Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-8
DOI:10.22034/cmde.2026.70939.3552

Gradient estimate for bounded solutions of nonlinear equations on almost Ricci solitons

Sakineh Hajiaghasi and Shahroud Azami*
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

Abstract

This paper investigates gradient estimates for bounded solutions of a class of nonlinear elliptic equations on almost Ricci soliton manifolds. The equation under consideration generalizes several previously studied models and incorporates a smooth nonlinear term depending on the solution. Under suitable curvature conditions and structural assumptions on the almost Ricci soliton, we establish explicit gradient bounds. These estimates extend known results for Ricci solitons and provide new analytical tools for the study of geometric evolution equations.

Keywords. Gradient almost Ricci solitons, Gradient estimates, Almost Ricci solitons.
1991 Mathematics Subject Classification. 53C25, 53C21, 53C99.

Uncorrected Pro

1. Introduction

A Riemannian manifold (N^m, g) equipped with a smooth vector field Y and a smooth function $\alpha : N^m \to \mathbb{R}$ is called an almost Ricci soliton if

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \alpha g, \tag{1.1}$$

where Ric denotes the Ricci curvature tensor and $\mathcal{L}_Y g$ is the Lie derivative of the metric g with respect to Y which it is defined by $(\mathcal{L}_Y g)(X, Z) = g(\nabla_X Y, Z) + g(X, \nabla_Z Y)$ for any vector fields X, Z. An almost Ricci soliton is called expanding, steady, or shrinking if $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$, respectively. When $Y = \nabla f$ for some smooth function f, the manifold is called a gradient almost Ricci soliton, and Equation (1.1) becomes

$$\mathrm{Ric} + \mathrm{Hess}\, f = \alpha g, \tag{1.2}$$

where Hess f denotes the Hessian of f.

The theory of almost Ricci solitons was introduced by Pigola et al. [17] as a natural generalization of the theory of classical Ricci solitons. Since then, substantial progress has been made in understanding their geometric properties, including the classification of compact examples and the study of warped-product structures; see, for example, [2–4, 8–10, 17].

Gradient estimates are fundamental tools in geometric analysis, with applications to elliptic and parabolic equations as well as geometric evolution equations (see for instance [14–16]). The classical gradient estimates of Cheng–Yau and Li–Yau have motivated numerous extensions to different geometric settings, including almost Ricci solitons; see [6, 11–13, 18]. In particular, gradient estimates have been obtained for drift equations of the form $\Delta u - \langle X, \nabla u\rangle = f$, the Allen–Cahn equation, and equations with logarithmic nonlinearities under suitable curvature assumptions [11–13].

In this paper, we study the nonlinear elliptic equation

$$\Delta u + A(u) = h, \tag{1.3}$$

where $A : \mathbb{R} \to \mathbb{R}$ and $h : N^m \to \mathbb{R}$ are smooth functions. This equation includes several important models studied in the literature, such as the Allen–Cahn equation and semilinear equations with logarithmic or power-type nonlinearities;

Received: 27 December 2025; Accepted: 10 November 2026.
∗ Corresponding author. Email: azami@sci.ikiu.ac.ir.

see, for example, [5, 19]. Our main objective is to establish local gradient estimates for bounded positive solutions of (1.3) on almost Ricci solitons under suitable geometric and analytic assumptions. In the Riemannian setting, such estimates are commonly derived by combining Sobolev-type inequalities, cutoff-function arguments, and Moser iteration; see, for example, [17].

The principal ingredients in our analysis are suitable isoperimetric and Sobolev inequalities, together with a volume comparison theorem for almost Ricci solitons. These results, established in the relevant geometric settings in [7, 20], are recalled in Section 2. Our main theorem, presented in Section 3, provides explicit local gradient bounds in terms of the L^2-norm of the solution and the L^{2q}-norm of the source term h.

