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Revisiting the viability problem for fractional differential equations: Corrections and sharp criteria

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Abstract

We revisit the viability problem for Caputo fractional differential equations with closed convex constraints. First, we construct a memory-preserving Euler approximation with a fixed lower terminal and derive a uniform distance estimate showing that the Euler trajectories approach the constraint set as the mesh tends to zero; compactness and consistency then yield a viable integral solution. This repairs the stepwise Euler philosophy used in early fractional viability arguments without discarding the Caputo prehistory. Second, we give an independent direct proof based on a convex fractional distance estimate, which yields the sharp nonincrease estimate for the distance to the constraint set and does not use the Euler approximation. The applicability of the criterion is illustrated by fractional population, potential-driven, and electrical circuit models.

Keywords. Caputo fractional differential equation, Viability, Projection inwardness condition.
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Uncorrected Proof

1. Introduction

We address viability for the initial value problem

$$D^q_{t_0} y(t) = f(t, y(t)), \quad y(t_0) = x_0, \tag{1.1}$$

where $q \in (0,1)$, $t_0 \in [a,b)$, $D^q_{t_0}$ denotes the Caputo fractional derivative of order q, and $f : [a,b) \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

The study of viability for differential equations seeks to determine when an evolution can remain within a given constraint set. Formally, for a given $\mathcal{G} = [a,b) \times G$ with $G \subset \mathbb{R}^n$, we say that $\mathcal{G}$ is viable with respect to (1.1) if, for every initial datum in $\mathcal{G}$, there exists at least one solution whose trajectory remains in G for a nontrivial time interval. This notion plays a fundamental role in control theory and constrained dynamics.

In the classical setting of ordinary differential equations (ODEs), i.e., when $q = 1$, the viability problem has been extensively studied. Under suitable regularity assumptions, viability can be characterized by a geometric tangency condition; modern presentations of this classical framework can be found in [1, 3].

For systems governed by fractional differential equations, the situation is significantly more intricate. The fractional derivative introduces nonlocal, memory-dependent behavior that prevents a direct extension of many classical

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arguments. In particular, the Caputo derivative incorporates the accumulated history of the system, creating both analytical challenges and additional modeling flexibility.

The literature on fractional extensions of classical viability criteria remains relatively limited. In particular, the approaches in [6, 15] treat the dynamics essentially as local and therefore do not fully account for the memory effects intrinsic to fractional models; as discussed in [2], this creates difficulties in the formulation of the viability problem.

The study in [2] provides a viability result for fractional differential inclusions, but its tangency condition is implicit and depends on the unknown solution itself, which makes the condition difficult to verify and apply in practice.

A more tractable approach was proposed in [4], where the authors established viability for a specific class of fractional systems; however, the result remains restricted to that particular setting. Related existence and viability results for fractional differential equations with initial conditions at inner points were obtained in [5]. The memory-dependent formulation was further emphasized in the memo-viability analysis of Girejko, Mozyrska, and Wyrwas [7]. More recently, viability has also been investigated for time-fractional functional equations driven by fractional Brownian motion [9]. These developments confirm that the appropriate treatment of the past history remains central to fractional viability theory.

The broader literature on fractional differential equations has also developed rapidly. Recent studies include numerical and qualitative analyses of fractional epidemic models [16], Ulam-type stability for ψ–Hilfer fractional integro-delay equations [14], and stability and boundedness results for generalized Caputo proportional fractional integro-differential systems with delays [13]. Although these papers address different questions, they illustrate the continuing importance of memory, stability, and qualitative behavior in modern fractional models.

In the present paper, we revisit these issues and formulate a directly verifiable viability criterion adapted to the Caputo setting. The principal contributions are as follows:

- We construct a memory-preserving Euler approximation with one fixed lower terminal t_0 and derive its nonlocal nodal recursion, so that the complete Caputo history is retained at every step.
- We derive a uniform distance estimate for the corrected Euler trajectories, with an error tending to zero with the mesh, and use compactness and consistency to obtain a viable integral limit.
- We give a second, independent proof of the same viability criterion by a convex fractional distance argument applied directly to an exact integral solution; this proof yields a sharp distance estimate and does not use Euler approximations.
- We explain how the first proof repairs the Euler philosophy of early fractional viability arguments and compare the construction with earlier approaches. The criterion is illustrated through fractional population, potential-driven, and resistor–capacitor/constant-phase-element (RC/CPE) circuit models.

A detailed comparison with earlier works is presented in Section 5.

The paper is organized as follows. Section 2 introduces the notation, the basic concepts of fractional calculus, the geometric tools, and the convex fractional distance estimate used later. Section 3 develops the memory-preserving Euler approximation, proves its compactness and consistency, and establishes a uniform distance estimate for the Euler trajectories. Section 4 states the viability criterion and proves it in two different ways: first through the corrected Euler construction and then by a sharp fractional distance estimate. Section 5 compares the construction with earlier approaches, and Section 6 presents an application to a fractional RC/CPE circuit.

Uncorrected Proof

2. Preliminaries

This section collects the definitions and preliminary facts used in the sequel. Standard references for fractional calculus include [10, 11], while the geometric notions related to projections and viability can be found in [1, 3].

2.1. Fractional calculus.

Definition 2.1 (see [10, 11])**.** The Riemann–Liouville fractional integral of order q of a function $y \in L^1(t_0, T; \mathbb{R}^n)$ is defined by

$$(I^q_{t_0} y)(t) := \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} y(s)\, ds, \tag{2.1}$$

for every $t \in [t_0, T]$.

