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Fractal–fractional modeling of chaotic inputs in neuronal systems: Rössler–FitzHugh–Nagumo framework

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Abstract

This study examines a fractal–fractional Rössler system and its role as a chaotic input to the FitzHugh–Nagumo (FHN) neuron model. The introduction of fractional order μ and fractal dimension ν allows the incorporation of memory effects and structural heterogeneity. Numerical results show that the Rössler system follows a period–doubling route to chaos, where increasing μ suppresses oscillations and delays chaos, while decreasing ν enhances irregular dynamics. When coupled to the FHN neuron, the chaotic input induces transitions from regular spiking to chaotic firing. All simulations are performed using a predictor–corrector numerical scheme, which demonstrates stable and accurate performance for fractal–fractional systems. These results highlight the effectiveness of fractal–fractional modeling in capturing realistic neuronal and chaotic dynamics.

Keywords. Fractal–fractional differential equations, chaotic systems, Rössler system, FitzHugh–Nagumo model, neuronal dynamics, iterative methods, chaos synchronization.

2010 Mathematics Subject Classification. 34A08, 34A34, 37N25, 92C20, 65L05.

Uncorrected

1. Introduction and Preliminaries

Chaos theory has emerged as a fundamental concept in nonlinear science, offering mathematical models that explain complex, seemingly random yet deterministic behaviors observed in both natural and engineered systems. Lorenz's historical discovery of deterministic chaos in atmospheric convection [20] and subsequent creation of the Rössler system [35] initiated systematic research on chaotic attractors, bifurcations, and synchronization phenomena. Chaotic systems have found applications in a wide range of fields—including physics, chemistry, biology, engineering, finance, and neuroscience—where irregular oscillations frequently influence the behavior of the system. [31, 33, 44].

Among the classical models, the Rössler system is notable for producing strange attractors using only a few nonlinear terms. Its mathematical structure allows analytical tractability while still exhibiting the hallmark features of chaos such as sensitive dependence on initial conditions, fractal attractors, and broadband spectra [43]. The Rössler system has been used to study nonlinear electronic circuits [23], chemical oscillations [30], population dynamics [24], and secure communications [32].

Recent research has also demonstrated the growing role of artificial intelligence and neural network techniques in the analysis of classical and fractional-order mathematical models. In particular, deep neural networks have been employed to investigate coupled nonlinear dynamical systems and to approximate solutions of fractional integro-differential equations, offering improved computational efficiency and accuracy [39, 42].

Fractional-order modeling combined with artificial intelligence has further been applied to a variety of real-world problems, especially in biological and medical sciences. Studies on epidemic compartmental models, infectious disease prediction, and psychological disorder modeling have shown that fractional operators effectively capture memory-dependent dynamics, while artificial intelligence tools enhance predictive performance and data interpretation [6, 40, 41].

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These contributions reflect the increasing interest in integrating fractional-order mathematical frameworks with modern computational approaches and highlight the broad applicability of fractional modeling across interdisciplinary domains. Recent studies have focused on the qualitative and theoretical analysis of fractional-order differential equations involving delay effects. Degree theory, boundary value problems, and variable-order fractional delay systems have been investigated to establish existence, uniqueness, and stability results [36, 37]. Moreover, nonlinear fractional models with discrete and proportional delays have been successfully applied to real-world biological systems, such as population dynamics and insect models [38].

Fractional calculus serves as a powerful generalization of classical calculus, where replacing integer-order derivatives with non-integer orders allows models to incorporate memory and hereditary effects that are absent in ordinary differential equations [19, 21, 34]. Fractional-order chaotic systems have exhibited new dynamics like richer bifurcation structures, long-term memory effects, and higher synchronization properties [4, 11, 45]. Incorporating fractional-order operators into classical chaotic models such as the Lorenz and Rössler systems has led to new frameworks that more accurately capture phenomena like viscoelasticity, anomalous diffusion, and biological signal transmission [16, 22].

Recent studies further demonstrate the growing interest in fractional and fractal–fractional formulations for modeling complex nonlinear systems. In particular, recent investigations have analyzed rich bifurcation structures and chaotic dynamics in neural and nonlinear dynamical systems, highlighting the role of non-integer order operators in capturing complex temporal behavior [26]. Fractional-order models have also been successfully applied to biological and epidemiological systems, where memory and hereditary effects play a crucial role in system evolution [14, 27, 28].

Moreover, advanced studies have explored fractional and fractal–fractional approaches in diverse applications, including complex dynamical processes, signal processing, and high-dimensional nonlinear systems, demonstrating their effectiveness in revealing intricate dynamical features beyond classical models [3, 29]. From a computational perspective, recent developments in numerical methods for fractional and nonlocal differential equations emphasize the necessity of robust and accurate numerical schemes tailored for such operators. These recent contributions provide strong motivation for the present study and position it within the current state of research.

While fractional operators effectively capture memory effects, many real-world systems also exhibit structural irregularities or fractal-like geometries. Examples include inhomogeneous biological tissues, porous media, and anomalous transport processes, which combine both memory and structural complexity. To address this, Atangana [8] introduced the concept of fractal–fractional derivatives, integrating fractional memory with fractal geometry. This new operator offers an improved framework for modeling dynamical systems in heterogeneous and complex media.

It has been successfully employed in numerous applications in recent years. For instance, fractal–fractional operators have been utilized in epidemiological studies of communicable diseases [1, 9], fluid mechanics [7] and finance [5, 12, 17]. For nonlinear oscillations, fractal–fractional generalizations of Lorenz and Chen systems have exhibited changed attractor geometries and different bifurcation diagrams, demonstrating that structural heterogeneity essentially transforms chaotic dynamics [15, 18].

The study of chaotic dynamics under fractal–fractional operators is particularly relevant in neuroscience. Neuronal oscillations often exhibit both long-term memory (e.g., due to ionic channel dynamics) and fractal-like spatial irregularity (e.g., dendritic branching and tissue heterogeneity). Coupling neuronal models with fractal–fractional chaotic systems has been proposed as a biologically realistic way to mimic complex synaptic drives [2, 10]. For instance, when the Rössler attractor is used as an external forcing to the FitzHugh–Nagumo model, one obtains irregular yet structured firing patterns, closely resembling real neuronal activity.

Despite these advances, a comprehensive investigation of the fractal–fractional Rössler system covering its mathematical formulation, stability, bifurcation structure, time series analysis, and biological applications remains limited. The motivation of this work is therefore threefold:

(1) To develop the fractal–fractional extension of the classical Rössler system using Fractal - Fractional derivative with Exponential kernel , thereby unifying memory and fractal effects in a single framework.
(2) To analyze its dynamical properties through equilibria, stability analysis, bifurcations, and time series simulation, highlighting the interplay between the fractional order and fractal dimension.
(3) To demonstrate an application in computational neuroscience, where the fractal–fractional Rössler system serves as a chaotic input to a FitzHugh–Nagumo neuron model, revealing biologically realistic variability.

Uncorrected Proof

This study contributes to the growing literature on generalized chaotic systems and fractional–fractal operators by showing how the combined effects of memory and heterogeneity enrich chaotic attractors and extend their applicability to biological and physical domains. The results indicates that the fractal–fractional Rössler system not only generalizes the classical model but also provides new perspectives for interdisciplinary research at the interface of mathematics, physics, and biology.