2. Preliminaries

In this section, we recall several fundamental results that will be used throughout the paper. The assumptions below are not arbitrary technical restrictions. The lower bound on the Bakry–Émery-type tensor controls the geometric error in the Bochner formula, while the decay condition on Y makes the Lie-derivative term integrable on small balls. The non-collapsing condition prevents the normalized local Sobolev constant from degenerating. Together these hypotheses allow the Sobolev inequality and volume comparison to be used with constants depending only on the stated geometric parameters. Proving our main results we may need following isoperimetric and sobolev inequality, which was established by Dai et al. in [7], for integral Ricci curvature, and generalized for the Bakry-Émery Ricci curvature in [20].

Theorem 2.1 (Isoperimetric inequality)**.** *Let N^m be an almost Ricci soliton satisfying the following conditions:*

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g \geq -\alpha g, \qquad |Y|(y) \leq \frac{K}{d(y,O)^\beta}, \tag{2.1}$$

for some constants $0 \leq \beta < 1$, $K \geq 0$, and a fixed origin $O \in N$. Assume further the volume non-collapsing condition

$$\mathrm{Vol}(B(y,1)) \geq \rho, \qquad \forall y \in N, \tag{2.2}$$

for some constant $\rho > 0$, and suppose $\alpha \leq m$. Then there exists a constant $r_1 = r_1(n, m, K, \beta, \rho)$ such that for any $r_2 \leq r_1$ and any $f \in C_0^\infty(B(y, r_2))$, the following isoperimetric constant estimate holds:

$$\mathrm{ID}_m^*(B(y,r_2)) \leq C(m) r_2.$$

Here the isoperimetric constant $\mathrm{ID}_m^*(B(y,r_2))$ is defined by

$$\mathrm{ID}_m^*(B(y,r_2)) = \mathrm{Vol}(B(y,r_2))^{1/m} \cdot \sup_{\mathcal{U}} \left\{ \frac{\mathrm{Vol}(\mathcal{U})^{(m-1)/m}}{\mathrm{Vol}(\partial \mathcal{U})} \right\},$$

where the supremum is taken over all subsets $\mathcal{U} \subset B(y,r_2)$ with $\partial\mathcal{U} \cap \partial B(y,r_2) = \emptyset$.

Under the same hypotheses as Theorem 2.1, we obtain the following Sobolev inequalities:

Theorem 2.2. *For any $f \in C_0^\infty(B(y,r_2))$ on an almost Ricci soliton N with $r_2 \leq r_1$, we have*

$$\left(\fint_{B(y,r_2)} |f|^{\frac{m}{m-1}} \, dg \right)^{\frac{m-1}{m}} \leq C(m) r_2 \fint_{B(y,r_2)} |\nabla f| \, dg, \tag{2.3}$$

and

$$\left(\fint_{B(y,r_2)} |f|^{\frac{2m}{m-2}} \, dg \right)^{\frac{m-2}{m}} \leq C(m) r_2^2 \fint_{B(y,r_2)} |\nabla f|^2 \, dg, \tag{2.4}$$

for $m > 2$. In the gradient case $Y = \nabla f$, we also have

$$\left(\fint_{B(y,r_2)} |f|^{\frac{m}{m-1}} \, dg \right)^{\frac{m-1}{m}} \leq C(m) r_2 \fint_{B(y,r_2)} |\nabla f| \, dg. \tag{2.5}$$

In addition we need the volume comparison theorem which was proved in [1] for almost Ricci solitons as follows:

Uncorrected Proof

Theorem 2.3. *Let N^m be an almost Ricci soliton with $\alpha \leq m$. Under conditions (2.1) and (2.2), for any $0 < \varrho_1 < \varrho_2 \leq 1$, we have*

$$\frac{\mathrm{Vol}(B(y,\varrho_2))}{\varrho_2^m} \leq e^{C(m,K,\beta,\rho)[m(\varrho_2^2-\varrho_1^2)+K(\varrho_2-\varrho_1)^{1-\beta}]} \cdot \frac{\mathrm{Vol}(B(y,\varrho_1))}{\varrho_1^m}. \tag{2.6}$$

The same estimate holds when $Y = \nabla f$.

3. Main result

Now, we present a new gradient estimates on almost Ricci solitons. We mainly follow the lines of proof in [1, 20].

