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Phase retrievability: Fusion frames, rank characterizations, dimension functions, and openness of dual frames

Masoumeh Allahgholizadeh and Mohammad Hossein Sattari[*]
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

Abstract

This paper develops a unified algebraic framework for phase retrieval by subspaces and fusion frames in finite-dimensional Hilbert spaces. The classical complement property, known to be both necessary and sufficient for vector phase retrieval, is shown to be insufficient when extended to collections of subspaces. To address this limitation, the authors introduce a fusion frame dimension function and establish a precise rank characterization, proving that its value coincides with the rank of an associated linear operator. This structural result leads to several consequences: the openness of the set of phase retrievable dual frames, a continuity criterion equivalent to phase retrievability. The presented framework clarifies the interplay among the complement property and operator-theoretic rank conditions, thereby extending the foundational theory of phase retrieval beyond the vector case to a broader class of subspace and fusion frame systems.

Keywords. Phase retrieval, Frames, Fusion frame, Frame dimension function.
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Uncorrected

1. Introduction

The phase retrieval problem is a fundamental inverse problem that arises in various areas of applied mathematics and engineering, including optics, crystallography, and signal processing. In its classical formulation, the goal is to recover a signal (or an element of a Hilbert space) from the magnitudes of its measurements, typically the absolute values of its inner products with a set of known vectors, while the phase information is lost. Mathematically, a family of vectors $\{f_i\}_{i\in I}$ is said to perform phase retrieval if for any two vectors x, y in a Hilbert space H, the equality $|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$ for all i implies that $x = cy$ for some unimodular scalar c. Over the past two decades, frame theory has emerged as a powerful mathematical framework for analyzing and constructing stable phase retrieval systems. In particular, the concepts of frame bounds, analysis and synthesis operators, and dual frames provide natural tools for understanding the redundancy and robustness of measurement schemes. A key milestone in the real case was the characterization of phase retrievable frames via the complement property. However, in complex Hilbert spaces, the situation is more delicate: the complement property is necessary but not sufficient, and additional algebraic conditions are required. More recently, the phase retrieval problem has been extended from vectors to subspaces, leading to the notion of fusion frames and projection based phase retrieval. In this setting, one observes the norms of orthogonal projections of a signal onto a family of subspaces rather than the absolute values of inner products with individual vectors. This generalization is not merely a mathematical curiosity; it arises naturally in applications such as distributed sensing, multi sensor networks, and quantum state tomography. As the authors note, the transition from vectors to subspaces introduces fundamental differences: even in real Hilbert spaces, the complement property is no longer equivalent to phase retrieval, and new analytical tools are required. The present paper investigates several key aspects of phase retrieval in the context of subspaces and fusion frames. The main contributions are as follows: Phase retrieval by subspaces: The authors examine necessary and sufficient conditions for a family of subspaces (or equivalently, their

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∗ Corresponding author. Email: sattari@azaruniv.ac.ir.

orthogonal projections) to perform phase retrieval in $\mathbb{R}^n$. We present a counterexample demonstrating (Example 3.2) showing that the complement property, while necessary, is not sufficient contrary to the vector case. Frame dimension function for fusion frames: Inspired by recent work on frame dimension functions, the authors extend this concept to fusion frames. We introduce $d_{\mathcal{P}}(x) = \dim \sum_{i=1}^{N} \|P_i x\| P_i(H)$ and establish a key rank equality $d_{\mathcal{P}}(x) = \operatorname{rank}(G_{\mathcal{P}}(x))$, where $G_{\mathcal{P}}(x) = \sum_{i=1}^{N} \|P_i x\| P_i$. This yields a unified algebraic characterization. Continuity and openness results: The authors prove that for a phase retrievable frame $\mathcal{F}$, the set of phase retrievable dual frames is open in the space of all dual frames (Theorem 4.2). Moreover, they show that the frame dimension function is continuous if and only if the frame does phase retrieval (Proposition 4.3). The paper is organized as follows. Section 2 recalls necessary preliminaries on frames, dual frames, and fusion frames. Section 3 studies phase retrieval by subspaces, including necessary conditions and counterexamples. Section 4 introduces the fusion frame dimension function and establishes its properties, including continuity and openness of phase retrievable dual frames.

2. Notations and preliminaries

We now give a brief review of the concepts of frame and fusion frame. We begin by defining the concept of a frame.

Definition 2.1. A sequence $\{f_i\}_{i\in I}$ is called a frame for a Hilbert space H, if there exist constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha \parallel f \parallel^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant \beta \parallel f \parallel^2, \tag{2.1}$$

for all $f \in H$.