Unlike existing studies that focus on either classical or purely fractional-order chaotic systems, the present work develops a fractal–fractional Rössler system with an exponential kernel, which simultaneously incorporates memory effects through the fractional order μ and structural heterogeneity through the fractal dimension ν. To the best of our knowledge, such a combined framework has not been previously investigated in the context of chaotic neuronal forcing.

The use of a fractional-order formulation is particularly well suited for neuronal modeling, as it allows the inclusion of long-term memory and hereditary effects associated with ion-channel dynamics and synaptic activity, which are not adequately captured by integer-order models. In addition, the incorporation of fractal characteristics reflects the intrinsic geometrical irregularity and heterogeneous structure of neural tissue.

The novelty of this study further lies in the application of a newly developed predictor–corrector numerical scheme for the analysis of the fractal–fractional Rössler system, as well as in its utilization as a chaotic drive for a fractal–fractional FitzHugh–Nagumo neuron model. This approach reveals previously unreported dynamical behaviors, including transitions between regular spiking, bursting, and chaotic firing, thereby providing a more realistic mathematical description of neuronal responses under complex and irregular inputs.

1.1. **Mathematical Preliminaries.** In this section, we present essential definitions and operators that form the mathematical foundation for the fractal–fractional Rössler system studied in this work. These include the notion of fractal derivatives, fractal–fractional derivatives with exponential kernels, and the corresponding integral operators.

Definition 1.1. Let f be a continuous function and $0 < \nu < 1$. Then the fractal derivative of f with order ν given as; [13]:

$$\frac{df(t)}{dt^\nu} = \lim_{t\to t_0} \frac{f(t) - f(t_0)}{t^\nu - t_0^\nu}, \qquad t > 0.$$

This operator generalizes the classical derivative by incorporating a fractal index ν, which measures local irregularity and scaling, making it suitable for signals evolving in heterogeneous or fractal-like media.

Definition 1.2. If f be a differentiable function, Let μ be a constant fractional order, such that $0 < \mu, \nu < 1$, then a fractional derivative of f with order μ and fractal dimension ν is defined as [8]:

$${}_a^{FFER}D_t^{\mu,\nu} f(t) = \frac{M(\mu)}{1-\mu} \frac{d}{dt^\nu} \int_a^t f(y) \exp\left(-\frac{\mu}{1-\mu}(t-y)\right) dy,$$

where μ denotes the fractional order (memory depth), ν the fractal index (structural irregularity) and $M(\mu)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 1.3. The *fractal–fractional integral* associated with the above operators is defined as [8]

$${}_0^{FFE}I_t^{\mu,\nu} f(t) = \frac{\nu\mu}{M(\mu)} \int_0^t y^{\nu-1} f(y)\, dy + \frac{\nu(1-\mu)t^{\nu-1} f(t)}{M(\mu)}.$$

2. Fractal–Fractional Rössler Dynamics: Numerical Simulation, Equilibria, Stability and Bifurcation Analysis

2.1. **Classical Rössler Model.** The Rössler system was first introduced by Otto Rössler in 1976 [35]. It is regarded as one of the simplest three-dimensional continuous-time chaotic systems. Despite its structural simplicity, the system exhibits a wide range of nonlinear dynamics, including fixed points, limit cycles, and chaotic attractors. The standard

Uncorrected Proof

TABLE 1. Variables and parameters in the fractal–fractional Rössler system.

Symbol	Description
$x(t)$	Fast-changing state variable, associated with rotational or oscillatory dynamics.
$y(t)$	Intermediate state variable that provides feedback and couples with $x(t)$.
$z(t)$	Slow-changing state variable that interacts nonlinearly with $x(t)$, creating chaotic trajectories.
a	Parameter controlling the growth or decay rate of $y(t)$, influencing oscillation frequency and stability.
b	Constant shifting the dynamics of $z(t)$, ensuring nontrivial oscillations and influencing periodic or chaotic behavior.
c	Bifurcation parameter that modulates nonlinear coupling between $x(t)$ and $z(t)$, critical for chaos generation.

integer-order Rössler system is governed by the following set of ordinary differential equations:

$$\dot{x}(t) = -y(t) - z(t), \tag{2.1}$$

$$\dot{y}(t) = x(t) + a\,y(t), \tag{2.2}$$

$$\dot{z}(t) = b + z(t)\big(x(t) - c\big), \tag{2.3}$$

where $x(t)$, $y(t)$, and $z(t)$ represent the system states, while a, b, and c are positive parameters that control the nature of trajectories. For example, the well-known chaotic attractor appears for the parameter values $a = 0.2$, $b = 0.2$, and $c = 5.7$.

In this classical framework, one can identify equilibrium points, analyze their stability through linearization, and study bifurcation phenomena by varying system parameters, particularly c. Such analyses provide insight into the onset of chaos and transitions to periodic oscillations.

2.2. **Fractal–Fractional Rössler System.** Fractional derivatives have been extensively employed to capture memory effects in dynamical systems. Nevertheless, many real-world physical and engineering systems also exhibit structural irregularities or fractal-like geometries that cannot be represented adequately by classical or fractional-order operators alone. To address this, Atangana [8] proposed the *fractal–fractional derivative*, which integrates both memory effects and fractal properties into a unified framework.

By generalizing the classical Rössler system to fractal–fractional derivatives, we define the *fractal–fractional Rössler system* as

$${}^{FF}D_t^{\mu,\nu}x(t) = -y(t) - z(t), \tag{2.4}$$

$${}^{FF}D_t^{\mu,\nu}y(t) = x(t) + a\,y(t), \tag{2.5}$$

$${}^{FF}D_t^{\mu,\nu}z(t) = b + z(t)\big(x(t) - c\big). \tag{2.6}$$

We consider the fractal–fractional Rössler system with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \tag{2.7}$$

where the constants $a, b, c \in \mathbb{R}$ are real numbers and $0 < \mu, \nu < 1$ are the fractional order and fractal dimension of the system. Let $M(\mu)$ be the normalization function of the fractal–fractional derivative and a uniform discretization grid by $t_n = nh$, where the time step is $h > 0$

Uncorrected Proof

2.3. **Numerical Approximation.** We consider the fractal–fractional Rössler system (2.6) with initial conditions (2.7) where the constants $a, b, c \in \mathbb{R}$ are real numbers and $0 < \mu, \nu < 1$ are the fractional order and fractal dimension of the system. Let $M(\mu)$ be the normalization function of the fractal–fractional derivative and a uniform discretization grid by $t_n = nh$, where the time step is $h > 0$.

