Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-25
DOI:10.22034/cmde.2026.75070.3917

Nematicons in liquid crystals by the extended sinh–Gordon equation expansion method

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Uncorrected Proof

Abstract

In the research paper, we try to illustrate the structure of new exact traveling wave solutions of nematicons in liquid crystals with four law nonlinearities including: Kerr, power, parabolic and dual-power by utilizing the extended sinh–Gordon equation expansion method. The aim of this research is not just to find the dark, bright, combined dark-bright, singular types, traveling and solitary solutions of nematicons in liquid crystals but also inquiry in this one method, showing the differences between the obtained solutions and other solutions that obtained by using different methods. Moreover, the constraints guarantee their existence. Finally, we believe that the executed method is powerful and efficient than other methods and the obtained solutions in this paper can help us to understand soliton molecules in liquid crystals.

Keywords. Solitons, Nematicons in liquid crystals, Kerr, Power, Parabolic and dual-powerm, Extended sinh–Gordon equation expansion method.
2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p.

1. Introduction

In recent years, nonlinear science has received more and more attention [5, 32, 33, 38]. Particularly, the nonlinear wave phenomena are a hot topic from natural science fields [26, 34, 39, 42, 44]. These often appear in nonlinear soliton types system which are the integrable system, especially in nonlinear optical fibers, crystals, metamaterials, metasurfaces and liquid crystals, etc [13, 25, 40, 46]. Nonlinear partial differential equations can describe wave phenomena such as solitons, chaotic waves, periodic waves, rogue waves and breathers. More specifically, soliton and other coherent wave solutions allow one to analyze localized pulses, oscillatory patterns, wave amplitudes, and wave interactions [1, 4, 11, 19–21]. As such, the analysis of coherent waves in nonlinear dispersive systems can yield valuable insights into both traditional physical wave propagation problems as well as potential biomedical ones [30, 31, 37, 50]. These days, NLPDEs are essential in domains including fluid mechanics, biology, and physics [14, 15, 24, 29, 51]. They are essential in contemporary applied mathematics because of their capacity to characterize realistic dynamical behavior, which includes modeling complex systems and predicting phenomena in various fields [35, 36]. This special phenomenon in area of liquid crystals has caused widespread interest among experts has a well-known name which is nematicons that was first introduced by Assanto [3, 7, 9, 10]. Many workers have done a lot of research in this direction including nematic liquid crystals in microcapillaries [18], twisted nematic liquid crystal [27], molecular deformations in a nematic liquid crystal system [28], molecular director configuration in a liquid crystal cell [41], soliton switching in inhomogeneous nonlocal media [43], nematicons in liquid crystals [45], beam propagation in nematic liquid crystals

Received: 02 October 2026; Accepted: 04 November 2026.

[47], Hyper-solitons in nematic liquid crystals [48], and Vector vortex solitons in nematic liquid crystals [49].
In this paper, we focus on the nematicons in liquid crystals, which can be used to describe dynamics of solitons in liquid crystals, in dimensionless form, is governed by the following coupled system of equations [49], which reads

$$i\Lambda_t + a\Lambda_{xx} + b\Theta\Lambda = 0, \tag{1.1}$$

$$c\Theta_{xx} + \lambda\Theta + \alpha F(|\Lambda|)^2 = 0. \tag{1.2}$$

The above system (1.1)-(1.2) has been studied by many scholars [43]-[49].
We mentioned the nematicons in liquid crystals with four law nonlinearity forms such as Kerr, power, parabolic and dual-power nonlinearity and study all considered cases by help of the extended sinh–Gordon equation expansion method (EShGEE) for more details refer to Refs. [12, 16, 17, 23]. All of these studying show the power of this model and the world's serious insight into more and more deeply in the properties of this model shows their full confidence that many of the properties have not yet been revealed.
In this paper, we perform an analytical technique of the nematicons in liquid crystals with four law nonlinearity forms using the extended sinh–Gordon equation expansion method. Numerous researchers have utilized the EShGEE to solve NLPDEs. The EShGEE has numerous benefits compared to other techniques, including its simple application and capacity to produce various precise wave solutions, like soliton, periodic, and rational forms.
In this research, we investigate the exact and solitary traveling wave for the nematicons in liquid crystals with four law nonlinearity forms via the aforementioned method. The remaining of this paper is arranged as follows: In Section 2, we give the basic structure of extended sinh–Gordon equation expansion method. In Section 3, we derive governing equation with decomposing the system to real and imaginary parts. In Section 4, we implement this method to obtain the exact traveling and solitary wave solutions of the nonlinear evolution equations pointed out above with four cases of law nonlinearities. In Section 5, we discuss the results that obtained by using this method and show the power of this method. In Section 6, the conclusions are given.

2. The extended sinh–Gordon equation expansion method

Consider the sinh–Gordon equation as:

$$u_{xt} = \alpha \sinh(u), \tag{2.1}$$

where u=$u(x,t)$ and α is a constant. By utilizing the following relations

$$u(x,t) = U(\xi),\ \xi = kx - \omega t,$$

then Eq. (2.1) reduces to the following nonlinear ordinary differential equation as:

$$U'' = -\frac{\alpha}{k\omega}\sinh(U). \tag{2.2}$$

By applying of the integrating process of Eq. (2.2) gets

$$\left[\left(\frac{U}{2}\right)'\right]^2 = -\frac{\alpha}{k\omega}\sinh^2\left(\frac{U}{2}\right) + p, \tag{2.3}$$

where p is an integration constant.
Putting $\frac{U}{2} = w(\xi)$, and $-\frac{\alpha}{k\omega} = q$ in Eq. (2.3), we gain

$$w' = \sqrt{p + q\sinh^2(w)}, \tag{2.4}$$

by choosing the parameters p and q in Eq. (2.4) can be found the further results of solutions [12, 16, 23, 29]:
Case-I: Suppose $p = 0$ and $q = 1$, then Eq. (2.4) reads

$$w' = \sinh(w). \tag{2.5}$$

Simplifying Eq. (2.5), the following results are gained [12]:

$$\sinh(w) = \pm i\,\mathrm{sech}(\xi), \cosh(w) = -\tanh(\xi), \tag{2.6}$$

Uncorrected Proof

and

$$\sinh(w) = \pm\operatorname{csch}(\xi), \cosh(w) = -\coth(\xi). \tag{2.7}$$

where, $i = \sqrt{-1}$ represent an imaginary number.
Case-II: Suppose $p = 1$ and $q = 1$, then Eq. (2.4) reads

$$w' = \cosh(w). \tag{2.8}$$

Once again, simplifying Eq. (2.8), the following results are achieved [12]:

$$\sinh(w) = \tan(\xi), \cosh(w) = \pm\sec(\xi), \tag{2.9}$$

and

$$\sinh(w) = -\cot(\xi), \cosh(w) = \pm\operatorname{csc}(\xi). \tag{2.10}$$

We assume that the given nonlinear PDE

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \ t > 0. \tag{2.11}$$

In Eq. (2.11) $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its various partial derivatives are involved. Consider the following transformation:

$$u(x,t) = U(\xi) \ \ \xi = kx - \omega t. \tag{2.12}$$

Implementing the transformation of (2.12) into Eq. (2.11), then Eq. (2.11) converted to the following NODE

$$G(U, \ -\omega U', \ kU', \ \omega^2 U'', \ k^2 U'', \dots) = 0, \tag{2.13}$$

where G is a polynomial of $U = U(\xi)$ and its derivatives with respect to ξ. Suppose the exact solution be as

$$U(w) = \sum_{j=1}^{N} \cosh^{j-1}(w) \left[B_j \sinh(w) + A_j \cosh(w)\right] + A_0. \tag{2.14}$$

Base on the relations (2.5)-(2.7), then (2.14) can be written in the following forms

$$U(\xi) = \sum_{j=1}^{N} (-\tanh(\xi))^{j-1} \left[\pm iB_j \ \operatorname{sech}(\xi) - A_j \tanh(\xi)\right] + A_0, \tag{2.15}$$

and

$$U(\xi) = \sum_{j=1}^{N} (-\coth(\xi))^{j-1} \left[\pm B_j \operatorname{csch}(\xi) - A_j \coth(\xi)\right] + A_0. \tag{2.16}$$

Similarly, base on the relations (2.8)-(2.10), then (2.14) can be written in the following forms

$$U(\xi) = \sum_{j=1}^{N} (\pm\sec(\xi))^{j-1} \left[B_j \tan(\xi) \pm A_j \sec(\xi)\right] + A_0, \tag{2.17}$$

and

$$U(\xi) = \sum_{j=1}^{N} (\pm\operatorname{csc}(\xi))^{j-1} \left[-B_j \cot(\xi) \pm A_j \operatorname{csc}(\xi)\right] + A_0, \tag{2.18}$$

the value of N can be determined by using the homogeneous balance principle.

