



Hedging of exotic options in Hawkes jump-diffusion models via Malliavin calculus

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Abstract

In financial mathematics, the calculation of the Greeks, particularly delta, is emphasized for its role in risk management. In this article, we employ Malliavin calculus to determine the delta of European and Asian options, where the underlying asset evolves according to a Hawkes jump-diffusion process. A central feature is that the Hawkes jump intensity is stochastic, which substantially affects Malliavin derivatives and the delta representation.

Keywords. Malliavin calculus, Stochastic intensity, Hawkes processes, Delta hedging.

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1. INTRODUCTION

Stochastic jump models with stochastic intensity describe the behavior of phenomena that can experience abrupt changes in value. They rely on stochastic processes to describe the evolution of a variable over time in a probabilistic framework. A central feature is the ability to accommodate sudden, unpredictable shifts, unlike traditional stochastic models such as Brownian motion, which assume smooth and continuous change over time. In these models, jumps occur at random times and with random sizes, usually modeled using a Poisson distribution. The stochastic intensity governs the jump rate and itself evolves as a stochastic process over time. These models are widely used in finance to capture both the gradual evolution of asset prices and the abrupt jumps caused by unexpected events. [9, 12, 13, 19, 27, 33, 34]. They have applications in a wide range of fields, including finance, insurance, and engineering (see, for example, [15, 50]). A key application of stochastic intensity jump models is in credit risk modeling for financial instruments. In this context, the jumps represent defaults or other credit events, while the stochastic intensity expresses the evolving rate at which such events occur.

Here, we use a special class of stochastic intensity models known as Hawkes processes. Hawkes introduced the Hawkes process [24, 25], as a model for event data arising from contagious phenomena. Today, Hawkes processes are applied across diverse fields: geology (earthquakes, volcanic eruptions), biology (population growth, the spread of infections, neuronal activity), computer science and social sciences (networks and social interactions), and finance (order-book dynamics, defaults and related topics) (see, e.g., [26, 48]).

The main feature of this class of processes is their self-exciting behavior, whereby the jump intensity rises instantaneously following each event. It is reasonable to limit such a self-exciting effect to decay over time. This behavior is captured by the exponential Hawkes model, which provides a tractable representation.

$$\lambda_t = \lambda_0 + \alpha \sum_{T_i < t} e^{-\beta(t-T_i)},$$

where $t \in \mathbb{R}_+ := [0, \infty)$ denotes time, T_i the random time of the i -th jump and λ_0, α, β are deterministic, constant and positive parameters. The exponential Hawkes model is mathematically tractable by virtue of its Markov property (see [21]).

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By definition, a Hawkes process with $\alpha = 0$ reduces to a Poisson process. It is therefore useful to study the behavior of the Hawkes process, particularly in terms of its deviation from a Poisson process when $\alpha \approx 0$. Conversely, a Hawkes process approaches a Poisson process as $\beta \rightarrow +\infty$. However, no results quantify their similarity for finite β . In the exponential Hawkes model, α represents the immediate impact of a jump on the intensity, and β governs its decay over time. A higher β indicates weaker effects of the occurred jumps. Mancini et al. [37] have fitted the JPM jumps to three types of Hawkes models, using maximum likelihood estimation, and obtained calibrated parameters. The stability condition $\alpha < \beta$ guarantees that the Hawkes process is non-explosive, meaning that the intensity does not grow without bound after a jump, which is a standard requirement in financial applications.