For the approximate computation of the solutions, a numerical scheme for fractal–fractional differential equations is utilized, namely the two–step predictor–corrector scheme proposed by Nagpurkar and Kachhia [25].
Predictor 1 steps:

$$x_{n+1}^{P_1} = x_n + \nu(t_n)^{\nu-1} f_1(t_n, x_n, y_n, z_n) \left[\frac{h\mu}{2M(\mu)} - \frac{1-\mu}{M(\mu)}\right], \tag{2.8}$$

$$y_{n+1}^{P_1} = y_n + \nu(t_n)^{\nu-1} f_2(t_n, x_n, y_n) \left[\frac{h\mu}{2M(\mu)} - \frac{1-\mu}{M(\mu)}\right], \tag{2.9}$$

$$z_{n+1}^{P_1} = z_n + \nu(t_n)^{\nu-1} f_3(t_n, x_n, y_n, z_n) \left[\frac{h\mu}{2M(\mu)} - \frac{1-\mu}{M(\mu)}\right]. \tag{2.10}$$

Predictor 2 steps:

$$x_{n+1}^{P_2} = \nu(t_{n+1})^{\nu-1} f_1(t_{n+1}, x_{n+1}^{P_1}, y_{n+1}^{P_1}, z_{n+1}^{P_1}) \left[\frac{h\mu}{2M(\mu)} + \frac{1-\mu}{M(\mu)}\right], \tag{2.11}$$

$$y_{n+1}^{P_2} = \nu(t_{n+1})^{\nu-1} f_2(t_{n+1}, x_{n+1}^{P_1}, y_{n+1}^{P_1}) \left[\frac{h\mu}{2M(\mu)} + \frac{1-\mu}{M(\mu)}\right], \tag{2.12}$$

$$z_{n+1}^{P_2} = \nu(t_{n+1})^{\nu-1} f_3(t_{n+1}, x_{n+1}^{P_1}, y_{n+1}^{P_1}, z_{n+1}^{P_1}) \left[\frac{h\mu}{2M(\mu)} + \frac{1-\mu}{M(\mu)}\right]. \tag{2.13}$$

Corrector steps:

$$x_{n+1}^{C} = x_{n+1}^{P_1} + \left(\frac{h\mu\nu}{2M(\mu)} + \frac{(1-\mu)\nu}{M(\mu)}\right)(t_{n+1})^{\nu-1} f_1\Big(t_{n+1},\, x_{n+1}^{P_1}+x_{n+1}^{P_2},\, y_{n+1}^{P_1}+y_{n+1}^{P_2},\, z_{n+1}^{P_1}+z_{n+1}^{P_2}\Big), \tag{2.14}$$

$$y_{n+1}^{C} = y_{n+1}^{P_1} + \left(\frac{h\mu\nu}{2M(\mu)} + \frac{(1-\mu)\nu}{M(\mu)}\right)(t_{n+1})^{\nu-1} f_2\Big(t_{n+1},\, x_{n+1}^{P_1}+x_{n+1}^{P_2},\, y_{n+1}^{P_1}+y_{n+1}^{P_2}\Big), \tag{2.15}$$

$$z_{n+1}^{C} = z_{n+1}^{P_1} + \left(\frac{h\mu\nu}{2M(\mu)} + \frac{(1-\mu)\nu}{M(\mu)}\right)(t_{n+1})^{\nu-1} f_3\Big(t_{n+1},\, x_{n+1}^{P_1}+x_{n+1}^{P_2},\, y_{n+1}^{P_1}+y_{n+1}^{P_2},\, z_{n+1}^{P_1}+z_{n+1}^{P_2}\Big). \tag{2.16}$$

The nonlinear functions are

$$f_1(t, x, y, z) = -y - z,$$
$$f_2(t, x, y) = x + a\,y,$$
$$f_3(t, x, y, z) = b + z(x - c).$$

This predictor–corrector scheme captures the nonlinear dynamics of the fractal–fractional Rössler system and enhances stability and accuracy for $0 < \mu, \nu < 1$.

2.4. **Equilibria and Jacobian and stability analysis.** The fractal–fractional Rössler system is

$${}^{FF}D_t^{\mu,\nu} x = -y - z, \tag{2.17}$$

$${}^{FF}D_t^{\mu,\nu} y = x + ay, \tag{2.18}$$

$${}^{FF}D_t^{\mu,\nu} z = b + z(x - c). \tag{2.19}$$

The equilibrium points (x^*, y^*, z^*) satisfy

$$-y^* - z^* = 0, \quad x^* + ay^* = 0, \quad b + z^*(x^* - c) = 0.$$

Uncorrected Proof

TABLE 2. Equilibria, Jacobian matrices, and eigenvalues of the fractal–fractional Rössler system for $a = 0.2, b = 0.2, c = 5.7$.

Equilibrium	**Coordinates** (x^*, y^*, z^*)	**Jacobian** J	**Eigenvalues**
E_1	$[5.693, \quad -28.465, \quad 28.465]$	$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 28.465 & 0 & -0.007 \end{bmatrix}$	$\{-0.000 \pm 5.428i,\ 0.193\}$
E_2	$[0.0070, \quad -0.0351, \quad 0.0351]$	$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0.0351 & 0 & -5.693 \end{bmatrix}$	$\{0.097 \pm 0.995i,\ -5.687\}$

TABLE 3. Local stability of the fractal–fractional Rössler system for $a = 0.2$, $b = 0.2$, $c = 5.7$ using Matignon's criterion ($|\arg(\lambda_i)| > \mu\pi/2$ for $0 < \mu \leq 1$).

Equilibrium	**Eigenvalues of** J	**Stability check**	**Result**
E_1 (5.693, −28.465, 28.465)	$\{-0.000 \pm 5.428\,i,\ 0.193\}$	One real eigenvalue $> 0 \Rightarrow \lvert\arg(\lambda)\rvert = 0 \not> \mu\pi/2$	Unstable
E_2 (0.0070, −0.0351, 0.0351)	$\{\,0.097 \pm 0.995\,i,\ -5.687\,\}$	$\Re(0.097 \pm 0.995\,i) > 0 \Rightarrow \lvert\arg(\lambda)\rvert < \mu\pi/2$	Unstable

Solving yields

$$x^* = \frac{c \pm \sqrt{c^2 - 4ab}}{2}, \quad y^* = -\frac{x^*}{a}, \quad z^* = \frac{x^*}{a}.$$

The Jacobian matrix at an equilibrium point is

$$J = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z^* & 0 & x^* - c \end{bmatrix}.$$

The system admits two equilibrium points. Their coordinates, Jacobians, and eigenvalues are summarized in Table 2.

We have at E_1 eigenvalues that include a pair of imaginary conjugate eigenvalues on the imaginary axis ($\pm 5.428i$) and a small positive real eigenvalue (0.193). It is an *unstable spiral center* with oscillatory divergence.

At E_2, we obtain a conjugate pair of eigenvalues with positive real part ($0.097 \pm 0.995i$) and a negative eigenvalue (-5.687). This is a *saddle-focus*, in which a trajectory is attracted along a single stable direction but repelled in a two-dimensional unstable manifold.

Thus, both equilibria are unstable, consistent with the presence in the system of Rössler of chaotic dynamics and strange attractors.

For the commensurate fractal–fractional Rössler system with order $\mu \in (0, 1)$, Local stability at equilibrium requires Matignon's criterion:

$$\left|\arg(\lambda_i)\right| > \frac{\mu\pi}{2} \quad \text{for every eigenvalue } \lambda_i \text{ of the Jacobian.}$$

Case E_1**:** The equilibrium $E_1 = (5.693, -28.465, 28.465)$ has Jacobian eigenvalues $\{-0.000 \pm 5.428i,\ 0.193\}$. Since one eigenvalue is real and positive, we obtain $|\arg(\lambda)| = 0$, which violates Matignon's condition for all $\mu \in (0, 1]$. Therefore E_1 is unstable (oscillatory repulsion with a weakly expanding direction).