Uncorrected Proof

3. The governing equation with decomposing to real and imaginary parts

The dynamics of nematicons in liquid crystals, in dimensionless form, is governed by the following coupled system of equations

$$i\Lambda_t + a\Lambda_{xx} + b\Theta\Lambda = 0, \tag{3.1}$$

$$c\Theta_{xx} + \lambda\Theta + \alpha F(|\Lambda|)^2 = 0. \tag{3.2}$$

In the above relations (3.1) and (3.2), the variable $\Lambda(x,t)$ represents the wave profile, while the second dependent variable $\Theta(x,t)$ is the tilt angle of the liquid crystal molecule. The first term in (3.1) represents temporal evolution of nematicons, while the second term is the group velocity dispersion. Moreover, α is the coefficient of nonlinear term. The functional F is the type of nonlinearity that will be studied. Moreover, a, b, c, λ and α are all real-valued constants.
In order to solve Eqs. (3.1) and (3.2), the starting hypothesis is [35, 36]

$$\Lambda(x,t) = P(\xi(x,t))\exp(i\Phi(x,t)), \tag{3.3}$$

$$\Theta(x,t) = Q(\xi(x,t)), \tag{3.4}$$

where

$$\xi(x,t) = k(x - \nu t), \tag{3.5}$$

and the phase component Φ is given by

$$\Phi(x,t) = -\kappa x + \omega t + \theta. \tag{3.6}$$

In (3.5) and (3.6), $u(x,t)$ represents the amplitude portion of the soliton, and k and ν are inverse width and velocity of soliton. From (3.3), κ is the frequency of the soliton, ω is the wave number of the soliton and finally θ is the phase constant. Inserting (3.3) into (3.1), (3.2) and then decomposing into real and imaginary parts yield a pair of relations as follow

$$ak^2P'' - (a\kappa^2 + \omega)P + bPQ = 0, \tag{3.7}$$

$$ck^2Q'' + \lambda Q + \alpha F(P^2) = 0, \tag{3.8}$$

and

$$-k\nu P' - 2ak\kappa P' = 0, \tag{3.9}$$

from the imaginary part that leads to the speed of the soliton being

$$\nu = -2a\kappa. \tag{3.10}$$

Nematicon solutions will now be derived for four forms of nonlinearity for the functional F.

4. Types of Nonlinearities

These are sequentially discussed in the continue with offering in the following subsections.

4.1. **Kerr law nonlinearity.**

The Kerr nonlinearity arises when $F(r) = r$. Therefore, (3.7) and (3.8) transform to

$$ak^2P'' - (a\kappa^2 + \omega)P + bPQ = 0, \tag{4.1}$$

$$ck^2Q'' + \lambda Q + \alpha P^2 = 0. \tag{4.2}$$

By balancing P'' with PQ and Q'' with P^2, and considering the below relations

$$N + 2 = N + M, \tag{4.3}$$

$$M + 2 = 2N, \tag{4.4}$$

we get the following findings

$$M = N = 2. \tag{4.5}$$

Uncorrected Proof

4.1.1. *For First Case.* Base on the Eqs. (2.14)-(2.16), the solutions of Eqs. (4.1) and (4.2) can be presented as

$$P(\xi)=\sum_{j=1}^{2}(-\tanh(\xi))^{j-1}\left[\pm iB_j\ \operatorname{sech}(\xi)-A_j\tanh(\xi)\right]+A_0, \tag{4.6}$$

$$P(\xi)=\sum_{j=1}^{2}(-\coth(\xi))^{j-1}\left[\pm B_j\operatorname{csch}(\xi)-A_j\coth(\xi)\right]+A_0, \tag{4.7}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+\cosh(w)[B_2\sinh(w)+A_2\cosh(w)]+A_0, \tag{4.8}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(-\tanh(\xi))^{j-1}\left[\pm iD_j\ \operatorname{sech}(\xi)-C_j\tanh(\xi)\right]+C_0, \tag{4.9}$$

$$Q(\xi)=\sum_{j=1}^{2}(-\coth(\xi))^{j-1}\left[\pm D_j\operatorname{csch}(\xi)-C_j\coth(\xi)\right]+C_0, \tag{4.10}$$

and so

$$Q(w)=D_1\sinh(w)+C_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.11}$$

where $A_2\neq 0$ or $B_2\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.

Plugging (4.8), (4.11) and those derivatives into Eqs. (4.1) and (4.2), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:

Set I-1:

$$\begin{aligned}&A_0=A_1=A_2=B_1=C_1=D_1=D_2=0,\quad C_0=C_0,\ B_2=3\lambda\sqrt{\frac{a}{bc\alpha}},\\ &C_2=-\frac{3a\lambda}{bc},k=\sqrt{\frac{\lambda}{2c}},\quad \kappa=\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}},\quad \omega=\omega.\end{aligned} \tag{4.12}$$

Thus, we infer the following dark and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,1}(x,t)&=\left\{3i\lambda\sqrt{\frac{a}{bc\alpha}}\frac{\sinh\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}{\cosh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}\right\}\times e^{i\left(-\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}x+\omega t+\theta\right)},\\ \Lambda_{1,2}(x,t)&=\left\{3\lambda\sqrt{\frac{a}{bc\alpha}}\frac{\cosh\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}{\sinh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}\right\}\times e^{i\left(-\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}x+\omega t+\theta\right)},\end{aligned} \tag{4.13}$$

and also

$$\begin{aligned}\Theta_{1,1}(x,t)&=\left\{C_0-\frac{3a\lambda}{bc}\tanh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)\right\},\\ \Theta_{1,2}(x,t)&=\left\{C_0-\frac{3a\lambda}{bc}\coth^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)\right\},\end{aligned} \tag{4.14}$$

these solutions are valid when $\lambda c>0$.

Uncorrected Proof

4.1.2. *For Second Case.* Base on the Eqs. (2.17)-(2.18), the solutions of Eqs. (4.1) and (4.2) can be presented as

$$P(\xi)=\sum_{j=1}^{2}(\pm\sec(\xi))^{j-1}\left[B_j\tan(\xi)\pm A_j\sec(\xi)\right]+A_0, \tag{4.15}$$

and

$$P(\xi)=\sum_{j=1}^{2}(\pm\csc(\xi))^{j-1}\left[-B_j\cot(\xi)\pm A_j\csc(\xi)\right]+A_0, \tag{4.16}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+\cosh(w)[B_2\sinh(w)+A_2\cosh(w)]+A_0, \tag{4.17}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\sec(\xi))^{j-1}\left[D_j\tan(\xi)\pm C_j\sec(\xi)\right]+C_0, \tag{4.18}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\csc(\xi))^{j-1}\left[-D_j\cot(\xi)\pm C_j\csc(\xi)\right]+C_0, \tag{4.19}$$

and so

$$Q(w)=D_1\sinh(w)+A_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.20}$$

where $A_2\neq 0$ or $B_2\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.
Plugging (4.17), (4.20) and those derivatives into Eqs. (4.1) and (4.2), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned}&A_0=A_1=A_2=B_1=C_1=D_1=D_2=0,\ C_0=C_0,\ B_2=3\lambda\sqrt{\frac{a}{bc\alpha}},\\ &C_2=\frac{3a\lambda}{bc},k=\sqrt{-\frac{\lambda}{2c}},\quad \kappa=\sqrt{\frac{2bcC_0+a\lambda-2c\omega}{2ac}},\quad \omega=\omega.\end{aligned} \tag{4.21}$$

Hence, we conclude the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{2,1}(x,t)&=\left\{-3\lambda\sqrt{\frac{a}{bc\alpha}}\frac{\sin\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}{\cos^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0+a\lambda-2c\omega}{2ac}}t\right]\right)}\right\}\times e^{i\left(-\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}x+\omega t+\theta\right)},\\ \Lambda_{2,2}(x,t)&=\left\{3\lambda\sqrt{\frac{a}{bc\alpha}}\frac{\cos\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}t\right]\right)}{\sin^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0+a\lambda-2c\omega}{2ac}}t\right]\right)}\right\}\times e^{i\left(-\sqrt{\frac{2bcC_0-5a\lambda-2c\omega}{2ac}}x+\omega t+\theta\right)},\end{aligned} \tag{4.22}$$

and also

$$\begin{aligned}\Theta_{2,1}(x,t)&=\left\{C_0+\frac{3a\lambda}{bc}\sec^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0+a\lambda-2c\omega}{2ac}}t\right]\right)\right\},\\ \Theta_{2,2}(x,t)&=\left\{C_0+\frac{3a\lambda}{bc}\csc^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+2a\sqrt{\frac{2bcC_0+a\lambda-2c\omega}{2ac}}t\right]\right)\right\},\end{aligned} \tag{4.23}$$

these solutions are valid when $\lambda c<0$.