If the right inequality in (2.1) is satisfied, $\{f_i\}_{i\in I}$ is a Bessel sequence. The constants α and β are called the frame bounds. In the N-dimensional case, frames are exactly the spanning sets, i.e., $\operatorname{span}\{f_i\}_{i=1}^{N} = H$.

We associate to a Bessel sequence $\{f_i\}_{i\in I}$ the synthesis operator

$$\mathcal{T} : l^2(I) \longrightarrow H, \mathcal{T}\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i,$$

and the analysis operator

$$\mathcal{T}^* : H \longrightarrow l^2(I), \mathcal{T}^*(f) = \{\langle f, f_i\rangle\}_{i\in I}.$$

For a frame $\{f_i\}_{i\in I}$ the operator

$$\mathcal{S} = \mathcal{T}\mathcal{T}^* : H \longrightarrow H.$$

Definition 2.2. Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space H, with synthesis operator $\mathcal{T}$. A frame $\{g_i\}_{i\in I}$ for H with synthesis operator $\mathcal{U}$ is dual frame of $\{f_i\}_{i\in I}$ if the following reconstruction formula holds:

$$f = \mathcal{U}\mathcal{T}^* f = \sum_{i\in I} \langle f, f_i\rangle g_i,$$

for all $f \in H$.

In particular, $\{S^{-1} f_i\}_{i\in I}$ is called the canonical dual frame of $\{f_i\}_{i\in I}$.

Definition 2.3. A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space H does phase retrieval if the equalities

$$|\langle x, f_i\rangle| = |\langle y, f_i\rangle|,$$

for every $i \in I$, imply that there is a unimodular scalar α so that $x = \alpha y$.

Uncorrected Proof

Note that if $\{f_i\}_{i\in I}$ does phase retrieval, then so does $\{a_i f_i\}_{i\in I}$ for any $0 < a_i < \infty$ for all $i \in I$.

For a finite-dimensional real Hilbert spaces, it is known [2] that a frame $\{f_i\}_{i=1}^N$ for H is phase retrieval if and only if

$$\text{span}\{\langle x, f_i\rangle f_i : i = 1, 2, ..., N\} = H,$$

holds for every nonzero vector $x \in H$. For the complex Hilbert space case, this condition is still necessary, but not sufficent[6].

Definition 2.4. A frame $\{f_i\}_{i=1}^N$ for a finite-dimensional Hilbert space H has the complement property if either $\text{span}\{f_i : i \in \Lambda\} = H$ or $\text{span}\{f_i : i \in \Lambda^c\} = H$ for any subset Λ of $\{1, 2, ..., N\}$.

Proposition 2.5. *[2]If a frame $\{f_i\}_{i=1}^N$ for a finite-dimensional Hilbert space H does phase retrieval, then it has complement property. In a real Hilbert space, if $\{f_i\}_{i=1}^N$ has the complement property, then it does phase retrieval.*

Proposition 2.6. *[7]Let $\{f_i\}_{i=1}^N$ be a frame for a finite-dimensional real Hilbert space H. Then the following are equivalent:*

- $\text{span}\{\langle x, f_i\rangle f_i : i = 1, 2, ..., N\} = H$ *for every nonzero vector* $x \in H$.
- $\text{rank}(\sum_{i=1}^N \mid \langle x, f_i\rangle \mid^2 f_i \otimes f_i) = dim H$ *for every nonzero vector* $x \in H$.
- $\{f_i\}_{i=1}^N$ *has the complement property.*

Note that for $x, y \in H$, $x \otimes y \in B(H)$ is the rank one operator defined by $z \longrightarrow \langle z, y\rangle x$ and $B(H)$ is the space of bounded linear operators on H.

Definition 2.7. For a frame $\mathcal{F} = \{f_i\}_{i=1}^N$ and a subset $\Lambda \subseteq [N]$, we use $\mathcal{F}_\Lambda$ to denote the subframe $\{f_i\}_{i\in\Lambda}$ for $H_\Lambda := \text{span}\mathcal{F}_\Lambda$, where $[N] = \{1, 2, ..., N\}$.