Case E_2**:** The equilibrium $E_2 = (0.0070, -0.0351, 0.0351)$ has eigenvalues $\{0.097 \pm 0.995i,\ -5.687\}$. Since the pair has positive real part, $|\arg(\lambda)| < \mu\pi/2$, violating the condition. Therefore E_2 is also unstable (a saddle–focus: just one strongly stable and one two-dimensional unstable manifold).

Therefore, both equilibria are unstable for any $\mu \in (0, 1]$, consistent with the emergence of chaotic attractors in the fractal–fractional Rössler system.

Uncorrected Proof

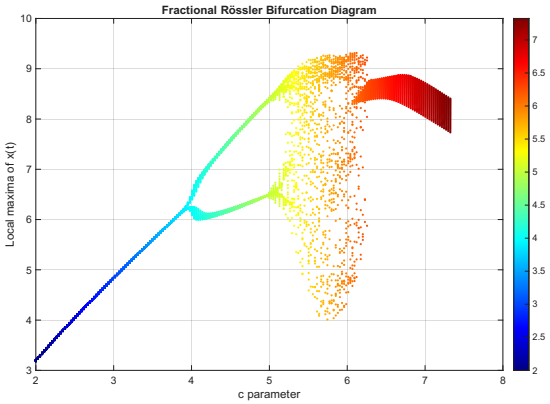


FIGURE 1. Bifurcation diagram of the fractal–fractional Rössler system showing local maxima of $x(t)$ versus parameter c. The diagram exhibits the classical period-doubling route to chaos as c increases, modified by the fractional order μ and fractal parameter ν.

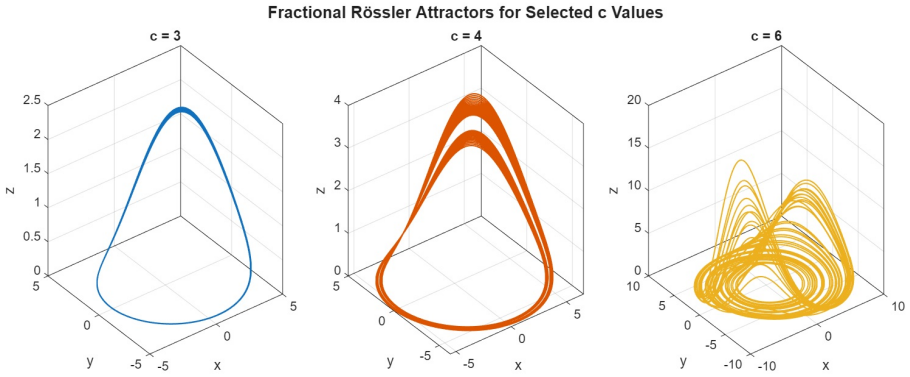


FIGURE 2. Phase portraits of the fractal–fractional Rössler system for selected values of c: (left) $c = 3$ showing a simple periodic orbit, (middle) $c = 4$ showing a double-period orbit, and (right) $c = 6$ showing a chaotic attractor.

2.5. **Bifurcation Analysis.** Bifurcation analysis reveals how the qualitative dynamics of the fractal–fractional Rössler system change when the system parameter c varies, while keeping $a = 0.2$ and $b = 0.2$ fixed. For each c, the system is numerically integrated using the recently proposed predictor–corrector scheme [25], and the local maxima of the $x(t)$ variable are recorded after discarding transients. This allows the construction of a bifurcation diagram that illustrates transitions from periodic oscillations to chaos. Figure 1 demonstrates that for small values of c (e.g., $c \approx 3$), the system exhibits a stable periodic orbit. As c increases, successive period-doubling bifurcations occur, leading to more complex oscillations and eventually to chaotic dynamics. The fractional order μ tends to suppress the onset of chaos by delaying bifurcation points, while the fractal parameter ν alters the density and geometry of the bifurcation branches, giving rise to more irregular transitions. Figure 2 illustrates representative attractors for specific c values. At $c = 3$, the system has a single closed orbit, indicating periodic behavior. At $c = 4$, the orbit splits into multiple loops, demonstrating the onset of period-doubling. By $c = 6$, the attractor becomes irregular and densely filled, confirming the chaotic regime predicted by the bifurcation diagram. Together, these results validate that the fractal–fractional Rössler system undergoes the classical route to chaos, while fractional order μ and fractal dimension ν reshape the bifurcation structure and attractor geometry.

Uncorrected Proof

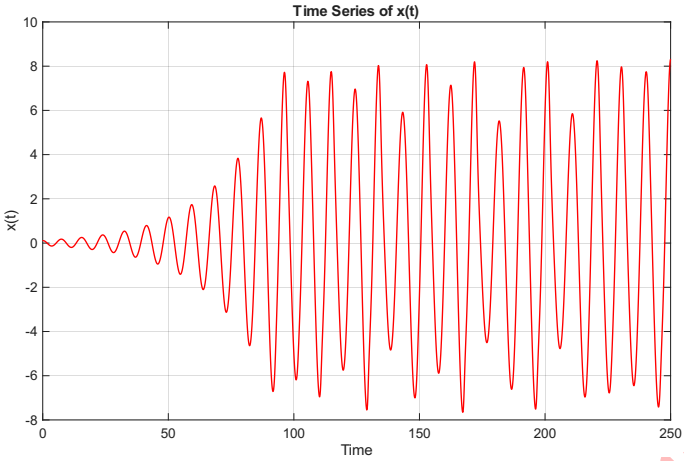


FIGURE 3. Time series of $x(t)$ for the fractal–fractional Rössler system.

Uncorrected Proof

2.6. **Time Series Analysis.** The time series plots of the state variables $x(t)$, $y(t)$, and $z(t)$ of the fractal–fractional Rössler system are shown in Figures 3–5. The simulations were carried out using the recently development new predictor–corrector scheme [25] with fractional order $\mu = 0.95$, fractal dimension $\nu = 0.93$, and system parameters $a = 0.2$, $b = 0.2$, and $c = 5.7$.

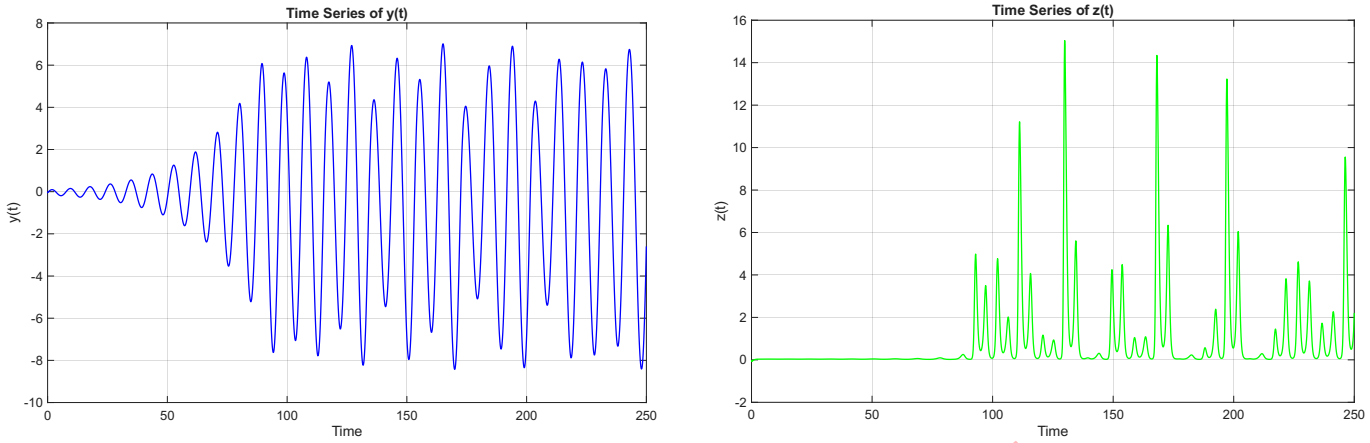
From the graphs, it can be observed that the function $x(t)$ displays oscillations whose amplitude increases and then damps to end up being chaotic variations. It is evident that the function $y(t)$ displays the oscillating pattern but with a notable phase lag relative to $x(t)$ and signifies the connection of the state functions. On the other hand, the development of $z(t)$ is very distinctive: instead of smooth oscillations, it produces sharp spikes of chaotic nature with areas of reduced activity between them. They are generated by the nonlinear feedback term $z(x - c)$ and are accountable for stretching and folding the orbits and sustaining chaotic motion.