Hawkes jump-diffusion models produce heavier-tailed distributions with higher peaks than Poisson jump-diffusion models, which leads to higher option prices under the Hawkes framework for deep out-of-the-money options. Asset price dynamics exhibiting jumps with self-exciting features have been studied, for example, by [20], which show that sequences of large price movements are triggered by preceding large moves. [22] propose a Bayesian learning approach to jump-cluster detection and provide evidence of jump clustering since the 1987 market crash, with even more pronounced clustering after the 2008 global financial crisis. [43] describe foreign exchange markets. Liu and Zhu in [35] present an analytical method for pricing variance swaps when the underlying asset follows a Hawkes jump-diffusion process with stochastic volatility and clustered jumps. In [36], the authors apply a Hawkes jump-diffusion model to evaluate foreign equity options and derive valuation formulas using the Fourier transform, accommodating both clustered jumps and cross-market jump propagation. They calculate the Greeks and hedging strategies for swap options within this framework. Brignone and Sgarra [11] introduce a method for pricing Asian options in risky-asset models with Hawkes processes and an exponential kernel. Ketelbuters and Hainaut [30] investigate CDS pricing in fractional Hawkes models. Recently, Chen et. al. [16] modeled the clustering behavior of price jumps and variance using high-frequency data within a labeled Hawkes process in a bivariate jump-diffusion setting.

Malliavin calculus plays a critical role in financial mathematics, especially in the pricing and risk management of financial derivatives. It is a powerful tool for computing the Greeks, which are measures of sensitivity of the derivative prices to changes in the underlying asset price, volatility, time to maturity, and interest rates, with greater precision. This calculus provides a rigorous framework for considering the stochastic behavior of the underlying assets and deriving derivative prices. It has been widely applied in the literature (see, for example, [3, 6–8, 17, 28, 31, 32, 51]). In Hawkes models, Torrisi [49] provides a Gaussian bound for the first chaos of these point processes by combining Stein's method and Malliavin calculus via a Clark–Ocone-type representation formula.

To the best of our knowledge, this is the first paper to propose a hedging strategy for exotic options in Hawkes models. Our contributions in this manuscript are as follows. We replace the stochastic integral with respect to a point process driven by a stochastic intensity with a stochastic integral with respect to a Lévy process whose Lévy measure incorporates the stochastic intensity (see Equations (2.2) and (3.2)). This reformulation enables us to express the solution of Hawkes SDEs in an exponential form of another process, which is convenient for obtaining an explicit expression for their Malliavin derivatives. Despite classical arguments, the Malliavin derivatives of the stochastic intensity appear in the formula (Equation (3.5)), which complicates the proofs of moment boundedness for the inverse of the solution (see Lemma 3.7). Invertibility of the Malliavin derivatives is required to show that new, notably advantageous Malliavin weights, used to obtain deltas for Asian options and European options, lie in the domain of the Skorokhod integral (see Lemmas 4.2 and 4.4). In addition, numerically, computing the delta requires establishing the convergence of the discretized method. In our numerical approach, for every discretized path, we update the point at the next time step using the stochastic intensity evaluated at the current time (see Section 5).

While Malliavin calculus has been extensively used for computing Greeks in jump-diffusion models (see, e.g., [6, 17, 32]), none of the previous works address the case where the jump intensity is itself a stochastic process driven by self-exciting dynamics. In particular, prior studies on Hawkes processes in finance have focused on pricing (e.g., [11] for Asian options, [35] for variance swaps) or on probabilistic properties ([49] provides Gaussian bounds). The appearance of the stochastic intensity in the Malliavin derivative (Equation (3.5)) introduces substantial technical challenges—including new exponential moment bounds (Lemma 3.6)—that are absent in classical settings. Also, we establish the new weight in (4.3) and (4.7), and show their Skorokhod integrability in Lemmas 4.2 and 4.4. Both the construction of these weights and the demonstration of their integrability are crucially tied to the stochastic intensity



and its Malliavin derivatives. Moreover, the convergence proof of our discretized scheme (Lemma 5.2) is, to our knowledge, the first of its kind for Hawkes-driven hedging. In summary, while Malliavin calculus for jump processes is well established, the combination of stochastic intensity, exotic options, and rigorous convergence analysis constitutes a significant and original contribution to the field.