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces in H, and let $\{\omega_i\}_{i\in I}$ be a family of weights. We will denote $\{W_i\}_{i\in I}$ with $\mathcal{W}$, $\{\omega_i\}_{i\in I}$ with ω and $\{(W_i, \omega_i)\}_{i\in I}$ with $(\mathcal{W}, \omega)$. If $\mathcal{T} \in L(H, K)$ we will write $(\mathcal{T}\mathcal{W}, \omega)$ for $\{(\mathcal{T}W_i, \omega_i)\}_{i\in I}$. For V a closed subspace of H, P_V is the orthogonal projection onto V.

Definition 2.8. We say that $(\mathcal{W}, \omega)$ is a fusion frame for H, if there exist constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha \parallel f \parallel^2 \leqslant \sum_{i\in I} \omega_i^2 \parallel P_{W_i}(f) \parallel^2 \leqslant \beta \parallel f \parallel^2,$$

for all $f \in H$.

We call α and β the fusion frame bounds.

Definition 2.9. Let $\{W_i\}_{i=1}^N$ be a collection of subspaces of finite-dimensional Hilbert space H and $\{P_i\}_{i=1}^N$ be the orthogonal projections of the subspaces respectively. Then we say $\{W_i\}_{i=1}^N$ does phase retrieval if for all $x, y \in H$ satisfying

$$\parallel P_i x \parallel = \parallel P_i y \parallel \quad (forall \quad i \in [N]),$$

then $x = cy$ for some scalar $\mid c \mid = 1$.

In such a case $\{P_i\}_{i=1}^N$ is also called phase retrieval.

3. Phase retrievability of subspaces

Since the introduction of the phase retrieval problem, its connection with the complement property has played a fundamental role in the theory. The real and complex settings, however, exhibit essential differences. In the real Hilbert space, for a collection of vectors, possessing the complement property is equivalent to performing phase retrieval. In contrast, in the complex Hilbert space, phase retrieval is strictly stronger than the complement property. When dealing with a family of subspaces, the situation changes even in the real case, where the two notions are no longer equivalent. Indeed, for a collection of subspaces (or, equivalently, orthogonal projections), phase retrieval constitutes a strictly stronger property than the complement property.

Uncorrected Proof

Definition 3.1. A family of subspaces $\{W_i\}_{i=1}^N$ for a finite-dimensional Hilbert space H has the complement property if either $\sum_{i\in\Lambda} W_i = H$ or $\sum_{i\in\Lambda^c} W_i = H$ for any subset Λ of $\{1, 2, ..., N\}$.

Theorem 3.2. *Let $\{W_i\}_{i=1}^N$ be a family of closed subspaces of $\mathbb{R}^n$. Suppose that $\{W_i\}_{i=1}^N$ performs phase retrieval in $\mathbb{R}^n$, then the collection $\{W_i\}_{i=1}^N$ necessarily satisfies the complement property.*

Proof. Let Λ be arbitrary subset of $\{1, 2, ..., N\}$ and $\{f_{i,j}\}_{j=1}^{J_i}$ be an orthonormal basis for $W_i (i \in [N])$. $\{f_{i,j}\}_{i\in[N]j\in[N_j]}$ allows phase retrieval in $\mathbb{R}^n$ by [5], and therefore we have $\text{span}\{f_{i,j}\}_{(i,j)\in\Theta} = \mathbb{R}^n$ or
$\text{span}\{f_{i,j}\}_{(i,j)\in\Theta^c} = \mathbb{R}^n$ where $\Theta = \{(i, j) : i \in \Lambda\}$.
Then $\sum_{i\in\Lambda} W_i = \mathbb{R}^n$ or $\sum_{i\in\Lambda^c} W_i = \mathbb{R}^n$ respectively. □

The example below provides a counterexample to the converse of the above theorem ,by Example 3.6 in [4].

Example 3.3. Let $\{W_i\}_{i=1}^5$ be the subspaces of $\mathbb{R}^3$
$W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$,
$W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_3 = 0\}$,
$W_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 + x_3 = 0\}$
$W_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + (1 - \sqrt{2})x_2 + 2x_3 = 0\}$,
$W_5 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.
In any partition of the subspaces $\{W_i\}_{i=1}^5$ into two subsets at least one subset contains two of the subspaces. Since each of these subspaces is two-dimensional, the span of such a subset of the subspaces necessarily coincides with the entire space.

Proposition 1.1 presented in[7] holds for vectors. In this paper, we aim to extend this theorem to projections. We show that one direction of the theorem remains valid for projections, while the converse does not hold in general.