For a better illustration of the global dynamics, a composite time series of $x(t)$, $y(t)$, and $z(t)$ is presented. It shows the mutual interaction of the three series: the fast oscillations of $x(t)$, the mid-range oscillations of $y(t)$, and the random spiking of $z(t)$. Each of these series separately confirms that the system is of deterministic chaos kind, where memory effects (due to the fractional order μ) and structural heterogeneity (due to the fractal parameter ν) both play their role in the resulting dynamics.

2.7. **Phase Portrait.** The phase portraits show the evolution of the Rössler system under different combinations of fractional order μ and fractal dimension ν. Introducing fractional and fractal effects modifies the attractor's shape, with the fractal–fractional case displaying the most complex and irregular trajectories.

Figure 6 shows the influence of fractional order μ and fractal dimension ν on the chaotic character of the Rössler system. When it is in the classical regime ($\mu = 1, \nu = 1$), the system has the standard Rössler attractor of large oscillation amplitude and intense chaotic motion. When the fractional order is reduced slightly ($\mu = 0.98, \nu = 1$), the attractor is reduced and oscillations are suppressed, showing that memory effects caused by the fractional derivative reduce chaotic strength. When it is just the fractal case ($\mu = 1, \nu = 0.88$), the attractor has original amplitude but is subject to significant geometrical distortion, showing the role of fractality in reshaping trajectory shape. When it is the fractal–fractional scenario ($\mu = 0.95, \nu = 0.93$), memory and fractality work together and produce a highly distorted attractor of reduced amplitude and knobby geometry. This confirms that the fractal–fractional system generalizes the classical Rössler system, where μ is primarily responsible for oscillation strength control and ν is accountable for the shape and structural irregularity of the chaotic attractor.





(A) Time series of $y(t)$. (B) Time series of $z(t)$.

FIGURE 4. Time series of $y(t)$ and $z(t)$ for the fractal–fractional Rössler system.

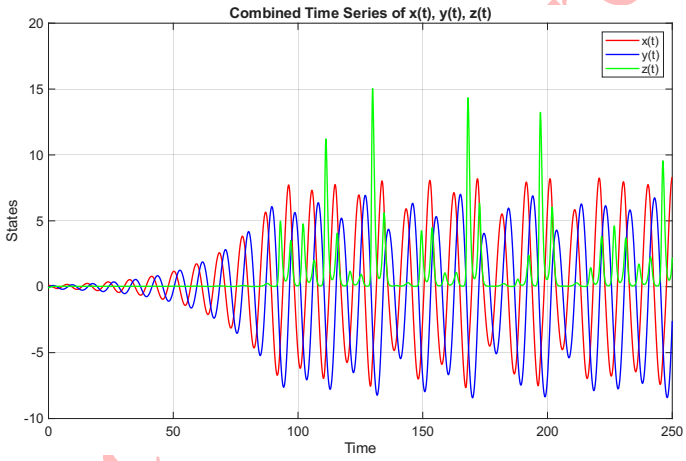
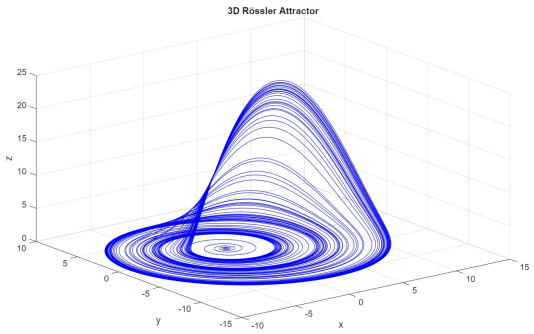


FIGURE 5. Combined time series of $x(t)$, $y(t)$, and $z(t)$ for the fractal–fractional Rössler system.

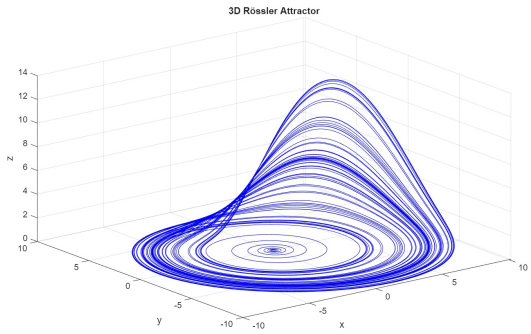
TABLE 4. Effect of fractional order μ and fractal dimension ν on the Rössler system dynamics.

Case	μ	ν	**Observed Behavior**
Classical Rössler	1	1	Standard chaotic attractor with large oscillation amplitude; baseline dynamics.
Fractional only	0.98	1	Attractor contracts; oscillations weakened due to memory effect of fractional order.
Fractal only	1	0.88	Attractor shape deforms structurally; amplitude preserved but trajectories display fractal irregularity.
Fractal–fractional	0.95	0.93	Combined effect: oscillations suppressed and attractor geometry distorted; richer and more complex chaotic dynamics.

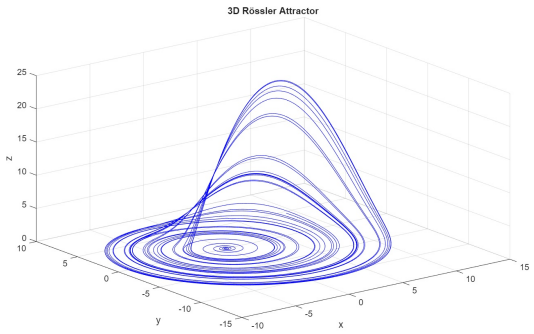
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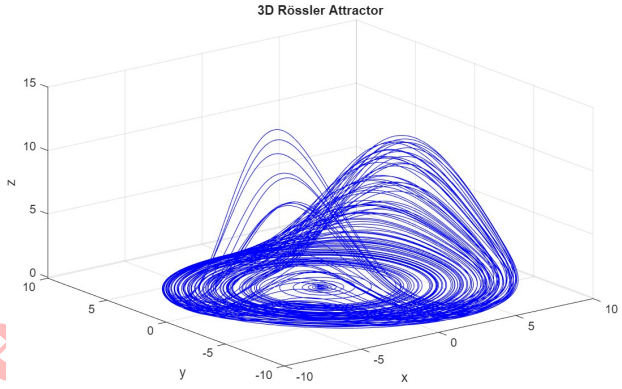
(A) $\mu = 1,\ \nu = 1$ (Classical Rössler).



