Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-24
DOI:10.22034/cmde.2026.70992.3560

A high-order numerical method for Caputo fractional Volterra integro-differential equations

Aynalem Tafere Chekole[1], Mesfin Mekuria Woldaregay[2], Kewani Welay Brhane[1], and Gemechis File Duressa[3]
1Department of Mathematics, Mekelle University, Mekelle, Ethiopia.
[2]Department of Applied Mathematics, Adama Science and Technology University, Adama, Ethiopia.
[3]Department of Mathematics, Jimma University, Jimma, Ethiopia.

Abstract

In this paper, we study fractional Volterra integro-differential equations with the Caputo derivative of order $\gamma \in (0,1)$. The domain is discretized using a uniform mesh. To approximate the solution, the fractional derivative is discretized using the L1-2-3 method, while Simpson's $\frac{1}{3}$ approximation is applied to approximate the Volterra integral term. The proposed method achieves third-order convergence for all $\gamma \in (0,1)$. The stability and convergence analysis are analyzed and realize that the approximate solution converges to the exact solution. The proposed method is compared with the results in the literature through examples supported by graphs and tables, and the results show that it achieves higher accuracy and order of convergence.

Keywords. Fractional integro-differential equations, Caputo derivative, L1-2-3 scheme, Convergence analysis.
2010 Mathematics Subject Classification. 34A08, 65L05, 65R20.

Uncorrected

1. Introduction

Fractional order derivatives extend the classical derivatives and are intrinsically nonlocal, as they are defined by integrals over the whole domain, in contrast to classical derivatives, which are local operators determined by an infinitesimal neighborhood around a point. The foundations of differential and integral calculus were developed by Newton and Leibniz at the end of the seventeenth century. Fractional calculus enables a more accurate and effective analysis of complex behaviors in systems exhibiting memory effects, anomalous diffusion, and long-range dependencies. Over time, this subject has developed into a more potent and effective mathematical tool for simulating various physical phenomena. To list a few applications, the theory has been used in fluid mechanics [21], viscoelasticity[4, 36], rigid porous materials [10], and an autonomous electric vehicle [37].

Recent years have seen increasing interest in integro-differential equations which serve as models in many scientific and engineering applications, such as population growth, reactor dynamics and one dimensional viscoelasticity. Moreover, the qualitative study of fractional differential and fractional order integro-differential has attracted significant attention due to their wide applicability in science and technology, including cryptography [29], options trading [13], neural networks [23], control theory [27] and viscoelasticity [20]. Fractional Volterra integro-differential equations (FVIDEs) are applicable in modeling various physical problems, including signal processing, acoustic waves, nonlinear oscillations of earthquakes, fluid-dynamic, traffic flow, nuclear reactor dynamics, and the strength of disordered materials [30, 9, 22, 7].

This article focuses on the following factional order Volterra integro-differential equation:

$$\begin{aligned} &D_z^{\gamma} w(z) + b(z)w(z) + \mu \int_0^z \mathbb{K}(z,p)\, w(p)\, dp = g(z), \quad z \in (0,1], \\ &w(0) = \phi_0, \end{aligned} \tag{1.1}$$

Received: 29 December 2025; Accepted: 07 November 2026.
∗ Corresponding author. Email: msfnmkr02@gmail.com.

where $b(z) > 0$ and $g(z)$ are smooth functions on the interval $[0, 1]$. D_z^γ represents the fractional differential operator in the Caputo sense of order $\gamma \in (0, 1)$. ϕ_0 is a real constant and $\mathbb{K}(z, p)$ is the kernel function, assumed to be continuous on $[0, 1] \times [0, 1]$, and μ is a parameter that typically depends on physical quantities arising in practical applications.

A number of researchers have shown great interest in obtaining numerical and exact solutions of FVIDEs. The existence and uniqueness of analytical solutions for fractional linear and nonlinear Volterra integro-differential equations (VIDEs) have been studied in [3, 19], while fractional linear and nonlinear Volterra-Fredholm integro-differential equations (VFIDEs) in the Caputo sense have been investigated in [26, 14, 18]. In [16, 39], the authors studied FVIDEs with Caputo fractional order and established the existence and convergence results of the solution. Many equations do not admit analytical solutions; therefore, numerical methods are required to obtain an approximate solution. Several analytical techniques have been developed for solving fractional integro-differential equations, including the Legendre collocation method [40], the variational iteration method [2, 1], the spectral collocation method [35], and the Homotopy analysis method [38]. Various numerical methods have been investigated for approximating solutions to VFIDEs and VIDEs. Some of these numerical approximation techniques are listed as follows. In [8], the Chebyshev collocation method was utilized to solve Volterra-type integro-differential equations, while Shahmorad [34] employed the Tau method for solving linear FVIDEs. Additionally, a numerical solution for FVIDEs of Caputo type, based on the modified variational iteration method, is discussed in [17]. A numerical method based on Homotopy perturbation method was proposed by Sayevand et al. [33] for solving FVIDEs. [24] employed a new numerical method combining trapezoidal and Simpsons rules for solving Volterra integral equations. Many initial and boundary value problems can be transformed into integral equations of the Volterra or Fredholm types when direct solutions are not possible.

Researchers have drawn attention to and increased their interest in approximating Caputo derivatives using higher-order schemes. The works in [6, 25, 28] approximate the Caputo fractional derivative using higher-order formulas. Recently, Santra and Mohapatra [32] solved time FVIDEs based on the $L1$ scheme. In [31], $L1$ scheme was employed, the integrand on each sub-interval is approximated using piecewise linear interpolation, achieving an order of accuracy of $2 - \gamma$, to solve Caputo fractional integro differential equations of Volterra type of the form in (1.1). In [12], time FVIDES were solved using the $L1$ and L1-2 schemes for the Caputo fractional derivative combined with the trapezoidal rule for the integral part. Furthermore, the authors in [11] employed the $L1$ and $L1-2$ methods, where the integrand is approximated using quadratic interpolation achieving second order of accuracy for the time fractional derivative, with trapezoidal rule for the integral part, to solve FVIDEs of the form in (1.1).

In the literature, existing numerical methods for the class of fractional differential equations described in problem (1.1) predominantly rely on $(2-\gamma)$ and second order accuracy on the L1 scheme and L1-2 respectively. Consequently, obtaining an approximate solution with higher-order accuracy has not yet been addressed in the literature. Motivated by this, we proposed a higher-order numerical scheme to approximate the solution of problem (1.1). The main contribution of this work is the design of a higher-order temporal discretization scheme for the time-fractional derivatives using the L1-2-3 method, which achieves $(4-\gamma)$ order accuracy, and the approximation of integral part using Simpson's rule. In this scheme, the integrand is approximated using linear interpolation on the first sub-interval, quadratic interpolation on the second, and cubic interpolation on the remaining sub-intervals. The convergence and stability analyses of the proposed method are discussed, and we also analyze the computational complexity and memory requirements. Finally, numerical examples are provided to illustrate the method's effectiveness , and comparisons are performed to confirm its accuracy.

The paper is organized as follows. Section 2 begins with preliminaries and some definitions. In Section 3 and Section 4, we discretize the semi-discrete scheme to formulate Problem (1.1) and establish the stability and convergence analyses. In Section 5, numerical examples and discussions are presented . Finally, in Section 6 the paper presents the conclusion of the proposed method and suggested directions for future work.

2. Preliminaries

In this section important definitions and theorems are included as follows:

Definition 2.1. Assume that $\mathrm{Re}(k) > 0$ for any complex number k. The Gamma function, $\Gamma(k)$, is defined as

$$\Gamma(k) = \int_0^\infty p^{k-1} e^{-p}\, dp.$$

Uncorrected Proof

Definition 2.2. Let $w(z)$ be a continuous function. The Riemann-Liouville fractional integral of order $\gamma \in \mathbb{R}^+$ is defined as

$$J^{\gamma} w(z) = \frac{1}{\Gamma(\gamma)} \int_0^z (z-p)^{\gamma-1} w(p)\, dp.$$

Definition 2.3. [32] For $\gamma \in \mathbb{R}^+$, the Caputo fractional derivative of order γ of $w(z)$ is defined as

$$D_z^{\gamma} w(z) = \left(J^{r-\gamma} \left(\frac{\partial^r w}{\partial z^r} \right) \right) z, \qquad 0 < z < 1,$$

where $r = \lceil \gamma \rceil$ represents the smallest integer greater than or equal to γ.

Theorem 2.4. *Let (r, c) be a complete metric space. Then every contraction mapping $Y : R \to R$ has a unique fixed point $r \in R$, that is*

$$Yr = r.$$

The following are the basic properties of fractional derivatives and integrals:

(1) If g is a constant function, then its fractional derivative satisfies $D_z^{\gamma} g = 0$.
(2) For $\delta \in \mathbb{R}$ and $r \in \mathbb{N}$, the fractional derivative of z^{δ} of order $\delta \in (r-1, r)$ is given by

$$D_z^{\gamma} z^{\delta} = \begin{cases} 0, & \delta \le r-1, \\ \dfrac{\Gamma(\delta+1)}{\Gamma(\delta-\gamma+1)} z^{\delta-\gamma}, & \delta > r-1. \end{cases}$$

(3) For $\gamma \in (0,1)$ we have

$$D_z^{\gamma} J^{\gamma} w(z) = w(z), \qquad J^{\gamma} D_z^{\gamma} w(z) = w(z) - w(0^+).$$

(4) Linearity:
   (a) $J^{\gamma}(c_2 w \pm c_3 w) = c_2 J^{\gamma} w \pm c_3 J^{\gamma} w$,
   (b) $D_z^{\gamma}(c_2 w \pm c_3 w) = c_2 D_z^{\gamma} w \pm c_3 D_z^{\gamma} w$,
   where $c_2, c_3 > 0$.

Theorem 2.5. *If the quantity $\frac{(\gamma+1)\,\|b\|_{\infty} + \mu\beta}{\Gamma(2+\gamma)} < 1$ and $|\mathbb{K}(z,p)| \le \beta$, then there exists a unique solution $w(z)$ for all $z \in [0,1]$ of Equation (1.1).*

Proof. Following the idea discussed in [15], apply the fractional integral J_z^{γ} to both sides of Equation (1.1). Then

$$w(z) = Hw(z), \quad z \in [0,1],$$

where $Hw(z) = \phi_0 + J_z^{\gamma} g(z) - J_z^{\gamma}\big[b(z)w(z)\big] - \mu J_z^{\gamma}\left[\int_0^z \mathbb{K}(z,p)\, w(p)\, dp\right]$. We want to show that H is a contraction. For any $w_2, w_3 \in C[0,1]$,

$$\begin{aligned}
|Hw_2(z) - Hw_3(z)| &\le J_z^{\gamma}\big(|b(z)|\,|w_2(z) - w_3(z)|\big) + \mu J_z^{\gamma}\left(\int_0^z |\mathbb{K}(z,p)|\,|w_2(p) - w_3(p)|\, dp\right) \\
&= \frac{1}{\Gamma(\gamma)} \int_0^z \frac{1}{(z-p)^{1-\gamma}} |b(z)|\,|w_2(p) - w_3(p)|\, dp \\
&+ \frac{\mu}{\Gamma(\gamma)} \int_0^z \frac{1}{(z-\omega)^{1-\gamma}} \left(\int_0^{\omega} |\mathbb{K}(\omega,p)|\,|w_2(p) - w_3(p)|\, dp\right) d\omega \\
&\le \frac{\|b\|_{\infty}}{\Gamma(1+\gamma)} |w_2(z) - w_3(z)| + \frac{\mu\beta}{\Gamma(\gamma)} \int_0^z (z-\omega)^{\gamma-1} \left(\int_0^{\omega} |w_2(p) - w_3(p)|\, dp\right) d\omega \\
&\le \frac{\|b\|_{\infty}}{\Gamma(1+\gamma)} |w_2 - w_3| + \frac{\mu\beta}{\Gamma(\gamma+2)} ||w_2 - w_3|| \\
&\le \frac{(1+\gamma)\|b\|_{\infty}}{(1+\gamma)\Gamma(1+\gamma)} |w_2 - w_3| + \frac{\mu\beta}{\Gamma(\gamma+2)} ||w_2 - w_3||
\end{aligned}$$

Uncorrected Proof

$$\leq \frac{(\gamma+1)\|b\|_\infty + \mu\beta}{\Gamma(2+\gamma)} \|w_2 - w_3\|.$$

By the hypothesis of the theorem, $\frac{(\gamma+1)\|b\|_\infty+\mu\beta}{\Gamma(2+\gamma)} < 1$, then we have $|Hw_2(z) - Hw_3(z)| \leq \|w_2 - w_3\|$. Therefore, H is a contraction mapping. As $(C[0,1], ||.||)$ is a Banach space, it follows from Banach's fixed point theorem that problem (1.1) possesses a unique solution in $C[0,1]$.

