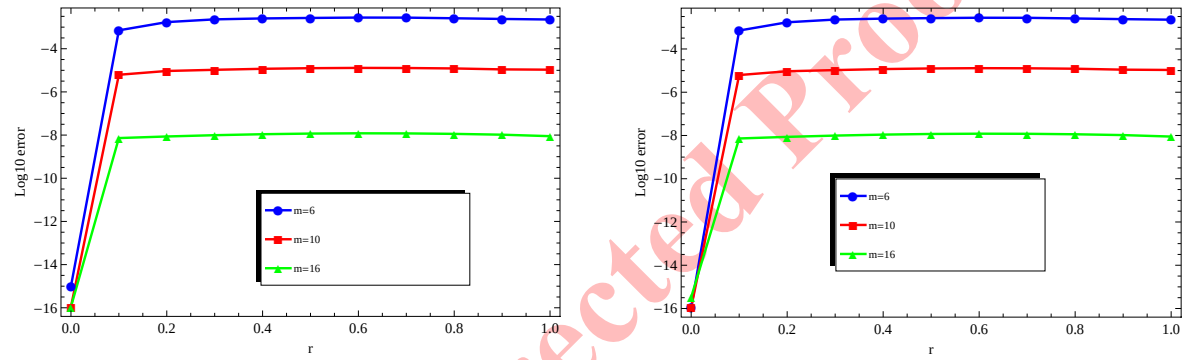


FIGURE 8. The Log10 error comparison at different values of m (7th approximate Left, 8th approximate Right) For Example 7.4.

Uncorrected Proof

TABLE 9. Absolute errors of the Seventh- and Eighth-kind Chebyshev at $\lambda = 2$ for Example 7.3.

r	$m = 12$		[47]		
	Error$_{\mathscr{Z}}$	Error$_E$	$m = 12$	$m = 14$	$m = 28$
0.1	2.70×10^{-14}	2.70×10^{-14}	1.94×10^{-08}	2.04×10^{-10}	5.99×10^{-13}
0.2	4.57×10^{-14}	4.58×10^{-14}	2.37×10^{-08}	2.44×10^{-10}	5.69×10^{-13}
0.3	5.62×10^{-14}	5.63×10^{-14}	2.59×10^{-08}	2.64×10^{-10}	4.70×10^{-13}
0.4	6.27×10^{-14}	6.28×10^{-14}	2.71×10^{-08}	2.74×10^{-10}	3.83×10^{-13}
0.5	6.70×10^{-14}	6.71×10^{-14}	2.80×10^{-08}	2.76×10^{-10}	3.16×10^{-13}
0.6	6.93×10^{-14}	6.94×10^{-14}	2.15×10^{-08}	2.10×10^{-10}	3.36×10^{-13}
0.7	6.94×10^{-14}	6.96×10^{-14}	1.58×10^{-08}	1.56×10^{-10}	3.66×10^{-13}
0.8	6.74×10^{-14}	6.76×10^{-14}	1.04×10^{-08}	1.06×10^{-10}	3.48×10^{-13}
0.9	6.27×10^{-14}	6.29×10^{-14}	4.99×10^{-09}	5.71×10^{-11}	1.97×10^{-13}
1.0	1.11×10^{-16}	1.11×10^{-16}	–	–	–

TABLE 10. Exact values at $\lambda = 2$ and approximations at $\lambda = 1.98$ and $\lambda = 1.88$ for the Seventh- and Eighth-kind Chebyshev functions (for Example 7.4).

r	$\lambda = 2$			$\lambda = 1.98$		$\lambda = 1.88$	
	Exact $\mathcal{U}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$	$\mathcal{U}_m^{(\mathscr{Z})}(r)$	$\mathcal{U}_m^{(E)}(r)$
0.1	0.00995004	0.0100107	0.0100107	0.0134258	0.00995004	0.0100107	0.0134258
0.2	0.0392027	0.0401549	0.0401549	0.0506355	0.0392027	0.0401549	0.0506355
0.3	0.0859803	0.0882912	0.0882912	0.107321	0.0859803	0.0882912	0.107321
0.4	0.147373	0.151142	0.151142	0.178397	0.147373	0.15114	0.178397
0.5	0.219396	0.224402	0.224402	0.258063	0.219396	0.224402	0.258063
0.6	0.297121	0.302882	0.302882	0.339870	0.297121	0.302882	0.339870
0.7	0.374773	0.380604	0.380604	0.416781	0.374773	0.380604	0.416781
0.8	0.445892	0.450933	0.450933	0.481236	0.445892	0.450933	0.481236
0.9	0.503504	0.506697	0.506697	0.525215	0.503504	0.506697	0.525215
1.0	0.540302	0.540302	0.540302	0.540302	0.540302	0.540302	0.540302