Theorem 3.1. *Let on an almost Ricci solitons N^m, (2.1) hold and $\alpha \leq n_1$. Let smooth functions w and h admit (1.3), $q > \frac{m}{2}$, $|A| \leq n_2$, and $|\nabla A| \leq n_3$, for constants n_2, n_3. Then there exists a positive constant $C = C(m, K, \beta, \rho, n_1, n_2, n_3)$, such that $\forall x \in N$, and $0 < r \leq C$, we get*

$$\sup_{B(y,\frac{1}{2}r)} |\nabla w|^2 \leq C(m,K,\beta,\rho,n_1,n_2,n_3)\big[(\|h\|^*_{2q,B(y,r)})^2 + r^{-2}(\|w\|^*_{2,B(y,r)})^2\big]. \tag{3.1}$$

Proof. Let $v = |\nabla w|^2 + \|f^2\|^*_{q,B(y,r)}$. So, the Bochner formula yields

$$\Delta v = 2|\nabla^2 w|^2 + 2 < \nabla w, \nabla \Delta w > + 2\mathrm{Ric}(\nabla w, \nabla w). \tag{3.2}$$

Putting (1.3) in (3.2) implies that

$$\begin{aligned}\Delta v &\geq 2 < \nabla w, \nabla h > -2 < \nabla w, \nabla A(w) > -2\alpha v - (\mathcal{L}_Y g)(\nabla w, \nabla w)\\ &\geq 2 < \nabla w, \nabla h > -2(n_1+n_3)v - (\mathcal{L}_Y g)(\nabla w, \nabla w).\end{aligned}$$

For every $q > 0$, it follows that

$$\begin{aligned}\Delta v^q &= qv^{q-1}\Delta v + (q-1)qv^{q-2}|\nabla v|^2\\ &\geq 2qv^{q-1} < \nabla w, \nabla h > -2qv^q(n_1+n_3) - qv^{q-1}(\mathcal{L}_Y g)(\nabla w, \nabla w) + \frac{q-1}{q}v^{-q}|\nabla v^q|^2.\end{aligned} \tag{3.3}$$

Set $\Omega = B(y,r)$. Using (3.3), $\forall \theta \in C_0^\infty(B_y(1))$ and $q \geq 1$, we deduce

$$\begin{aligned}\int_\Omega |\nabla(\theta v^q)|^2 &= \int_\Omega |\theta \nabla v^q + v^q \nabla \theta|^2\\ &= \int_\Omega v^{2q}|\nabla\theta|^2 - \theta^2 v^q \Delta v^q\\ &\leq \int_\Omega v^{2q}|\nabla\theta|^2 - 2q\theta^2 v^{2q-1} w_i h_i + 2(m_1+m_3)q\theta^2 v^{2q} + q\theta^2 v^{2q-1}(\mathcal{L}_Y g)_{ij} w_i w_j.\end{aligned} \tag{3.4}$$

Using

$$(L_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i,$$

we first estimate the last term in (3.3). Since

$$\nabla_i Y_j \, w_i w_j \leq |\nabla Y| \, |w|^2,$$

Young's inequality implies that, for every $\varepsilon > 0$,

$$q\theta^2 v^{2q-1}(L_Y g)_{ij} w_i w_j \leq \varepsilon q\theta^2 v^{2q-1}|\nabla w|^2 + C(\varepsilon)q\theta^2 v^{2q-1}|Y|^2.$$

Moreover, the mixed term generated by differentiating θv^q is estimated by

$$2q\theta v^{2q-1} w_i \nabla_i \theta \leq \frac{q}{2}\theta^2 v^{2q-2}|\nabla w|^2 + 2qv^{2q}|\nabla\theta|^2.$$