□

3. Numerical scheme formulation

Consider $R \in \mathbb{N}$ and on a uniform grid $z_r = r\sigma$ for $r = 0, 1, ..., R$, with step length $\sigma = \frac{1}{R}$. Denote $w_r = w(z_r)$ for $r = 1, 2, \ldots, R$. For a function $w(z) \in C^4[0,1]$, the Caputo fractional derivative of order $\gamma \in (0,1)$ at the node $z_r, r = 1, 2, \ldots R$ is

$$\begin{aligned} D_z^\gamma w(z_r) &= \frac{1}{\Gamma(1-\gamma)} \int_0^{z_r} \frac{w'(p)}{(z_r - p)^\gamma} dp \\ &= \frac{1}{\Gamma(1-\gamma)} \sum_{q=0}^{r-1} \int_{p=z_q}^{z_{q+1}} \frac{w'(p)}{(z_r - p)^\gamma} dp \\ &= \frac{1}{\Gamma(1-\gamma)} \int_{z_0}^{z_1} \frac{w'(p)}{(z_2 - p)^\gamma} dp + \frac{1}{\Gamma(1-\gamma)} \int_{z_1}^{z_2} \frac{w'(p)}{(z_r - p)^\gamma} dp + \frac{1}{\Gamma(1-\gamma)} \sum_{p=2}^{r-1} \int_{z_p}^{z_{p+1}} \frac{w'(p)}{(z_r - p)^\gamma} dp. \end{aligned}$$

On the first small interval $[z_0, z_1]$, we use linear interpolation to approximate $w(z)$. The linear approximation for w on $[z_0, z_1]$ is

$$w'(p) \approx \frac{w_1 - w_0}{\sigma}.$$

Thus,

$$\frac{1}{\Gamma(1-\gamma)} \int_{z_0}^{z_1} (z_r - p)^{-\gamma} w'(p) dp \approx \frac{1}{\Gamma(1-\gamma)} \int_{z_0}^{z_1} \frac{w_1 - w_0}{\sigma} (z_r - p)^{-\gamma} dp.$$

We evaluate the integral using integration by parts and perform change of variables, yields:

$$\frac{1}{\Gamma(1-\gamma)} \int_{z_0}^{z_1} (z_r - p)^{-\gamma} w'(p)\, dp = \sigma^{-\gamma} \frac{c_{r-1}}{\Gamma(2-\gamma)} (w_1 - w_0), \tag{3.1}$$

where, $c_{r-1} = -(r-1)^{1-\gamma} + r^{1-\gamma}$.
On the second small interval $[z_1, z_2]$, we use the quadratic Lagrange interpolant through $(z_0, w_0), (z_1, w_1), (z_2, w_2)$. For the uniform grid $z_q = q\sigma$ the basis polynomials (with nodes $0, \sigma, 2\sigma$) are

$$\begin{aligned} L_0(p) &= \frac{(p-\sigma)(p-2\sigma)}{2\sigma^2}, \quad L_1(p) = -\frac{p(p-2\sigma)}{\sigma^2}, \quad L_2(p) = \frac{p(p-\sigma)}{2\sigma^2}, \\ L_0'(p) &= \frac{2p-3\sigma}{2\sigma^2}, \quad L_1'(p) = -\frac{2(p-\sigma)}{\sigma^2}, \quad L_2'(p) = \frac{2p-\sigma}{2\sigma^2}. \end{aligned}$$

So the quadratic interpolant is

$$\begin{aligned} y_2(p) &= w_0 \frac{(p-2\sigma)(p-\sigma)}{2\sigma^2} + w_1 \frac{p(p-2\sigma)}{\sigma^2} + w_2 \frac{p(p-\sigma)}{2\sigma^2}, \\ y_2'(p) &= \frac{1}{2\sigma^2}\Big((2p-3\sigma)w_0 - 4(p-\sigma)w_1 + (2p-\sigma)w_2\Big). \end{aligned} \tag{3.2}$$

Approximate $w'(p) \approx y_2'(p)$ on $[z_1, z_2]$ and substitute into the integral we have

$$\frac{1}{\Gamma(1-\gamma)} \int_{z_1}^{z_2} (z_r - p)^{-\gamma} w'(p)\, dp \approx \frac{1}{\Gamma(1-\gamma)} \int_{z_1}^{z_2} (z_r - p)^{-\gamma} y_2'(p)\, dp.$$

Uncorrected Proof

Substituting Equation (3.2) in to the integrand and perform a change of variables for $w \in \Big[(r-2)\sigma,(r-1)\sigma\Big]$, we obtain

$$\frac{1}{\Gamma(1-\gamma)}\int_{z_1}^{z_2}(z_r-p)^{-\gamma}w'(p)\,dp \approx \sigma^{-\gamma}\frac{1}{\Gamma(2-\gamma)}\Big((c_{r-2}+d_{r-2})\,w_2-(c_{r-2}+2d_{r-2})\,w_1+d_{r-2}\,w_0\Big), \tag{3.3}$$

where, $c_{r-2}=(r-1)^{1-\gamma}-(r-2)^{1-\gamma}$, and $d_{r-2}=\frac{(r-1)^{2-\gamma}-(r-2)^{2-\gamma}}{2-\gamma}+\frac{(r-2)^{1-\gamma}-(r-1)^{1-\gamma}}{2}$.

For $(q\geq 3)$, a cubic interpolation function $y_3(z)$ is constructed based on the four points $(z_q,w(z_q))$, $(z_{q-1},w(z_{q-1}))$, $(z_{q-2},w(z_{q-2}))$, $(z_{q-3},w(z_{q-3}))$.

$$y_3(z)=\sum_{j=0}^{3}w(z_{q-j})\prod_{m=0,q\neq j}^{3}\frac{z-z_{q-m}}{z_{q-1}-z_{q-m}},$$

thus, we have

$$\begin{aligned}(y_3(z))'=&\,w(z_{q-3})\frac{(z-z_{q-2})(z_{q-1}+z_q-2z)+(z_q-z)(z-z_{q-1})}{6\sigma^3}\\&+w(z_{q-2})\frac{(z-z_{q-3})(2z-z_{q-1}-z_q)+(z-z_q)(z-z_{q-1})}{2\sigma^3}\\&+w(z_{q-1})\frac{(z-z_{q-2})(z_{q-3}+z_q-2z)+(z_q-z)(z-z_{q-3})}{2\sigma^3}\\&+w(z_q)\frac{(z-z_{q-2})(2z-z_{q-2}-z_{q-3})+(z-z_{q-1})(z-z_{q-3})}{6\sigma^3},\end{aligned}$$

and $w(z)-y_3(z)=\frac{w^{(4)}(\xi_q)}{12}(z-z_{q-3})(z-z_{q-2})(z-z_{q-1})(z-z_q)$, $\xi_q\in(z_{q-3},z_q)$.

Use $y_3(z)$ in the Caputo fractional formula to approximate $w(z)$ on the interval $[z_{q-1},z_q]$.

$$\frac{1}{\Gamma(-\gamma+1)}\sum_{q=3}^{r}\int_{z_{q-1}}^{z_q}(z_q-p)^{-\gamma}w'(p)dp\approx\frac{1}{\Gamma(-\gamma+1)}\sum_{q=3}^{r}\int_{z_{q-1}}^{z_q}(z_r-p)^{-\gamma}y_3'(p)dp.$$

The cubic Lagrange interpolation on the four nodes, z_q,z_{q-1},z_{q-2} and z_{q-3}, is

$$y_3(p)=\sum_{j=0}^{3}w_{\,q-j}\,L_j(p),\quad y_3'(p)=\sum_{j=0}^{3}w_{\,q-j}\,L_j'(p).$$

where $L_j(p)$ are the cubic Lagrange basis polynomials associated with $z_{q-3},z_{q-2},z_{q-1},z_q$. Substitute the interpolant and change the variable on the integrand

$$\frac{1}{\Gamma(1-\gamma)}\int_{z_{q-1}}^{z_q}(z_r-p)^{-\gamma}w'(p)\,dp\approx\frac{1}{\Gamma(1-\gamma)}\int_{z_{q-1}}^{z_q}(z_r-p)^{-\gamma}y_3'(p)\,dp.$$

By using a change of variables and substitution in the integral, we obtain

$$\frac{1}{\Gamma(1-\gamma)}\int_{z_{q-1}}^{z_q}(z_r-p)^{-\gamma}w'(p)\,dp\approx\frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)}\Big(m_{1,r-q}w_q+m_{2,r-q}w_{q-1}+m_{3,r-q}w_{q-2}+m_{4,r-q}w_{q-3}\Big), \tag{3.4}$$

Uncorrected Proof

where, the weights $m_{h,r-q}$ (for $h = 1, 2, 3, 4$) are

$$\begin{aligned}
m_{1,r-q} &= \frac{1}{6}\Big[2(r-q+1)^{1-\gamma} - 11(r-q)^{1-\gamma}\Big] - \frac{1}{2-\gamma}\Big[2(r-q)^{2-\gamma} - (r-q+1)^{2-\gamma}\Big] \\
&\quad - \frac{1}{(2-\gamma)(3-\gamma)}\Big[(r-q)^{3-\gamma} - (r-q+1)^{3-\gamma}\Big], \\
m_{2,r-q} &= \frac{1}{2}\Big[6(r-q)^{1-\gamma} + (r-q+1)^{1-\gamma}\Big] + \frac{1}{2-\gamma}\Big[5(r-q)^{2-\gamma} - 2(r-q+1)^{2-\gamma}\Big] \\
&\quad + \frac{3}{(2-\gamma)(3-\gamma)}\Big[(r-q)^{3-\gamma} - (r-q+1)^{3-\gamma}\Big], \\
m_{3,r-q} &= -\frac{1}{2}\Big[3(r-q)^{1-\gamma} + 2(r-q+1)^{1-\gamma}\Big] - \frac{1}{2-\gamma}\Big[4(r-q)^{2-\gamma} - (r-q+1)^{2-\gamma}\Big] \\
&\quad - \frac{3}{(2-\gamma)(3-\gamma)}\Big[(r-q)^{3-\gamma} - (r-q+1)^{3-\gamma}\Big], \\
m_{4,r-q} &= \frac{1}{6}\Big[2(r-q)^{1-\gamma} + (r-q+1)^{1-\gamma}\Big] + \frac{1}{2-\gamma}(r-q)^{2-\gamma} \\
&\quad + \frac{1}{(2-\gamma)(3-\gamma)}\Big[(r-q)^{3-\gamma} - (r-q+1)^{3-\gamma}\Big].
\end{aligned}$$

From Equations (3.1), (3.3), and (3.4), the Caputo derivative of order γ $(0 < \gamma < 1)$ is defined as

$$\begin{aligned}
D_z^{\gamma} w(z_r) &= \frac{1}{\Gamma(1-\gamma)} \int_0^{z_r} \frac{1}{(z_r - p)^{\gamma}} w'(p)dp \\
&= \frac{1}{\Gamma(1-\gamma)} \left(\int_{z_0}^{z_1} \frac{1}{(z_r - p)^{\gamma}} w'(p)dp + \int_{z_1}^{z_2} \frac{1}{(z_r - p)^{\gamma}} w'(p)dp + \sum_{q=3}^{r} \int_{z_{q-1}}^{z_q} \frac{1}{(z_r - p)^{\gamma}} w'(p)dp \right) \\
&= \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \sum_{q=0}^{r} k_q w_{r-q} + T_r^{(L123)} = D_R^{\gamma} w(z_r) + T_r^{(L123)}.
\end{aligned} \tag{3.5}$$

The coefficients k_q for different values of r are listed below.

For $r = 1$, $\begin{cases} k_0 = c_0, \\ k_1 = -c_0, \end{cases}$ $r = 2$, $\begin{cases} k_0 = c_0 + d_0, \\ k_1 = c_1 - c_0 - 2d_0, \\ k_2 = d_0 - c_1, \end{cases}$

$r = 3$, $\begin{cases} k_0 = m_{1,0}, \\ k_1 = m_{2,0} + c_1 + d_1, \\ k_2 = m_{3,0} + c_2 - c_1 - 2d_1, \\ k_3 = m_{4,0} - c_2 + d_1, \end{cases}$ $r = 4$, $\begin{cases} k_0 = m_{1,0}, \\ k_1 = m_{1,1} + m_{2,0}, \\ k_2 = m_{2,1} + m_{3,0} + c_2 + d_2, \\ k_3 = m_{3,1} + m_{4,0} + c_3 - c_2 - 2d_2, \\ k_4 = m_{4,1} - c_3 + d_2, \end{cases}$

$r = 5$, $\begin{cases} k_0 = m_{1,0}, \\ k_1 = m_{1,1} + m_{2,0}, \\ k_2 = m_{1,2} + m_{2,1} + m_{3,0}, \\ k_3 = m_{2,2} + m_{3,1} + m_{4,0} + c_3 + d_3, \\ k_4 = m_{3,2} + m_{4,1} + c_4 - c_3 - 2d_3, \\ k_5 = m_{4,2} - c_4 + d_3, \end{cases}$

Uncorrected Proof

and for $r \geq 6$, the coefficients are given in relations as follows

$$\begin{cases} k_0 = m_{1,0}, \\ k_1 = m_{1,1} + m_{2,0}, \\ k_2 = m_{1,2} + m_{2,1} + m_{3,0}, \\ k_q = m_{1,q} + m_{2,q-1} + m_{3,q-2} + m_{4,q-3} \quad \text{for} \quad q \in [3, r-3], \\ k_{r-2} = c_{r-2} + d_{r-2} + m_{2,r-3} + m_{3,r-4} + m_{4,r-5}, \\ k_{r-1} = m_{3,r-3} + m_{4,r-4} + c_{r-1} - c_{r-2} - 2d_{r-2}, \\ k_r = m_{4,r-3} - c_{r-1} + d_{r-2}. \end{cases}$$