Theorem 3.4. *Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathbb{R}^n$ with respective projections $\{P_i\}_{i=1}^N$. If $\{P_i\}_{i=1}^N$ does phase retrieval then for any nonzero vectors $x, y \in \mathbb{R}^n$*
then $\sum_{i=1}^N ||P_i x||^2 ||P_i y||^2 > 0$.

Proof. We prove by contradiction. Assume there are $x, y \in \mathbb{R}^n$, $x \neq 0$, $y \neq 0$, such that $\sum_{i=1}^N ||P_i x||^2 ||P_i y||^2 = 0$. Then for any $i \in I = \{1, 2, ..., N\}$, $||P_i x|| = 0$ or $||P_i y|| = 0$. Let $I_1 = \{i : \ P_i x = 0\}$, and $I_2 = \{1, 2, ..., N\}\backslash \text{I}_1$. Therefore, x is orthogonal to all $\{P_i y\}$ for all $i \in I$. That is impossible by Theorem 3.1 in[4]. □

The following example demonstrates that, unlike the case for certain classes of vectors, the converse of the above theorem does not hold for these projections. In fact, this example, which is adapted from Example 3.6 in [4], involves projections that are not phase retrieval, despite having orthogonal complement projections, that still enable phase retrieval.

Example 3.5. Given $\{W_i\}_{i=1}^5$ in $\mathbb{R}^3$ be the subspaces that appear in Example 3.3 with corresponding orthogonal projections P_i,
$P_1 x = (x_1, x_2, 0)$
$P_2 x = (\frac{x_1 - x_3}{2}, x_2, \frac{x_3 - x_1}{2})$
$P_3 x = (x_1, \frac{x_2 - x_3}{2}, \frac{x_3 - x_2}{2})$
$P_4 x = (x_1 - \frac{1}{8-2\sqrt{2}}(x_1 + (1 - \sqrt{2})x_2 + 2x_3), x_2 - \frac{1}{8-2\sqrt{2}}((1 - \sqrt{2})x_1 + (1 - \sqrt{2})^2 x_2 + 2(1 - \sqrt{2})x_3)$,
$x_3 - \frac{1}{8-2\sqrt{2}}(2x_1 + 2(1 - \sqrt{2})x_2 + 4x_3))$
$P_5 x = (\frac{2x_1 - x_2 - x_3}{3}, \frac{2x_2 - x_1 - x_3}{3}, \frac{2x_3 - x_1 - x_2}{3})$
$\{W_i\}_{i=1}^5$ fail phase retrieval, but if for all $i \in \{1, 2, ..., 5\}$ we have $P_i x = 0$ or $P_i y = 0$, then x or y becomes zero. For, we may take $P_1 x = 0$, if there is a $i \geq 2$ with $P_i x = 0$ then $x = 0$ and so we can assume $P_2 y = P_3 y = P_4 y = P_5 y = 0$ which this one implies $y = 0$.

Uncorrected Proof

4. Fusion frame dimension function

Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a frame for a n-dimensional Hilbert space H. The frame dimension function of a frame $\mathcal{F}$ for H, $d_{\mathcal{F}}$ is defined for all $0 \neq x \in H$ as:

$$d_{\mathcal{F}}(x) = \dim \text{span}\{\langle x, f_i\rangle f_i : i = 1, 2, ..., N\}.$$

Let $\{x_1, x_2, ..., x_k\}$ are vectors in the Hilbert space H and $A = \text{span}\{x_1, x_2, ..., x_k\}$ then
$\mathcal{G}A = \text{span}\{\mathcal{G}x_1, \mathcal{G}x_2, ..., \mathcal{G}x_k\}$ where $\mathcal{G} : H \to H$ is a bounded, invertible operator.
First we show $\dim \text{span}\{x_1, x_2, ..., x_k\} = \dim \text{span}\{\mathcal{G}x_1, \mathcal{G}x_2, ..., \mathcal{G}x_k\}$. In fact, if $\{z_1, z_2, ..., z_k\}$ is a basis for A, then $\{\mathcal{G}z_1, \mathcal{G}z_2, ..., \mathcal{G}z_k\}$ is a basis for $\mathcal{G}A$, since $\mathcal{G}(\sum_{i=1}^k c_i z_i) = \sum_{i=1}^k c_i \mathcal{G}z_i = 0$ if and only if $\sum_{i=1}^k c_i z_i = 0$.