Uncorrected Proof

Combining these estimates with (3.3), and choosing the constants in Young's inequality appropriately, gives

$$\int_\Omega \eta^2 v^{2q-1}(\mathcal{L}_Y g)_{ij} w_i w_j \le \int_\Omega \frac{8q-1}{4q} v^{2q}|\nabla\theta|^2 + \frac{2(2q-1)^2+5q}{2q}\theta^2 v^{2q}|Y|^2 + \frac{1}{2q}|\nabla(\theta v^q)|^2 + \frac{1}{2}\theta^2 v^{2q-1}h^2 + \frac{1}{2}n_2\theta^2 v^{2q-1} + \frac{1}{2}n_2\theta^2 v^{2q-1}|\nabla w|^2. \tag{3.5}$$

On the other hand, integration by parts gives

$$-2q\int_\Omega \theta^2 v^{2q-1} w_i h_i \le \int_\Omega \frac{4(2q-1)^2+1}{2q}\theta^2 v^{2q-1}h^2 + \frac{1}{2q}v^{2q}|\nabla\theta|^2 + \frac{1}{8q}|\nabla(\theta v^q)|^2. \tag{3.6}$$

Putting (3.5), and (3.6) into (3.4), follows that

$$\int_\Omega |\nabla(\theta v^q)|^2 \le \int_\Omega 16qv^{2q}|\nabla\theta|^2 + 70q^2\theta^2 v^{2q-1}h^2 + 30q^2\theta^2 v^{2q}|Y|^2 + 8q(n_1+n_3)\theta^2 v^{2q} + 2qn_2\theta^2 v^{2q}|Y|^2 + 2qn_2\theta^2 v^{2q-1}. \tag{3.7}$$

Let $\psi_j(s)$ be a cut-off function such that for $r_j = (\frac{1}{2}, \frac{1}{2^{j+2}})$, $j = 0, 1, 2, ...$, $\psi_j(s) \equiv 1$ if $t \in [0, r_{j+1}]$, $supp\psi_j \subseteq [0, r_j]$ and $-\frac{52^j}{r} \le \psi_j' \le 0$. Let $\theta_j(y) = \psi_j(s)$. Hence, (3.7) yields

$$\int_{B(y,r_j)} |\nabla(\theta_j v^q)|^2 \le \int_{B(y,r_j)} 16qv^{2q}|\nabla\theta_j|^2 + 70q^2\theta_j^2 v^{2q-1}h^2 + [30q^2 + 2qn_2]\theta_j^2 v^{2q}|Y|^2 + 8q(n_1+n_3)\theta_j^2 v^{2q} + 2qn_2\theta_j^2 v^{2q-1}. \tag{3.8}$$

From volume comparison Theorem 2.3, for $\frac{r}{2} \le r_j \le \frac{3r}{4}$, we get

$$\begin{aligned}
70q^2 \fint_{B(y,r_j)} \theta_j^2 v^{2q-1}h^2 &\le \frac{70q^2}{\|h^2\|^*_{q,\Omega}} \fint_{B(y,r_j)} \theta_j^2 v^{2q}h^2 \\
&\le C(m,\alpha,K,\beta,\rho)q^2\left(\fint_{B(y,r_j)} (\theta_j v^q)^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}} \\
&\le C(m,\alpha,K,\beta,\rho)q^2\left(\fint_{B(y,r_j)} (\theta_j v^q)^{z\cdot\frac{2q}{q-1}\cdot a}\right)^{\frac{q-1}{q^a}} \\
&\quad\times\left(\fint_{B(y,r_j)} (\theta_j v^q)^{(1-z)\cdot\frac{2q}{q-1}\cdot\frac{a}{a-1}}\right)^{\frac{(q-1)(a-1)}{q^a}}.
\end{aligned}$$

Using the Young's inequality in the last line, gives

$$\begin{aligned}
70q^2 \fint_{B(y,r_j)} \theta_j^2 v^{2q-1}h^2 &\le \epsilon\left(\fint_{B(y,r_j)} (\theta_j v^q)^{z\cdot\frac{2q}{q-1}\cdot a}\right)^{\frac{q-1}{q^{wz}}} + \epsilon^{-\frac{z}{1-z}} C^{\frac{1}{1-z}} q^{\frac{2}{1-z}} \\
&\quad\times\left(\fint_{B(y,r_j)} (\theta_j v^q)^{(1-z)\cdot\frac{2q}{q-1}\cdot\frac{a}{a-1}}\right)^{\frac{(q-1)(a-1)}{q^{a(1-z)}}}.
\end{aligned}$$