Lemma 3.1. *[6] For $r \geq 3$ and any $\gamma \in (0,1)$ the weighted coefficients $k_q (0 \leq q \leq r)$ in (3.5) possess the following properties:*

(1) $k_0 = \frac{1}{3} + \frac{1}{2-\gamma} + \frac{1}{(2-\gamma)(3-\gamma)} \in \left(1, \frac{1}{16}\right)$,
(2) $k_q < 0, \quad for \quad q \geq 1, \quad and \quad k_2 > 0$,
(3) $\sum_{q=0}^{r} k_q = 0$,
(4) *There exists $C_k > 0$ such that $|k_q| \leq C_k q^{-\gamma}$ for $q \geq 1$.*

The integral part on the partition $[z_0, z_r]$ depends on the number of mesh points, which is given by

$$\int_{z_0}^{z_r} \mathbb{K}(z_r, p) w(p) dp = \sum_{q=0}^{r-1} \int_{z_q}^{z_{q+1}} \mathbb{K}(z_r, p) w(p) dp.$$

If r is even, we apply Simpson's $\frac{1}{3}$ rule approximated by

$$\begin{aligned} \int_{z_0}^{z_r} \mathbb{K}(z_r, p) w(p) dp \approx & \frac{\sigma}{3} \left[\mathbb{K}(z_r, z_0) w(z_0) + \mathbb{K}(z_r, z_r) w(z_r) + 4 \sum_{q=1}^{\frac{r}{2}} \mathbb{K}(z_r, z_{2q-1}) w(z_{2q-1}) + 2 \sum_{q=1}^{\frac{r}{2}-1} \mathbb{K}(z_r, z_{2q}) w(z_{2q}) \right] + T_r^{(S)} \\ = & J_R w(z_r) + T_r^{(S)}. \end{aligned} \tag{3.6}$$

If r is odd, we use the combination of Simpson's $\frac{1}{3}$ and trapezoidal rule by splitting the interval $[0, z_r]$ into $[0, z_{r-1}]$ and $[z_{r-1}, z_r]$. The interval $[0, z_{r-1}]$ is divided in to an even number of sub-intervals.

$$\begin{aligned} \int_{z_0}^{z_r} \mathbb{K}(z_r, p) w(p) dp = & \int_{0}^{z_{r-1}} \mathbb{K}(z_r, p) w(p) dp + \int_{z_{r-1}}^{z_r} \mathbb{K}(z_r, p) w(p) dp \\ = & \frac{\sigma}{3} \Big[\mathbb{K}(z_{r-1}, z_0) w(z_0) + \mathbb{K}(z_{r-1}, z_{r-1}) w(z_{r-1}) + 4 \sum_{q=1}^{\frac{r-1}{2}} \mathbb{K}(z_r, z_q) w(z_q) \\ & + 2 \sum_{q=1}^{\frac{r-1}{2}-1} \mathbb{K}(z_r, z_q) w(z_q) \Big] + \frac{\sigma}{2} \left[\mathbb{K}(z_r, z_{r-1}) w(z_{r-1}) + \mathbb{K}(z_r, z_r) w(z_r) \right] + T_r^{(S)} \\ = & J_R w(z_r) + T_r^{(S)}, \quad r = 1, 2, \ldots, R. \end{aligned} \tag{3.7}$$

Substituting Equations (3.5) and (3.6) into (1.1), we get the following scheme.

$$\begin{aligned} & \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \sum_{q=0}^{r} w(z_{r-q}) k_q + b_r w_r + \frac{\mu\sigma}{3} \Big[\mathbb{K}(z_r, z_0) w(z_0) + \mathbb{K}(z_r, z_r) w(z_r) \\ & + 4 \sum_{q=1}^{\frac{r}{2}} \mathbb{K}(z_r, z_{2q-1}) w(z_{2q-1}) + 2 \sum_{q=1}^{\frac{r}{2}-1} \mathbb{K}(z_r, z_{2q}) w(z_{2q}) \Big] = g(z_r) + T_r^{(CS)}, \\ & w(0) = \phi_0, \quad \text{for} \quad r = 1, 2, \ldots R, \end{aligned} \tag{3.8}$$

Uncorrected Proof

where $T_r^{(CS)}$ is the remainder term defined as $T_r^{(CS)} = T_r^{(L123)} + T_r^{(S)}$. Now, rejecting $T_r^{(CS)}$, the discrete problem takes the following form:

$$\begin{aligned}
&\frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)}\big[k_0\hat{W}_r + k_1\hat{W}_{r-1} + \sum_{q=3}^{r-2} k_{r-q}\hat{W}_q + k_{r-2}\hat{W}_2 + k_{r-1}\hat{W}_1 + k_r\hat{W}_0\big] + b_r\hat{W}_r \\
&+ \frac{\mu\sigma}{3}[\mathbb{K}(z_r,z_0)\hat{W}_0 + \mathbb{K}(z_r,z_r)\hat{W}_r + 4\sum_{q=1}^{\frac{r}{2}} \mathbb{K}(z_r,z_{2q-1})\hat{W}_{2q-1} + 2\sum_{q=1}^{\frac{r}{2}-1} \mathbb{K}(z_r,z_{2q})\hat{W}_2q] = g(z_r), \\
&\hat{W}_0 = \phi_0, \quad r = 1,2,\ldots,R.
\end{aligned} \tag{3.9}$$

Where, $(\hat{W}_r)_{r=0}^R$ denotes the approximate solution of (1.1) at each $w(z_r)$.

Similarly, from Equations (3.5), (3.7), and (1.1) we obtain:

$$\begin{aligned}
&\frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)}\big[k_0\hat{W}_r + k_1\hat{W}_{r-1} + \sum_{q=3}^{r-2} k_{r-q}\hat{W}_q + k_{r-2}\hat{W}_2 + k_{r-1}\hat{W}_1 + k_r\hat{W}_0\big] \\
&+ \big(b_r + \frac{\mu\sigma}{2}\mathbb{K}(z_r,z_r)\big)\hat{W}_r + \frac{\mu\sigma}{3}\big[\mathbb{K}(z_{r-1},z_0)\hat{W}(z_0) + \mathbb{K}(z_{r-1},z_{r-1})\hat{W}(z_{r-1})\big] \\
&+ 4\sum_{q=1}^{\frac{r-1}{2}} \mathbb{K}(z_r,z_q)\hat{W}(z_q) + 2\sum_{q=2}^{\frac{r-1}{2}-1} \mathbb{K}(z_r,z_q)\hat{W}(z_q)\big] + \frac{\sigma}{2}\big[\mathbb{K}(z_r,z_{r-1})\hat{W}(z_{r-1})\big] = g(z_r), \\
&\hat{W}_0 = \phi_0, \quad r = 1,2,\ldots,R.
\end{aligned} \tag{3.10}$$

At each level of Equation (3.9), we have R unknowns $\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_R$ and R corresponding equations. Let $\varepsilon = \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)}, S = \frac{\mu\sigma}{3}$. Then the system can be written in the following matrix form:

$$H\,\hat{W} = \hat{G},$$

where, $\hat{W} = (\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_R)^T$,

$$H = \begin{pmatrix}
\varepsilon k_0 + b_1 + S\mathbb{K}_{1,1} & 0 & 0 & \cdots & 0 \\
\varepsilon k_1 + 4S\mathbb{K}_{2,1} & \varepsilon k_0 + b_2 + S\mathbb{K}_{2,2} & 0 & \cdots & 0 \\
\varepsilon k_2 + 4S\mathbb{K}_{3,1} & \varepsilon k_1 + 2S\mathbb{K}_{3,2} & \varepsilon k_0 + b_3 + S\mathbb{K}_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\varepsilon k_{R-1} + 4SK_{R,1} & \varepsilon k_{R-2} + 2S\mathbb{K}_{R,2} & \varepsilon k_{R-3} + S\,\mathbb{K}_{R,3} & \cdots & \varepsilon k_0 + b_R + S\mathbb{K}_{R,R}
\end{pmatrix},$$

and

$$\hat{G} = \begin{pmatrix}
g_1 - (\varepsilon k_1 + S\mathbb{K}_{1,0})\phi_0 \\
g_2 - (\varepsilon k_2 + S\mathbb{K}_{2,0})\phi_0 \\
g_3 - (\varepsilon k_3 + S\mathbb{K}_{3,0})\phi_0 \\
\vdots \\
g_R - (\varepsilon k_R + S\mathbb{K}_{R,0})\phi_0
\end{pmatrix}.$$

4. Stability and connvergence analysis

In this section, we investigate the stability of the full scheme defined in (3.9), drive the truncation errors for the Caputo fractional derivative and Simpson's rule, and prove the convergence of the scheme.

Uncorrected Proof

Lemma 4.1. *For any discrete function* $\{\hat{W}_r\}_{r=0}^{R}$ *satisfying* $\hat{W}_0 = 0$, *and for* $\gamma \in (0,1)$, *we have*

$$D_R^\gamma |\hat{W}_r| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} k_0 |\hat{W}_r|, \quad r = 1, 2, \ldots, R.$$

Proof. From the definition of Caputo fractional derivative

$$D_R^\gamma |\hat{W}_r| = \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \left[k_0 |\hat{W}_r| + \sum_{q=1}^{r-1} k_q |\hat{W}_{r-q}| + k_r |\hat{W}_0| \right].$$

Since $\hat{W}_0 = 0$ this simplifies

$$D_R^\gamma |\hat{W}_r| = \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \left[k_0 |\hat{W}_r| + \sum_{q=1}^{r-1} k_q |\hat{W}_{r-q}| \right].$$

Since $|\hat{W}_{r-q}| \leq |\hat{W}_r|$ and $k_q < 0$ for $q \geq 1$ (except $k_2 > 0$ for $\gamma \in (0,1)$), by Lemma 3.1 we obtain

$$\sum_{q=1}^{r-1} k_q |\hat{W}_{r-q}| \geq \sum_{q=1}^{r-1} k_q |\hat{W}_r| = |\hat{W}_r| \sum_{q=1}^{q-1} k_q.$$

Using Lemma 3.1, $\sum_{q=0}^{r} k_q = 0$, so $\sum_{q=1}^{r} k_q = -k_0$ and $\sum_{q=1}^{r-1} k_q = -k_0 - k_r$. Thus,

$$D_R^\gamma |\hat{W}_r| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \left[k_0 |\hat{W}_r| + (-k_0 - k_r) |\hat{W}_r| \right] = \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} (-k_r) |\hat{W}_r|.$$

Since $-k_r \geq k_0$ and $\sum_{q=1}^{r-1} k_q + k_0 = -k_r$ it follows;

$$D_R^\gamma |\hat{W}_r| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} (-k_r) |\hat{W}_r| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} (k_0) |\hat{W}_r|.$$

Therefore, $D_R^\gamma |\hat{W}_r| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} (k_0) |\hat{W}_r|$. □

Lemma 4.2. *For any discrete function* $\{\hat{W}_r\}_{r=0}^{R}$ *satisfying* $\hat{W}_0 = 0$, *and for* $\gamma \in (0,1)$, *it holds*

$$|\hat{W}_r| \leq \max_{q=1,2,\ldots,r} \left\{ \frac{\sigma^\gamma \Gamma(2-\gamma)}{k_0} D_R^\gamma |\hat{W}_q| \right\}, \quad \text{for } r = 1, 2, \ldots, R.$$

Proof. Suppose $\max_{k=1,2,\ldots,r} |\hat{W}_r| = |\hat{W}_p|$ for some $p \in \{1, 2, \ldots, r\}$. By Lemma 4.1,

$$D_R^\gamma |\hat{W}_p| \geq \frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} k_0 |\hat{W}_p|, \quad |\hat{W}_p| \leq \frac{\sigma^\gamma \Gamma(2-\gamma)}{k_0} D_R^\gamma |\hat{W}_p|.$$

Since $|\hat{W}_r| \leq |\hat{W}_p|$ for all $r = 1, 2, \ldots, R$, we obtain $|\hat{W}_r| \leq \max_{q=1,2,\ldots,r} \left\{ \frac{\sigma^\gamma \Gamma(2-\gamma)}{k_0} D_R^\gamma |\hat{W}_q| \right\}$. □

Theorem 4.3. *Let* $\{\hat{W}_r\}$ *be the numerical solution of scheme* (3.9), *and let* $\hat{e}_r = w(z_r) - \hat{W}_r$ *be the error at* z_r. *Then the error satisfies the following:*

$$\|\hat{E}\|_\infty = \max_{0 \leq r \leq R} |\hat{e}_r| \leq C^\star \|T^{(CS)}\|_\infty,$$

where $E = (e_0, e_1, \ldots, e_R)^T$, $T^{(CS)} = (T_0^{(CS)}, T_1^{(CS)}, \ldots, T_R^{(CS)})^T$ *is the total truncation error, and* $C^\star$ *is a positive constant independent on* σ.