Uncorrected Proof

4.2. **Power law nonlinearity.** The power law nonlinearity arises when $F(r) = r^n$. Therefore, (3.7) and (3.8) transform to

$$ak^2P'' - (a\kappa^2 + \omega)P + bPQ = 0, \tag{4.24}$$

$$ck^2Q'' + \lambda Q + \alpha P^{2n} = 0. \tag{4.25}$$

Suppose

$$P = R^{\frac{1}{n}} = 0, \tag{4.26}$$

so that (4.24) and (4.25) transform to

$$ak^2(nRR'' + (1-n)(R')^2) - n^2(a\kappa^2 + \omega)R^2 + n^2bR^2Q = 0, \tag{4.27}$$

$$ck^2Q'' + \lambda Q + \alpha R^2 = 0. \tag{4.28}$$

By balancing RR'' with R^2Q and Q'' with R^2, and considering the below relations

$$N + 2 = N + M, \tag{4.29}$$

$$M + 2 = 2N, \tag{4.30}$$

we get the following

$$M = N = 2. \tag{4.31}$$

4.2.1. *For First Case.* By regarding on the Eqs. (2.14)-(2.16), the solutions of Eqs. (4.27) and (4.28) can be presented as

$$P(\xi) = \sum_{j=1}^{2} (-\tanh(\xi))^{j-1} [\pm iB_j \operatorname{sech}(\xi) - A_j \tanh(\xi)] + A_0, \tag{4.32}$$

$$P(\xi) = \sum_{j=1}^{2} (-\coth(\xi))^{j-1} [\pm B_j \operatorname{csch}(\xi) - A_j \coth(\xi)] + A_0, \tag{4.33}$$

and so

$$P(w) = B_1 \sinh(w) + A_1 \cosh(w) + \cosh(w)[B_2 \sinh(w) + A_2 \cosh(w)] + A_0, \tag{4.34}$$

and

$$Q(\xi) = \sum_{j=1}^{2} (-\tanh(\xi))^{j-1} [\pm iD_j \operatorname{sech}(\xi) - C_j \tanh(\xi)] + C_0, \tag{4.35}$$

$$Q(\xi) = \sum_{j=1}^{2} (-\coth(\xi))^{j-1} [\pm D_j \operatorname{csch}(\xi) - C_j \coth(\xi)] + C_0, \tag{4.36}$$

and so

$$Q(w) = D_1 \sinh(w) + C_1 \cosh(w) + \cosh(w)[D_2 \sinh(w) + C_2 \cosh(w)] + C_0, \tag{4.37}$$

where $A_2 \neq 0$ or $B_2 \neq 0$ and $C_2 \neq 0$ or $D_2 \neq 0$.
Plugging (4.34), (4.37) and those derivatives into Eqs. (4.27) and (4.28), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned} &A_0 = -\frac{\lambda}{3n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}, \quad A_1 = 0, A_2 = \frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}, \quad B_1 = B_2 = C_1 = D_1 = D_2 = 0, \\ &C_2 = -\frac{\lambda a(n+2)}{2bcn^2}, \quad k = \sqrt{\frac{\lambda}{4c}}, \quad \kappa = \kappa, \quad \omega = -\frac{6ac\kappa^2n^2 - 6bcn^2C_0 + a\lambda n + 8a\lambda}{6cn^2}. \end{aligned} \tag{4.38}$$

Uncorrected Proof

Whereas, we infer the following dark and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,3}(x,t) &= \left\{-\frac{\lambda}{3n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}\tanh^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\\ &\times e^{i\left(-\kappa x-\frac{6ac\kappa^2n^2-6bcn^2C_0+a\lambda n+8a\lambda}{6cn^2}t+\theta\right)},\\ \Lambda_{1,4}(x,t) &= \left\{-\frac{\lambda}{3n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}\coth^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\\ &\times e^{i\left(-\kappa x-\frac{6ac\kappa^2n^2-6bcn^2C_0+a\lambda n+8a\lambda}{6cn^2}t+\theta\right)},\end{aligned} \tag{4.39}$$

and also

$$\begin{aligned}\Theta_{2,3}(x,t) &= \left\{C_0-\frac{\lambda a(n+2)}{2bcn^2}\tanh^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},\\ \Theta_{2,4}(x,t) &= \left\{C_0-\frac{\lambda a(n+2)}{2bcn^2}\coth^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},\end{aligned} \tag{4.40}$$

these solutions are valid when $\lambda c>0$.

Set I-2:

$$\begin{aligned}&A_0=-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}},\quad A_1=0,\ A_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}},\quad B_1=B_2=C_1=D_1=D_2=0,\\ &C_2=\frac{\lambda a(n+2)}{2bcn^2},\quad k=\sqrt{-\frac{\lambda}{4c}},\quad \kappa=\kappa,\quad \omega=-\frac{2ac\kappa^2n-2bcnC_0-a\lambda}{2nc}.\end{aligned} \tag{4.41}$$

Therefore, we conclude the following dark and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,5}(x,t) &= \left\{-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}\tanh^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x-\frac{2ac\kappa^2n-2bcnC_0-a\lambda}{2nc}t+\theta\right)},\\ \Lambda_{1,6}(x,t) &= \left\{-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4\alpha bc}}\coth^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x-\frac{2ac\kappa^2n-2bcnC_0-a\lambda}{2nc}t+\theta\right)},\end{aligned} \tag{4.42}$$

and also

$$\begin{aligned}\Theta_{2,5}(x,t) &= \left\{C_0+\frac{\lambda a(n+2)}{2bcn^2}\tanh^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},\\ \Theta_{2,6}(x,t) &= \left\{C_0+\frac{\lambda a(n+2)}{2bcn^2}\coth^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},\end{aligned} \tag{4.43}$$

these solutions are valid when $\lambda c<0$.

Set I-3:

$$\begin{aligned}&A_0=-\frac{2\lambda}{3n}\sqrt{\frac{3a(n+2)}{\alpha bc}},\quad A_1=0,\ A_2=B_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}},\quad B_1=C_1=D_1=0,\\ &C_2=D_2=-\frac{\lambda a(n+2)}{bcn^2},\quad k=\sqrt{\frac{\lambda}{c}},\quad \kappa=\kappa,\quad \omega=-\frac{3a\kappa^2n^2c-3bn^2C_0c+2a\lambda n+7a\lambda}{3n^2c}.\end{aligned} \tag{4.44}$$

Uncorrected Proof

Thus, we deduce the following combined dark-bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{1,7}(x,t)=\left\{-\frac{2\lambda}{3n}\sqrt{\frac{3a(n+2)}{\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\tanh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)\right.$$
$$\left.+i\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\frac{\sinh\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\cosh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right\}^{\frac{1}{n}}e^{i\left(-\kappa x-\frac{3a\kappa^2n^2c-3bn^2C_0c+2a\lambda n+7a\lambda}{3n^2c}t+\theta\right)},\tag{4.45}$$

and

$$\Lambda_{1,8}(x,t)=\left\{-\frac{2\lambda}{3n}\sqrt{\frac{3a(n+2)}{\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\coth^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)\right.$$
$$\left.-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\frac{\cosh\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\sinh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right\}^{\frac{1}{n}}e^{i\left(-\kappa x-\frac{3a\kappa^2n^2c-3bn^2C_0c+2a\lambda n+7a\lambda}{3n^2c}t+\theta\right)},\tag{4.46}$$

and also

$$\Theta_{2,7}(x,t)=\left\{C_0-\frac{\lambda a(n+2)}{bcn^2}\tanh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)-i\frac{\lambda a(n+2)}{bcn^2}\frac{\sinh\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\cosh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right\},$$
$$\Theta_{2,8}(x,t)=\left\{C_0-\frac{\lambda a(n+2)}{bcn^2}\coth^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)+\frac{\lambda a(n+2)}{bcn^2}\frac{\cosh\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\sinh^2\left(\sqrt{\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right\},\tag{4.47}$$

these solutions are valid when $\lambda c>0$.