Now, consider a frame $\mathcal{F} = \{f_1, f_2, ..., f_N\}$ for H. We set $\mathcal{F}_0 = \mathcal{GF} = \{\mathcal{G}f_1, \mathcal{G}f_2, ..., \mathcal{G}f_N\}$, for every $x \in H$, take $\Lambda = \{i : \langle x, \mathcal{G}f_i\rangle \neq 0\}$ and so $d_{\mathcal{F}_0}(x) = \dim \text{span}\{\langle x, \mathcal{G}f_i\rangle \mathcal{G}f_i : i \in \Lambda\} = \dim \text{span}\{\mathcal{G}f_i : i \in \Lambda\} = \dim \text{span}\{f_i : i \in \Lambda\} = \dim \text{span}\{\langle \mathcal{G}^* x, f_i\rangle f_i : i \in \Lambda\} = \dim \text{span}\{\langle \mathcal{G}^* x, f_i\rangle f_i : i = 1, 2, ..., N\} = d_{\mathcal{F}}(\mathcal{G}^* x)$. Therefore $\text{range}(\mathcal{F}) = \text{range}(\mathcal{GF})$, particularly for $\mathcal{G} = \mathcal{S}_{\mathcal{F}}^{-1}$, $\mathcal{GF}$ the canonical dual associated with $\mathcal{F}$.

As we know, the function $d_{\mathcal{F}}$ determines important properties of the frame $\mathcal{F}$; for example, the phase retrieval property of $\mathcal{F}$ is equivalent to having $d_{\mathcal{F}}(x) = n$ for every x, where $n = \dim H$.
Next, we show that every point in H is a local minimum for the function $d_{\mathcal{F}}$.

Theorem 4.1. *If $0 \neq x \in H$, then one can find a neighborhood around x, such that $d_{\mathcal{F}}(x)$ attains its minimum value on the neighborhood.*

Proof. Suppose $d_{\mathcal{F}}(x) = r$, for each $q > 0$, let $N_q(x)$ be the open ball of radius q centered at x, defined by $N_q(x) = \{y : \| x - y \| < q\}$, we put $\Lambda_x = \{i : \langle x, f_i\rangle \neq 0\}$, and so $r = \dim \text{span}\{f_i : i \in \Lambda_x\}$.
Let $\Pi : y \longrightarrow \prod_{i\in\Lambda_x} \langle y, f_i\rangle$, since Π is continuous and $\langle x, f_i\rangle \neq 0$ for all $i \in \Lambda_x$ there exist a neighborhood $N_q(x)$ such that $\langle y, f_i\rangle \neq 0$ for all $i \in \Lambda_x$ and $y \in N_q(x)$. Hence $\Lambda_x \subset \{i : \langle y, f_i\rangle \neq 0\} = \Lambda_y$ which implies that $\mathcal{F}_x = \text{span}_j \langle x, f_j\rangle f_j = \text{span}_{j\in\Lambda_x} f_j \subset \text{span}_{j\in\Lambda_y} f_j = \text{span}_j \langle y, f_j\rangle f_j = \mathcal{F}_y$ consequently $d_{\mathcal{F}}(x) = \dim \mathcal{F}_x \leq \dim \mathcal{F}_y = d_{\mathcal{F}}(y)$ $(y \in N_q(x))$. □

In Theorem 2.3 of [1], it is shown that if $\mathcal{F} = \{f_i\}_{i\in I_{2n-1}}$ is a phase retrievable frame for $\mathbb{R}^n$ consisting of $2n - 1$ elements, then the set of all dual frames of $\mathcal{F}$ that are phase retrievable is open in the space of all dual frames of $\mathcal{F}$. In what follows, for a phase retrievable frame $\mathcal{F} = \{f_i\}_{i=1}^N$, we show that the above result holds by a proof different from the one given in [1]. Note that the principal advantage of this proof is that it imposes no constraint on the cardinality of $\mathcal{F}$.

Theorem 4.2. *Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a phase retrievable frame for a n-dimensional real Hilbert space H. Then $PD_{\mathcal{F}}$ is an open subset of $D_{\mathcal{F}}$, where $D_{\mathcal{F}}$ denotes the set of all dual frames of $\mathcal{F}$ and $PD_{\mathcal{F}}$ denotes the set of all phase retrievable dual frames.*