Here Γ denotes the gamma function, defined by

$$\Gamma(u) = \int_0^\infty s^{u-1} e^{-s}\, ds, \qquad \Re(u) > 0\,. \tag{2.2}$$

Definition 2.2 (see [10, 11]). Let $y : [t_0, T] \to \mathbb{R}^n$ be absolutely continuous. The Caputo fractional derivative of order q is defined by

$$(D^q_{t_0} y)(t) := (I^{1-q}_{t_0} y')(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} y'(s)\, ds, \tag{2.3}$$

for every $t \in (t_0, T]$.

We first recall the classical composition property between the Riemann–Liouville integral and the Caputo derivative.

Proposition 2.3 (see [10]). *If $y : [t_0, T] \to \mathbb{R}^n$ is absolutely continuous, then*

$$I^q_{t_0} \left(D^q_{t_0} y\right)(t) = y(t) - y(t_0), \tag{2.4}$$

for all $t \in [t_0, T]$.

For the trajectories used later, it is convenient to extend this notation to the fractional-integral class required by the Euler construction and by the convex fractional Lyapunov estimates of [8]. Let $\psi \in L^\infty(t_0, T; \mathbb{R}^n)$ and suppose that $y(t) - y(t_0) = (I^q_{t_0}\psi)(t)$. Then $y - y(t_0) \in I^q_{t_0} L^\infty(t_0, T; \mathbb{R}^n)$, and we write, in the corresponding integral Caputo sense,

$$D^q_{t_0} y(t) = \psi(t) \quad \text{for a.e. } t \in [t_0, T]. \tag{2.5}$$

Consequently, $I^q_{t_0} D^q_{t_0} y(t) = y(t) - y(t_0)$. Whenever y has the regularity required in the preceding definition, this integral interpretation agrees with the classical Caputo derivative. In particular, it applies to the Volterra representations used below, including the Euler approximations for which the corresponding forcing is piecewise constant.

Definition 2.4 (see [10, 11]). Let f be continuous. A continuous function $y : [t_0, T] \to \mathbb{R}^n$ is called an integral solution of problem (1.1) if

$$y(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} f(\tau, y(\tau))\, d\tau, \qquad t \in [t_0, T]. \tag{2.6}$$

Equivalently, $y - x_0 \in I^q_{t_0} C([t_0, T]; \mathbb{R}^n)$ and, in the integral Caputo sense described above,

$$D^q_{t_0} y(t) = f(t, y(t)). \tag{2.7}$$

Throughout this paper, unless otherwise specified, the term *solution* of problem (1.1) means an integral solution in the preceding sense.

2.2. **Geometric notions.** The following elementary facts concerning metric projections are standard; see, for example, [1, 3]. The Euclidean space $\mathbb{R}^n$ is endowed with the standard inner product $\langle \cdot, \cdot \rangle$, which defines the standard Euclidean norm $\| \cdot \|$. Accordingly, the distance from a vector $\mathbf{v} \in \mathbb{R}^n$ to a subset $G \subset \mathbb{R}^n$ is defined by

$$\operatorname{dist}(\mathbf{v}, G) := \inf_{\mathbf{g} \in G} \|\mathbf{v} - \mathbf{g}\|. \tag{2.8}$$

If $x \in \mathbb{R}^n$ and $\rho > 0$, then $B(x, \rho)$ denotes the closed ball

$$B(x, \rho) = \{z \in \mathbb{R}^n; \|z - x\| \le \rho\}. \tag{2.9}$$

If $G \subset \mathbb{R}^n$ is nonempty and closed, then, for each $x \in \mathbb{R}^n$, a point $z \in G$ is called a projection of x onto G if

$$\|x - z\| \le \|x - g\| \quad \text{for all } g \in G. \tag{2.10}$$

If the set G is also *convex*, then the projection is *unique*, and the projection map $P_G : \mathbb{R}^n \to G$ is well-defined. In that case, the projection $z = P_G(x)$ satisfies the following variational inequality:

$$\langle x - z, g - z \rangle \le 0, \quad \text{for all } g \in G. \tag{2.11}$$

Uncorrected Proof

2.3. **A convex fractional distance estimate.** We record the following convex fractional distance estimate, which will be used in the sequel. For related fractional Lyapunov methods, see [12].

Lemma 2.5 (Convex fractional distance estimate)**.** *Let $G \subset \mathbb{R}^n$ be nonempty, closed and convex, and define $\Phi : \mathbb{R}^n \to [0,\infty)$ by*

$$\Phi(x) := \frac{1}{2}\operatorname{dist}(x,G)^2, \qquad x \in \mathbb{R}^n. \tag{2.12}$$

Then Φ is convex and continuously differentiable, with $\nabla\Phi(x) = x - P_G(x)$ for every $x \in \mathbb{R}^n$. Let $z : [t_0,T] \to \mathbb{R}^n$ satisfy $z(t) = z_0 + (I^q_{t_0} v)(t)$ with $v \in L^\infty(t_0,T;\mathbb{R}^n)$. Then $\Phi(z(\cdot)) - \Phi(z_0) \in I^q_{t_0} L^\infty(t_0,T;\mathbb{R})$ and, in the integral Caputo sense introduced above,

$$D^q_{t_0}\big[\Phi(z(\cdot)) - \Phi(z_0)\big](t) \le \langle z(t) - P_G(z(t)), v(t)\rangle\,, \tag{2.13}$$

for almost every $t \in [t_0,T]$.