Set I-4:

$$A_0=-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}},\quad A_1=0,\ A_2=B_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}},\quad B_1=C_1=D_1=0,$$
$$C_2=D_2=\frac{\lambda a(n+2)}{bcn^2},\quad k=\sqrt{-\frac{\lambda}{c}},\quad \kappa=\kappa,\quad \omega=-\frac{a\kappa^2n^2c-bn^2C_0c-a\lambda n-7a\lambda}{n^2c}.\tag{4.48}$$

Hence, we deduce the following combined dark-bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{1,9}(x,t)=\left\{-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\tanh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)\right.$$
$$\left.+i\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\frac{\sinh\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\cosh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right\}^{\frac{1}{n}}e^{i\left(-\kappa x-\frac{a\kappa^2n^2c-bn^2C_0c-a\lambda n-7a\lambda}{n^2c}t+\theta\right)},\tag{4.49}$$

and

$$\Lambda_{1,10}(x,t)=\left\{-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\coth^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)\right.$$

Uncorrected Proof

$$-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{\alpha bc}}\frac{\cosh\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\sinh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\Bigg\}^{\frac{1}{n}} e^{i\left(-\kappa x-\frac{a\kappa^2n^2c-bn^2C_0c-a\lambda n-7a\lambda}{n^2c}t+\theta\right)}, \tag{4.50}$$

and also

$$\begin{aligned}\Theta_{2,9}(x,t)&=\left\{C_0+\frac{\lambda a(n+2)}{bcn^2}\left[\tanh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)+i\frac{\sinh\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\cosh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right]\right\},\\ \Theta_{2,10}(x,t)&=\left\{C_0+\frac{\lambda a(n+2)}{bcn^2}\left[\coth^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)-\frac{\cosh\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}{\sinh^2\left(\sqrt{-\frac{\lambda}{c}}\left[x+2a\kappa t\right]\right)}\right]\right\},\end{aligned} \tag{4.51}$$

these solutions are valid when $\lambda c<0$.

4.2.2. *For Second Case.* By regarding on the Eqs. (2.17)-(2.18), the solutions of Eqs. (4.27) and (4.28) can be presented as

$$P(\xi)=\sum_{j=1}^{2}(\pm\sec(\xi))^{j-1}\left[B_j\tan(\xi)\pm A_j\sec(\xi)\right]+A_0, \tag{4.52}$$

and

$$P(\xi)=\sum_{j=1}^{2}(\pm\csc(\xi))^{j-1}\left[-B_j\cot(\xi)\pm A_j\csc(\xi)\right]+A_0, \tag{4.53}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+\cosh(w)[B_2\sinh(w)+A_2\cosh(w)]+A_0, \tag{4.54}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\sec(\xi))^{j-1}\left[D_j\tan(\xi)\pm C_j\sec(\xi)\right]+C_0, \tag{4.55}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\csc(\xi))^{j-1}\left[-D_j\cot(\xi)\pm C_j\csc(\xi)\right]+C_0, \tag{4.56}$$

and so

$$Q(w)=D_1\sinh(w)+A_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.57}$$

where $A_2\neq 0$ or $B_2\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.
Plugging (4.54), (4.57) and those derivatives into Eqs. (4.27) and (4.28), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned}&A_0=A_1=B_2=B_1=C_1=D_1=D_2=0,\quad A_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}},\\ &C_2=-\frac{a(n+2)\lambda}{2bcn^2},\quad k=\sqrt{\frac{\lambda}{4c}},\quad \kappa=\kappa,\quad \omega=-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}.\end{aligned} \tag{4.58}$$

Uncorrected Proof

Whereas, we conclude the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{2,3}(x,t)=\left\{\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}}\sec^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}t+\theta\right)},$$
$$\Lambda_{2,4}(x,t)=\left\{\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}}\csc^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}t+\theta\right)}, \tag{4.59}$$

and also

$$\Theta_{2,3}(x,t)=\left\{C_0-\frac{a(n+2)\lambda}{2bcn^2}\sec^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},$$
$$\Theta_{2,4}(x,t)=\left\{C_0-\frac{a(n+2)\lambda}{2bcn^2}\csc^2\left(\sqrt{\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}, \tag{4.60}$$

these solutions are valid when $\lambda c>0$.

Set I-2:

$$A_0=-\frac{\lambda a(n+2)}{bcn\alpha}\sqrt{\frac{bc\alpha}{3a(n+2)}},\quad A_1=B_2=B_1=C_1=D_1=D_2=0,\quad \kappa=\kappa,$$
$$A_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}},\ C_2=\frac{a(n+2)\lambda}{2bcn^2},\quad k=\sqrt{-\frac{\lambda}{4c}},\quad \omega=-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda(n-1)}{3cn^2}. \tag{4.61}$$

Therefore, we deduce the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{2,5}(x,t)=\left\{-\frac{\lambda a(n+2)}{bcn\alpha}\sqrt{\frac{bc\alpha}{3a(n+2)}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}}\sec^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}}$$
$$\times e^{i\left(-\kappa x-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda n+a\lambda}{3cn^2}t+\theta\right)},$$
$$\Lambda_{2,6}(x,t)=\left\{-\frac{\lambda a(n+2)}{bcn\alpha}\sqrt{\frac{bc\alpha}{3a(n+2)}}+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{4bc\alpha}}\csc^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}^{\frac{1}{n}} \tag{4.62}$$
$$\times e^{i\left(-\kappa x-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda n+a\lambda}{3cn^2}t+\theta\right)},$$

and also

$$\Theta_{2,5}(x,t)=\left\{C_0+\frac{a(n+2)\lambda}{2bcn^2}\sec^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\},$$
$$\Theta_{2,6}(x,t)=\left\{C_0+\frac{a(n+2)\lambda}{2bcn^2}\csc^2\left(\sqrt{-\frac{\lambda}{4c}}\,[x+2a\kappa t]\right)\right\}, \tag{4.63}$$

these solutions are valid when $\lambda c<0$.

Set I-3:

$$A_0=A_1=B_1=C_1=D_1=D_2=0,\quad A_2=B_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}},$$
$$C_2=D_2=-\frac{a(n+2)\lambda}{bcn^2},\quad k=\sqrt{\frac{\lambda}{c}},\quad \kappa=\kappa,\quad \omega=-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}. \tag{4.64}$$

Uncorrected Proof

Hence, we infer the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{2,7}(x,t) &= \left\{\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}}\right\}^{\frac{1}{n}}\left\{\sec^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)-\frac{\sin\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\cos^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right\}^{\frac{1}{n}}\\ &\quad\times e^{i\left(-\kappa x-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}t+\theta\right)},\\ \Lambda_{2,8}(x,t) &= \left\{\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}}\right\}^{\frac{1}{n}}\left\{\csc^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)+\frac{\cos\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\sin^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right\}^{\frac{1}{n}}\\ &\quad\times e^{i\left(-\kappa x-\frac{ac\kappa^2n^2-bcn^2C_0+a\lambda}{cn^2}t+\theta\right)},\end{aligned} \tag{4.65}$$

and also

$$\begin{aligned}\Theta_{2,7}(x,t) &= \left\{C_0-\frac{a(n+2)\lambda}{bcn^2}\left[\sec^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)-\frac{\sin\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\cos^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right]\right\},\\ \Theta_{2,8}(x,t) &= \left\{C_0-\frac{a(n+2)\lambda}{bcn^2}\left[\sec^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)-\frac{\cos\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\sin^2\left(\sqrt{\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right]\right\},\end{aligned} \tag{4.66}$$

these solutions are valid when $\lambda c>0$.

Set I-4:

$$\begin{aligned}&A_0=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}},\quad A_1=B_1=C_1=D_1=D_2=0,\\ &C_0=C_0,\ A_2=-\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}},\ C_2=\frac{a(n+2)\lambda}{bcn^2},\\ &B_2=\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}},\ D_2=-\frac{a(n+2)\lambda}{bcn^2},k=\sqrt{-\frac{\lambda}{c}},\\ &\kappa=\kappa,\quad \omega=-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda n+a\lambda}{3cn^2}.\end{aligned} \tag{4.67}$$

Thus, we conclude the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{2,9}(x,t) &= \left\{\sec^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)-\frac{\sin\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\cos^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right\}^{\frac{1}{n}}\\ &\quad\times\left\{1+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}}\right\}^{\frac{1}{n}}e^{i\left(-\kappa x-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda n+a\lambda}{3cn^2}t+\theta\right)},\end{aligned}$$

Uncorrected Proof

$$\Lambda_{2,10}(x,t)=\left\{\csc^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)+\frac{\cos\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\sin^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right\}^{\frac{1}{n}}$$
$$\times\left\{1+\frac{\lambda}{n}\sqrt{\frac{3a(n+2)}{bc\alpha}}\right\}^{\frac{1}{n}}e^{i\left(-\kappa x-\frac{3ac\kappa^2n^2-3bcn^2C_0-a\lambda n+a\lambda}{3cn^2}t+\theta\right)}, \tag{4.68}$$

and also

$$\Theta_{2,9}(x,t)=\left\{C_0+\frac{a(n+2)\lambda}{bcn^2}\left[\sec^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)-\frac{\sin\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\cos^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right]\right\},$$
$$\Theta_{2,10}(x,t)=\left\{C_0+\frac{a(n+2)\lambda}{bcn^2}\left[\csc^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)+\frac{\cos\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}{\sin^2\left(\sqrt{-\frac{\lambda}{c}}\,[x+2a\kappa t]\right)}\right]\right\}, \tag{4.69}$$

these solutions are valid when $\lambda c<0$.