Taking $z = \frac{m}{2q}$, and $a = \frac{2q-2}{n-2}$, it follows

$$70q^2 \fint_{B(y,r_j)} \theta_j^2 v^{2q-1}h^2 \le \epsilon\left(\fint_{B(y,r_j)} (\theta_j v^q)^{\frac{2n}{m-2}}\right)^{\frac{m-2}{m}} + \epsilon^{-\frac{m}{2q-m}} C^{\frac{2q}{2q-m}} q^{\frac{4q}{2q-m}} \fint_{B(y,r_j)} \theta_j^2 v^{2q}. \tag{3.9}$$

Uncorrected Proof

Similarly, for $q \in (\frac{m}{2}, \frac{m}{2\alpha})$, we have

$$30q^2 \oint_{B(y,r_j)} \theta_j^2 v^{2q}|Y|^2 \le \epsilon r_j^{-2\alpha}\Big(\oint_{B(y,r_j)} (\theta_j v^q)^{\frac{2n}{m-2}}\Big)^{\frac{m-2}{m}} + \epsilon^{-\frac{m}{2q-m}} q^{\frac{4q}{2q-m}} C^{\frac{2q}{2q-m}} r_j^{-2\beta} \oint_{B(y,r_j)} \theta_j^2 v^{2q}, \tag{3.10}$$

and

$$2qm_2 \oint_{B(y,r_j)} \theta_j^2 v^{2q}|Y|^2 \le \Big[\epsilon r_j^{-2\beta}\Big(\oint_{B(y,r_j)} (\theta_j v^q)^{\frac{2n}{m-2}}\Big)^{\frac{m-2}{m}} + \epsilon^{-\frac{m}{2q-m}} q^{\frac{2q}{2q-m}} C_1^{\frac{2q}{2q-m}} r_j^{-2\beta} \oint_{B(y,r_j)} \theta_j^2 v^{2q}\Big]. \tag{3.11}$$

Here $C_1 = C_1(m, K, \beta, \rho, n_1, n_2)$. From (3.8), the left-hand side is bounded by the sum of the terms containing

$$|\nabla\theta_j|^2, \qquad \theta_j^2 v^{2q-1} h^2, \qquad \theta_j^2 v^{2q}|Y|^2, \qquad \text{and} \qquad \theta_j^2 v^{2q-1}.$$

The term involving h^2 is estimated by the volume comparison theorem and Hölder's inequality. More precisely, since

$$\frac{r}{2} \le r_j \le \frac{3r}{4},$$

we have

$$\int_{B(y,r_j)} \theta_j^2 v^{2q-1} h^2 \le C r_j^{-2\beta} \left(\int_{B(y,r_j)} (\theta_j v^q)^{\frac{2n}{m-2}}\right)^{\frac{m-2}{m}},$$

after applying Young's inequality with

$$z = \frac{m}{2q}, \qquad a = \frac{2q-2}{n-2}.$$

The same argument, applied to the terms involving $|Y|^2$, gives the estimates in (3.9) and (3.10). Inserting (3.9), (3.10) and (3.11) in (3.8), and applying Sobolev inequality, it follows that

$$\begin{aligned}
\Big(\oint_{B(y,r_j)} (\theta_j v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} &\le C(m) r_j^2 \oint_{B(y,r_j)} |\nabla(\theta_j v^q)|^2 \\
&\le C(m) r_j^2 \Big[16q v^{2q}|\nabla\theta_j|^2 + 8q(n_1+n_3)\theta_j^2 v^{2q} + 2qn_2\theta_j^2 v^{2q-1} \\
&+ C(n)\epsilon r_j^{2-2\alpha}\Big(\oint_{\Omega} (\theta v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} \\
&+ C(m)\epsilon^{-\frac{z}{1-z}} C^{\frac{2q}{2q-m}} q^{\frac{4q}{2q-m}} r_j^{2-2\alpha} \oint_{B(y,r_j)} \theta_j^2 v^{2q} \\
&+ C(m) r_j^2 \epsilon \Big(\oint_{B(y,r_j)} (\theta_j v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} \\
&+ C(m) r_j^2 \epsilon^{-\frac{z}{1-z}} \big(q^{\frac{4q}{2q-m}} + q^{\frac{2q}{2q-m}}\big) \oint_{B(y,r_j)} \theta_j^2 v^{2q}.
\end{aligned} \tag{3.12}$$