Proof. For a nonempty closed convex subset of $\mathbb{R}^n$, the squared distance function is convex and continuously differentiable and satisfies $\nabla\Phi = I - P_G$; see, for instance, [3]. Moreover, $I - P_G$ is nonexpansive, so $\nabla\Phi$ is globally Lipschitz.

Set $w(t) := z(t) - z_0 = (I^q_{t_0} v)(t)$ and $V_{z_0}(\xi) := \Phi(z_0+\xi) - \Phi(z_0)$. Then V_{z_0} is convex and continuously differentiable, $V_{z_0}(0) = 0$, and $\nabla V_{z_0}(\xi) = \nabla\Phi(z_0+\xi)$; in particular, its gradient is globally Lipschitz.

To apply the result of [8] in its original form, translate the initial time to zero. For $\tau \in [0, T-t_0]$, define $\widetilde{w}(\tau) := w(t_0+\tau)$ and $\widetilde{v}(\tau) := v(t_0+\tau)$. Then $\widetilde{w} = I^q_0 \widetilde{v}$, with $\widetilde{v} \in L^\infty(0,T-t_0;\mathbb{R}^n)$. Lemma 3.1 of [8], applied to $\widetilde{w}$ and V_{z_0}, yields $V_{z_0}(\widetilde{w}(\cdot)) \in I^q_0 L^\infty(0,T-t_0;\mathbb{R})$ and

$$D^q_0 V_{z_0}(\widetilde{w}(\tau)) \le \langle \nabla V_{z_0}(\widetilde{w}(\tau)), \widetilde{v}(\tau)\rangle \quad \text{for a.e. } \tau \in [0,T-t_0]. \tag{2.14}$$

Translating back to $[t_0,T]$ gives $V_{z_0}(w(\cdot)) \in I^q_{t_0} L^\infty(t_0,T;\mathbb{R})$ and the corresponding inequality in the integral Caputo sense. Since $V_{z_0}(w(t)) = \Phi(z(t)) - \Phi(z_0)$ and $\nabla V_{z_0}(w(t)) = z(t) - P_G(z(t))$, we obtain (2.13). □

3. Memory-Preserving Euler Approximation for the Fractional Initial Value Problem

The construction developed in this section is inspired by the Euler approximation philosophy used in viability theory and by the Volterra integral representation of Caputo fractional differential equations. Its purpose is not to introduce a new notion of solution, but to construct an approximation that preserves the complete fractional history from the fixed lower terminal t_0. To this end, we approximate the right-hand side by left-endpoint values without restarting the fractional derivative at the intermediate nodes.

Let $\pi = \{t_0, t_1, \dots, t_N\}$ be a partition of $[t_0,T]$, where $t_0 < t_1 < \cdots < t_N = T$. Its mesh size is denoted by $\mu_\pi := \max\{t_i - t_{i-1} : 1 \le i \le N\}$.

We initiate the construction on the first subinterval $[t_0,t_1]$ by setting

$$D^q_{t_0} y_\pi(t) = f(t_0,x_0), \qquad y_\pi(t_0) = x_0. \tag{3.1}$$

Since the right-hand side is constant, the corresponding integral solution is

$$y_\pi(t) = x_0 + \frac{f(t_0,x_0)}{\Gamma(1+q)}(t-t_0)^q, \qquad t \in [t_0,t_1]. \tag{3.2}$$

Hence, $x_1 := y_\pi(t_1) = x_0 + \dfrac{f(t_0,x_0)}{\Gamma(1+q)}(t_1-t_0)^q$.

On the second interval, the lower terminal remains equal to t_0 and the contribution generated on $[t_0,t_1]$ must be retained. Introduce, on $[t_0,t_2)$, the piecewise-constant forcing

$$g_\pi(s) = \begin{cases} f(t_0,x_0), & s \in [t_0,t_1), \\ f(t_1,x_1), & s \in [t_1,t_2). \end{cases} \tag{3.3}$$

Uncorrected Proof

Then, for $t \in [t_1, t_2]$, we define

$$y_\pi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g_\pi(s)\, ds, \tag{3.4}$$

that is,

$$y_\pi(t) = x_0 + \frac{1}{\Gamma(q)} \left[\int_{t_0}^{t_1} (t-s)^{q-1} f(t_0, x_0)\, ds + \int_{t_1}^{t} (t-s)^{q-1} f(t_1, x_1)\, ds \right]. \tag{3.5}$$

Consequently,

$$y_\pi(t) = x_0 + \frac{1}{\Gamma(q+1)} \Big[\big((t-t_0)^q - (t-t_1)^q\big) f(t_0, x_0) + (t-t_1)^q f(t_1, x_1) \Big], \tag{3.6}$$

and therefore

$$x_2 := y_\pi(t_2) = x_0 + \frac{1}{\Gamma(q+1)} \sum_{j=0}^{1} \Big[(t_2 - t_j)^q - (t_2 - t_{j+1})^q \Big] f(t_j, x_j). \tag{3.7}$$

Proceeding inductively, let $x_i := y_\pi(t_i)$ for $i = 0, 1, \ldots, N$. Define the piecewise-constant functions $\theta_\pi(s) = t_j$ and $\widehat{y}_\pi(s) = x_j$ for $s \in [t_j, t_{j+1})$, and set $g_\pi(s) := f\big(\theta_\pi(s), \widehat{y}_\pi(s)\big)$. The Euler approximation is then defined globally by the Volterra representation

$$\begin{aligned} y_\pi(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g_\pi(s)\, ds \\ &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f\big(\theta_\pi(s), \widehat{y}_\pi(s)\big)\, ds. \end{aligned} \tag{3.8}$$

For $t \in [t_i, t_{i+1}]$, splitting the integral over the preceding subintervals gives the explicit representation

$$y_\pi(t) = x_0 + \frac{1}{\Gamma(q+1)} \left[\sum_{j=0}^{i-1} \Big((t-t_j)^q - (t-t_{j+1})^q \Big) f(t_j, x_j) + (t-t_i)^q f(t_i, x_i) \right], \tag{3.9}$$

where the sum is understood to be zero when $i = 0$.