Proof. It is known that every dual frame is of the form $\{\mathcal{S}^{-1}f_i + u_i\}_{i=1}^N$ where $\{u_i\}_{i=1}^N$ is a Bessel sequence that satisfies $\sum_{i=1}^N \langle f, u_i\rangle f_i = 0$ for all $f \in H$, [1].
Suppose $G = \{\mathcal{S}^{-1}f_i + u_i\}$ belongs to $PD_{\mathcal{F}}$. Then for every $0 \neq x \in H$, we have $H = G_x = \text{span}_i\{\langle x, \mathcal{S}^{-1}f_i + u_i\rangle(\mathcal{S}^{-1}f_i + u_i)\}$. Choose $\Lambda \subset \{i : \langle x, \mathcal{S}^{-1}f_i + u_i\rangle \neq 0\}$ with $| \Lambda | = n$, such that $\{\mathcal{S}^{-1}f_i + u_i\}_{i\in\Lambda}$ is linearly independent. By the continuity of the determinant function, $\delta_0 > 0$ can be selected so that, for every $v_i \in H$ with property $\| u_i - v_i \| < \delta_0$, $\{\mathcal{S}^{-1}f_i + u_i\}_{i\in\Lambda}$ is linearly independent. On the other hand, for each $i \in \Lambda$, the function $\Gamma_i(v_i) = \langle x, \mathcal{S}^{-1}f_i + v_i\rangle$ is continuous on H and $\Gamma_i(u_i) \neq 0$. Therefore, $\delta_i > 0$ can be chosen such that, for every $v_i \in H$, if $\| u_i - v_i \| < \delta_i$ then $\Gamma_i(v_i) \neq 0$. With a suitable choice of $\delta = \min\{\delta_i : i \in \Lambda \cup \{0\}\}$ if $\| u_i - v_i \| < \delta$ holds for every $i \in \Lambda$, then $\{\mathcal{S}^{-1}f_i + v_i\}_{i\in\Lambda}$ is linearly independent and $\Gamma_i(v_i) = \langle x, \mathcal{S}^{-1}f_i + v_i\rangle \neq 0$. Thus, for every $K = \{\mathcal{S}^{-1}f_i + v_i\}$ belong to $D_{\mathcal{F}}$, with $\| u_i - v_i \| < \delta_i$ we have $K_x = \text{span}_i\{\langle x, \mathcal{S}^{-1}f_i + v_i\rangle(\mathcal{S}^{-1}f_i + v_i)\} \supset \text{span}_{i\in\Lambda}\{\langle x, \mathcal{S}^{-1}f_i + v_i\rangle(\mathcal{S}^{-1}f_i + v_i)\} = H$ therefore $K_x = H$ which shows that $K \in PD_{\mathcal{F}}$. Thus for each $K \in D_{\mathcal{F}}$, on a neighborhood of G, we have $K \in PD_{\mathcal{F}}$ which implies that $PD_{\mathcal{F}}$ is an open set. □

Uncorrected Proof

We begin by observing, through a straightforward argument, that whenever $\mathcal{F} = \{f_i\}_{i=1}^N$ constitutes a frame for $\mathbb{R}^n$, then $n \in \text{range}(d_\mathcal{F})$. This is because for every $1 \leq i \leq N$ $\ker f_i = \{x : \langle x, f_i\rangle = 0\}$ is a subspace of dimension $n-1$ and so $\cup_{i=1}^N \ker f_i \neq \mathbb{R}^n$. Consequently, there exists an element $x \in \mathbb{R}^n$ such that for every $1 \leq i \leq N$, $\langle x, f_i\rangle \neq 0$. In this case $\text{span}_{1\leq i\leq N}\{\langle x, f_i\rangle f_i\} = \text{span}_{1\leq i\leq N}\{f_i\} = \mathbb{R}^n$ which implies $d_\mathcal{F}(x) = n$, therefore $n \in \text{range}(d_\mathcal{F})$.

The continuity of this function can also be characterized in terms of phase retrievability.

Proposition 4.3. *The function $d_\mathcal{F} : H \longrightarrow \mathbb{R}$ is continuous if and only if the frame $\mathcal{F}$ possesses the phase retrieval property.*

Proof. Since continuous functions map connected sets to connected sets, $d_\mathcal{F}$ is continuous if and only if $d_\mathcal{F}$ is a constant function, on the other hand, since always $n \in \text{range}(d_\mathcal{F})$, the constancy of $d_\mathcal{F}$ means that $d_\mathcal{F}(x) = n$ for every $x \in H$, which, in turn, is equivalent to the phase retrieval property for $\mathcal{F}$. □

It is well known that a finite collection of vectors forms a frame for $\mathbb{R}^n$ if and only if this collection spans $\mathbb{R}^n$. To extend this discussion to fusion frames, we formulate the following proposition, which provides a natural analogue in the setting of fusion frames.