Proof. The error equation for e_r is the following:

$$\frac{\sigma^{-\gamma}}{\Gamma(2-\gamma)} \sum_{q=0}^{r} k_q e_{r-q} + b_r e_r + \mu J_R e_r = T_r^{(CS)}, \quad r = 1, 2, \ldots, R,$$

Uncorrected Proof

with $\hat{e}_0 = 0$. Multiply both sides by e_r we get

$$\left(\frac{\sigma^{-\gamma} g_0}{\Gamma(2-\gamma)} + b_r\right)|e_r|^2 + \frac{1}{\sigma^{\gamma}\Gamma(2-\gamma)}\sum_{q=1}^{r-1} k_q \hat{e}_{r-q} e_r + \mu J_R(e_r) e_r = T_r^{(CS)} e_r.$$

Since $b_r > 0$ and $K(z,p) > 0$, we have

$$\left(\frac{\sigma^{-\gamma} k_0}{\Gamma(2-\gamma)} + \right)|e_r|^2 + \frac{\sigma^{-\gamma}}{\Gamma(2-\alpha)}\sum_{q=1}^{r-1} k_q \hat{e}_{r-q} e_r \le T_r^{(CS)} e_r.$$

This implies $D_R^{\gamma}|e_r| \le |T_r^{(CS)}|$. Now, apply Lemma 4.2 to the mesh function $\{|e_r|\}$ yields

$$|e_r| \le \max_{q=1,2,\ldots,r}\left\{\frac{\sigma^{\gamma}\Gamma(2-\gamma)}{k_0} D_R^{\gamma}|e_q|\right\} \le \max_{q=1,2,\ldots,r}\left\{\frac{\sigma^{\gamma}\Gamma(2-\gamma)}{k_0}|T_r^{(CS)}|\right\} \le \frac{\sigma^{\gamma}\Gamma(2-\gamma)}{k_0}\|T^{(CS)}\|_{\infty}.$$

Since $\sigma^{\gamma} \le 1^{\gamma} = 1$ and $\Gamma(2-\gamma)/k_0$ is bounded for $\gamma \in (0,1)$, $\exists C^{\star}(C^{\star} > 0)$ such that

$$\|\hat{E}\|_{\infty} \le C^{\star}\|R^{(CS)}\|_{\infty}.$$

□

Lemma 4.4. *Assume that $w(z) \in C^4[0, z_r]$. For any γ with $0 < \gamma < 1$, the truncation errors $|T_r^{(L123)}|$ defined in Equation (3.5) satisfy the following bound:*

(1) $|T_1^{(L123)}| \le c_1 \max_{z_0\le z\le z_1}|w''(z)|\,\sigma^{2-\gamma}, \quad c_1 > 0,\ r = 1,$
(2) $|T_2^{(L123)}| \le c_2 \max_{z_0\le z\le z_2}|w'''(z)|\,\sigma^{3-\gamma}, \quad c_2 > 0,\ r = 2,$
(3) $|T_r^{(L123)}| \le \frac{1}{\Gamma(1-\gamma)}\left\{\frac{2\gamma}{3}\max_{z_0\le z\le z_2}|w'''(z)|(z_r - z_2)^{-\gamma-1}\sigma^4 + \left(\frac{1}{12} + \frac{3\gamma^2}{2(1-\gamma)(2-\gamma)}\right)\max_{z_0\le z\le z_r}|w^{(4)}(z)|\sigma^{4-\gamma}\right\}.$

Proof. When $r = 1$, using linear interpolant function $w_1(z)$ on $[z_0, z_1]$, satisfying $y_1(z_0) = w(z_0), y_1(z_1) = w(z_1)$. Set $z_0 = 0$, $z_1 = \sigma$ and Taylor expansion of w around $z_0 = 0$, for each $p \in [0,\sigma]$ there exist ξ_p, $\xi_1 \in (0,\sigma)$ such that

$$w'(p) = w'(0) + w''(\xi_p)p, \qquad w_1 - w_0 = w'(0)\sigma + \tfrac{1}{2}w''(\xi_1)\sigma^2.$$

The derivative of the linear interpolant is $y_1'(p) = \frac{w_1 - w_0}{\sigma} = w'(0) + \frac{1}{2}w''(\xi_1)\,\sigma$. And the difference of the first derivative function and derivative of its interploant is

$$w'(p) - y_1'(p) = w''(\xi_p)p - \tfrac{1}{2}w''(\xi_1)\sigma.$$

Let $M_2 := \max_{z_0\le z\le z_1}|w''(z)|$. Since $0 \le p \le \sigma$, we have the uniform bound

$$|w'(p) - y_1'(p)| \le |w''(\xi_p)|\,p + \tfrac{1}{2}|w''(\xi_1)|\sigma \le \tfrac{3}{2}\,M_2\,\sigma.$$

From Equation (3.5) we have

$$|T_1^{(L123)}| = \frac{1}{\Gamma(1-\gamma)}\int_0^{\sigma}(\sigma - p)^{-\gamma}\,|w'(p) - y_1'(p)|\,dp \le \frac{\frac{3}{2}\,M_2\,\sigma}{\Gamma(1-\gamma)}\int_0^{\sigma}(\sigma-p)^{-\gamma}\,dp.$$

Using integration by substitution, $\int_0^{\sigma}(\sigma-p)^{-\gamma}\,dp = \frac{\sigma^{1-\gamma}}{1-\gamma}$, we get

$$|T_1^{(L123)}| \le \frac{3}{2\Gamma(2-\gamma)}\,M_2\,\sigma^{2-\gamma}.$$

Hence, $\exists c_1 (c_1 > 0)$ such that

$$|T_1^{(L123)}| \ \le\ c_1 \max_{z_0\le z\le z_1}|w''(z)|\,\sigma^{2-\gamma}.$$

When $r = 2$, we consider the Caputo derivative at z_2.

$$D_z^{\gamma}w(z_2) = \frac{1}{\Gamma(1-\gamma)}\int_0^{z_2}(z_2 - p)^{-\gamma}w'(p)\,dp.$$

Uncorrected Proof

We approximate w on the whole interval $[z_0, z_2] = [0, 2\sigma]$ by a quadratic interpolant $y_2(z)$ which satisfies

$$y_2(z_q) = w(z_q), \qquad q = 0, 1, 2.$$

Thus, the truncation error for $r = 2$ can be written as

$$T_2^{(L123)} = \frac{1}{\Gamma(1-\gamma)} \int_0^{2\sigma} (2\sigma - p)^{-\gamma} \big(w'(p) - y_2'(p)\big)\, dp.$$

The remainder for quadratic interpolation on the nodes z_0, z_1 and z_2 yields, for each $p \in [0, 2\sigma]$, some $\xi_p \in (0, 2\sigma)$,

$$w(p) - y_2(p) = \frac{w^{(3)}(\xi_p)}{3!}\,(p - z_2)(p - z_1)(p - z_0),$$
$$w'(p) - y_2'(p) = \frac{w^{(3)}(\xi_p)}{6}\,\frac{d}{dp}\Big[(p - z_2)(p - z_1)(p - z_0)\Big].$$

In the interval $[0, 2\sigma]$ each factor $(p - z_q)$ has an absolute value at most 2σ, hence each quadratic term is bounded by $(2\sigma)^2 = 4\sigma^2$. Therefore,

$$\left|\frac{d}{dp}\big[(p - z_2)(p - z_1)(p - z_0)\big]\right| \le (3)(4\sigma^2) = 12\sigma^2.$$

Let $M_3 := \max_{0\le z\le 2\sigma} |w^{(3)}(z)|$, and then the uniform bound on the derivative remainder is

$$|w'(p) - y_2'(p)| \le \frac{W_3}{6} \cdot 12\sigma^2 = 2\, M_3\, \sigma^2, \quad \forall p \in [0, 2\sigma].$$

Hence,

$$|T_2^{(L123)}| \le \frac{1}{\Gamma(1-\gamma)} \int_0^{2\sigma} (2\sigma - p)^{-\gamma}\, |w'(p) - y_2'(p)|\, dp \le \frac{2M_3\sigma^2}{\Gamma(1-\gamma)} \int_0^{2\sigma} (2\sigma - p)^{-\gamma}\, dp.$$

Using integration by substitution, $\int_0^{2\sigma} (2\sigma - p)^{-\gamma}\, dp = \frac{(2\sigma)^{1-\gamma}}{1-\gamma}$, we get

$$|T_2^{(L123)}| \le \frac{2^{2-\gamma}}{(1-\gamma)} \cdot \frac{M_3}{\Gamma(1-\gamma)}\, \sigma^{3-\gamma}.$$

Hence, $\exists c_2 (c_2 > 0)$ doesn't dependent on σ such that

$$|T_2^{(L123)}| \le c_2 \max_{z_0\le z\le z_2} |w^{(3)}(z)|\, \sigma^{3-\gamma}.$$

When $r \ge 3$, using a quadratic interpolation polynomial $y_2(z)$ constructed from the points $(z_0, w(z_0))$, $(z_1, w(z_1))$, $(z_2, w(z_2))$ we approximate $w(z)$ over the interval $[z_0, z_2]$.

$$\begin{aligned} T_r^{(L123)} &= |D_z^\gamma w(z_r) - D_R^\gamma w(z_r)| \\ &= \frac{1}{\Gamma(1-\gamma)} \left[\int_{z_0}^{z_2} [w(p) - y_2(p)](z_r - p)^{-\gamma}\, dp + \sum_{q=3}^{r} \int_{z_{q-1}}^{z_q} [w(p) - y_3(p)](z_r - p)^{-\gamma}\, dp\right] \\ &= \frac{1}{\Gamma(1-\gamma)} \left\{ [w(p) - y_2(p)](z_r - p)^{-\gamma}\Big|_{z_0}^{z_2} - \gamma \int_{z_0}^{z_2} [w(p) - y_2(p)](z_r - p)^{-\gamma-1} dp \right. \\ &\left. + \sum_{q=3}^{r} \left([w(p) - y_3(p)](z_r - p)^{-\gamma}\Big|_{z_{q-1}}^{z_q} - \gamma \int_{z_{q-1}}^{z_q} [w(p) - y_3(p)](z_r - p)^{-\gamma-1} dp \right) \right\}. \end{aligned}$$

By applying the Lagrange interpolation error theorem, we obtain

$$\begin{aligned} \left|\int_{z_0}^{z_2} [w(p) - y_2(p)](z_r - p)^{-\gamma-1} dp\right| &= \frac{w'''(\xi)}{6} \left|\int_{z_0}^{z_2} (p - z_2)(p - z_1)(p - z_0)(z_r - p)^{-\gamma-1} dp\right| \\ &\le \frac{2}{3}\, |w'''(\xi)|\, (z_r - z_2)^{-\gamma-1} \sigma^4, \end{aligned}$$

Uncorrected Proof

where $\xi \in (z_0, z_2)$. Thus,

$$\left|\sum_{q=3}^{r-1}\int_{z_{q-1}}^{z_q}[w(p)-y_3(p)](z_r-p)^{-\gamma-1}dp\right| = \left|\sum_{q=3}^{r-1}\int_{z_{q-1}}^{z_q}\frac{w^{(4)}(\eta_q)}{12}(z_r-p)^{-\gamma-1}\prod_{k=0}^{3}(p-z_{q-k})\,dp\right|$$
$$\leq \frac{1}{12}|w^{(4)}(\eta)|\sigma^4\int_{z_2}^{z_{r-1}}(z_r-p)^{-\gamma-1}dp \leq \frac{1}{12\gamma}|w^{(4)}(\eta)|\sigma^{4-\gamma},$$

where $\eta \in (z_2, z_{r-1})$. In addition,

$$\left|\int_{z_{r-1}}^{z_r}(z_r-p)^{-\gamma}[w(p)-y_3(p)]'\,dp\right| = \left|\gamma\int_{z_{r-1}}^{z_r}(z_r-p)^{-\gamma-1}[w(p)-y_3(p)]\,dp\right|$$
$$\leq \frac{3\gamma}{2(1-\gamma)(2-\gamma)}|w^{(4)}(\eta_r)|\sigma^{4-\gamma},$$

where $\eta_r \in (z_{r-1}, z_r)$. Then we have the following;

$$\begin{aligned}
|T_r^{(L123)}| \leq& \frac{2}{3}\left|w'''(\xi)\right|(z_r-z_2)^{-\gamma-1}\sigma^4 + \frac{1}{12}\gamma|w^{(4)}(\eta)|\sigma^{4-\gamma} + \frac{3\gamma}{2(1-\gamma)(2-\gamma)}|w^{(4)}(\eta_r)|\sigma^{4-\gamma}\\
\leq& \frac{2}{3}\max_{z_0\leq z\leq z_2}\left|w'''(\xi)\right|(z_r-z_2)^{-\gamma-1}\sigma^4 + \frac{1}{12\gamma}\max_{z_0\leq z\leq z_r}|w^{(4)}(\eta)|\sigma^{4-\gamma}\\
&+\frac{3\gamma}{2(1-\gamma)(2-\gamma)}\max_{z_0\leq z\leq z_r}|w^{(4)}(\eta_r)|\sigma^{4-\gamma}\\
\leq& \frac{1}{\Gamma(1-\gamma)}\left(\frac{2\gamma}{3}\max_{z_0\leq z\leq z_2}|w'''(z)|(z_r-z_2)^{-\gamma-1}\sigma^4\right.\\
&\left.+\left(\frac{1}{12}+\frac{3\gamma^2}{2(1-\gamma)(2-\gamma)}\right)\max_{z_0\leq z\leq z_r}|w^{(4)}(z)|\sigma^{4-\gamma}\right).
\end{aligned}$$

□

Lemma 4.5. *The remainder $T_r^{(S)}$ of the Simpsons' $\frac{1}{3}$ approximation on $C(0,1)$ satisfies*

$$\left|T_r^{(S)}\right| = |(J_z - J_R)w(z_r)| \leq C\,\sigma^4, \qquad r = 1, 2, \ldots, R,$$

where $J_z w(z_r)$ denotes the exact integral and $J_R w(z_r)$ its Simpson's approximation.