Choosing a sufficiently small $\varepsilon = \varepsilon(m)$ and noting that $\beta < 1$ and $r_j \le r \le 1$ the terms on the right-hand side involving negative powers of the radius can be absorbed into the general constant C. This leads:

$$\Big(\oint_{B(y,r_j)} (\theta_j v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} \le C(m, K, \beta, \rho, n_1, n_2, n_3) r_j^2 \oint_{B(y,r_j)} q v^{2q}|\nabla\theta_j|^2 + q\theta_j^2 v^{2q} + q\theta_j^2 v^{2q-1}. \tag{3.13}$$

Uncorrected Proof

Again Volume comparison Theorem 2.3 leads to the following

$$\Big(\fint_{B(y,r_{j+1})}(v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} \le C(m,K,\beta,\rho,n_1)\Big(\fint_{B(y,r_j)}(\theta_j v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}}$$
$$\le C(m,K,\beta,\rho,n_1,n_2,n_3)\fint_{B(y,r_j)} 2^{2i}qv^{2q}+2qv^{2q}.$$

By selecting $q=\frac{\mu^j}{2}$ for $j=0,1,2,\cdots$, where $\mu=\frac{m}{m-2}$, we get

$$\Big(\fint_{B(y,r_{j+1})}v^{\mu^{j+1}}\Big)^{\frac{m-2}{m}} = \Big(\fint_{B(y,r_{j+1})}(v^q)^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}}$$
$$\le C(m,K,\beta,\rho,n_1,n_2,n_3)(2^{2j-1}\mu^j+2\mu^j)\fint_{B(y,r_j)}v^{\mu^j}$$
$$\le C(m,K,\beta,\rho,n_1,n_2,n_3)2(2^{2j-2}+1)2^{2j}\fint_{B(y,r_j)}v^{\mu^j},$$

here we apply $\mu\le 3$. So

$$\|v\|^*_{\mu^{j+1},B(y,r_{j+1})} \le C^{\mu^{-j}}(2^{4j-1}+2^{2j+1})^{\mu^{-j}}\|v\|^*_{\mu^j,B(y,r_j)}. \tag{3.14}$$

Iterating this inequality from $j=0$ to $j=N$, we obtain

$$\|v\|^*_{\mu^{N+1},B(y,r_{N+1})} \le C^{\sum_{j=0}^N \mu^{-j}}\prod_{j=0}^N\big(2^{4j-1}+2^{2j+1}\big)^{\mu^{-(j+1)}}\|v\|^*_{1,B(y,r)}.$$

Because $\mu>1$, the geometric series $\sum_{j=0}^\infty \mu^{-j}$, converges, and the product on the right-hand side is bounded by a constant depending only on $m,K,\beta,\rho,n_1,n_2,n_3$. Letting $N\to\infty$, we have

$$\mu^{N+1}\to\infty, \qquad r_{N+1}\to\frac{r}{2}.$$

Applying the Nash-Moser iteration described above, we conclude

$$\sup_{B(y,\frac{r}{2})} v \le C(m,K,\beta,\rho,n_1,n_2,n_3)\|v\|^*_{1,B(y,\frac{3r}{4})}. \tag{3.15}$$

Since

$$\int_\Omega \theta^2|\nabla w|^2 = \int_\Omega -\theta^2 w(h+A(w))+2\theta w\nabla w\nabla\theta$$
$$\le \int_\Omega \frac{1}{2}w^2\theta^2+\frac{1}{2}h^2\theta^2+\theta^2 n_2+\theta^2|\nabla w|^2+w^2|\nabla\theta|^2.$$