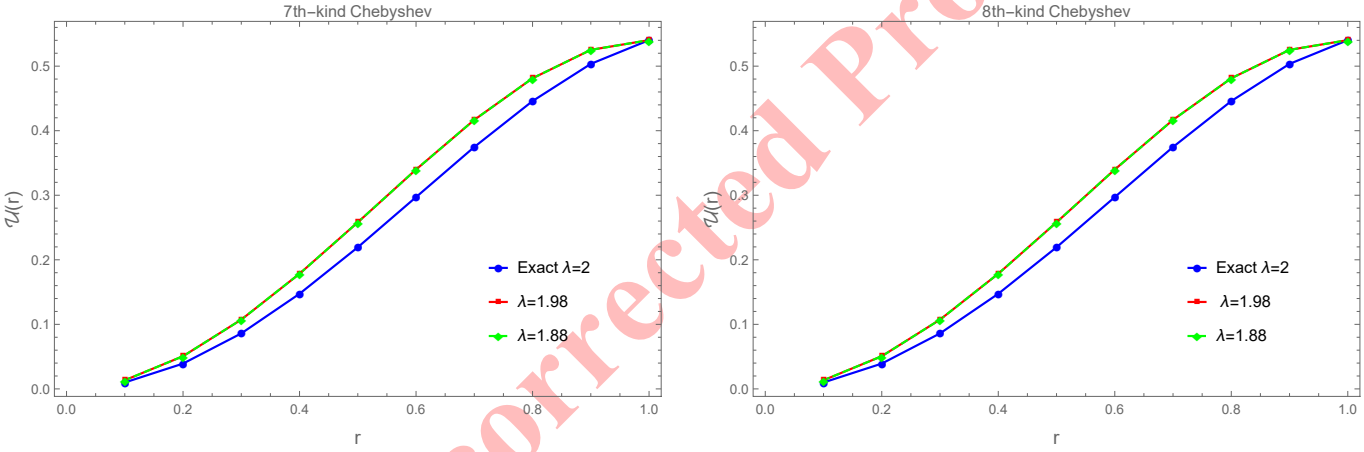


FIGURE 9. Comparison of exact and approximate solutions for Example 7.4 using the 7th and 8th-kind Chebyshev polynomials with different values of λ.

Uncorrected Proof

8. CONCLUSION

This paper presented a spectral collocation framework using Caputo operational matrices of shifted seventh- and eighth-kind Chebyshev polynomials. The compact operational matrices for both 7th and 8th polynomials transformed fractional differential equations (linear and non-linear) into algebraic systems, enabling efficient computations for linear and nonlinear cases. All matrix relations for different terms of the FDEs in the Chebyshev numerical collocation approach are investigated and presented in this study along with the proofs of the convergence and stability theorems and relations. Comparative results with classical schemes showed that higher-kind Chebyshev bases improve accuracy, spectral convergence, and stability. These findings establish generalized Chebyshev families as powerful alternatives to traditional polynomial bases in fractional calculus. The method also shows promise for applications in optimal control, delay systems, and physics-informed neural networks. Future work will address multi-term, variable-order, and space–time fractional problems.

References

[1] M. A. Abd El Salam, M. A. Ramadan, M. A. Nassar, P. Agarwal, and Y.-M. Chu, *Matrix computational collocation approach based on rational Chebyshev functions for nonlinear differential equations*, Advances in Difference Equations, *2021*(1) (2021), 331.

[2] W. M. Abd-Elhameed, Y. H. Youssri, and A. G. Atta, *Adopted spectral tau approach for the time-fractional diffusion equation via seventh-kind Chebyshev polynomials*, Advances in Continuous and Discrete Models, *2024* (2024), 102.

[3] W. M. Abd-Elhameed, Y. H. Youssri, A. K. Amin, and A. G. Atta, *Eighth-kind Chebyshev polynomials collocation algorithm for the nonlinear time-fractional generalized Kawahara equation*, Fractal and Fractional, *7*(9) (2023), 652.

[4] M. Abdelhakem, M. Fawzy, M. El-Kady, and H. Moussa, *An efficient technique for approximated BVPs via the second derivative Legendre polynomials pseudo-Galerkin method: Certain types of applications*, Results in Physics, *43* (2022), 106067.

[5] J. Ablinger, *Discovering and proving infinite Pochhammer sum identities*, Experimental Mathematics, *31*(1) (2022), 309–323.