4.3. **Parabolic law nonlinearity.** The power law nonlinearity arises when $F(r)=c_1r+c_2r^2$. Therefore, (3.7) and (3.8) transform to

$$ak^2P''-(a\kappa^2+\omega)P+bPQ=0, \tag{4.70}$$

$$ck^2Q''+\lambda Q+\alpha(c_1P^2+c_2P^4)=0. \tag{4.71}$$

By balancing P'' with PQ and Q'' with P^4, and considering the below relations

$$N+2=N+M, \tag{4.72}$$

$$M+2=4N, \tag{4.73}$$

we get the following

$$M=2,\quad N=1. \tag{4.74}$$

4.3.1. *For First Case.* Base on the Eqs. (2.14)-(2.16), the solutions of Eqs. (4.70) and (4.71) can be found as

$$P(\xi)=\sum_{j=1}^{1}(-\tanh(\xi))^{j-1}\left[\pm iB_j\ \mathrm{sech}(\xi)-A_j\tanh(\xi)\right]+A_0, \tag{4.75}$$

$$P(\xi)=\sum_{j=1}^{1}(-\coth(\xi))^{j-1}\left[\pm B_j\mathrm{csch}(\xi)-A_j\coth(\xi)\right]+A_0, \tag{4.76}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+A_0, \tag{4.77}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(-\tanh(\xi))^{j-1}\left[\pm iD_j\ \mathrm{sech}(\xi)-C_j\tanh(\xi)\right]+C_0, \tag{4.78}$$

Uncorrected Proof

$$Q(\xi)=\sum_{j=1}^{2}(-\coth(\xi))^{j-1}\left[\pm D_j \operatorname{csch}(\xi)-C_j\coth(\xi)\right]+C_0, \tag{4.79}$$

and so

$$Q(w)=D_1\sinh(w)+C_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.80}$$

where $A_1\neq 0$ or $B_1\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.
Plugging (4.77), (4.80), and those derivatives into Eqs. (4.70) and (4.71), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned} &a=\frac{3c_1^2c\alpha b}{c_2(8ck^2-\lambda)^2},\quad A_1=k\sqrt{-\frac{6cc_1}{8ck^2c_2-\lambda c_2}}, A_0=B_1=C_1=D_1=D_2=0,\ \kappa=\kappa,\\ &C_2=-\frac{6c_1^2c\alpha k^2}{c_2(8ck^2-\lambda)^2},\quad \omega=-\frac{b(-64c^2k^4C_0c_2+6\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+16ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(8ck^2-\lambda)^2}. \end{aligned} \tag{4.81}$$

Hence, we infer the following dark and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned} \Lambda_{1,1}(x,t)&=\left\{-k\sqrt{-\frac{6cc_1}{8ck^2c_2-\lambda c_2}}\tanh(k[x+2a\kappa t])\right\}\times e^{i\left(-\kappa x-\frac{b(-64c^2k^4C_0c_2+6\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+16ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(8ck^2-\lambda)^2}t+\theta\right)},\\ \Lambda_{1,2}(x,t)&=\left\{-k\sqrt{-\frac{6cc_1}{8ck^2c_2-\lambda c_2}}\coth(k[x+2a\kappa t])\right\}\times e^{i\left(-\kappa x-\frac{b(-64c^2k^4C_0c_2+6\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+16ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(8ck^2-\lambda)^2}t+\theta\right)}, \end{aligned} \tag{4.82}$$

and also

$$\begin{aligned} \Theta_{2,1}(x,t)&=\left\{C_0-\frac{6c_1^2c\alpha k^2}{c_2(8ck^2-\lambda)^2}\tanh^2(k[x+2a\kappa t])\right\},\\ \Theta_{2,2}(x,t)&=\left\{C_0-\frac{6c_1^2c\alpha k^2}{c_2(8ck^2-\lambda)^2}\coth^2(k[x+2a\kappa t])\right\}, \end{aligned} \tag{4.83}$$

these solutions are valid when $\lambda\neq 8ck^2$ and $cc_1c_2(8ck^2-\lambda)<0$.
Set I-2:

$$\begin{aligned} &a=\frac{3c_1^2c\alpha b}{c_2(4ck^2+\lambda)^2},\quad B_1=k\sqrt{\frac{6cc_1}{4ck^2c_2+\lambda c_2}}, A_0=A_1=C_1=D_1=D_2=0,\\ &C_2=-\frac{6c_1^2c\alpha k^2}{c_2(4ck^2+\lambda)^2}, \omega=-\frac{b(-16c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2-8ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(4ck^2+\lambda)^2}. \end{aligned} \tag{4.84}$$

Whereas, we conclude the following bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned} \Lambda_{1,3}(x,t)&=\left\{-ik\sqrt{\frac{6cc_1}{4ck^2c_2+\lambda c_2}}sech(k[x+2a\kappa t])\right\}\times e^{i\left(-\kappa x-\frac{b(-16c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2-8ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(4ck^2+\lambda)^2}t+\theta\right)},\\ \Lambda_{1,4}(x,t)&=\left\{k\sqrt{\frac{6cc_1}{4ck^2c_2+\lambda c_2}}csch(k[x+2a\kappa t])\right\}\times e^{i\left(-\kappa x-\frac{b(-16c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2-8ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(4ck^2+\lambda)^2}t+\theta\right)}, \end{aligned} \tag{4.85}$$

and also

$$\begin{aligned} \Theta_{2,3}(x,t)&=\left\{C_0-\frac{6c_1^2c\alpha k^2}{c_2(4ck^2+\lambda)^2}\tanh^2(k[x+2a\kappa t])\right\},\\ \Theta_{2,4}(x,t)&=\left\{C_0-\frac{6c_1^2c\alpha k^2}{c_2(4ck^2+\lambda)^2}\coth^2(k[x+2a\kappa t])\right\}, \end{aligned} \tag{4.86}$$

Uncorrected Proof

these solutions are valid when $\lambda \neq -4ck^2$ and $cc_1c_2(4ck^2+\lambda) > 0$.

Set I-3:

$$\begin{aligned} a &= \frac{3c_1^2 c\alpha b}{c_2(2ck^2-\lambda)^2}, \quad A_1 = B_1 = k\sqrt{-\frac{3cc_1}{4ck^2c_2-2\lambda c_2}}, A_0 = C_1 = D_1 = 0, \\ C_2 &= D_2 = -\frac{3c_1^2 c\alpha k^2}{c_2(2ck^2-\lambda)^2}, \ \omega = -\frac{b(-4c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+4ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(2ck^2-\lambda)^2}. \end{aligned} \tag{4.87}$$

Thus, we infer the following combined dark-bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned} \Lambda_{1,5}(x,t) &= -k\sqrt{-\frac{3cc_1}{4ck^2c_2-2\lambda c_2}}\left\{\tanh\left(k\left[x+2a\kappa t\right]\right)+i\ sech\left(k\left[x+2a\kappa t\right]\right)\right\} \\ &\quad\times e^{i\left(-\kappa x-\frac{b(-4c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+4ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(2ck^2-\lambda)^2}t+\theta\right)}, \\ \Lambda_{1,6}(x,t) &= -k\sqrt{-\frac{3cc_1}{4ck^2c_2-2\lambda c_2}}\left\{\coth\left(k\left[x+2a\kappa t\right]\right)-csch\left(k\left[x+2a\kappa t\right]\right)\right\} \\ &\quad\times e^{i\left(-\kappa x-\frac{b(-4c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+4ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(2ck^2-\lambda)^2}t+\theta\right)}, \end{aligned} \tag{4.88}$$

and also

$$\begin{aligned} \Theta_{2,5}(x,t) &= \left\{C_0-\frac{3c_1^2c\alpha k^2}{c_2(2ck^2-\lambda)^2}\left[\tanh^2\left(k[x+2a\kappa t]\right)-i\frac{\sinh\left(k[x+2a\kappa t]\right)}{\cosh^2\left(k[x+2a\kappa t]\right)}\right]\right\}, \\ \Theta_{2,6}(x,t) &= \left\{C_0-\frac{3c_1^2c\alpha k^2}{c_2(2ck^2-\lambda)^2}\left[\coth^2\left(k[x+2a\kappa t]\right)-\frac{\cosh\left(k[x+2a\kappa t]\right)}{\sinh^2\left(k[x+2a\kappa t]\right)}\right]\right\}, \end{aligned} \tag{4.89}$$

these solutions are valid when $\lambda \neq 2ck^2$ and $cc_1c_2(2ck^2-\lambda) > 0$.