This leads

$$\fint_\Omega \theta^2|\nabla w|^2 \le 100r^{-2}(\|w\|^*_{2,\Omega})^2+4\|h^2\|^*_{q,\Omega}+n_2.$$

Thus

$$\|v\|^*_{1,B(y,\frac{3}{4}r)} \le \frac{Vol(\Omega)}{Vol(B(y,\frac{3}{4}r))}\fint_\Omega \theta^2(|\nabla w|^2+\|h^2\|^*_{q,\Omega})$$
$$\le C(m,K,\beta,\rho,n_1,n_2,n_3)\big[r^{-2}(\|w\|^*_{2,\Omega})^2+(\|h\|^*_{2q,\Omega})^2\big].$$

Uncorrected Proof

Combining (3.15) with the above equation, we have

$$\sup_{B(x,\frac{1}{2}r)} |\nabla w|^2 \le \|v\|_{\infty,B(x,\frac{1}{2}r)} \le C\big[r^{-2}(\|w\|^*_{2,\Omega})^2 + (\|h\|^*_{2q,\Omega})^2\big],$$

here $C = C(m, K, \beta, \rho, n_1, n_2, n_3)$. This completes the proof. □

Remark 3.2. Note that the same results hold without non-collapsing condition when $\alpha = 0$.

Corollary 3.3. *Consider a gradient ARS satisfying the subsequent requirement:*

$$\mathrm{Ric} + Hessf \ge -\alpha g.$$

Moreover,

$$|f(y) - f(z)| \le K_1 d(y,z)^\beta, \text{ and } \sup_{y\in M, 0\le r\le 1} \big(r^\gamma \|\nabla f\|^*_{q,\Omega}\big) \le K_2.$$

Then there exist a constant $C = C(n, K_1, K_2, \beta, \gamma, m_1, m_2, m_3)$, such that for the solution of (1.3), we have

$$\sup_{B(y,\frac{r}{2})} |\nabla w|^2 \le C(m, K_1, K_2, \beta, \gamma, n_1, n_2, n_3)\big[r^{-2}(\|w\|^*_{2,\Omega})^2 + (\|f\|^*_{2q,\Omega})^2\big],$$

for any $q > \frac{n}{2}$.

Remark 3.4. The estimate is local: it converts integral control of a bounded solution and its source into pointwise control of its gradient on a smaller ball. In particular, families of solutions with uniformly bounded normalized L^2-norms and source terms in uniformly bounded L^{2q}-balls are locally equi-Lipschitz under the stated geometric hypotheses. Also, the gradient estimates established in Theorem 3.1 have several potential applications in geometric analysis:

Liouville-type theorems: Under suitable growth conditions on u and h, the estimate (3.1) can be used to prove that bounded solutions of (1.3) must be constant on complete non-compact almost Ricci solitons. Indeed, if $\|w\|^*_{2,\Omega}$ and $\|h\|^*_{2q,\Omega}$ remain bounded as $r \to \infty$, then the gradient estimate forces $\nabla w \equiv 0$.

Stability of solutions: The gradient estimate provides quantitative control on the variation of solutions, which is useful in establishing compactness and stability results for families of solutions on varying almost Ricci soliton backgrounds.

Geometric evolution equations: Equation (1.3) arises naturally in the study of various geometric flows, including the Ricci flow and its variants. The gradient estimates obtained here can be applied to analyze the long-time behavior of solutions to these flows.

Comparison principles: The explicit dependence of the constant on the geometric parameters of the almost Ricci soliton allows for comparison of solutions on different manifolds, facilitating the study of stability and rigidity phenomena.

Uncorrected Proof

4. Conclusion

In this study, we established local gradient estimates for positive bounded solutions of the nonlinear equation on almost Ricci solitons. By employing Moser iteration techniques and geometric inequalities, we derived in Theorem 3.1 an effective upper bound for the gradient, providing a robust tool for analyzing the geometric rigidity and Liouville-type properties of these manifolds. While our results characterize the equi-Lipschitz control under specified curvature and potential function constraints, they highlight the sensitivity of these estimates to the boundedness of the solution and the non-collapsing condition. These findings provide a foundational basis for future investigations, particularly in extending the current framework to parabolic operators or exploring solution behavior in the presence of singular points, offering promising directions for further research in geometric analysis.