[6] A. Aghazadeh and Y. Mahmoudi, *On approximating eigenvalues and eigen-functions of fractional order Sturm–Liouville problems*, Computational Methods for Differential Equations, *11*(4) (2023), 811–821.

[7] A. Ahmad, M. Sulaiman, A. J. Aljohani, A. Alhindi, and H. Alrabaiah, *Design of an efficient algorithm for solution of Bratu differential equations*, Ain Shams Engineering Journal, *12*(2) (2021), 2211–2225.

[8] K. K. Ali, M. A. Abd El Salam, E. M. H. Mohamed, B. Samet, S. Kumar, and M. S. Osman, *Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series*, Advances in Difference Equations, *2020*(1) (2020), 494.

[9] K. K. Ali, M. A. Abd El Salam, and E. M. H. Mohamed, *A numerical technique for a general form of nonlinear fractional-order differential equations with the linear functional argument*, International Journal of Nonlinear Sciences and Numerical Simulation, *22*(1) (2021), 83–91.

[10] M. Al-Smadi, S. Momani, N. Djeddi, A. El-Ajou, and Z. Al-Zhour, *Adaptation of reproducing kernel method in solving Atangana–Baleanu fractional Bratu model*, International Journal of Dynamics and Control, *11*(1) (2023), 136–148.

[11] Y. A. Amer, K. R. Mohammed, M. A. Abd El Salam, and A. A. Mahdi, *Spectral matrix computational tau approach for fractional differential equations via fifth-kind Chebyshev polynomials*, Boletim da Sociedade Paranaense de Matemática, *43*(2) (2025), 1–31.

[12] Y. A. Amer, K. R. Raslan, M. A. El Salam, and A. A. Mahdi, *Efficient estimation of Lucas polynomial derivatives through combined BVP and IVP*, Bulletin of Faculty of Science, Zagazig University, *2025*(2) (2025), 61–76.

[13] S. Aydinlik and A. Kiris, *A high-order numerical method for solving nonlinear Lane-Emden type equations arising in astrophysics*, Astrophysics and Space Science, *363*(12) (2018), 264.

[14] H. Azin, A. Habibirad, E. Hesameddini, and M. H. Heydari, *An efficient collocation scheme for new type of variable-order fractional Lane–Emden equation*, Iranian Journal of Numerical Analysis and Optimization, *14*(4) (2024), 1224–1246.

[15] M. G. Bin-Saad, M. J. Shahwan, J. Younis, H. Aydi, and M. A. Abd El Salam, *On Gaussian hypergeometric functions of three variables: Some new integral representations*, Journal of Mathematics, *2022*(1) (2022), 1914498.

[16] H. S. Cohl and R. S. Costas-Santos, *On the relation between Gegenbauer polynomials and the Ferrers function of the first kind*, Analysis Mathematica, *48*(3) (2022), 695–716.

[17] S. A. Dahy, H. M. El-Hawary, A. Fahim, and T. Aboelenen, *High-order spectral collocation method using tempered fractional Sturm–Liouville eigenproblems*, Computational and Applied Mathematics, *42*(8) (2023), 338.

[18] S. K. Damarla and M. Kundu, *Novel hybrid function operational matrices of fractional integration: An application for solving multi-order fractional differential equations*, arXiv preprint arXiv:1801.06973 (2018).

[19] K. Diethelm, R. Garrappa, A. Giusti, and M. Stynes, *Why fractional derivatives with nonsingular kernels should not be used*, Fractional Calculus and Applied Analysis, *23*(3) (2020), 610–634.