Proof. Following the procedure of [5], Let the function $v(p) = \mathbb{K}(z_r, p)w(p)$. Then the Taylor series expansion $v(p)$ about $p = z_r$ is:

$$v(p) = v_r + (p-z_r)v_r' + \frac{1}{2!}(p-z_r)^2v_r'' + \frac{1}{3!}(p-z_r)^3v_r''' + \frac{1}{4!}(p-z_r)^4v_r'''' + \cdots.$$

Assume the number of sub-interval n is even and integrate each term over $p \in [z_{2r}, z_{2r+2}], r = 0, 1, 2, \ldots, \frac{R-2}{2}$, where $z_{2r} - z_{2r+2} = 2\sigma$.

$$\begin{aligned}
\int_{z_0}^{z_2}v(p)dp \approx& \int_{z_0}^{z_2}\left[v_0 + (p-z_0)v_0' + \frac{1}{2!}(p-z_0)^2v_0'' + \frac{1}{3!}(p-z_0)^3v_0''' + \frac{1}{4!}(p-z_0)^4v_0'''' + \ldots\right]dp,\\
=& v_0\int_{z_0}^{z_2}dp + v_0'\int_{z_0}^{z_2}(p-z_0)dp + v_0''\int_{z_0}^{z_2}\frac{(p-z_0)^2}{2}dp + v_0'''\int_{z_0}^{z_2}\frac{(p-z_0)^3}{3!}dp + \cdots.
\end{aligned}$$

Using integration by substitution, we have

$$\int_{z_0}^{z_2}v(p)dp = 2\sigma v_0 + 2\sigma^2v_0' + \frac{4}{3}\sigma^3v_0'' + \frac{2}{3}\sigma^4v_0''' + \frac{4}{15}\sigma^5v_0^{(4)} + \cdots. \tag{4.1}$$

The values of v at nodes $0, \sigma, 2\sigma$ are

$$v(0) = v(0),$$

Uncorrected Proof

$$\begin{aligned} v(\sigma) =& v(0) + \sigma v_0' + \frac{\sigma^2}{2} v_0'' + \frac{\sigma^3}{6} v_0''' + \frac{\sigma^4}{24} v_0'''' + \dots, \\ v(2\sigma) =& v(0) + 2\sigma v_0' + \frac{(2\sigma)^2}{2} v_0'' + \frac{(2\sigma)^3}{6} v_0''' + \frac{(2\sigma)^4}{24} v_0'''' + \dots \\ =& v(0) + 2\sigma v_0' + 2\sigma^2 v_0'' + \frac{4}{3}\sigma^3 v_0''' + \frac{2}{3}\sigma^4 6 v_0^{(4)} + \frac{1}{120}(2\sigma)^5 v_0^{(5)}(\eta_0), \quad \eta_0 \in (0, 2\sigma). \end{aligned}$$

The Simpson's $\frac{1}{3}$ approximation on $[0, 2\sigma]$ is

$$\int_0^{2\sigma} v(p)dp = \frac{\sigma}{3}\big[v(0) + 4v(\sigma) + v(2\sigma)\big].$$

Substituting the the values $v(0), v(\sigma), v(2\sigma)$ and rearranging the terms, we obtain

$$\int_0^{2\sigma} v(p)dp = 2\sigma v_0 + 2\sigma^2 v_0' + \frac{4}{3}\sigma^3 v_0'' + \frac{2}{3}\sigma^4 v_0''' + \frac{5}{18}\sigma^5 v_0^{(4)} + \cdots. \tag{4.2}$$

From Equations (4.1) and (4.2), we have

$$\begin{aligned} \int_{z_0}^{z_2} \big(v(p) - \frac{\sigma}{3}\big[v(0) + 4v(\sigma) + v(2\tau)\big]\big)dp =& (2\sigma v_0 + 2\sigma^2 v_0' + \frac{4}{3}\sigma^3 v_0'' + \frac{2}{3}\sigma^4 v_0''' + \frac{4}{15}\sigma^5 v_0'''' + \dots) \\ & - (2\sigma v_0 + 2\sigma^2 v_0' + \frac{4}{3}\sigma^3 v_0'' + \frac{2}{3}^4 v_0''' + \frac{5}{18}\sigma^5 v_0^{(4)} + \dots) \\ =& (\frac{4}{15} - \frac{5}{18})\sigma^5 v_0^{(4)} = -\frac{1}{90}\sigma^5 v^{(4)}(\eta_0). \end{aligned}$$

The local truncation error over $[z_{2m}, z_{2m+2}]$ is $T_r^{(S)} = -\frac{1}{90}\sigma^5 v^{(4)}(\eta_m)$, $\eta_m \in [z_{2m}, z_{2m+2}]$.

In general, the Simpson's $\frac{1}{3}$ requires an even number of sub-intervals. Let the interval $[0, 1]$ be divided into r sub-intervals, with r even, nodes $0 = z_0 < z_1 < z_2 < \dots, < z_r = 1$.
The error in the interval $[0, 1]$ is

$$|T_r^{(S)}| = \Big| - \sum_{q=0}^{\frac{r}{2}-1} \frac{1}{90}\sigma^5 v^{(4)}(\eta_m)\Big|.$$

Let $M_4 = max_{z\in[z_0,z_2]}|w^{(4)}(z)|$. Then $|T_r^{(S)}| = \sum_{q=0}^{\frac{r}{2}-1} \frac{1}{90}\sigma^5 M_4 = \frac{\sigma^5}{90}(\frac{r}{2})M_4 = \frac{\sigma^5}{180}(r)M_4$. For $z_r \in [0, 1] \implies z_r < 1$ and $z_r = r\sigma < 1$. Then $r < \frac{1}{\sigma}$.

$$|T_r^{(S)}| = \frac{\sigma^5}{180} r M_4 < \frac{\sigma^5}{180}(\frac{1}{\sigma})W_4 = \frac{1}{180}M_4\sigma^4.$$

Hence, $|T_r^{(S)}| \le C\sigma^4, \quad C > 0.$ □

Theorem 4.6. *If $\{w(z_r)\}_{r=1}^R$ is the true solution and $\{\hat{W}_r\}_{r=1}^R$ is the numerical solution obtained using the scheme (3.9) of problem (1.1), then the following relationship holds.*

$$\max_{r=1,2,\dots,R} |w(z_r) - \hat{W}_r| \le C^\star \sigma^{4-\gamma}, \quad C^\star > 0.$$

Proof. Using the definition of triangle inequality,

$$|T_r^{(CS)}| \le |T_r^{(L123)}| + |T_r^{(S)}|.$$

From Lemmas 4.4 and 4.5, we have $|R_r^{(L123)}| \le C_1\sigma^{4-\gamma}$ and $|R_r^{(S)}| \le C_2\sigma^4$, for $c_1, c_2 > 0$.

Then,

$$|T_r^{(CS)}| \le |T_r^{(L123)}| + |T_r^{(S)}| \le C^\star(\sigma^{4-\gamma} + \sigma^4).$$

Using Theorem 4.3, we obtain $\|E\|_\infty \le C^\star\|T\|_\infty \le C^\star(\sigma^{4-\gamma} + \sigma^4)$. Hence, $|E\|_\infty \le C^\star\sigma^{4-\gamma}$. □

Uncorrected Proof

5. Results and discussion

This section uses the following numerical experiment to confirm the accuracy and effectiveness of the proposed approach. The mathematical software MATLAB is used to carry out the numerical experiment.

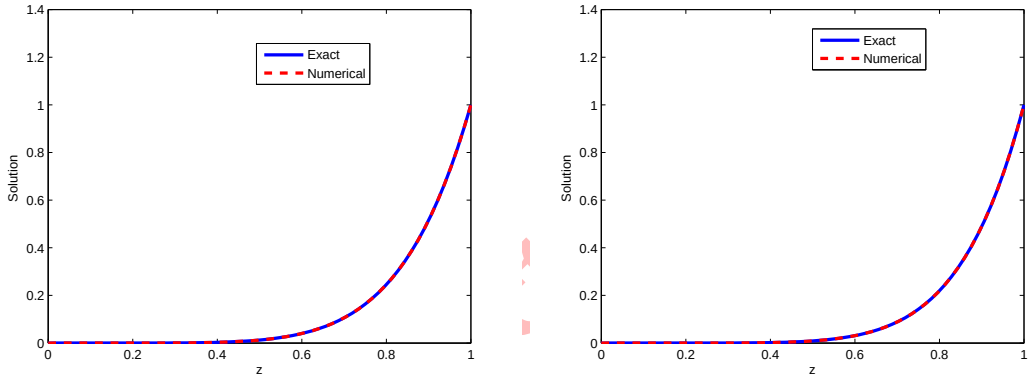
Example 5.1. Consider the following fractional integro-differential equation of the form

$$D_z^{\gamma} w(z) + w(z)^2 + \int_0^z (sin(zp)e^{zp} + z(cos(z) - 2))w(p)\, dp = g(z), \quad z \in (0, 1],$$

with initial condition $w(0) = 0$, where $\gamma \in (0, 1)$. We take the source term $g(z)$ with its exact solution $w(z) = z^{6+\gamma}$.

Let $z_r = r\sigma$ with $\sigma = 1/R$, and let $\hat{W}_r$ denote the numerical approximation of $w(z_r)$. The global error (φ_R) and associated order of rate of convergence (θ_R) are estimated as:

$$\varphi_R = \max_{0 \le r \le R} \left| w(z_r) - \hat{W}_r \right|, \qquad \theta_R = \log_2 \left(\frac{\varphi_R}{\varphi_{2R}} \right). \tag{5.1}$$



(A) $\gamma = 0.3$, $R = 64$. (B) $\gamma = 0.8$, $R = 64$.

Figure 1. Comparison of exact and numerical solutions of Example 5.1 for different γ values.

Comparisons between the exact and numerical solutions with γ values 0.3 and 0.8 with $R = 64$ for Example 5.1 are shown in Figure 1. Seeing these figures, the numerical solution overlaps the exact solution. One can conclude that the numerical solutions converge to the exact solutions. In Figure 2, we show the numerical solutions of different γ values with a fixed number of mesh point. Table 1 displays the comparisons of maximum absolute error and order of convergence of Example 5.1 with different values of γ. As we observe, the error decreases as R increases, demonstrating that the proposed method is convergent and achieves third-order accuracy.

Example 5.2. Let $\gamma \in (0, 1)$. Consider the following test problem:

$$D_z^{\gamma} w(z) + (e^z + w(z))w(z) + \int_0^z \left(\sin(z - p) + \frac{1}{1 + zp} \right) w^2(p)\, dp = g(z), \quad z \in (0, 1],$$

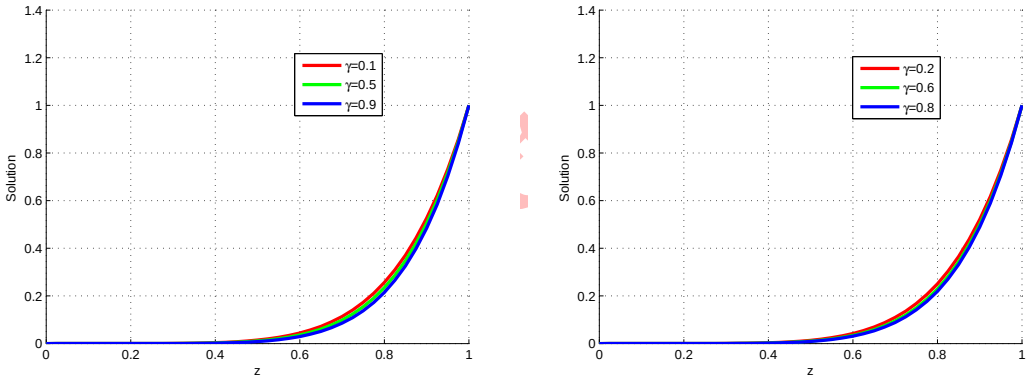
with initial condition $w(0) = 0$. Choose $g(z)$, so that the exact solution is $w(z) = z^{4-\gamma}$.

We show the exact solution and the numerical solutions in Figure 3 with $R = 50$ for $\gamma = 0.1, 0.2$. Figure 4a shows the numerical solutions of different γ values with fixed $R = 80$, and Figure 4b represents the numerical solutions for different mesh points with $\gamma = 0.6$. Maximum absolute error and order of convergence for multiple values of γ of Example 5.2 are displayed in Table 2, and we observe that the proposed method converges.

Uncorrected Proof

TABLE 1. Maximum absolute error and order of convergence of Example 5.1 for different γ values.