Proposition 4.4. *If $W_1, W_2, ..., W_N$, are subspaces that span the entire ambient space; that is, $W_1 + W_2 + ... + W_N = \mathbb{R}^n$, then for any choice of positive weights $v_1, v_2, ..., v_N$, $\{(W_i, v_i)\}_{i=1}^N$, forms a fusion frame.*

Proof. Let $v_1, v_2, ..., v_N$ be positive numbers, we introduce two constants A and B such that the following relation for any $f \in \mathbb{R}^n$ holds.

$$||f||^2 \leq \sum_{i=1}^N v_i ||P_{W_i} f||^2 \leq B||f||^2. \tag{4.1}$$

The triangle inequality implies that $B = \sum_{i=1}^N v_i$ holds for the right hand side of the above inequality. We now show that, for every vector g with $||g|| = 1$, we have

$$\sum_{i=1}^N v_i ||P_{W_i} g||^2 > 0.$$

Otherwise, there is g with unit norm that $P_{W_i} g = 0$, for any $i \in \{1, 2, ..., N\}$. Consequently, $g \in \cap W_i^\perp$, since there exists some $g_i \in W_i$ such that $g = \sum_{i=1}^N g_i$
$1 = ||g||^2 = \langle g, g\rangle = \langle g, \sum_{i=1}^N g_i\rangle = \sum_{i=1}^N \langle g, g_i\rangle = 0$, which is contradiction. Therefore by choosing

$$A = \inf\{\sum_{i=1}^N v_i ||P_{W_i} g||^2 : ||g|| = 1\},$$

the compactness of the unit sphere $\mathbb{R}^n$ implies $A > 0$, now for any $0 \neq f \in \mathbb{R}^n$ and by choosing $g = \frac{f}{||f||}$ appropriately the left hand side of (4.1) follows from the definition of A. □

Definition 4.5. Let $\{P_i\}_{i=1}^N$ denote the collection of orthogonal projections associated with the fusion frame $\{W_i\}_{i=1}^N$, for an n-dimensional real Hilbert space H. The frame dimension function of $\mathcal{P} = \{P_i\}_{i=1}^N$ is the function $d_\mathcal{P}(x) = \dim \sum_{i=1}^N ||P_i x|| P_i(H)$ for $0 \neq x \in H$.

In what follows, we derive an analogous result for projections, parallel to the one previously established for vectors.

We show that $n \in \text{range}(d_\mathcal{P})$. Since for all i, $\ker P_i = P_i(H)^\perp \subsetneq H$ then $\cup_{i=1}^N \ker P_i \neq H$. Therefore there exists $x \in H \setminus \cup_{i=1}^N \ker P_i$, and so $P_i(x) \neq 0$ for every $1 \leq i \leq N$, hence we obtain $d_\mathcal{P}(x) = \dim \sum_{i=1}^N ||P_i x|| P_i(H) =$

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$\dim \sum_{i=1}^{N} P_i(H) = \dim H = n$.

Let $\mathcal{F} = \{f_i\}_{i=1}^{N}$ be a frame for a Hilbert space H. Inspired by Proposition 1.1 in [7], we introduce the function $G_{\mathcal{F}}(x)$ in the following form: $G_{\mathcal{F}}(x) = \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 f_i \otimes f_i$, as noted in [7], Property 5 of Theorem 2.1 in [3] implies that

$$d_{\mathcal{F}}(x) = \operatorname{rank}(G_{\mathcal{F}}(x)). \tag{4.2}$$

For the sake of completeness, a proof of this equality is given. For every $0 \neq x \in H$, we put

$$M = G_{\mathcal{F}}(x)(H) = \{\sum_{i=1}^{N} |\langle x, f_i\rangle|^2 \langle y, f_i\rangle f_i : y \in H\}.$$

As previously noted, for $\Lambda = \{i : \langle x, f_i\rangle \neq 0\}$

$$d_{\mathcal{F}}(x) = \dim \operatorname{span}\{\langle x, f_i\rangle f_i : 1 \leq i \leq N\} = \dim \operatorname{span}\{f_i : i \in \Lambda\} = \dim H_\Lambda,$$

clearly $M \subseteq H_\Lambda$. We next show that $f_i \in M$ for every $i \in \Lambda$ and as a direct consequence of this $d_{\mathcal{F}}(x) = \operatorname{rank}(G_{\mathcal{F}}(x))$. For each $i \in \Lambda$, we set $g_i = \langle x, f_i\rangle f_i$ and $\mathcal{F}' = \{g_i\}_{i\in\Lambda}$.