Acknowledgment

The authors would like to express their sincere gratitude to the Editor and the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly helped to improve the quality, clarity, and overall strength of this paper.

References

[1] S. Azami and S. Hajiaghasi, *New volume comparison with almost Ricci soliton*, Communications of the Korean Mathematical Society, *37*(3) (2022), 839-849.
[2] A. Barros, R. Batista, and J. E. Ribeiro, *Rigidity of gradient almost Ricci soliton*, Illinois Journal of Mathematics, *56*(4) (2012), 1267-1279.
[3] A. Barros and E. Ribeiro Jr, *Some characterizations for compact almost Ricci solitons*, Proc. Am. Math. Soc., *140* (2012), 213-223.
[4] G. Catino, *Generalized quasi-Einstein manifold with harmonic weyl tensor*, Math. Z., *271* (2012), 751-756.
[5] G. Catino and D. D. Monticelli, *Semilinear elliptic equations on manifolds with nonnegative Ricci curvature*, Journal of the European Mathematical Society, *28*(1) (2024), 359-392.
[6] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., *28*(3) (1975), 333-354.
[7] X. Dai, G. Wei, and Z. Zhang, *Local sobolev constant estimate for integral Ricci curvature bounds*, Adv. Math., *325*(2018), 1-33.
[8] S. Deshmukh, H. Alsodais, and N. Bin Turki, *Some results on Ricci almost solitons*, Symmetry, *13*(3) (2021), 430.
[9] S. Deshmukh and H. Alsodais, *A note on almost Ricci soliton*, Analysis and Mathematical Physics, *10*(4) (2020), 76.
[10] S. Deshmukh, *Almost Ricci solitons isometric to spheres*, International Journal or Geometric Methods in Modern Physics, *16*(5) (2019) 1950073.
[11] S. Hajiaghasi and S. Azami, *A Gradient Bound for the Allen-Cahn Equation Under Almost Ricci Solitons*, Sahand Commun. Math. Anal., *21*(1) (2024), 83-98.
[12] S. Hajiaghasi and S. Azami, *Gradient estimates for a nonlinear equation under the almost Ricci soliton condition*, Computational Methods for Differential Equations, *12*(4) (2024), 710-718.
[13] S. Hajiaghasi and S. Azami, *Gradient estimate of Poisson equation under the almost Ricci solitons*, Boundary Value Problems, *2022*(1) (2022), 104.
[14] Q. Han and F. Li, *Elliptic partial differential equation*, American Mathematical Society, *1* (2011).
[15] W. Hebisch and L. Saloff-Costa, *On the relation between elliptic and parabolic Harnack inequations*, Annales de l'institut Fouriere, *51*(5) (2001), 1437-1481.
[16] P. Petersen and G. F. Wei, *Analysis and geometry on manifolds with integral Ricci curvature bounds*,Transactions of the American Mathematical Society, *353* (2000), 457-478.
[17] S. Pigola, M. Rigoli, M. Rimoldi and A. G. Setti, *Ricci almost solitons*, Annali della Scuola normale superiore di Pisa-Classe di scienze, *10*(4) (2011), 757-799.
[18] W. Tokara, L. Adriano, R. Pina and M. Barboza, *Gradient estimates on warped product gradient almost Ricci solitons*, arXiv preprint arXiv:1905.00068 (2019).
[19] Y. Wang, *Gradient estimates and Liouville theorems for a class of semilinear elliptic equations*, Communications on Pure and Applied Analysis, *24*(2) (2025), 315-330.
[20] Q. S. Zhang and M. Zhu, *New volume comparison results and applications to degeneration of Riemannian metrics*, Advances in Mathematics, *352* (2019), 1096-1154.

Uncorrected Proof