γ/R	32	64	128	256	512	1024
0.1	5.708×10^{-5}	7.517×10^{-6}	9.551×10^{-7}	1.193×10^{-7}	1.479×10^{-8}	1.842×10^{-9}
	2.925	2.976	3.001	3.012	3.005	
0.2	5.538×10^{-5}	6.870×10^{-6}	8.348×10^{-7}	1.008×10^{-7}	1.216×10^{-8}	1.499×10^{-9}
	3.011	3.041	3.050	3.051	3.020	
0.3	6.052×10^{-5}	6.960×10^{-6}	7.956×10^{-7}	9.171×10^{-8}	1.071×10^{-8}	1.258×10^{-9}
	3.120	3.129	3.117	3.098	3.089	
0.4	7.706×10^{-5}	8.325×10^{-6}	8.937×10^{-7}	9.734×10^{-8}	1.084×10^{-8}	1.228×10^{-9}
	3.210	3.220	3.199	3.167	3.142	
0.5	1.117×10^{-4}	1.182×10^{-5}	1.224×10^{-6}	1.273×10^{-7}	1.347×10^{-8}	1.456×10^{-9}
	3.240	3.271	3.265	3.241	3.209	
0.6	1.762×10^{-4}	1.890×10^{-5}	1.971×10^{-6}	2.034×10^{-7}	2.105×10^{-8}	2.199×10^{-9}
	3.221	3.261	3.277	3.273	3.258	
0.7	2.926×10^{-4}	3.251×10^{-5}	3.496×10^{-6}	3.706×10^{-7}	3.912×10^{-8}	4.123×10^{-9}
	3.170	3.217	3.238	3.244	3.246	
0.8	4.800×10^{-4}	5.655×10^{-5}	6.428×10^{-6}	7.180×10^{-7}	7.954×10^{-8}	8.792×10^{-9}
	3.085	3.137	3.162	3.174	3.177	
0.9	7.729×10^{-4}	9.728×10^{-5}	1.179×10^{-5}	1.404×10^{-6}	1.655×10^{-7}	1.943×10^{-8}
	2.990	3.044	3.071	3.084	3.091	



(A) $R = 40$. (B) $R = 40$.

FIGURE 2. Numerical solutions of Example 5.1 for different γ values with $R = 40$.

Example 5.3. Consider the following VIDE's of fractional order:

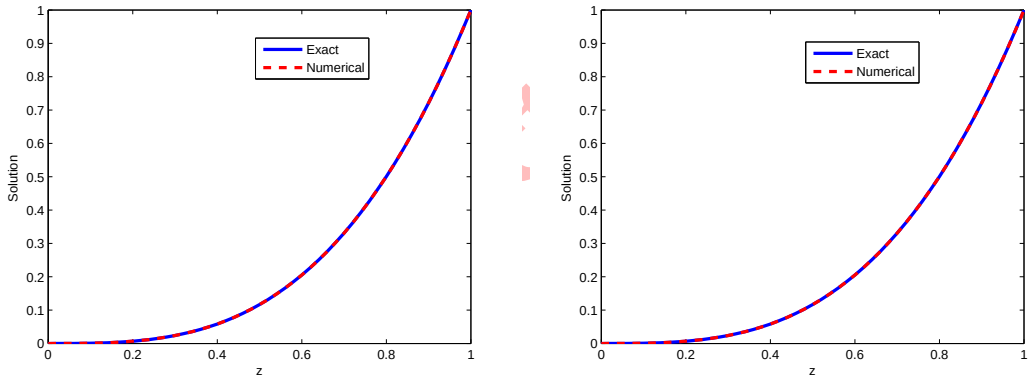
$$D_z^{\gamma} w(z) + w(z) + \int_0^z (3z + 4p)\, w(p)\, dp = g(z), \quad z \in (0, 1],$$

with initial condition $w(0) = 0$, and $g(z)$ is given by: $g(z) = \frac{\Gamma(4+\gamma)}{6} z^3 + z^{3+\gamma} + z^{5+\gamma} \left(\frac{3}{4+\gamma} + \frac{4}{5+\gamma} \right)$. The exact solution of Example 5.3 is $w(z) = z^{3+\gamma}$.

Figure 5a and 5b illustrate the exact and approximate solutions of Example 5.3, using $\gamma = 0.4$ and $\gamma = 0.7$, respectively, with $R = 64$. As we can see from these figures, it follows that the approximate solution approach the exact solution more closely. Figure 6 presents the approximate solutions for Example 5.3 with $R = 100$ at different values of γ. The numerical results for different values are shown in Figures 7a and 7b. Specifically, Figure 7a displays comparison of the exact solution and approximate solution for different mesh $R = 64, 128, 256$ with $\gamma = 0.2$, while

TABLE 2. Maximum absolute error and order of convergence of Example 5.2 for different γ values.

γ/R	32	64	128	256	512	1024
0.1	9.201×10^{-7}	1.258×10^{-7}	1.632×10^{-8}	2.068×10^{-9}	2.616×10^{-10}	4.636×10^{-11}
	2.871	2.946	2.980	2.983	2.496	
0.2	9.333×10^{-7}	1.213×10^{-7}	1.522×10^{-8}	1.879×10^{-9}	2.339×10^{-10}	5.057×10^{-11}
	2.944	2.994	3.018	3.006	2.210	
0.3	1.722×10^{-6}	1.463×10^{-7}	1.391×10^{-8}	1.666×10^{-9}	1.979×10^{-10}	2.698×10^{-11}
	3.557	3.395	3.061	3.073	2.875	
0.4	3.628×10^{-6}	3.308×10^{-7}	2.944×10^{-8}	2.575×10^{-9}	2.224×10^{-10}	1.969×10^{-11}
	3.455	3.490	3.515	3.534	3.497	
0.5	7.031×10^{-6}	6.835×10^{-7}	6.463×10^{-8}	5.997×10^{-9}	5.494×10^{-10}	5.045×10^{-11}
	3.363	3.403	3.430	3.448	3.445	
0.6	1.290×10^{-5}	1.332×10^{-6}	1.370×10^{-7}	1.375×10^{-8}	1.354×10^{-9}	1.316×10^{-10}
	3.276	3.281	3.316	3.344	3.363	
0.7	2.305×10^{-5}	2.638×10^{-6}	2.876×10^{-7}	3.050×10^{-8}	3.181×10^{-9}	3.283×10^{-10}
	3.127	3.197	3.237	3.261	3.276	
0.8	4.215×10^{-5}	5.072×10^{-6}	5.835×10^{-7}	6.553×10^{-8}	7.261×10^{-9}	7.983×10^{-10}
	3.055	3.120	3.154	3.174	3.185	
0.9	7.509×10^{-5}	9.513×10^{-6}	1.158×10^{-6}	1.381×10^{-7}	1.633×10^{-8}	1.922×10^{-9}
	2.981	3.038	3.068	3.080	3.087	



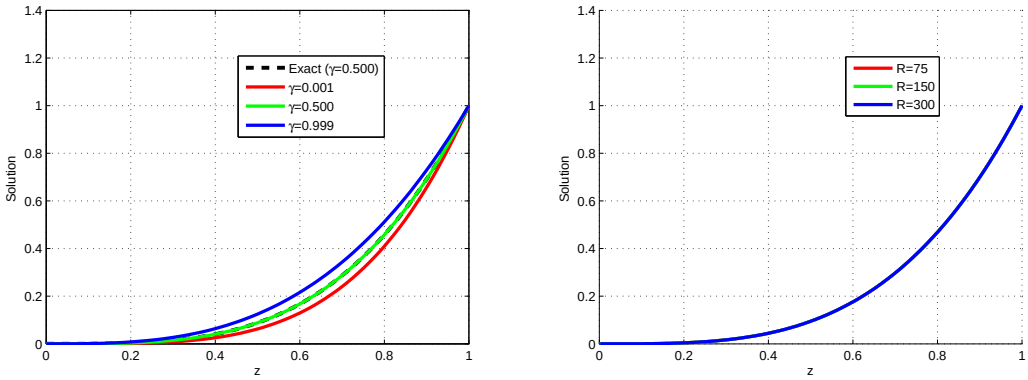
(A) $\gamma = 0.1$, $R = 50$. (B) $\gamma = 0.9$, $R = 50$.

FIGURE 3. Comparison of exact and numerical solutions of Example 5.2 for different γ values.

Figure 7b shows with $\gamma = 0.8$ of Example 5.3. We use the formula defined in (5.1) to calculate the maximum absolute error for Example 5.3, and its associated orders of convergence. Table 3 summarize the maximum absolute error and order of convergence of the proposed method and results reported in [11] of Example 5.3, using the same number of mesh points and the corresponding values of $\gamma = 0.1, 0.3, 0.5, 0.7, 0.9$. We observe that the proposed method achieves third order accuracy, whereas the results in [11] exhibits first and second order accuracy. Hence, the proposed method provides better accuracy.

In Table 4, we display the comparison of maximum absolute error, convergence order, and CPU time (in seconds) for Example 5.3. As we see from the table, the proposed method achieves third-order convergence for all tested fractional orders $\gamma = 0.2, 0.4, 0.6, 0.8$, and the error decreases when R increases, demonstrating the convergence of the proposed method. Let us consider the reported CPU times alongside errors and convergence orders at $R = 1024$ and $\gamma = 0.6$. The proposed method requires 7.962 seconds, compared to 5.077 seconds for the L1 method and 7.132

Proof



(A) $R = 80$. (B) $\gamma = 0.6$.

FIGURE 4. Numerical solutions of Example 5.2 for different γ values and different mesh points.

TABLE 3. Comparison of maximum absolute error and order of convergence of Example 5.3 for different γ values.

γ/R	32	64	128	256	512	1024
Proposed method						
0.1	4.251×10^{-5}	5.307×10^{-6}	6.540×10^{-7}	8.012×10^{-8}	9.779×10^{-9}	1.136×10^{-9}
	3.002	3.021	3.029	3.034	3.106	
0.3	3.317×10^{-5}	4.007×10^{-6}	4.808×10^{-7}	5.768×10^{-8}	6.945×10^{-9}	8.135×10^{-10}
	3.049	3.059	3.059	3.054	3.094	
0.5	2.974×10^{-5}	3.642×10^{-6}	4.440×10^{-7}	5.426×10^{-8}	6.658×10^{-9}	7.920×10^{-10}
	3.030	3.036	3.033	3.027	3.072	
0.7	2.681×10^{-5}	3.444×10^{-6}	4.334×10^{-7}	5.415×10^{-8}	6.757×10^{-9}	8.830×10^{-10}
	2.960	2.991	3.001	3.003	2.936	
0.9	3.636×10^{-5}	4.151×10^{-6}	4.771×10^{-7}	5.506×10^{-8}	6.369×10^{-9}	7.376×10^{-10}
	3.131	3.121	3.115	3.112	3.110	
Ref. [11]						
0.1	5.356×10^{-4}	1.340×10^{-4}	3.350×10^{-5}	8.376×10^{-6}	2.094×10^{-6}	5.235×10^{-7}
	1.999	2.000	2.000	2.000	2.000	
0.3	5.437×10^{-4}	1.366×10^{-4}	3.423×10^{-5}	8.571×10^{-6}	2.145×10^{-6}	5.365×10^{-7}
	1.993	1.996	1.998	1.999	1.999	
0.5	4.958×10^{-4}	1.272×10^{-4}	3.238×10^{-5}	8.195×10^{-6}	2.066×10^{-6}	5.197×10^{-7}
	1.962	1.974	1.982	1.988	1.991	
0.7	2.666×10^{-4}	7.750×10^{-5}	2.159×10^{-5}	5.850×10^{-6}	1.555×10^{-6}	4.074×10^{-7}
	1.782	1.844	1.884	1.912	1.932	
0.9	7.177×10^{-4}	1.690×10^{-4}	3.928×10^{-5}	9.069×10^{-6}	2.087×10^{-6}	4.791×10^{-7}
	2.086	2.105	2.115	2.120	2.123	

Uncorrected Proof

seconds for the L1-2 method. Thus, the proposed method requires approximately 57% more CPU time than the L1 method (ratio ≈ 1.57) and 12% more than the L1-2 method (ratio ≈ 1.12). The proposed method provides significantly higher accuracy per computational cost. At $\gamma = 0.6$, the CPU time quadruples each time R doubles ($0.031 \to 0.124 \to 0.497 \to 1.989 \to 7.962$ seconds). This shows the time complexity of the proposed method is $O(R^2)$. The $O(R)$ memory complexity follows from the necessity of storing all previous time steps. The method's superior third-order accuracy leads lower overall memory usage in practical applications. The following example of type (1.1) is considered, where the analytical solution is unknown. For such equations, the maximum error and the convergence

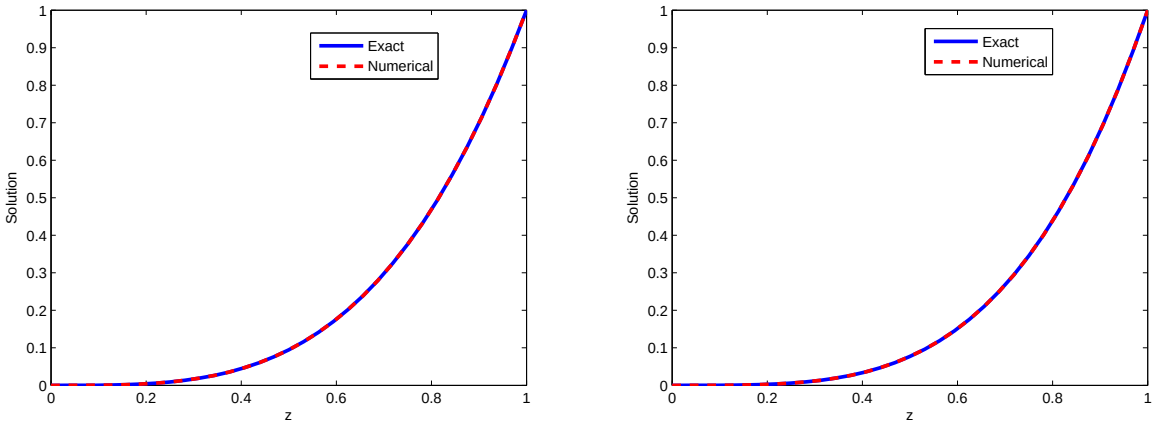
TABLE 4. Comparison of maximum absolute error, convergence order, and central processing unit (CPU) time (in seconds) for Example 5.3 at different γ and R values.