(B) $\mu = 0.98,\ \nu = 1$ (Fractional only).



(C) $\mu = 1,\ \nu = 0.88$ (Fractal only).



(D) $\mu = 0.95,\ \nu = 0.93$ (Fractal–fractional).

FIGURE 6. Phase portraits of the Rössler system with initial condition $x(0) = 0.1,\ y(0) = 0,\ z(0) = 0$ and parameters $a = 0.2, b = 0.2, c = 5.7$ under different combinations of fractional order μ and fractal dimension ν, showing the influence of memory and fractality on chaotic dynamics.

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3. Application to Neuronal Modeling

Chaotic signals are often employed as realistic forcing inputs in biological models to mimic the complex, irregular drives arising from heterogeneous media and neural networks. we couple the fractal–fractional Rössler system with a simple neuron model, namely the FitzHugh–Nagumo (FHN) equations, which represent an archetypal excitable system widely used in computational neuroscience. This framework highlights how memory effects (fractional order μ) and structural heterogeneity (fractal dimension ν) influence neuronal dynamics when the neuron is subject to chaotic synaptic-like inputs.

3.1. The FitzHugh–Nagumo Model.

The classical FitzHugh–Nagumo model is a reduced version of the Hodgkin–Huxley equations and describes neuronal excitability using two state variables: a fast activator $v(t)$, representing the membrane potential, and a slow recovery variable $w(t)$, representing ion-channel dynamics. In its standard form, the equations are

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I(t),$$
$$\frac{dw}{dt} = \varepsilon\big(v + a - b\,w\big),$$

where $I(t)$ denotes an external stimulus, a and b are model parameters, and ε is a small positive parameter that separates the time scales of v and w.

3.2. **Fractal–Fractional FitzHugh–Nagumo Model with Chaotic Forcing.** To capture hereditary and structural effects in excitable media, we generalize the neuron dynamics using fractal–fractional derivatives with exponential kernel. Moreover, we couple the neuron to a chaotic drive provided by the fractal–fractional Rössler system. Let $(x_R(t), y_R(t), z_R(t))$ denote the Rössler state variables, computed for given (μ, ν). Using the x_R component as the input, the coupled system becomes

$$ {}^{FF}D_t^{\mu,\nu} v(t) = v(t) - \frac{v(t)^3}{3} - w(t) + K\, s(t), \tag{3.1} $$

$$ {}^{FF}D_t^{\mu,\nu} w(t) = \varepsilon\big(v(t) + a - b\, w(t)\big), \tag{3.2} $$

with

$$ s(t) = \alpha_s\, x_R(t) \quad \text{or} \quad s(t) = \alpha_s\big(x_R(t) - \overline{x_R}\big). $$

Here K denotes the coupling gain, and α_s is a scaling factor to normalize the input amplitude. For simplicity, we assume the same (μ, ν) values for both the driving Rössler system and the neuron; however, distinct orders may also be considered to model different memory and fractal characteristics in the drive and the response.

TABLE 5. Biological interpretation and suggested parameters for the coupled fractal–fractional FitzHugh–Nagumo and Rössler system.

Symbol / Variable	Interpretation	Suggested value / note
$v(t)$	Fast activator: membrane potential	Dynamical variable
$w(t)$	Slow recovery variable: ion-channel gating	Dynamical variable
$x_R(t)$	Rössler chaotic drive (external input)	Dynamical input signal
μ	Fractional order (memory in ion kinetics)	$\mu = 1.0$ (classical), $\mu < 1$ (with memory)
ν	Fractal dimension (tissue heterogeneity)	$\nu = 1.0$ (classical), $\nu < 1$ (fractal)
a (FHN)	Bias parameter	0.7 (excitable regime)
b (FHN)	Recovery scaling	0.8–1.0
ε	Time-scale separation	0.08–0.1 (small)
K	Coupling gain (input strength)	0.05–0.5 (tunable)
α_s	Scaling factor for input x_R	chosen to normalize amplitude
Initial conds.	Neuron state initial conditions	$(v(0), w(0)) = (0.1, 0)$ or $(1, 0)$
Rössler params	Chaotic regime parameters	$(a_R, b_R, c_R) = (0.2, 0.2, 5.7)$

3.3. **Simulation of the FitzHugh–Nagumo Model with Chaotic Forcing.** We consider the fractal–fractional FitzHugh–Nagumo system given by

$$ {}^{FF}D_t^{\mu,\nu} v(t) = v(t) - \frac{v(t)^3}{3} - w(t) + K\, s(t), \tag{3.3} $$

$$ {}^{FF}D_t^{\mu,\nu} w(t) = \varepsilon\big(v(t) + a - b\, w(t)\big), \tag{3.4} $$

with initial conditions

$$ v(0) = v_0, \quad w(0) = w_0. $$

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The time domain is discretized as $t_n = nh$ with uniform step size $h > 0$. The normalization function associated with the fractal–fractional operator is denoted by $M(\mu)$. The nonlinear terms are

$$f_v(t, v, w) = v - \frac{v^3}{3} - w + K\, s(t),$$

$$f_w(t, v, w) = \varepsilon\big(v + a - b\, w\big).$$

To approximate the solution, we use the predictor–corrector method developed by Nagpurkar and Kachhia [25]. At each step $t_n \to t_{n+1}$, the updates are carried out as follows.
Predictor 1 (P_1):

$$v_{n+1}^{P_1} = v_n + \nu\, t_n^{\nu-1}\, f_v(t_n, v_n, w_n) \left(\frac{h\mu}{2M(\mu)} - \frac{1-\mu}{M(\mu)} \right), \tag{3.5}$$

$$w_{n+1}^{P_1} = w_n + \nu\, t_n^{\nu-1}\, f_w(t_n, v_n, w_n) \left(\frac{h\mu}{2M(\mu)} - \frac{1-\mu}{M(\mu)} \right). \tag{3.6}$$

Predictor 2 (P_2):

$$v_{n+1}^{P_2} = \nu\, t_{n+1}^{\nu-1}\, f_v(t_{n+1}, v_{n+1}^{P_1}, w_{n+1}^{P_1}) \left(\frac{h\mu}{2M(\mu)} + \frac{1-\mu}{M(\mu)} \right), \tag{3.7}$$

$$w_{n+1}^{P_2} = \nu\, t_{n+1}^{\nu-1}\, f_w(t_{n+1}, v_{n+1}^{P_1}, w_{n+1}^{P_1}) \left(\frac{h\mu}{2M(\mu)} + \frac{1-\mu}{M(\mu)} \right). \tag{3.8}$$

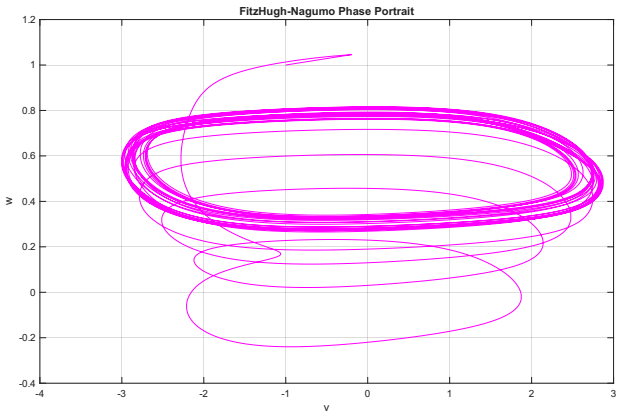
Corrector (C):.