This article is organized as follows. Section 2 recalls the definition of Hawkes processes and examines the properties of their stochastic intensity. Section 3 introduces the main SDEs and establishes properties of their solutions. In keeping with the Malliavin derivative framework, we employ a novel technique to establish boundedness of the inverse derivatives—a requisite step for ensuring that the Malliavin weight arising in the delta computation belongs to the domain of the Skorokhod integral. Section 4 presents our main results on delta computation via innovation quadratures for both European and Asian options. The convergence of our numerical methods for computing the delta is proved in Section 5. Section 6 provides numerical illustrations and verification of our results. Most proofs of lemmas and the preliminaries on the Malliavin calculus are deferred to the Appendices.

Readers primarily interested in the computational implementation and numerical results may proceed directly to Sections 4 and 6. However, the statements of the lemmas in Sections 3 and 5 are essential for understanding the theoretical foundation of the Malliavin weights and the convergence of the numerical scheme.

TABLE 1. Key notation used throughout the manuscript.

Symbol	Meaning
$\bar{N}(ds, dz)$	Background Poisson measure on $\mathbb{R} \times \mathbb{R}_+$
$N(dt)$	Hawkes jump counting measure
$\lambda(t)$	Stochastic intensity of the Hawkes process
$\epsilon_{(u,z)}$	Dirac measure at point (u, z)
$D_{u,z}^{\bar{N}}$	Malliavin derivative with respect to the background Poisson measure $\bar{N}$
$N_{(u,z)}(ds)$	Perturbed jump measure appearing in the Malliavin derivative of λ
$\mathcal{B}_1, \mathcal{B}_2$	Integrals of indicators over the sets B and B^c for European options
$\mathcal{C}_1, \mathcal{C}_2$	Integrals of indicators over the sets C_Y and C_Y^c for Asian options
$\delta^{\bar{N}}$	Skorokhod integral with respect to $\bar{N}$
δ^W	Skorokhod integral with respect to Brownian motion

2. HAWKES PROCESSES AND PROPERTIES OF STOCHASTIC INTENSITIES

In this section, we give some preliminaries on point processes with stochastic intensity, as mentioned by Brémaud in Chapter 5 of [10]. In particular, Hawkes processes are a class of multivariate point processes introduced by Hawkes in [24, 25] that are characterized by a stochastic intensity vector. Some of their applications can be seen in [4]. We refer the reader to [49] for more details.

The process $N = \{N(A)\}_{A \in \mathcal{B}(\mathbb{R})} := \{\sum_{n \in \mathbb{Z}} 1_A(T_n)\}_{A \in \mathcal{B}(\mathbb{R})}$ called the point process with increasing random times $\{T_n\}_{n \in \mathbb{Z}}$. We suppose that N is locally finite and simple, and $\mathcal{F}_t := \{\mathcal{F}_t\}_{t \in \mathbb{R}} \subset \mathcal{F}$ be a filtration such that $\mathcal{F}_t \supseteq \mathcal{F}_t^N$ for any $t \in \mathbb{R}$, where $\mathcal{F}^N := \{\mathcal{F}_t^N\}_{t \in \mathbb{R}}$ is the natural filtration of the point process N , that is,

$$\mathcal{F}_t^N := \sigma\{N(A) : A \in \mathcal{B}(\mathbb{R}), A \subseteq (-\infty, t]\},$$

Given a non-negative $\mathcal{F}_t$ -adapted stochastic process $\{\lambda(t)\}_{t \in \mathbb{R}}$ such that

$$\int_a^b \lambda(t) dt < \infty, \text{ a.s. for all } a, b \in \mathbb{R}, \quad (2.1)$$

we call $\{\lambda(t)\}_{t \in \mathbb{R}}$ as a $\mathcal{F}$ -stochastic intensity of N , if for any $a, b \in \mathbb{R}$, almost surely $\mathbb{E}[N((a, b]) | \mathcal{F}_a] = \mathbb{E}[\int_a^b \lambda(t) dt | \mathcal{F}_a]$.