Since span$\mathcal{F}' = H_\Lambda$, thus $\mathcal{F}'$ is a frame for H_Λ, and the corresponding frame operator is given by

$$S'_{\mathcal{F}}(y) = \sum_{i\in\Lambda} \langle y, g_i\rangle g_i = \sum_{i\in\Lambda} |\langle x, f_i\rangle|^2 \langle y, f_i\rangle f_i.$$

The surjectivity of the operator S', for every $i \in \Lambda$, f_i belong to H_Λ, implies that there exists $y \in H_\Lambda$ such that $f_i = S_{\mathcal{F}'}(y) \in M$.

Whenever $\mathcal{P} = \{P_i\}_{i=1}^{N}$ is a fusion frame for $\mathbb{R}^n$, we set $G_{\mathcal{P}}(x) = \sum_{i=1}^{N} ||P_i x|| P_i$ where $x \in H$. We will have the same as (4.2) for the fusion frame $\mathcal{P}$:

$$d_{\mathcal{P}}(x) = \operatorname{rank}(G_{\mathcal{P}}(x)). \tag{4.3}$$

To prove this equality, let $M_{\mathcal{P}} = \{\sum_{i=1}^{N} ||P_i x|| P_i y : y \in H\}$, $\Lambda = \{i : ||P_i x|| \neq 0\}$ and $H_\Lambda = \{\sum_{i\in\Lambda} ||P_i x|| P_i(H)\}$ and so $\operatorname{rank}(G_{\mathcal{P}}(x)) = \dim M_{\mathcal{P}}$ and $M_{\mathcal{P}} \subseteq H_\Lambda$.
In what follows, we apply the same method used for the frame case to the setting of fusion frames as well, we show that $H_\Lambda \subseteq M_{\mathcal{P}}$ for $i \in \Lambda$. We consider $\mathcal{P}' = \{||P_i x|| P_i : i \in \Lambda\}$, by the definition of H_Λ, $\mathcal{P}'$ is a fusion frame for H_Λ, in particular, for all weights $v_i = 1$ where $i \in \Lambda$, this is especially the case because the frame operator $S' : H_\Lambda \to H_\Lambda$, $S'(f) = \sum_{i\in\Lambda} ||P_i x|| P_i(f)$ $(f \in H_\Lambda)$ is surjective. If $y \in H_\Lambda$ therefore there exists f such that $S'(f) = y$ then $y = \sum_{i\in\Lambda} ||P_i x|| P_i(f) \in M_{\mathcal{P}}$.

$$d_{\mathcal{P}}(x) = \dim H_\Lambda = \dim M_{\mathcal{P}} = \operatorname{rank}(G_{\mathcal{P}}(x)).$$

Note that none of the equality (4.3) presented above will hold under the given definition of $d_{\mathcal{P}}(x) = \dim \operatorname{span}\{P_i x : i \in \Lambda\}$.

5. Conclusion

We showed that the complement property is necessary but not sufficient for phase retrieval by subspaces. For fusion frames, the dimension function equals the rank of its associated operator, providing an algebraic characterization. We also established the openness of phase-retrievable dual frames and a continuity criterion for the dimension function in the frame setting.

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References

[1] F. Arabyani-Neyshaburi, A. Arefijamaal, and R. A. Kamyabi-Gol, *Characterization of (weak) phase retrieval dual frames*, Complex Anal. Oper. Theory, 17 (2023), 1–18.
[2] R. Balan, P. G. Casazza, and D. Edidin, *On signal reconstruction without phase*, Appl. Comput. Harmon. Anal., 20 (3) (2006), 345–356.
[3] R. Balan and Y. Wang, *Invertibility and robustness of phaseless reconstruction*, Appl. Comput. Harmon. Anal., 38(3) (2015), 469–488.
[4] S. Botelho-Andrade, et al. *Phase retrieval by hyperplanes*, Contemp. Math., 706 (2018), 21–31.
[5] J. Cahill, P. G. Casazza, J. Peterson, and L. Woodland, *Phase retrieval by projections*, Houst. J. Math., 42(2) (2016), 537–558.
[6] A. Conca, D. Edidin, M. Hering, and C. Vinzant, *An algebraic characterization of injectivity in phase retrieval*, Appl. Comput. Harmon. Anal., 38(2) (2015), 346–356.
[7] D. Han and K. Liu, *Frame dimension functions and phase retrievability*, Adv. Oper. Theory, 9 (2024), 1–19.

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