$$v_{n+1} = v_{n+1}^{P_1} + \left(\frac{h\mu\nu}{2M(\mu)} + \frac{(1-\mu)\nu}{M(\mu)} \right) t_{n+1}^{\nu-1}\, f_v\Big(t_{n+1}, v_{n+1}^{P_1} + v_{n+1}^{P_2}, w_{n+1}^{P_1} + w_{n+1}^{P_2}\Big), \tag{3.9}$$

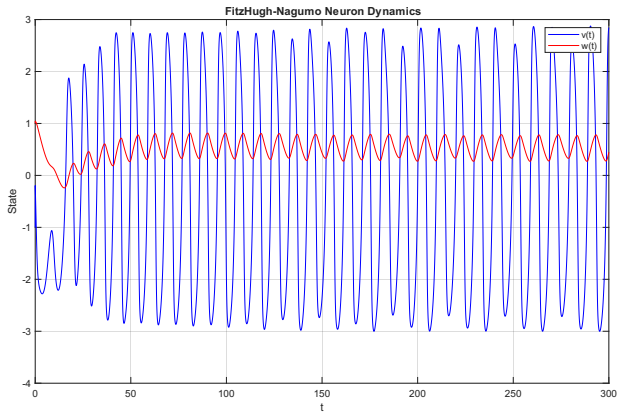
$$w_{n+1} = w_{n+1}^{P_1} + \left(\frac{h\mu\nu}{2M(\mu)} + \frac{(1-\mu)\nu}{M(\mu)} \right) t_{n+1}^{\nu-1}\, f_w\Big(t_{n+1}, v_{n+1}^{P_1} + v_{n+1}^{P_2}, w_{n+1}^{P_1} + w_{n+1}^{P_2}\Big). \tag{3.10}$$

3.4. **Synchronization Analysis between Rössler Drive and FitzHugh–Nagumo Neuron.** To quantify the interaction between the chaotic Rössler drive and the fractal–fractional FitzHugh–Nagumo (FHN) neuron, we perform synchronization analysis using correlation measures, phase difference, and time series overlays. These diagnostics highlight how memory effects (μ) and structural irregularity (ν) influence the entrainment of neuronal dynamics by chaotic inputs.

Uncorrected Proof



(A) Phase portrait of the FitzHugh–Nagumo neuron in the (v, w) plane.



(B) Time series of $v(t)$ (blue) and $w(t)$ (red) showing oscillatory firing patterns.

FIGURE 7. Numerical simulation results of the fractal–fractional FitzHugh–Nagumo model driven by the Rössler chaotic signal. (a) The phase portrait shows the oscillatory trajectory in the (v, w) phase plane, illustrating the excitable behavior of the neuron. (b) The time series depicts periodic spiking in the membrane potential $v(t)$ and the slower recovery oscillations in $w(t)$. Together, these confirm that the model captures both fast activation and slow recovery under chaotic forcing, with modifications induced by the fractional order μ and fractal parameter ν.

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3.5. **Bifurcation Analysis of the Fractal–Fractional FitzHugh–Nagumo Model.** To further investigate the nonlinear dynamics of the coupled system, we perform a bifurcation analysis of the fractal–fractional FitzHugh–Nagumo (FHN) model driven by the Rössler attractor. The bifurcation parameter is chosen as c, which governs the nonlinear feedback in the Rössler subsystem. For each value of c, the fractal–fractional FitzHugh–Nagumo system is numerically integrated using the predictor–corrector scheme, and the local maxima of the membrane potential $v(t)$ are recorded after discarding transients. The resulting bifurcation diagram is shown in Figure 9.

There is a rich variety of behaviors depicted by the diagram. For very small values of c (for example, $c \approx 2.5$), the neuron is observed to have periodic oscillations with sharp maxima corresponding to regular spiking. For bigger values of c, a sequence of period-doubling bifurcations occur and higher-order oscillatory firing behaviors are obtained. At $c \approx 5.5$, the system is within a chaotic region where the maxima of $v(t)$ are tightly distributed corresponding to irregular spiking activity similar to biological neurons under non-homogeneous inputs.

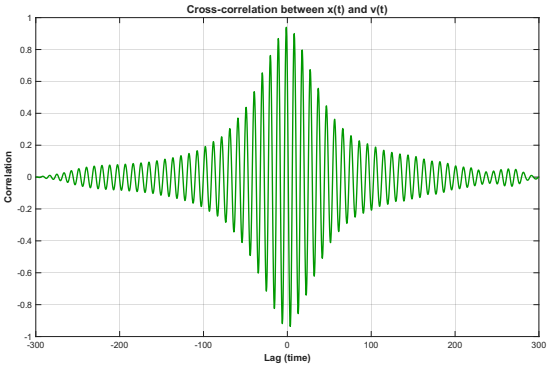
Fractional order μ and fractal dimension ν are prominent features of these bifurcation patterns. Reductions of μ tend generally to slow the development of chaos and serve the role of a memory effect dampening oscillations. Disruptions of ν, by contrast, add structural heterogeneities bending branches of the bifurcation diagram and boosting irregularity. Memory and fractality thus together dictate the excitability threshold and the neuron's stability.

In short, this analysis shows that the fractal–fractional FitzHugh–Nagumo model not only inherits the chaotic variability of the input Rössler system but it also converts it through memory and fractal parameters to produce biologically plausible transitions between regular spiking and bursting and chaotic firing.

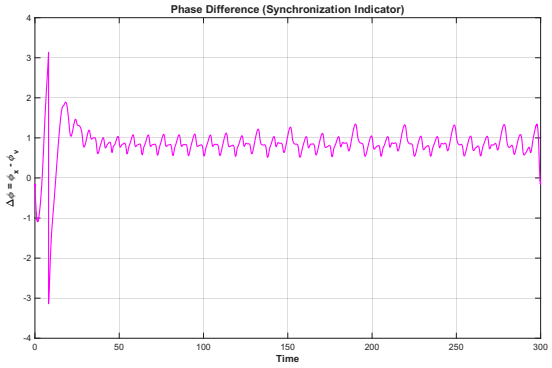
4. Results and Discussion

This section presents the numerical simulations of the fractal–fractional Rössler system and its influence as a chaotic input to the FitzHugh–Nagumo (FHN) neuron model. All computations were performed using the predictor–corrector method with variable fractional order μ and fractal dimension ν, which represent memory effects and structural heterogeneity, respectively.

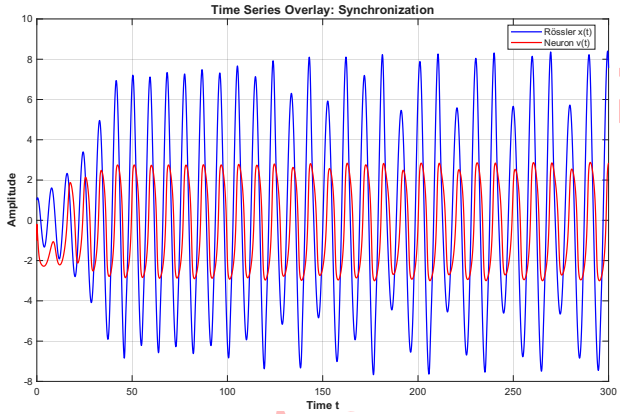
The dynamics of the fractal–fractional Rössler system are illustrated through time series and phase portraits. The state variables $x(t)$, $y(t)$, and $z(t)$ exhibit irregular oscillations, characteristic of chaotic behavior. The phase portraits



(A) Cross-correlation between Rössler $x(t)$ and neuron $v(t)$ signals.