4.3.2. *For Second Case.* By regarding on the Eqs. (2.17)-(2.18), the solutions of Eqs. (4.70) and (4.71) can be written as

$$P(\xi) = \sum_{j=1}^{1}\left(\pm\sec(\xi)\right)^{j-1}\left[B_j\tan(\xi)\pm A_j\sec(\xi)\right]+A_0, \tag{4.90}$$

and

$$P(\xi) = \sum_{j=1}^{1}\left(\pm\csc(\xi)\right)^{j-1}\left[-B_j\cot(\xi)\pm A_j\csc(\xi)\right]+A_0, \tag{4.91}$$

and so

$$P(w) = B_1\sinh(w)+A_1\cosh(w)+\cosh(w)[B_2\sinh(w)+A_2\cosh(w)]+A_0, \tag{4.92}$$

and

$$Q(\xi) = \sum_{j=1}^{2}\left(\pm\sec(\xi)\right)^{j-1}\left[D_j\tan(\xi)\pm C_j\sec(\xi)\right]+C_0, \tag{4.93}$$

and

$$Q(\xi) = \sum_{j=1}^{2}\left(\pm\csc(\xi)\right)^{j-1}\left[-D_j\cot(\xi)\pm C_j\csc(\xi)\right]+C_0, \tag{4.94}$$

and so

$$Q(w) = D_1\sinh(w)+A_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.95}$$

Uncorrected Proof

where $A_1 \neq 0$ or $B_1 \neq 0$ and $C_2 \neq 0$ or $D_2 \neq 0$.
Plugging (4.92), (4.95) and those derivatives into Eqs. (4.70) and (4.71), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned} a &= \frac{3c_1^2 c\alpha b}{c_2(4ck^2-\lambda)^2}, \quad A_1 = k\sqrt{-\frac{6cc_1}{4ck^2c_2-\lambda c_2}}, A_0 = B_1 = C_1 = D_1 = D_2 = 0, \\ C_2 &= -\frac{6c_1^2 c\alpha k^2}{c_2(4ck^2-\lambda)^2}, \quad \omega = -\frac{b(-16c^2k^4C_0c_2 + 3\alpha ck^2c_1^2 + 3\alpha c\kappa^2 c_1^2 + 8ck^2\lambda C_0c_2 - \lambda^2 C_0c_2)}{c_2(4ck^2-\lambda)^2}. \end{aligned} \tag{4.96}$$

Therefore, we infer the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned} \Lambda_{2,7}(x,t) &= \left\{-k\sqrt{-\frac{6cc_1}{4ck^2c_2-\lambda c_2}}\sec\left(k\left[x+2a\kappa t\right]\right)\right\} \times e^{i\left(-\kappa x - \frac{b(-16c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+8ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(4ck^2-\lambda)^2}t+\theta\right)}, \\ \Lambda_{2,8}(x,t) &= \left\{-k\sqrt{-\frac{6cc_1}{4ck^2c_2-\lambda c_2}}\csc\left(k\left[x+2a\kappa t\right]\right)\right\} \times e^{i\left(-\kappa x - \frac{b(-16c^2k^4C_0c_2+3\alpha ck^2c_1^2+3\alpha c\kappa^2c_1^2+8ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(4ck^2-\lambda)^2}t+\theta\right)}, \end{aligned} \tag{4.97}$$

and also

$$\begin{aligned} \Theta_{2,7}(x,t) &= \left\{C_0 - \frac{6c_1^2c\alpha k^2}{c_2(4ck^2-\lambda)^2}\sec^2\left(k\left[x+2a\kappa t\right]\right)\right\}, \\ \Theta_{2,8}(x,t) &= \left\{C_0 - \frac{6c_1^2c\alpha k^2}{c_2(4ck^2-\lambda)^2}\csc^2\left(k\left[x+2a\kappa t\right]\right)\right\}, \end{aligned} \tag{4.98}$$

these solutions are valid when $\lambda \neq 4ck^2$ and $cc_1c_2(4ck^2-\lambda) < 0$.
Set I-2:

$$\begin{aligned} a &= \frac{3c_1^2c\alpha b}{c_2(8ck^2+\lambda)^2}, \quad B_1 = k\sqrt{\frac{6cc_1}{8ck^2c_2+\lambda c_2}}, \quad A_0 = A_1 = C_1 = D_1 = D_2 = 0, \\ C_2 &= -\frac{6c_1^2c\alpha k^2}{c_2(8ck^2+\lambda)^2}, \quad \omega = -\frac{b(-64c^2k^4C_0c_2 + 3\alpha ck^2c_1^2 - 16ck^2\lambda C_0c_2 - \lambda^2C_0c_2)}{c_2(8ck^2+\lambda)^2}. \end{aligned} \tag{4.99}$$

Thus, we deduce the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned} \Lambda_{2,9}(x,t) &= \left\{k\sqrt{\frac{6cc_1}{8ck^2c_2+\lambda c_2}}\tan\left(k\left[x+2a\kappa t\right]\right)\right\} \times e^{i\left(-\kappa x - \frac{b(-64c^2k^4C_0c_2+3\alpha ck^2c_1^2-16ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(8ck^2+\lambda)^2}t+\theta\right)}, \\ \Lambda_{2,10}(x,t) &= \left\{k\sqrt{\frac{6cc_1}{8ck^2c_2+\lambda c_2}}\cot\left(k\left[x+2a\kappa t\right]\right)\right\} \times e^{i\left(-\kappa x - \frac{b(-64c^2k^4C_0c_2+3\alpha ck^2c_1^2-16ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(8ck^2+\lambda)^2}t+\theta\right)}, \end{aligned} \tag{4.100}$$

and also

$$\begin{aligned} \Theta_{2,9}(x,t) &= \left\{C_0 - \frac{6c_1^2c\alpha k^2}{c_2(8ck^2+\lambda)^2}\sec^2\left(k\left[x+2a\kappa t\right]\right)\right\}, \\ \Theta_{2,10}(x,t) &= \left\{C_0 - \frac{6c_1^2c\alpha k^2}{c_2(8ck^2+\lambda)^2}\csc^2\left(k\left[x+2a\kappa t\right]\right)\right\}, \end{aligned} \tag{4.101}$$

these solutions are valid when $\lambda \neq -8ck^2$ and $cc_1c_2(8ck^2+\lambda) > 0$.
Set I-3:

$$\begin{aligned} a &= \frac{3c_1^2c\alpha b}{c_2(2ck^2+\lambda)^2}, \quad A_1 = B_1 = k\sqrt{\frac{3cc_1}{2ck^2c_2+\lambda c_2}}, \quad A_0 = C_1 = D_1 = 0, \\ C_2 &= D_2 = -\frac{3c_1^2c\alpha k^2}{c_2(2ck^2+\lambda)^2}, \quad \omega = -\frac{b(-4c^2k^4C_0c_2 + 3\alpha ck^2c_1^2 - 4ck^2\lambda C_0c_2 - \lambda^2C_0c_2)}{c_2(2ck^2+\lambda)^2}. \end{aligned} \tag{4.102}$$

Uncorrected Proof

Hence, we conclude the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{2,11}(x,t) &= k\sqrt{\frac{3cc_1}{2ck^2c_2+\lambda c_2}}\left\{\tan\left(k\left[x+2a\kappa t\right]\right)-\sec\left(k\left[x+2a\kappa t\right]\right)\right\}\\ &\quad\times e^{i\left(-\kappa x-\frac{b(-4c^2k^4C_0c_2+3\alpha ck^2c_1^2-4ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(2ck^2+\lambda)^2}t+\theta\right)},\\ \Lambda_{2,12}(x,t) &= k\sqrt{\frac{3cc_1}{2ck^2c_2+\lambda c_2}}\left\{-\cot\left(k\left[x+2a\kappa t\right]\right)-\csc\left(k\left[x+2a\kappa t\right]\right)\right\}\\ &\quad\times e^{i\left(-\kappa x-\frac{b(-4c^2k^4C_0c_2+3\alpha ck^2c_1^2-4ck^2\lambda C_0c_2-\lambda^2C_0c_2)}{c_2(2ck^2+\lambda)^2}t+\theta\right)},\end{aligned} \tag{4.103}$$

and also

$$\begin{aligned}\Theta_{2,11}(x,t) &= \left\{C_0-\frac{3c_1^2c\alpha k^2}{c_2(2ck^2+\lambda)^2}\left[\sec^2\left(k\left[x+2a\kappa t\right]\right)-\frac{\sin\left(k\left[x+2a\kappa t\right]\right)}{\cos^2\left(k\left[x+2a\kappa t\right]\right)}\right]\right\},\\ \Theta_{2,12}(x,t) &= \left\{C_0-\frac{3c_1^2c\alpha k^2}{c_2(2ck^2+\lambda)^2}\left[\csc^2\left(k\left[x+2a\kappa t\right]\right)-\frac{\cos\left(k\left[x+2a\kappa t\right]\right)}{\sin^2\left(k\left[x+2a\kappa t\right]\right)}\right]\right\},\end{aligned} \tag{4.104}$$

these solutions are valid when $\lambda\neq-2ck^2$ and $cc_1c_2(2ck^2+\lambda)>0$.