γ/R	64	128	256	512	1024
Proposed method					
0.2	4.512×10^{-6}	5.446×10^{-7}	6.549×10^{-8}	7.855×10^{-9}	8.780×10^{-10}
	3.051	3.056	3.060	3.161	
	0.028 sec	0.112 sec	0.447 sec	1.789 sec	7.162 sec
0.4	3.747×10^{-6}	4.521×10^{-7}	5.469×10^{-8}	6.649×10^{-9}	8.115×10^{-10}
	3.051	3.047	3.040	3.035	
	0.029 sec	0.115 sec	0.461 sec	1.845 sec	7.384 sec
0.6	3.591×10^{-6}	4.434×10^{-7}	5.471×10^{-8}	6.764×10^{-9}	8.172×10^{-10}
	3.018	3.019	3.016	3.049	
	0.031 sec	0.124 sec	0.497 sec	1.989 sec	7.962 sec
0.8	2.884×10^{-6}	3.773×10^{-7}	4.840×10^{-8}	6.145×10^{-9}	7.778×10^{-10}
	2.934	2.963	2.977	2.982	
	0.030 sec	0.119 sec	0.476 sec	1.906 sec	7.628 sec
Ref. [11]: L1 method					
0.2	7.316×10^{-5}	1.505×10^{-5}	3.397×10^{-6}	1.094×10^{-6}	3.459×10^{-7}
	2.282	2.147	1.635	1.661	
	0.019 sec	0.076 sec	0.305 sec	1.221 sec	4.886 sec
0.4	2.708×10^{-4}	9.873×10^{-5}	3.508×10^{-5}	1.223×10^{-5}	4.209×10^{-6}
	1.455	1.493	1.520	1.539	
	0.018 sec	0.074 sec	0.298 sec	1.193 sec	4.775 sec
0.6	1.482×10^{-3}	5.866×10^{-4}	2.285×10^{-4}	8.811×10^{-5}	3.377×10^{-5}
	1.337	1.360	1.375	1.384	
	0.020 sec	0.079 sec	0.317 sec	1.268 sec	5.077 sec
0.8	5.864×10^{-3}	2.598×10^{-3}	1.142×10^{-3}	4.996×10^{-4}	2.181×10^{-4}
	1.175	1.186	1.192	1.196	
	0.019 sec	0.077 sec	0.310 sec	1.242 sec	4.972 sec
Ref. [11]: L1-2 method					
0.2	1.361×10^{-4}	3.406×10^{-5}	8.518×10^{-6}	2.130×10^{-6}	5.326×10^{-7}
	1.999	1.999	2.000	2.000	
	0.027 sec	0.108 sec	0.433 sec	1.734 sec	6.938 sec
0.4	1.343×10^{-4}	3.381×10^{-5}	8.492×10^{-6}	2.129×10^{-6}	5.334×10^{-7}
	1.990	1.993	1.996	1.997	
	0.026 sec	0.106 sec	0.425 sec	1.702 sec	6.812 sec
0.6	1.112×10^{-4}	2.900×10^{-5}	7.475×10^{-6}	1.912×10^{-6}	4.861×10^{-7}
	1.939	1.956	1.967	1.976	
	0.028 sec	0.111 sec	0.445 sec	1.782 sec	7.132 sec
0.8	7.464×10^{-5}	1.593×10^{-5}	3.369×10^{-6}	7.147×10^{-7}	2.148×10^{-7}
	2.228	2.241	2.237	1.734	
	0.027 sec	0.109 sec	0.438 sec	1.754 sec	7.020 sec

Uncorrected Proof

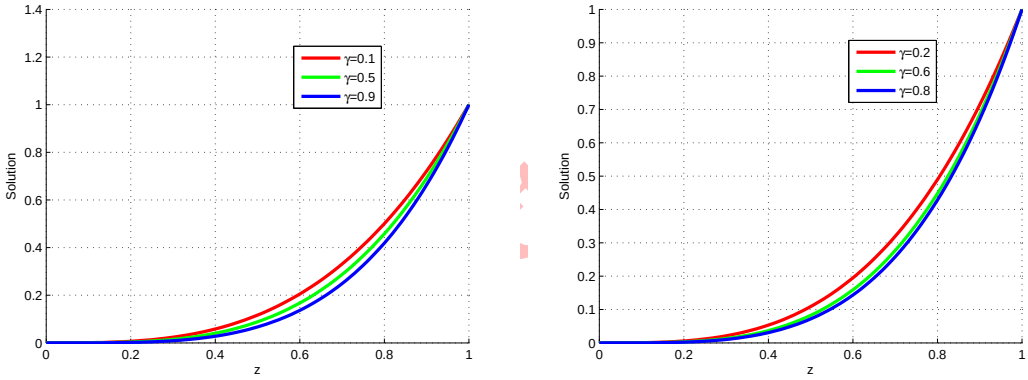
rate are computed using the double mesh principle. The double mesh principle is employed to compute the error and the order of convergence. Suppose that $\hat{W}_r$ and $\hat{W}_{2r}$ denotes the numerical solutions on meshs with grid points R and $2R$, respectively. Accordingly, the error and the associated order of convergence are estimated as follows:

$$\widehat{\varphi}_R = \max_{0 \le r \le R} \left| \hat{W}_{2r} - \hat{W}_r \right|, \qquad \widehat{\theta_R} = \log_2 \left(\frac{\widehat{\varphi_R}}{\widehat{\varphi_{2R}}} \right). \tag{5.2}$$



(A) $\gamma = 0.4,\ R = 32.$ (B) $\gamma = 0.7,\ R = 32.$

FIGURE 5. Comparison of exact and numerical solutions of Example 5.3 for different γ values.



(A) $R = 100.$ (B) $R = 100.$

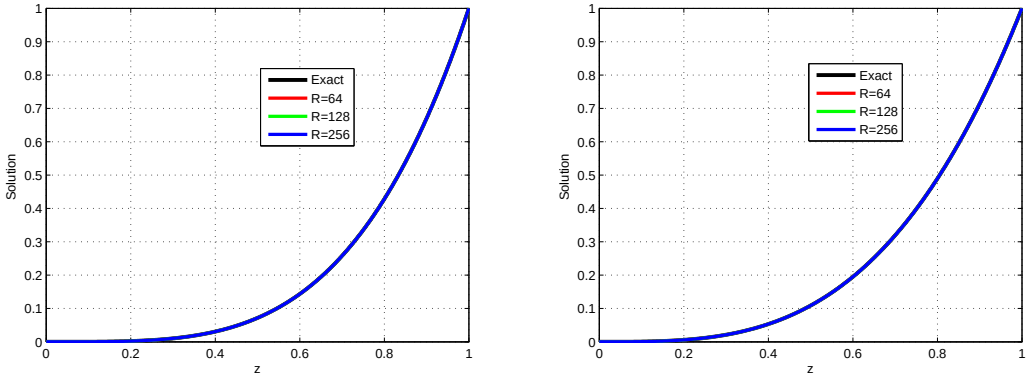
FIGURE 6. Numerical solutions of Example 5.3 for different γ values with $R = 100$.

Example 5.4. Consider the following fractional order integro PDE:

$$D_z^{\gamma} w(z) + (1 - z)\, w(z) + \int_0^z \left(p^3 e^z + z \sin^2(z)\right) w(p)\, dp = z^3, \quad z \in (0, 1],$$

with initial condition $w(0) = 0$.

Using the scheme (3.9), we solve the problem and present the results in graphs and tables. A comparison between the exact and approximate solutions are illustrated in Figure 8. From these figure, the approximate solution is very close to the exact solution. This indicates that the proposed method is convergent. In Figure 9 we present the numerical solutions of Example 5.4 with different values of γ and R, and one can observe that for all R the approximate solution are in good agreement. This indicates that the proposed method is consistent and fits well. Since the exact solution of Example 5.4 is unknown, so the convergence rate and the maximum error are computed using the formula given in (5.2). In Table 5 and Table 6, we compare the maximum absolute error and the rate of convergence in the proposed method with those reported in [11], using the values of $\gamma = 0.3, 0.6, 0.9$ and $\gamma = 0.2, 0.4, 0.6, 0.8$, respectively. We



(A) $\gamma = 0.2$. (B) $\gamma = 0.8$.

FIGURE 7. Numerical solutions of Example 5.3 with different mesh points.

TABLE 5. Comparison of maximum absolute error and order of convergence of Example 5.4 for different γ values.

γ/R	32	64	128	256	512	1024
Proposed method						
0.1	3.240×10^{-5}	3.761×10^{-6}	4.488×10^{-7}	5.379×10^{-8}	6.518×10^{-9}	8.495×10^{-10}
	3.107	3.067	3.061	3.045	2.940	
0.3	2.991×10^{-5}	3.328×10^{-6}	3.695×10^{-7}	4.186×10^{-8}	4.873×10^{-9}	8.226×10^{-10}
	3.168	3.171	3.142	3.102	2.567	
0.5	2.826×10^{-5}	3.379×10^{-6}	3.826×10^{-7}	4.282×10^{-8}	4.850×10^{-9}	8.213×10^{-10}
	3.064	3.143	3.159	3.142	2.562	
0.7	2.634×10^{-5}	3.295×10^{-6}	3.856×10^{-7}	4.396×10^{-8}	4.959×10^{-9}	2.898×10^{-10}
	2.999	3.095	3.133	3.148	4.097	
0.9	1.310×10^{-5}	2.001×10^{-6}	2.723×10^{-7}	3.455×10^{-8}	4.245×10^{-9}	6.705×10^{-10}
	2.711	2.877	2.978	3.025	2.662	
Ref. [11]						
0.1	3.925×10^{-4}	9.614×10^{-5}	2.370×10^{-5}	5.873×10^{-6}	1.460×10^{-6}	3.639×10^{-7}
	2.029	2.020	2.013	2.008	2.005	
0.3	3.345×10^{-4}	7.993×10^{-5}	1.919×10^{-5}	4.652×10^{-6}	1.138×10^{-6}	2.804×10^{-7}
	2.065	2.058	2.044	2.031	2.021	
0.5	2.197×10^{-4}	5.422×10^{-5}	1.312×10^{-5}	3.168×10^{-6}	7.686×10^{-7}	1.876×10^{-7}
	2.019	2.047	2.050	2.043	2.035	
0.7	1.019×10^{-4}	2.059×10^{-5}	4.641×10^{-6}	1.279×10^{-6}	3.398×10^{-7}	8.853×10^{-8}
	2.307	2.149	1.859	1.913	1.940	
0.9	2.703×10^{-4}	6.364×10^{-5}	1.485×10^{-5}	3.451×10^{-6}	8.005×10^{-7}	1.855×10^{-7}
	2.087	2.100	2.105	2.108	2.109	

Uncorrected Proof

observe that the proposed method achieves third-order convergence and gives better accuracy than the results in Ref [11].

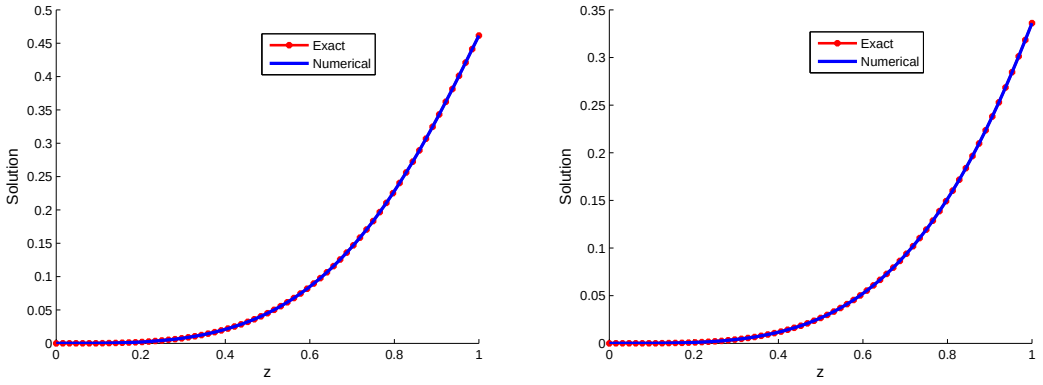
6. CONCLUSION

This work presents a higher order numerical method for fractional Volterra integro-differential equations of the form in (1.1) in the Caputo sense. The problem is solved using the proposed L1-2-3 method for Caputo fractional derivative, combined with Simpson's $\frac{1}{3}$ rule for the Volterra integral, on a uniform mesh. The convergence and stability of the

TABLE 6. Comparison of maximum absolute error and order of convergence of Example 5.4 for different γ values with the result in [11].