(B) Phase difference $\Delta\phi = \phi_x - \phi_v$ as a synchronization indicator.



(C) Time series overlay of Rössler $x(t)$ (blue) and neuron $v(t)$ (red), showing partial synchronization.

FIGURE 8. Synchronization analysis of the fractal–fractional FitzHugh–Nagumo neuron driven by the chaotic Rössler system. (a) The cross-correlation function exhibits oscillatory peaks, indicating strong coupling and phase locking between $x(t)$ and $v(t)$. (b) The phase difference $\Delta\phi$ fluctuates around zero after transient dynamics, confirming phase synchronization. (c) The direct time-series comparison demonstrates that the neuron output $v(t)$ follows the drive $x(t)$ with reduced amplitude, revealing partial but robust synchronization.

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reveal spiral attractors that are stretched and folded as the values of μ and ν are varied. The bifurcation diagram with respect to the parameter c shows a transition from periodic oscillations to chaos via period-doubling bifurcations. While this route mirrors that of the classical Rössler system, the onset and extent of chaotic regions are strongly influenced by μ and ν. A smaller μ delays the onset of bifurcations, promoting more stable oscillations, whereas variations in ν reshape the chaotic windows, increasing the complexity of the dynamics.

The response of the FHN neuron under chaotic input from the Rössler attractor is also investigated. The bifurcation diagram of the membrane potential $v(t)$ with respect to the parameter c shows that for small values of c, the neuron exhibits regular spiking activity. As c increases, successive period-doublings lead to more complex firing patterns and eventually chaotic behavior. This phenomenon resembles the activity of biological neurons subjected to heterogeneous synaptic inputs. The impact of the fractional order and fractal dimension is particularly significant: decreasing μ reduces high-frequency oscillations, stabilizing the neuron's firing, whereas lowering ν introduces mixed-mode oscillations,

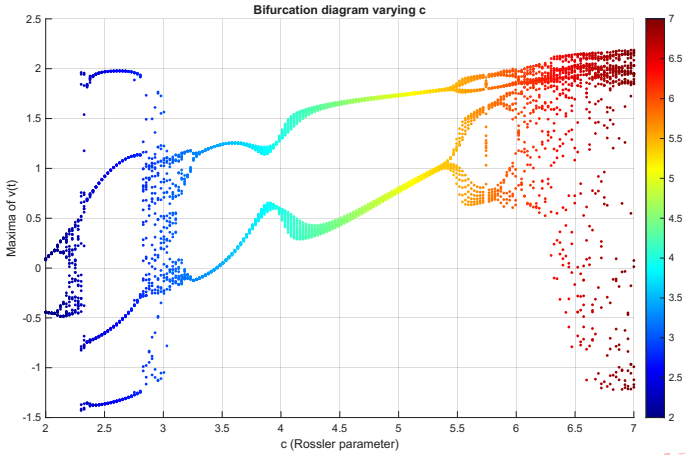


FIGURE 9. Bifurcation diagram of the fractal–fractional FitzHugh–Nagumo model under Rössler chaotic input, showing the maxima of $v(t)$ versus the Rössler parameter c.

increasing irregularity. These results indicate that memory effects tend to stabilize neuronal behavior, while structural heterogeneity promotes complex and irregular firing patterns.

The findings have biological and physical relevance. Chaotic spiking in real neurons is often associated with noise, irregular structural patterns, or memory-dependent ion channel dynamics. The fractal–fractional model captures these features mathematically in a unified way. The observed transition from regular to chaotic firing may correspond to pathological conditions such as epileptic seizures, where normal spiking behavior becomes disordered. From a physical standpoint, the synchronization between the chaotic Rössler input and the neuronal response suggests potential applications in secure communication systems, where chaos-based signals enhance robustness and unpredictability. Furthermore, the ability to modulate neuronal excitability by adjusting μ and ν opens avenues for designing fractional-order controllers in bioengineering applications.

In summary, the fractal–fractional Rössler system demonstrates a period-doubling route to chaos with dynamics shaped by the fractional order and fractal dimension. The FHN neuron model, when driven by this chaotic input, exhibits transitions from spiking to bursting and chaotic firing patterns, reflecting realistic neuronal behavior. The interplay between memory and fractality highlights how these features respectively stabilize and destabilize oscillations, providing a deeper understanding of complex biological rhythms and suggesting practical uses in engineering and communication systems. Overall, incorporating memory effects and structural heterogeneity enriches the dynamics of chaotic systems and neuronal models, enhancing their relevance for both theoretical and applied research.

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5. Conclusion

This study presented a comprehensive investigation of a fractal–fractional Rössler system and its application as a chaotic driving mechanism for a FitzHugh–Nagumo neuron model. Numerical simulations demonstrated that the proposed fractal–fractional Rössler system exhibits a clear period-doubling route to chaos as the bifurcation parameter c varies, consistent with the classical Rössler dynamics, while showing significant modifications due to the fractional order μ and fractal dimension ν. The time-series analysis and phase portraits confirmed that decreasing μ suppresses oscillation amplitude and delays the onset of chaos, whereas decreasing ν increases geometric distortion and irregularity of the chaotic attractors.

The equilibrium and stability analysis revealed that all equilibrium points remain unstable under the fractal–fractional formulation, supporting the persistence of chaotic behavior across the considered parameter ranges. Bifurcation diagrams further highlighted that the combined influence of memory and fractality reshapes the chaotic windows and alters the density of bifurcation branches, providing greater control over the system dynamics than classical integer-order models.

When the chaotic output of the fractal–fractional Rössler system was used as an external input to the FitzHugh–Nagumo neuron model, the neuronal dynamics exhibited rich firing patterns ranging from regular spiking to bursting and chaotic firing. The calculated time series, phase portraits, and bifurcation diagrams of the membrane potential showed that larger values of μ stabilize neuronal activity by reducing irregular oscillations, while smaller values of ν promote complex and irregular firing behavior. Synchronization analysis further demonstrated partial phase locking between the chaotic drive and the neuronal response, confirming effective coupling and biologically plausible entrainment.

From a numerical perspective, the proposed predictor–corrector scheme showed strong performance in accurately capturing long-time dynamics, chaotic attractors, and neuronal firing patterns without numerical instability. Compared to classical and existing fractional numerical methods, the scheme efficiently handles the nonlocal memory effects and fractal characteristics inherent in the model, while maintaining computational simplicity and flexibility with respect to parameter variation.

Overall, the calculated results confirm that the fractal–fractional framework provides a more realistic and versatile modeling approach for chaotic and neuronal systems than integer-order formulations. The ability to regulate memory and structural heterogeneity through μ and ν offers valuable insight into complex biological rhythms and pathological neural activity, and the proposed numerical scheme serves as a reliable computational tool for future studies involving fractal–fractional dynamical systems.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Data Availability

No data was used for this study. This work is entirely theoretical and original, and no external datasets were generated or analyzed.

Computational tools and software

All numerical simulations and graphical analyses were performed using **MATLAB** R2023a.

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Uncorrected Proof