In particular, evaluating at $t = t_{i+1}$ yields the nonlocal nodal recursion

$$x_{i+1} = x_0 + \frac{1}{\Gamma(q+1)} \sum_{j=0}^{i} \Big[(t_{i+1} - t_j)^q - (t_{i+1} - t_{j+1})^q \Big] f(t_j, x_j). \tag{3.10}$$

Thus, unlike the classical Euler recursion, the new nodal value x_{i+1} depends on the entire discrete history $x_0, \ldots, x_i$, which reflects the nonlocal character of the Caputo derivative.

Since $g_\pi \in L^\infty([t_0, T]; \mathbb{R}^n)$, the representation (3.8) shows that $y_\pi - x_0 = I^q_{t_0} g_\pi$. Hence, in the integral Caputo sense introduced in Section 2, $D^q_{t_0} y_\pi(t) = g_\pi(t)$ for a.e. $t \in [t_0, T]$. Thus the approximation preserves the fixed lower terminal t_0 and the complete fractional history.

Remark 3.1. The preceding construction corrects the Euler procedure used in the fractional viability approach of Vasundhara Devi and Lakshmikantham [15]. In that construction, on each subinterval $[t_i, t_{i+1}]$, the Caputo derivative is restarted with lower terminal t_i. This reinitialization does not retain the contribution of the previous intervals and therefore does not preserve the memory of the original problem with fixed lower terminal t_0. In contrast, the present construction keeps the lower terminal t_0 fixed on the whole interval and incorporates the complete preceding history through the nonlocal nodal recursion (3.10).

We can now formalize the preceding construction.

Definition 3.2. Let $\pi = \{t_0, t_1, \ldots, t_N\}$ be a partition of $[t_0, T]$. A function $y_\pi : [t_0, T] \to \mathbb{R}^n$ is called an *Euler approximate solution* of (1.1) associated with π if its nodal values $x_i := y_\pi(t_i)$ are generated by the recursion (3.10), and y_π is given on each interval $[t_i, t_{i+1}]$ by (3.9), equivalently by the global Volterra representation (3.8).

Uncorrected Proof

The following proposition establishes the relative compactness of the Euler-approximate solutions to (1.1).

Proposition 3.3. *Assume that f satisfies the linear growth condition*

$$\|f(t,y)\| \le L(1+\|y\|), \qquad (t,y) \in [a,b) \times \mathbb{R}^n, \tag{3.11}$$

for some $L > 0$. Then the family of Euler-approximate solutions associated with all partitions of $[t_0, T]$ is relatively compact in the space of continuous functions from $[t_0, T]$ into $\mathbb{R}^n$. In particular, every sequence of Euler-approximate solutions admits a uniformly convergent subsequence.

Proof. We consider the family of Euler-approximate solutions associated with partitions π and use the integral representation (3.8).

Step 1: Uniform boundedness. Choose $\delta > 0$ such that $c_\delta := L\delta^q/\Gamma(q+1) < 1$. Let $t_0 = s_0 < s_1 < \cdots < s_K = T$ be a finite subdivision satisfying $s_{m+1} - s_m \le \delta$. On the first interval, set $U_1 := \sup\{\|y_\pi(t)\| : t_0 \le t \le s_1\}$. Since every value entering $\widehat{y}_\pi(s)$ is a nodal value of y_π at an earlier time, the Volterra representation and the linear growth condition yield

$$U_1 \le \|x_0\| + c_\delta(1 + U_1). \tag{3.12}$$

Hence $U_1 \le (\|x_0\| + c_\delta)/(1 - c_\delta) =: M_1$, where M_1 is independent of the partition.

Assume that a partition-independent bound M_m has been obtained on $[t_0, s_m]$. For $t \in [s_m, s_{m+1}]$, split the integral in (3.8) at s_m. The contribution of $[t_0, s_m]$ is bounded by $H_m := L(1 + M_m)(s_m - t_0)^q/\Gamma(q+1)$, whereas the contribution over $[s_m, t]$ is bounded by $c_\delta(1 + U_{m+1})$, where $U_{m+1} := \sup\{\|y_\pi(t)\| : t_0 \le t \le s_{m+1}\}$. Consequently,

$$U_{m+1} \le \max\left\{M_m,\ \|x_0\| + H_m + c_\delta(1 + U_{m+1})\right\}, \tag{3.13}$$

and therefore

$$U_{m+1} \le \max\left\{M_m,\ \frac{\|x_0\| + H_m + c_\delta}{1 - c_\delta}\right\} =: M_{m+1}. \tag{3.14}$$

Since only finitely many intervals are required to cover $[t_0, T]$, the induction yields a constant $M > 0$, independent of the partition, such that

$$\|y_\pi(t)\| \le M, \qquad t \in [t_0, T]. \tag{3.15}$$

In particular, $\|x_i\| \le M$ for every node t_i.