γ/R	64	128	256	512	1024
Proposed method					
0.2	3.321×10^{-6}	3.790×10^{-7}	4.408×10^{-8}	5.255×10^{-9}	5.704×10^{-10}
	3.132	3.104	3.069	3.204	
0.4	3.383×10^{-6}	3.778×10^{-7}	4.246×10^{-8}	4.859×10^{-9}	4.200×10^{-10}
	3.16	3.154	3.127	3.532	
0.6	3.383×10^{-6}	3.839×10^{-7}	4.337×10^{-8}	4.889×10^{-9}	7.413×10^{-10}
	3.139	3.146	3.149	2.721	
0.8	2.885×10^{-6}	3.572×10^{-7}	4.243×10^{-8}	4.949×10^{-9}	5.917×10^{-10}
	3.014	3.074	3.100	3.064	
Ref. [11] for $L1$					
0.2	9.858×10^{-5}	2.700×10^{-5}	7.408×10^{-6}	2.039×10^{-6}	5.628×10^{-7}
	1.868	1.866	1.862	1.857	
0.4	2.034×10^{-4}	7.097×10^{-5}	2.437×10^{-5}	8.275×10^{-6}	2.789×10^{-6}
	1.519	1.542	1.558	1.569	
0.6	6.198×10^{-4}	2.418×10^{-4}	9.329×10^{-5}	3.575×10^{-5}	1.365×10^{-5}
	1.358	1.374	1.384	1.389	
0.8	1.577×10^{-3}	6.966×10^{-4}	3.057×10^{-4}	1.337×10^{-4}	5.833×10^{-5}
	1.179	1.188	1.193	1.196	
Ref. [11] for L1-2					
0.2	8.968×10^{-5}	2.179×10^{-5}	5.340×10^{-6}	1.318×10^{-6}	3.269×10^{-7}
	2.041	2.028	2.018	2.012	
0.4	6.810×10^{-5}	1.629×10^{-5}	3.919×10^{-6}	9.516×10^{-7}	2.329×10^{-7}
	2.064	2.055	2.042	2.031	
0.6	3.731×10^{-5}	9.428×10^{-6}	2.339×10^{-6}	5.773×10^{-7}	1.425×10^{-7}
	1.985	2.011	2.019	2.019	
	3.658×10^{-5}	7.899×10^{-6}	1.696×10^{-6}	3.630×10^{-7}	7.746×10^{-8}
0.8	2.211	2.219	2.224	2.229	

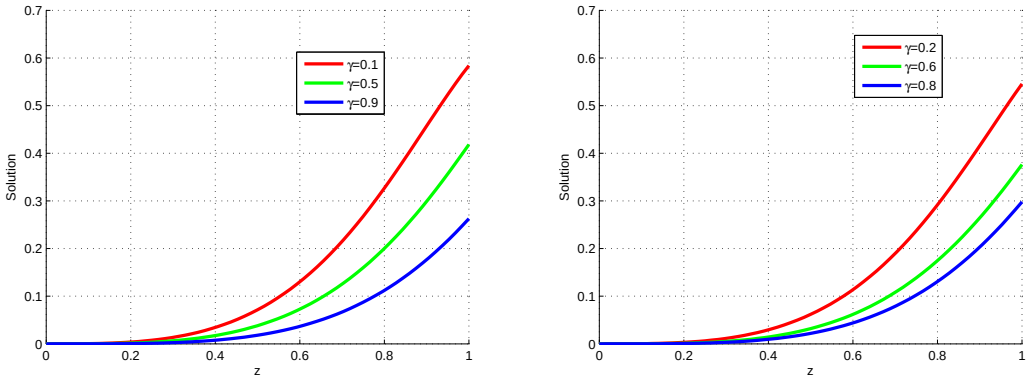
rrected Proof



(A) $\gamma = 0.4$, $R = 64$. (B) $\gamma = 0.7$, $R = 64$.

FIGURE 8. Comparison of exact and numerical solutions of Example 5.4.

proposed method have been thoroughly analyzed, and the method achieves third-order accuracy. As the mesh number R increases, the approximate solution becomes progressively closer to the exact solution. The proposed method



(A) $R = 64$. (B) $R = 64$.

FIGURE 9. Numerical solutions of Example 5.4 for different γ values with $R = 64$.

has $O(R^2)$ time complexity and $O(R)$ memory complexity. The method's efficiency is verified through illustrative examples, and the results are compared with those reported in other works. The proposed method is also effective for solving problems that do not have an exact solution. In the future, we plan to apply the method on nonuniform meshes to solve weak singularities and to extend the method to multi-term fractional derivatives for solving the model problem (1.1) and combined Fredholm-Volterra integro differential equations.

AVAILABILITY OF DATA

The data supporting the findings of the study will be made available by the corresponding author upon reasonable request.

CONFLICTS OF INTEREST

The authors have declared no conflicts of interest pertaining to the creation of this manuscript.

FUNDING

The authors affirm that no grants, funds, or other types of assistance were obtained to help them write this work.

AUTHORS CONTRIBUTION

All authors contributed equally and approved the article's submission.

REFERENCES

[1] S. Abbasbandy and E. Shivanian, *Application of the variational iteration method to nonlinear Volterra's integro-differential equations*, Zeitschrift für Naturforschung A, *63*(9) (2008), 538–542.
[2] S. Abbasbandy and E. Shivanian, *Application of the variational iteration method for system of nonlinear Volterra's integro-differential equations*, Mathematical and computational applications, *14*(2) (2009), 147–158.
[3] B. N. Abood, *Approximate solutions and existence of solution for a Caputo nonlocal fractional volterra fredholm integro-differential equation*, International Journal of Applied Mathematics, *33*(6) (2020), 1049.
[4] R. L. Bagley and P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, Journal of rheology, *27*(3) (1983), 201–210.
[5] S. R. Badeye, M. M. Woldaregay, and T. G. Dinka, *Solving singularly perturbed fredholm integro-differential equation using exact finite difference method*, BMC Research Notes, *16*(1) (2023), 233.

Uncorrected Proof

[6] J. Cao, C. Li, and Y. Chen, *High-order approximation to Caputo derivatives and Caputo-type advection-diffusion equations (II)*, Fractional calculus and Applied analysis, *18*(3) (2015), 735–761.
[7] A. Carpinteri, B. Chiaia, and P. Cornetti, *Static–kinematic duality and the principle of virtual work in the mechanics of fractal media*, Computer methods in applied mechanics and engineering, *191*(1-2) (2001), 3–19.
[8] Deepmala, V. N. Mishra, H. R. Marasi, H. Shabanian, and M. Nosraty, *Solution of voltrafredholm integro-differential equations using chebyshev collocation method*, Global Journal of Technology and Optimization, *8*(210) (2017), 1–4.
[9] K. Diethelm and N. J. Ford, *The analysis of fractional differential equations*, Lecture notes in mathematics, Springer, 2010.
[10] Z. E. A. Fellah, C. Depollier, and M. Fellah, *Application of fractional calculus to the sound waves propagation in rigid porous materials: validation via ultrasonic measurements*, Acta Acustica united with Acustica, *88*(1) (2002), 34–39.
[11] B. Ghosh and J. Mohapatra, *Analysis of finite difference schemes for Volterra integro-differential equations involving arbitrary order derivatives*, Journal of Applied Mathematics and Computing, *69*(2) (2023), 1865–1886.
[12] B. Ghosh and J. Mohapatra, *Efficient numerical schemes based on the cubic B-spline collocation method for time-fractional partial integro-differential equations of Volterra type*, Journal of Applied Mathematics and Computing, *70*(1) (2024), 741–769.
[13] A. Golbabai, O. Nikan, and T. Nikazad, *Numerical analysis of time fractional Black–Scholes European option pricing model arising in financial market*, Computational and Applied Mathematics, *38*(4) (2019), 173.
[14] M. E. Gordji, H. Baghani, and O. Baghani, *On existence and uniqueness of solutions of a nonlinear integral equation*, Journal of Applied Mathematics, *2011*(1) (2011), 743923.
[15] A. A. Hamoud, K. P. Ghadle, M. Sh. B. Issa, and Giniswamy, *Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations*, International Journal of Applied Mathematics, *31*(3) (2018), 333–348.
[16] A. A. Hamoud, K. P. Ghadle, and P. A. Pathade, *An existence and convergence results for Caputo fractional Volterra integro-differential equations*, Jordan Journal of Mathematics and Statistics, *12*(3) (2019), 307–327.
[17] A. A. Hamoud, K. P. Ghadle, et al., *Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations*, Journal of Applied and Computational Mechanics, *5*(1) (2019), 58–69.
[18] A. Hamoud, *Existence and uniqueness of solutions for fractional neutral volterra-fredholm integro differential equations*, Advances in the Theory of Nonlinear Analysis and its Application, *4*(4) (2020), 321–331.
[19] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi, and C. Cattani, *Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations*, Communications in Nonlinear Science and Numerical Simulation, *19*(1) (2014), 37–48.
[20] M. A. Ichou, H. El Amri, and A. Ezziani, *On existence and uniqueness of solution for space–time fractional Zener model*, Acta Applicandae Mathematicae, *170*(1) (2020), 593–609.
[21] V. V. Kulish and J. L. Lage, *Application of fractional calculus to fluid mechanics*, J. Fluids Eng., *124*(3) (2002), 803–806.
[22] S. Larsson, M. Racheva, and F. Saedpanah, *Discontinuous Galerkin method for an integro-differential equation modeling dynamic fractional order viscoelasticity*, Computer Methods in Applied Mechanics and Engineering, *283* (2015), 196–209.
[23] V. Mehandiratta, M. Mehra, and G. Leugering, *An approach based on Haar wavelet for the approximation of fractional calculus with application to initial and boundary value problems*, Mathematical Methods in the Applied Sciences, *44*(4) (2021), 3195–3213.
[24] A. Mennouni, *A new numerical approximation for Volterra integral equations combining two quadrature rules*, Applied Mathematics and Computation, *218*(5) (2011), 1962–1969.
[25] R. Mokhtari and F. Mostajeran, *A high order formula to approximate the Caputo fractional derivative*, Communications on Applied Mathematics and Computation, *2*(1) (2020), 1–29.

Uncorrected Proof

[26] S. Momani and M. A. Noor, *Numerical methods for fourth-order fractional integro-differential equations*, Applied Mathematics and Computation, *182*(1) (2006), 754–760.
[27] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-order systems and controls: fundamentals and applications*, Springer, 2010.
[28] M. Ramezani, R. Mokhtari, and G. Haase, *Some high order formulae for approximating Caputo fractional derivatives*, Applied Numerical Mathematics, *153* (2020), 300–318.
[29] M. Roohi, C. Zhang, and Y. Chen, *Adaptive model-free synchronization of different fractional-order neural networks with an application in cryptography*, Nonlinear Dynamics, *100*(4) (2020), 3979–4001.
[30] Y. A. Rossikhin and M. V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev., *50*(1) (1997), 15–67.
[31] S. Santra and J. Mohapatra, *Numerical analysis of Volterra integro-differential equations with Caputo fractional derivative*, Iranian Journal of Science and Technology, Transactions A: Science, *45*(5) (2021), 1815–1824.
[32] S. Santra and J. Mohapatra, *A novel finite difference technique with error estimate for time fractional partial integro-differential equation of Volterra type*, Journal of Computational and Applied Mathematics, *400* (2022), 113746.
[33] K. Sayevand, M. Fardi, E. Moradi, and F. H. Boroujeni, *Convergence analysis of homotopy perturbation method for Volterra integro-differential equations of fractional order*, Alexandria Engineering Journal, *52*(4) (2013), 807–812.
[34] S. Shahmorad, *Numerical solution of the general form linear Fredholm–Volterra integro-differential equations by the Tau method with an error estimation*, Applied Mathematics and Computation, *167*(2) (2005), 1418–1429.
[35] S. Shahmorad and M. H. Ostadzad, *An operational matrix method for solving delay Fredholm and Volterra integro–differential equations*, International Journal of Computational Methods, *13*(06) (2016), 1650040.
[36] E. Soczkiewicz, *Application of fractional calculus in the theory of viscoelasticity*, Molecular and Quantum Acoustics, *23* (2002), 397–404.
[37] M. Suárez, *Scientific representation: Against similarity and isomorphism*, International studies in the philosophy of science, *17*(3) (2003), 225–244.
[38] H. Vosughi, E. Shivanian, and S. Abbasbandy, *A new analytical technique to solve Volterra's integral equations*, Mathematical methods in the applied sciences, *34*(10) (2011), 1243–1253.
[39] T. Wongyat and W. Sintunavarat, *The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fractional differential equations via w-distances*, Advances in Difference Equations, *2017*(1) (2017), 211.
[40] Ş. Yüzbaşı and M. Karaçayır, *A numerical approach for solving high-order linear delay Volterra integro-differential equations*, International Journal of Computational Methods, *15*(05) (2018), 1850042.

Uncorrected Proof