Uncorrected Proof

[20] A. Draux, M. Sadik, and B. Moalla, *Markov–Bernstein inequalities for generalized Gegenbauer weight*, Applied Numerical Mathematics, *61*(12) (2011), 1301–1321.
[21] A. H. Hadian-Rasanan, D. Rahmati, S. Gorgin, and K. Parand, *A single layer fractional orthogonal neural network for solving various types of Lane–Emden equation*, New Astronomy, *75* (2020), 101307.
[22] M. S. Hadi, M. Lakestani, and B. Nemati Saray, *The wavelet Galerkin method for fractional delay differential equations*, Calcolo, *62*(4) (2025), Article 48.
[23] M. H. Heydari and M. Hosseininia, *An efficient Jacobi spectral method for variable-order time fractional 2D Wu-Zhang system*, Computers and Mathematics with Applications, *140* (2023), 89–106.
[24] G. J. Heckman and E. M. Opdam, *Jacobi polynomials and hypergeometric functions associated with root systems*, in: T. H. Koornwinder and J. V. Stokman (eds.), *Encyclopedia of Special Functions, Part II: Multivariable Special Functions*, Cambridge University Press, Cambridge (2021).
[25] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
[26] M. Lakestani, R. Tuntaş, and M. Dehghan, *Constructing efficient bases from B-Spline functions and solving fractional optimal control problems*, Journal of Applied Mathematics and Computing, *72*(3) (2026), Article 120.
[27] L. Li and J.-G. Liu, *A generalized definition of Caputo derivatives and its application to fractional ODEs*, SIAM Journal on Mathematical Analysis, *50*(3) (2018), 2867–2900.
[28] Z. Lu and Y. Zhu, *Comparison principles for fractional differential equations with the Caputo derivatives*, Advances in Difference Equations, *2018*(1) (2018), 237.
[29] Y. Luchko, *On the Caputo fractional derivative and the functions with non-singular kernels*, Mathematics, *8*(4) (2020), 660.
[30] I. Malmir, *Caputo fractional derivative operational matrices of Legendre and Chebyshev wavelets in fractional delay optimal control*, Numerical Algebra, Control and Optimization, *12*(2) (2022), 395–426.
[31] F. S. M. Nasrudin, C. Phang, S. Mahadi, and N. Arbin, *Operational matrix for solving fractional differential equations with Erdelyi-Kober differential operator*, AIP Conference Proceedings, *2905* (2024), 030008.
[32] R. Najafi, F. Bahrami, and P. Vafadar, *Lie symmetry analysis, and exact solutions to the time–fractional Black–Scholes equation of the Caputo–type*, Computational Methods for Differential Equations, *12*(4) (2024), 638–650.
[33] M. Obeid, M. A. Abd El Salam, and M. S. Mohamed, *A novel generalized symmetric spectral Galerkin numerical approach for solving fractional differential equations with singular kernel*, AIMS Mathematics, *8*(7) (2023), 16724–16747.
[34] N. Peykrayegan, M. Ghovatmand, and M. H. Noori Skandari, *On the convergence of Jacobi-Gauss collocation method for linear fractional delay differential equations*, Mathematical Methods in the Applied Sciences, *44*(2) (2021), 2237–2253.
[35] S. Raubitzek, K. Mallinger, and T. Neubauer, *Combining fractional derivatives and machine learning: A review*, Entropy, *25*(1) (2022), 35.
[36] S. Ranjbari, M. Baghmisheh, M. Jahangiri Rad, and B. Nemati Saray, *On the wavelet Galerkin method for solving the fractional Fredholm integro-differential equations*, Computational Methods for Differential Equations, *13*(3) (2025), 885–903.
[37] J. Shen, T. Tang, and L.-L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer, 2011.
[38] H. Singh and H. M. Srivastava, *Jacobi collocation method for the approximate solution of some fractional-order Riccati differential equations with variable coefficients*, Physica A: Statistical Mechanics and Its Applications, *523* (2019), 1130–1149.
[39] J. Stewart, *Essential Calculus: Early Transcendentals*, Brooks–Cole, Cengage Learning, Belmont, CA, 2013.
[40] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, 1939.
[41] H. Sun, A. Chang, Y. Zhang, and W. Chen, *A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications*, Fractional Calculus and Applied Analysis, *22*(1) (2019), 27–59.
[42] T. Taheri, A. A. Aghaei, and K. Parand, *Accelerating fractional PINNs using operational matrices of derivative*, arXiv preprint arXiv:2401.14081 (2024).

Uncorrected Proof

[43] Y. Xu, *An integral formula for generalized Gegenbauer polynomials and Jacobi polynomials*, Advances in Applied Mathematics, *29*(2) (2002), 328–343.
[44] Y. H. Youssri and A. G. Atta, *Spectral collocation approach via normalized shifted Jacobi polynomials for the nonlinear Lane-Emden equation with fractal-fractional derivative*, Fractal and Fractional, *7*(2) (2023), 133.
[45] D. Zaidi, I. Talib, M. B. Riaz, and P. Agarwal, *Novel derivative operational matrix in Caputo sense with applications*, Journal of Taibah University for Science, *18*(1) (2024), 2333061.
[46] W. Zeng and C. Xu, *An accurate and efficient space-time Galerkin spectral method for the subdiffusion equation*, Science China Mathematics, *67*(10) (2024), 2387–2408.
[47] F. Zhou, L.-X. Yu, Z.-B. Ma, and Z.-G. Yu, *Granulomatous lobular mastitis*, Chronic Diseases and Translational Medicine, *2*(1) (2016), 17–21.

Uncorrected Proof