Step 2: Equicontinuity. By (3.15) and the linear growth condition, with $C_f := L(1 + M)$ we have

$$\|f(\theta_\pi(s), \widehat{y}_\pi(s))\| \le C_f, \qquad s \in [t_0, T]. \tag{3.16}$$

Let $t > s$. From (3.8),

$$\|y_\pi(t) - y_\pi(s)\| \le \frac{C_f}{\Gamma(q)} \int_{t_0}^{s} \left((s-r)^{q-1} - (t-r)^{q-1}\right) dr + \frac{C_f}{\Gamma(q)} \int_s^t (t-r)^{q-1}\, dr. \tag{3.17}$$

Since $0 < q < 1$, direct integration gives

$$\|y_\pi(t) - y_\pi(s)\| \le \frac{2C_f}{\Gamma(q+1)} |t - s|^q. \tag{3.18}$$

Thus the family $(y_\pi)_\pi$ is uniformly bounded and equicontinuous on $[t_0, T]$. The Arzelà–Ascoli theorem therefore implies relative compactness in $C([t_0, T]; \mathbb{R}^n)$, and every sequence of Euler approximate solutions admits a uniformly convergent subsequence. □

Theorem 3.4 (Consistency of the memory-preserving Euler approximation)**.** *Let $f : [a,b) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy the linear growth condition. Let $(\pi_j)_j$ be a sequence of partitions of $[t_0, T]$ whose mesh sizes μ_j tend to zero, and let $(y_j)_j$ be the corresponding Euler approximate solutions. Then $(y_j)_j$ is relatively compact in $C([t_0, T]; \mathbb{R}^n)$, and every uniformly convergent subsequence converges to an integral solution of (1.1).*

Uncorrected Proof

Proof. Relative compactness follows from the preceding proposition. After passing to a subsequence if necessary,

$$y_j \longrightarrow y \qquad \text{uniformly on } [t_0, T]. \tag{3.19}$$

For $s \in [t_i^j, t_{i+1}^j)$, define $\theta_j(s) = t_i^j$ and $\widehat{y}_j(s) = y_j(t_i^j)$, and set $\theta_j(T) := T$, $\widehat{y}_j(T) := y_j(T)$. By the uniform boundedness established in the preceding proposition, all y_j take their values in a fixed bounded set. Moreover,

$$\sup_{s\in[t_0,T]} \|\widehat{y}_j(s) - y(s)\| \le \|y_j - y\|_\infty + \omega_y(\mu_j), \tag{3.20}$$

where ω_y is the modulus of continuity of y. Hence $\widehat{y}_j \to y$ uniformly and $\sup_{s\in[t_0,T]} |\theta_j(s) - s| \le \mu_j \to 0$.

All pairs $(\theta_j(s), \widehat{y}_j(s))$ and $(s, y(s))$ belong to a compact set $K = [t_0, T] \times \overline{B}(0, M)$. Since f is continuous, it is uniformly continuous on K, and therefore

$$\sup_{s\in[t_0,T]} \|f(\theta_j(s), \widehat{y}_j(s)) - f(s, y(s))\| \longrightarrow 0. \tag{3.21}$$

Using (3.8), uniformly for $t \in [t_0, T]$,

$$\left\| y_j(t) - \left(x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, y(s))\, ds \right) \right\| \le \frac{(T-t_0)^q}{\Gamma(q+1)} \sup_{s\in[t_0,T]} \|f(\theta_j(s), \widehat{y}_j(s)) - f(s, y(s))\|, \tag{3.22}$$

which tends to zero. Since $y_j \to y$ uniformly, we conclude that

$$y(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, y(s))\, ds, \qquad t \in [t_0, T]. \tag{3.23}$$

Thus y is an integral solution of (1.1). □

We next quantify how closely the memory-preserving Euler trajectories remain to a closed convex constraint set.

Proposition 3.5 (Distance estimate for the Euler trajectories). *Let $G \subset \mathbb{R}^n$ be nonempty, closed and convex, and let $x_0 \in G$. Assume that f satisfies the linear growth condition and*

$$\langle f(t,x), x - P_G(x)\rangle \le 0, \qquad (t,x) \in [a,b) \times \mathbb{R}^n. \tag{3.24}$$

Then there exists a constant $C > 0$, independent of the partition π, such that every Euler approximate solution y_π on $[t_0, T]$ satisfies

$$\sup_{t\in[t_0,T]} \operatorname{dist}(y_\pi(t), G) \le C\, \mu_\pi^{q/2}. \tag{3.25}$$

Consequently, the Euler trajectories approach the constraint set G uniformly as $\mu_\pi \to 0$.

Proof. By the uniform boundedness established in Section 3, there is a constant $M > 0$, independent of π, such that $\|y_\pi(t)\| \le M$ and, since $\|\widehat{y_\pi}(t)\| \le M$,

$$\|g_\pi(t)\| = \left\| f\big(\theta_\pi(t), \widehat{y_\pi}(t)\big) \right\| \le C_f := L(1+M), \tag{3.26}$$

for almost every t. Moreover, the equicontinuity estimate proved in Section 3 gives, for $t \in [t_i, t_{i+1})$,

$$\|y_\pi(t) - \widehat{y}_\pi(t)\| = \|y_\pi(t) - y_\pi(t_i)\| \le K \mu_\pi^q, \qquad K := \frac{2C_f}{\Gamma(q+1)}. \tag{3.27}$$

Let

$$R_G(x) := x - P_G(x). \tag{3.28}$$

Since the metric projection onto a closed convex set is firmly nonexpansive, $R_G = I - P_G$ is nonexpansive. Therefore

$$\|R_G(y_\pi(t)) - R_G(\widehat{y}_\pi(t))\| \le \|y_\pi(t) - \widehat{y}_\pi(t)\|. \tag{3.29}$$

By the assumed projection inequality, applied at $(\theta_\pi(t), \widehat{y}_\pi(t))$,

$$\langle g_\pi(t), R_G(\widehat{y}_\pi(t))\rangle \le 0. \tag{3.30}$$

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Hence, using (3.27),

$$\begin{aligned}\langle g_\pi(t), R_G(y_\pi(t))\rangle &= \langle g_\pi(t), R_G(\widehat{y}_\pi(t))\rangle + \langle g_\pi(t), R_G(y_\pi(t)) - R_G(\widehat{y}_\pi(t))\rangle \\ &\le C_f K \mu_\pi^q, \end{aligned} \tag{3.31}$$

for almost every $t \in [t_0, T]$.