4.4. **Dual-power law nonlinearity.** The power law nonlinearity arises when $F(r)=c_1r^n+c_2r^{2n}$. Therefore, (3.7) and (3.8) transform to

$$ak^2P''-(a\kappa^2+\omega)P+bPQ=0, \tag{4.105}$$

$$ck^2Q''+\lambda Q+\alpha(c_1P^{2n}+c_2P^{4n})=0. \tag{4.106}$$

Suppose

$$P=R^{\frac{1}{n}}, \tag{4.107}$$

so that (4.105) and (4.106) reduce to

$$ak^2(nRR''+(1-n)R'^2)-n^2(a\kappa^2+\omega)R^2+n^2bR^2Q=0, \tag{4.108}$$

$$ck^2Q''+\lambda Q+\alpha R^4=0. \tag{4.109}$$

By balancing RR'' with R^2Q and Q'' with R^4, and considering the below relations

$$2N+2=2N+M, \tag{4.110}$$

$$M+2=4N, \tag{4.111}$$

we get the following

$$M=2,\quad N=1. \tag{4.112}$$

4.4.1. *For First Case.* By regarding on the Eqs. (2.14)-(2.16), the solutions of Eqs. (4.108) and (4.109) can be presented as

$$P(\xi)=\sum_{j=1}^{1}(-\tanh(\xi))^{j-1}\left[\pm iB_j\ \mathrm{sech}(\xi)-A_j\tanh(\xi)\right]+A_0, \tag{4.113}$$

$$P(\xi)=\sum_{j=1}^{1}(-\coth(\xi))^{j-1}\left[\pm B_j\mathrm{csch}(\xi)-A_j\coth(\xi)\right]+A_0, \tag{4.114}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+A_0, \tag{4.115}$$

Uncorrected Proof

and

$$Q(\xi)=\sum_{j=1}^{2}(-\tanh(\xi))^{j-1}\left[\pm iD_j\ \mathrm{sech}(\xi)-C_j\tanh(\xi)\right]+C_0, \tag{4.116}$$

$$Q(\xi)=\sum_{j=1}^{2}(-\coth(\xi))^{j-1}\left[\pm D_j\mathrm{csch}(\xi)-C_j\coth(\xi)\right]+C_0, \tag{4.117}$$

and so

$$Q(w)=D_1\sinh(w)+C_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.118}$$

where $A_1\neq 0$ or $B_1\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.
Plugging (4.115), (4.118) and those derivatives into Eqs. (4.108) and (4.109), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{aligned}&A_0=0,\quad A_1=B_1=\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}},C_1=D_1=0,\quad \omega=\omega,\\ &C_2=D_2=-\frac{a(n+1)\lambda}{4bcn^2},\quad k=\sqrt{\frac{\lambda}{2c}},\quad \kappa=\frac{1}{n}\sqrt{\frac{8bcn^2C_0-8cn^2\omega-a\lambda n-3a\lambda}{8ac}}.\end{aligned} \tag{4.119}$$

Therefore, we conclude the following combined dark-bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,1}(x,t)=&\left\{-\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\right\}^{\frac{1}{n}}\left\{\tanh\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\ &\left.+i\,sech\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right\}^{\frac{1}{n}}e^{i\left(-\frac{1}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}x+\omega t+\theta\right)},\\ \Lambda_{1,2}(x,t)=&\left\{-\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\right\}^{\frac{1}{n}}\left\{\coth\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\ &\left.-csch\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right\}^{\frac{1}{n}}e^{i\left(-\frac{1}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}x+\omega t+\theta\right)},\end{aligned} \tag{4.120}$$

and also

$$\begin{aligned}\Theta_{2,1}(x,t)=&C_0-\frac{a(n+1)\lambda}{4bcn^2}\left\{\tanh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\ &\left.+i\,\frac{\sinh\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}{\cosh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}\right\},\\ \Theta_{2,2}(x,t)=&C_0-\frac{a(n+1)\lambda}{4bcn^2}\left\{\coth^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\ &\left.-\frac{\cosh\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}{\sinh^2\left(\sqrt{\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}\right\},\end{aligned} \tag{4.121}$$

Uncorrected Proof

these solutions are valid when $\lambda c > 0$.

Set I-2:

$$B_1 = \sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}}, A_0 = A_1 = C_1 = D_1 = D_2 = 0,$$
$$C_2 = \frac{a(n+1)\lambda}{4bcn^2}, \quad k = \sqrt{-\frac{\lambda}{4c}}, \quad \kappa = \sqrt{\frac{4bcnC_0 - 4cn\omega + a\lambda}{4acn}}, \quad \omega = \omega. \tag{4.122}$$

Hence, we infer the following bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{1,3}(x,t) = \left\{-i\sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}}sech\left(\sqrt{-\frac{\lambda}{4c}}\left[x + 2a\sqrt{\frac{4bcnC_0 - 4cn\omega + a\lambda}{4acn}}\,t\right]\right)\right\}^{\frac{1}{n}} \times e^{i\left(-\sqrt{\frac{4bcnC_0-4cn\omega+a\lambda}{4acn}}x+\omega t+\theta\right)},$$
$$\Lambda_{1,4}(x,t) = \left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}}csch\left(\sqrt{-\frac{\lambda}{4c}}\left[x + 2a\sqrt{\frac{4bcnC_0 - 4cn\omega + a\lambda}{4acn}}\,t\right]\right)\right\}^{\frac{1}{n}} \times e^{i\left(-\sqrt{\frac{4bcnC_0-4cn\omega+a\lambda}{4acn}}x+\omega t+\theta\right)}, \tag{4.123}$$

and also

$$\Theta_{2,3}(x,t) = \left\{C_0 + \frac{a(n+1)\lambda}{4bcn^2}\tanh^2\left(\sqrt{-\frac{\lambda}{4c}}\left[x + 2a\sqrt{\frac{4bcnC_0 - 4cn\omega + a\lambda}{4acn}}\,t\right]\right)\right\},$$
$$\Theta_{2,4}(x,t) = \left\{C_0 + \frac{a(n+1)\lambda}{4bcn^2}\coth^2\left(\sqrt{-\frac{\lambda}{4c}}\left[x + 2a\sqrt{\frac{4bcnC_0 - 4cn\omega + a\lambda}{4acn}}\,t\right]\right)\right\}, \tag{4.124}$$

these solutions are valid when $\lambda c < 0$.

Set I-3:

$$A_1 = \sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}, A_0 = B_1 = C_1 = D_1 = D_2 = 0,$$
$$C_2 = -\frac{a(n+1)\lambda}{8bcn^2}, \quad k = \sqrt{\frac{\lambda}{8c}}, \quad \kappa = \frac{1}{n}\sqrt{\frac{4bcn^2C_0 - 4cn^2\omega - a\lambda}{4ac}}, \quad \omega = \omega. \tag{4.125}$$

Whereas, we conclude the following bright and singular soliton solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{1,5}(x,t) = \left\{-\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\tanh\left(\sqrt{\frac{\lambda}{8c}}\left[x + \frac{2a}{n}\sqrt{\frac{4bcn^2C_0 - 4cn^2\omega - a\lambda}{4ac}}\,t\right]\right)\right\}^{\frac{1}{n}}$$
$$\times e^{i\left(-\frac{1}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}x+\omega t+\theta\right)},$$
$$\Lambda_{1,6}(x,t) = \left\{-\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\coth\left(\sqrt{\frac{\lambda}{8c}}\left[x + \frac{2a}{n}\sqrt{\frac{4bcn^2C_0 - 4cn^2\omega - a\lambda}{4ac}}\,t\right]\right)\right\}^{\frac{1}{n}} \tag{4.126}$$
$$\times e^{i\left(-\frac{1}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}x+\omega t+\theta\right)},$$

and also

$$\Theta_{2,5}(x,t) = \left\{C_0 - \frac{a(n+1)\lambda}{8bcn^2}\tanh^2\left(\sqrt{\frac{\lambda}{8c}}\left[x + \frac{2a}{n}\sqrt{\frac{4bcn^2C_0 - 4cn^2\omega - a\lambda}{4ac}}\,t\right]\right)\right\},$$
$$\Theta_{2,4}(x,t) = \left\{C_0 - \frac{a(n+1)\lambda}{8bcn^2}\coth^2\left(\sqrt{\frac{\lambda}{8c}}\left[x + \frac{2a}{n}\sqrt{\frac{4bcn^2C_0 - 4cn^2\omega - a\lambda}{4ac}}\,t\right]\right)\right\}, \tag{4.127}$$

these solutions are valid when $\lambda c > 0$.