Since

$$y_\pi(t) = x_0 + (I_{t_0}^q g_\pi)(t), \tag{3.32}$$

Lemma 2.5 yields

$$D_{t_0}^q \big[\Phi(y_\pi(\cdot)) - \Phi(x_0)\big](t) \le C_f K \mu_\pi^q, \quad \text{a.e.} \tag{3.33}$$

Set $w(t) := \Phi(y_\pi(t)) - \Phi(x_0)$. By Lemma 2.5, $w \in I_{t_0}^q L^\infty(t_0, T; \mathbb{R})$, so $I_{t_0}^q D_{t_0}^q w = w$ in the integral Caputo sense. Applying $I_{t_0}^q$ to the preceding inequality and using the positivity of the fractional kernel gives

$$\Phi(y_\pi(t)) - \Phi(x_0) \le \frac{C_f K \mu_\pi^q}{\Gamma(q+1)} (t - t_0)^q. \tag{3.34}$$

Because $x_0 \in G$, $\Phi(x_0) = 0$, and thus

$$\frac{1}{2} \operatorname{dist}(y_\pi(t), G)^2 \le \frac{C_f K (T - t_0)^q}{\Gamma(q+1)} \mu_\pi^q. \tag{3.35}$$

Taking square roots proves (3.25), with

$$C = \left(\frac{2 C_f K (T - t_0)^q}{\Gamma(q+1)} \right)^{1/2}. \tag{3.36}$$

□

Role of the Euler construction. The preceding distance estimate, together with compactness and consistency, will be used in the first proof of the viability theorem. This provides the corrected Euler route while retaining the complete Caputo history. The theorem will also admit a second, independent proof based on a sharp fractional distance estimate.

4. Viability results

We recall the notion of viability for cylindrical constraints; see, for instance, [3]. Let $G \subset \mathbb{R}^n$ be nonempty, closed and convex, and set $\mathcal{G} = [a, b) \times G$.

Definition 4.1. The set $\mathcal{G}$ is said to be *viable* with respect to (1.1) if, for every $(t_0, x_0) \in \mathcal{G}$ and every $T \in (t_0, b)$, there exists an integral solution $y : [t_0, T] \to \mathbb{R}^n$ of (1.1) such that $y(t) \in G$ for every $t \in [t_0, T]$.

We shall use the following projection inwardness condition:

$$\langle f(t, x), x - P_G(x)\rangle \le 0, \qquad (t, x) \in [a, b) \times \mathbb{R}^n, \tag{4.1}$$

where $P_G(x)$ denotes the metric projection of x onto G.

We now present examples of dynamics satisfying the projection inwardness condition.

Example 4.2 (A fractional saturated logistic model)**.** Consider the Caputo fractional system

$$D_{t_0}^q y_i(t) = r_i \frac{y_i^+(t)}{1 + y_i^+(t)} \left(1 - \frac{y_i(t)}{K_i}\right), \qquad i = 1, \dots, n, \tag{4.2}$$

where $y_i^+ = \max\{y_i, 0\}$, $r_i > 0$, and $K_i > 0$, and let $G = \prod_{i=1}^n [0, K_i]$. The right-hand side is continuous and has at most linear growth. For $x \in \mathbb{R}^n$, let $z = P_G(x)$. Componentwise, if $x_i < 0$, then $z_i = 0$ and $f_i(x) = 0$; if $0 \le x_i \le K_i$, then $z_i = x_i$; and if $x_i > K_i$, then $z_i = K_i$ and $f_i(x) < 0$. Thus $f_i(x)(x_i - z_i) \le 0$ for every i, and consequently

$$\langle f(x), x - P_G(x)\rangle \le 0. \tag{4.3}$$

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Therefore the projection inwardness condition holds for the population box G.

Example 4.3 (A fractional potential-driven system)**.** Consider

$$D^q_{t_0} y(t) = -\nabla V(y(t)), \tag{4.4}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and convex, and assume that ∇V has at most linear growth. For a fixed $c \in \mathbb{R}$, suppose that the sublevel set $G := \{x \in \mathbb{R}^n : V(x) \le c\}$ is nonempty. Since V is continuous and convex, G is closed and convex.

Let $x \in \mathbb{R}^n$ and $z = P_G(x)$. If $x \in G$, then $z = x$ and the projection inwardness inequality is immediate. If $x \notin G$, convexity gives $V(z) \ge V(x) + \langle \nabla V(x), z - x\rangle$. Since $V(z) \le c < V(x)$, it follows that $\langle \nabla V(x), x - z\rangle \ge V(x) - V(z) > 0$, and therefore

$$\langle -\nabla V(x), x - z\rangle < 0. \tag{4.5}$$

Hence the projection inwardness condition holds for the sublevel set G.

We now state the main viability result.

Theorem 4.4. *If f is continuous, satisfies the linear growth condition and the projection inwardness condition, then $\mathcal{G}$ is viable with respect to (1.1).*

First proof: memory-preserving Euler approximation. Let $(t_0, x_0) \in \mathcal{G}$ and let $T \in (t_0, b)$. Choose a sequence of partitions $(\pi_j)_j$ with $\mu_j \to 0$, and let y_j be the corresponding Euler approximate solutions. By the compactness proposition of Section 3, after passing to a subsequence,

$$y_j \longrightarrow y \qquad \text{uniformly on } [t_0, T]. \tag{4.6}$$

By Theorem 3.4, the limit y is an integral solution of (1.1). Since $x_0 \in G$, Proposition 3.5 gives

$$\sup_{t\in[t_0,T]} \operatorname{dist}(y_j(t), G) \longrightarrow 0. \tag{4.7}$$

Moreover,

$$\sup_{t\in[t_0,T]} \operatorname{dist}(y(t), G) \le \|y - y_j\|_\infty + \sup_{t\in[t_0,T]} \operatorname{dist}(y_j(t), G). \tag{4.8}$$

Both terms on the right tend to zero. Hence $\operatorname{dist}(y(t), G) = 0$ for every $t \in [t_0, T]$, and since G is closed, $y(t) \in G$ on $[t_0, T]$. Thus $\mathcal{G}$ is viable. □

The preceding proof makes the Euler approximation an intrinsic part of the viability argument and provides the corrected route inspired by the early stepwise construction. We now give a second proof that is independent of Euler approximations and yields a sharp fractional distance estimate.

Second proof: sharp fractional distance estimate. Let $(t_0, x_0) \in \mathcal{G}$ and let $T \in (t_0, b)$. Standard existence theory for the Volterra formulation gives an integral solution y on $[t_0, T]$ under the stated assumptions; see [10, 11]. Thus

$$y(t) = x_0 + I^q_{t_0}\big(f(\cdot, y(\cdot))\big)(t). \tag{4.9}$$

Since $f(\cdot, y(\cdot))$ is continuous, Lemma 2.5, with $z = y$ and $v(t) = f(t, y(t))$, gives

$$D^q_{t_0}\big[\Phi(y(\cdot)) - \Phi(x_0)\big](t) \le \langle y(t) - P_G(y(t)), f(t, y(t))\rangle \le 0, \tag{4.10}$$

for almost every $t \in [t_0, T]$. By the conclusion of Lemma 2.5, the function $w(t) := \Phi(y(t)) - \Phi(x_0)$ belongs to $I^q_{t_0} L^\infty(t_0, T; \mathbb{R})$. Hence $I^q_{t_0} D^q_{t_0} w = w$ in the integral Caputo sense, and the positivity of the fractional kernel yields

$$\Phi(y(t)) \le \Phi(x_0), \qquad t \in [t_0, T]. \tag{4.11}$$

Equivalently,

$$\operatorname{dist}(y(t), G) \le \operatorname{dist}(x_0, G), \qquad t \in [t_0, T]. \tag{4.12}$$

Notice that the argument up to this point does not use the assumption $x_0 \in G$; hence the preceding estimate is valid for arbitrary initial data $x_0 \in \mathbb{R}^n$. Moreover, the constant 1 is sharp in general: if $f \equiv 0$, then $y(t) \equiv x_0$ and equality

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holds for every t. Returning to the viability problem, since $x_0 \in G$, the right-hand side is zero. Therefore $y(t) \in G$ for every $t \in [t_0, T]$, and $\mathcal{G}$ is viable. □

5. Early approaches and limitations

We review early attempts to extend viability theory to cylindrical domains for the fractional initial-value problem (1.1) with $0 < q < 1$, and we identify the principal gaps in these constructions.

5.1. **The approach of Vasundhara Devi and Lakshmikantham.** In [15], the authors employ a discrete scheme that, at each step, resets the lower terminal by solving

$$D^q_{t_i} y(t) = f(t_i, x_i), \qquad y(t_i) = x_i, \tag{5.1}$$

on each subinterval $[t_i, t_{i+1}]$. This stepwise reinitialization erases the memory intrinsic to (1.1) and therefore does not approximate the original problem with lower terminal t_0. As noted in Section 3, the Euler-type construction used here keeps one fixed lower terminal and one global history. More precisely, define the piecewise-constant forcing

$$g_\pi(t) := f(t_i, x_i), \qquad t \in [t_i, t_{i+1}), \qquad x_i = y_\pi(t_i), \tag{5.2}$$

and set, on the whole interval,

$$y_\pi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g_\pi(s)\, ds. \tag{5.3}$$

Thus the values $x_i = y_\pi(t_i)$ are nodal values of a single history-preserving trajectory with lower terminal t_0, rather than new initial data at the intermediate nodes. Evaluating this global Volterra approximation at t_{i+1} yields the nonlocal update

$$x_{i+1} = x_0 + \frac{1}{\Gamma(q+1)} \sum_{j=0}^{i} \Big[(t_{i+1} - t_j)^q - (t_{i+1} - t_{j+1})^q \Big] f(t_j, x_j), \tag{5.4}$$

thereby retaining the full history from t_0.