Uncorrected Proof

4.4.2. *For Second Case.* By regarding on the Eqs. (2.17)-(2.18), the solutions of Eqs. (4.108) and (4.109) can be found as

$$P(\xi)=\sum_{j=1}^{1}(\pm\sec(\xi))^{j-1}\left[B_j\tan(\xi)\pm A_j\sec(\xi)\right]+A_0, \tag{4.128}$$

and

$$P(\xi)=\sum_{j=1}^{1}(\pm\csc(\xi))^{j-1}\left[-B_j\cot(\xi)\pm A_j\csc(\xi)\right]+A_0, \tag{4.129}$$

and so

$$P(w)=B_1\sinh(w)+A_1\cosh(w)+\cosh(w)[B_2\sinh(w)+A_2\cosh(w)]+A_0, \tag{4.130}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\sec(\xi))^{j-1}\left[D_j\tan(\xi)\pm C_j\sec(\xi)\right]+C_0, \tag{4.131}$$

and

$$Q(\xi)=\sum_{j=1}^{2}(\pm\csc(\xi))^{j-1}\left[-D_j\cot(\xi)\pm C_j\csc(\xi)\right]+C_0, \tag{4.132}$$

and so

$$Q(w)=D_1\sinh(w)+A_1\cosh(w)+\cosh(w)[D_2\sinh(w)+C_2\cosh(w)]+C_0, \tag{4.133}$$

where $A_1\neq 0$ or $B_1\neq 0$ and $C_2\neq 0$ or $D_2\neq 0$.
Plugging (4.130), (4.133) and those derivatives into Eqs. (4.108) and (4.109), we can obtain the following results by solving the nonlinear algebra system with the help of symbolic computation package as:
Set I-1:

$$\begin{gathered}A_1=-B_1=\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}},A_0=C_1=D_1=0,\ \omega=\omega,\\ C_2=-D_2=\frac{a(n+1)\lambda}{4bcn^2},\quad k=\sqrt{-\frac{\lambda}{2c}},\quad \kappa=\frac{1}{n}\sqrt{\frac{8bcn^2C_0-8cn^2\omega-a\lambda n-3a\lambda}{8ac}}.\end{gathered} \tag{4.134}$$

Therefore, we infer the following combined periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,7}(x,t)=&\left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\right\}^{\frac{1}{n}}\left\{\tan\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\&\left.+\sec\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right\}^{\frac{1}{n}}e^{i\left(-\frac{1}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}x+\omega t+\theta\right)},\\ \Lambda_{1,8}(x,t)=&\left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\right\}^{\frac{1}{n}}\left\{-\cot\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.\\&\left.+\csc\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right\}^{\frac{1}{n}}e^{i\left(-\frac{1}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}x+\omega t+\theta\right)},\end{aligned} \tag{4.135}$$

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and also

$$\Theta_{2,7}(x,t)=C_0+\frac{a(n+1)\lambda}{4bcn^2}\left\{\tan^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)\right.$$
$$\left.+\frac{\sin\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}{\cos^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8cn^2(bC_0-\omega)-a\lambda(n+3)}{8ac}}t\right]\right)}\right\},$$
$$\Theta_{2,8}(x,t)=C_0+\frac{a(n+1)\lambda}{4bcn^2}\left\{\cot^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcn^2C_0-8cn^2\omega-a\lambda n-3a\lambda}{8ac}}t\right]\right)\right.$$
$$\left.-\frac{\cos\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcn^2C_0-8cn^2\omega-a\lambda n-3a\lambda}{8ac}}t\right]\right)}{\sin^2\left(\sqrt{-\frac{\lambda}{2c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcn^2C_0-8cn^2\omega-a\lambda n-3a\lambda}{8ac}}t\right]\right)}\right\}, \tag{4.136}$$

these solutions are valid when $\lambda c<0$.

Set I-2:

$$B_1=\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}},A_0=A_1=C_1=D_1=D_2=0,$$
$$C_2=\frac{a(n+1)\lambda}{8bcn^2},\quad k=\sqrt{-\frac{\lambda}{8c}},\quad \kappa=\frac{1}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}},\quad \omega=\omega. \tag{4.137}$$

Hence, we conclude the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\Lambda_{1,9}(x,t)=\left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\tan\left(\sqrt{-\frac{\lambda}{8c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}\,t\right]\right)\right\}^{\frac{1}{n}}$$
$$\times e^{i\left(-\frac{1}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}x+\omega t+\theta\right)},$$
$$\Lambda_{1,10}(x,t)=\left\{-\sqrt[4]{\frac{3a\lambda^2(n+1)}{32bcn^2\alpha}}\cot\left(\sqrt{-\frac{\lambda}{8c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}\,t\right]\right)\right\}^{\frac{1}{n}} \tag{4.138}$$
$$\times e^{i\left(-\frac{1}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}x+\omega t+\theta\right)},$$

and also

$$\Theta_{2,9}(x,t)=\left\{C_0+\frac{a(n+1)\lambda}{8bcn^2}\sec^2\left(\sqrt{-\frac{\lambda}{8c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}\,t\right]\right)\right\},$$
$$\Theta_{2,10}(x,t)=\left\{C_0+\frac{a(n+1)\lambda}{8bcn^2}\csc^2\left(\sqrt{-\frac{\lambda}{8c}}\left[x+\frac{2a}{n}\sqrt{\frac{8bcnC_0-8cn\omega+a\lambda n-a\lambda}{8ac}}\,t\right]\right)\right\}, \tag{4.139}$$

these solutions are valid when $\lambda c<0$.

Set I-3:

$$A_1=\sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}},A_0=B_1=C_1=D_1=D_2=0,$$
$$C_2=-\frac{a(n+1)\lambda}{4bcn^2},\quad k=\sqrt{\frac{\lambda}{4c}},\quad \kappa=\frac{1}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}},\quad \omega=\omega. \tag{4.140}$$

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Whereas, we obtain the following periodic and singular periodic solutions for the Eqs. (3.1) and (3.2) respectively:

$$\begin{aligned}\Lambda_{1,11}(x,t) &= \left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}}\sec\left(\sqrt{\frac{\lambda}{4c}}\left[x+\frac{2a}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}\,t\right]\right)\right\}^{\frac{1}{n}}\\ &\quad\times e^{i\left(-\frac{1}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}x+\omega t+\theta\right)},\\ \Lambda_{1,12}(x,t) &= \left\{\sqrt[4]{\frac{3a\lambda^2(n+1)}{8bcn^2\alpha}}\csc\left(\sqrt{\frac{\lambda}{4c}}\left[x+\frac{2a}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}\,t\right]\right)\right\}^{\frac{1}{n}}\\ &\quad\times e^{i\left(-\frac{1}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}x+\omega t+\theta\right)},\end{aligned} \tag{4.141}$$

$$\begin{aligned}\Theta_{2,11}(x,t) &= \left\{C_0-\frac{a(n+1)\lambda}{4bcn^2}\sec^2\left(\sqrt{\frac{\lambda}{4c}}\left[x+\frac{2a}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}\,t\right]\right)\right\},\\ \Theta_{2,12}(x,t) &= \left\{C_0-\frac{a(n+1)\lambda}{4bcn^2}\csc^2\left(\sqrt{\frac{\lambda}{4c}}\left[x+\frac{2a}{n}\sqrt{\frac{4bcn^2C_0-4cn^2\omega-a\lambda}{4ac}}\,t\right]\right)\right\},\end{aligned} \tag{4.142}$$

these solutions are valid when $\lambda c > 0$.

5. Results

By implementation of the extended sinh–Gordon equation expansion method on a model with four law nonlinearities we get many new forms of exact traveling wave solutions. By making a simple comparison between the results, we can see in [22] has been obtained only solutions to all cases in which are given in solutions including (34)-(38) for Kerr nonlinearity, solutions (67)-(76) for power nonlinearity, solutions (95)-(100) for power nonlinearity and solution (118) for dual-power nonlinearity. Also, authors of [6] used the extended trial equation method and gained shock waves, singular solitons, snoidal waves, periodic singular waves and plane waves solutions. While with observing all obtained results in this paper, we can note how the closed between results that obtained by using this method and other methods.

6. Conclusion

The structure of new exact traveling wave solutions of nematicons in liquid crystals with four law nonlinearity including Kerr, power, parabolic and dual-power by utilizing the extended sinh–Gordon equation expansion method were illustrated. The dark, bright, combined dark-bright, singular types, traveling and solitary solutions of nematicons in liquid crystals were obtained. To describe the nonlinear dynamics of complex physical phenomena, the extended sinh–Gordon equation expansion method offers a reliable and efficient framework. By introducing new analytical solutions, this work significantly advances the understanding of wave dynamics, thereby establishing a foundation for future theoretical and applied investigations of soliton molecules in liquid crystals.

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