The correction is not merely formal. As observed in [2], the local distance estimates used in the early stepwise argument omit a remainder depending on the prehistory; in a fractional system this contribution cannot in general be discarded. The first proof of Theorem 4.4 avoids such a local restarted estimate. It works with the single global trajectory y_π above and proves instead the uniform distance estimate

$$\sup_{t \in [t_0, T]} \operatorname{dist}(y_\pi(t), G) \le C \mu_\pi^{q/2} \longrightarrow 0. \tag{5.5}$$

Thus the Euler strategy is retained, but the full Caputo memory is present throughout the approximation and the passage to the viable limit.

5.2. **The approach of Girejko, Mozyrska and Wyrwas.** The viability proof in [6] relies on the construction of approximate solutions via a lemma based on the Brezis–Browder principle and, in the final step, invokes a concatenation argument to pass from tangency to a global trajectory. Such a concatenation property is valid for classical ODEs but fails for fractional dynamics: the Caputo derivative at time t depends on the entire past on $[t_0, t]$, so splicing solutions at an intermediate time does not preserve the dynamics unless the whole prehistory is carried along. Consequently, the last step of the argument in [6] is not justified (see also [2]).

6. Application: Safe-voltage viability for a fractional RC/CPE circuit

We consider a simple electrochemical circuit consisting of a resistor R in parallel with a constant-phase element (CPE) of pseudo-capacitance C_α and order $q \in (0, 1)$. The node voltage $v(t)$ across the CPE satisfies the fractional dynamics

$$D^q_{t_0} v(t) \;=\; -\frac{1}{R\, C_\alpha}\, v(t) \;+\; \frac{1}{C_\alpha}\, u(t), \quad \text{with} \quad v(t_0) = v_0, \tag{6.1}$$

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where $u(t)$ is the injected current (control) assumed to be continuous over $[a,b)$. This model captures the constant-phase behavior observed in porous electrodes and supercapacitors and cannot be represented exactly by any finite-dimensional ODE. Set

$$\lambda := \frac{1}{R\,C_\alpha} > 0, \qquad \kappa := \frac{1}{C_\alpha} > 0, \tag{6.2}$$

so the system fits (1.1) with $y = v$ and

$$f(t,x) = -\lambda\, x + \kappa\, u(t). \tag{6.3}$$

Assume a physically reasonable input constraint

$$0 \;\le\; u(t) \;\le\; \bar{u}, \qquad \text{for all } t \in [a,b), \tag{6.4}$$

and define the safe operating set (viability constraint)

$$G := [0, V_{\max}] \subset \mathbb{R}. \tag{6.5}$$

The projection map P_G is given by

$$P_G(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le V_{\max}, \\ V_{\max}, & x > V_{\max}. \end{cases} \tag{6.6}$$

To apply Theorem 4.4, we first verify that the projection inwardness condition holds. For any $(t,x) \in [a,b) \times \mathbb{R}$, let $z = P_G(x)$. In one dimension, the condition becomes

$$\langle f(t,x),\, x - z \rangle \;=\; \big(-\lambda\, x + \kappa\, u(t)\big)\,(x - z) \;\le 0. \tag{6.7}$$

We have:

- If $x \in [0, V_{\max}]$, then $z = x$ and $\langle f(t,x),\, x - z\rangle = 0$.
- If $x < 0$, then $z = 0$ and $(x - z) = x < 0$. Since $u(t) \ge 0$, we obtain

  $$f(t,x) = -\lambda\, x + \kappa\, u(t) \;\ge\; -\lambda\, x \;>\; 0, \tag{6.8}$$

  hence $\langle f(t,x),\, x - z\rangle \le 0$.
- If $x > V_{\max}$, then $z = V_{\max}$ and $(x - z) > 0$. Using $u(t) \le \bar{u}$,

  $$f(t,x) = -\lambda\, x + \kappa\, u(t) \;\le\; -\lambda\, V_{\max} + \kappa\, \bar{u}. \tag{6.9}$$

  Therefore, the projection inwardness condition holds provided that

  $$\bar{u} \;\le\; \frac{\lambda}{\kappa}\, V_{\max} \;=\; \frac{V_{\max}}{R}, \tag{6.10}$$

  (equivalently, the input current bound does not exceed the steady inward flux at $v = V_{\max}$).

It remains to verify the linear growth condition. Indeed,

$$|f(t,x)| \le \lambda |x| + \kappa\, \bar{u}, \tag{6.11}$$

for every $(t,x) \in [a,b) \times \mathbb{R}$, and f is continuous.

By Theorem 4.4, if the input satisfies

$$0 \le u(t) \le \frac{V_{\max}}{R}, \tag{6.12}$$

for every $t \in [a,b)$, then the set $\mathcal{G} = [a,b) \times [0, V_{\max}]$ is viable with respect to (6.1). Hence, for every initial value $v_0 \in [0, V_{\max}]$, there exists an integral solution that remains in $[0, V_{\max}]$ for all $t \in [a,b)$.

Uncorrected Proof

7. Conclusion

We have established a viability criterion for Caputo fractional differential equations while explicitly accounting for the nonlocal character of the dynamics. A memory-preserving Euler construction with fixed lower terminal was used to prove a uniform distance estimate for the Euler trajectories and, after compactness and consistency, the existence of a viable integral limit. This provides a corrected Euler route to viability without suppressing the fractional prehistory. The same criterion was then proved independently by a sharp convex fractional distance argument, yielding the optimal constant 1 in the distance nonincrease estimate. The framework was compared with earlier fractional viability approaches and illustrated by models written directly in the form of (1.1). The criterion established here applies to fixed, closed, convex constraint sets in finite-dimensional spaces. Possible extensions include noncylindrical constraints, fractional differential inclusions, and systems with delay or stochastic memory.

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Uncorrected Proof