In this article, given a Poisson process $\bar{N}$ in $\mathbb{R} \times [0, \infty)$ with mean measure $dtdz$, we consider the point process

$$N(dt) := \bar{N}(dt \times (0, \lambda(t)]),$$



where $\{\lambda(t)\}_{t \in \mathbb{R}}$ is a non-negative process of the exponential form

$$\begin{aligned} \lambda(t) &:= \lambda_0 + \int_0^t \alpha e^{-\beta(t-s)} dN_s = \lambda_0 + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)} \\ &= \lambda_0 + \int_0^t \int_{\mathbb{R}_+} \alpha e^{-\beta(t-s)} 1_{(0, \lambda(s)]}(z) \bar{N}(dz, ds) =: \varphi_t(\bar{N}|_{(-\infty, t)}). \end{aligned} \quad (2.2)$$

in which $\varphi_t(\bar{N}|_{(-\infty, t)}) = \varphi(t, \bar{N}|_{(-\infty, t)})$ and $\varphi : \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}_+ = \mathbb{R} \cap [0, \infty)$ is a measurable functional such that for all $a, b \in \mathbb{R}$, $\lambda(t)$ satisfying (2.1). Here, $\mathcal{N}$ denotes the space of simple and locally finite counting measures on $\mathbb{R} \times \mathbb{R}_+$ endowed with the vague topology. For simplicity, with a little abuse of notation, we denote by $\bar{N}|_{(-\infty, t)}$ the restriction of $\bar{N}$ to $(-\infty, t) \times \mathbb{R}_+$, that is,

$$\bar{N}|_{((-\infty, t))}(A) := \bar{N}(A \cap ((-\infty, t) \times \mathbb{R}_+)), \quad A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+).$$

According to Lemma 2.1 in [49], N has $\mathcal{F}^{\bar{N}}$ -stochastic intensity $\{\lambda(t)\}$, where $\mathcal{F}^{\bar{N}} = \{\mathcal{F}_t^{\bar{N}}\}_{t \in \mathbb{R}}$ denote the natural filtration of $\bar{N}$, i.e.

$$\mathcal{F}_t^{\bar{N}} = \sigma\{\bar{N}(A \times B) : A \in \mathcal{B}(\mathbb{R}), B \in \mathcal{B}([0, \infty)), A \subseteq (-\infty, t]\}.$$

The process $\lambda(t)$ can be also demonstrated as the solution of the SDE

$$d\lambda(t) = \beta(\lambda_0 - \lambda(t))dt + \alpha dN_t.$$

Note that the stability condition $\alpha \int_0^\infty e^{-\beta s} ds = \frac{\alpha}{\beta} < 1$ should be held, (see [5], Section 2). In addition, the process N is called a Hawkes process. For every $0 \leq t < s \leq T$, due to the proof of Theorem 3.1 in [49],

$$\begin{aligned} \mathbb{E}\left(\int_t^s N(du) | \mathcal{F}_t^{\bar{N}}\right) &= \mathbb{E}\left(\int_t^s \int_{\mathbb{R}_+} \bar{N}(du \times (0, \lambda(u)]) | \mathcal{F}_t^{\bar{N}}\right) = \mathbb{E}\left(\int_t^s \int_{\mathbb{R}_+} 1_{(0, \lambda(u)]}(z) \bar{N}(du, dz) | \mathcal{F}_t^{\bar{N}}\right) \\ &= \mathbb{E}\left(\int_t^s \int_{\mathbb{R}_+} 1_{(0, \lambda(u)]}(z) dz du | \mathcal{F}_t^{\bar{N}}\right) = \mathbb{E}\left(\int_t^s \lambda(u) du | \mathcal{F}_t^{\bar{N}}\right). \end{aligned}$$

In addition, according to Theorem 5.1.31 [10], for every $0 \leq t \leq s \leq T$ and for every $\mathcal{F}_t^{\bar{N}}$ -predictable function k such that $\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+} (k(s, z)) 1_{(0, \lambda(s)]}(z) dz ds\right) < \infty$,

$$\mathbb{E}\left(\int_t^s \int_{\mathbb{R}_+} k(u, z) N(du) | \mathcal{F}_t^{\bar{N}}\right) = \mathbb{E}\left(\int_t^s \int_{\mathbb{R}_+} k(u, z) 1_{(0, \lambda(u)]}(z) dz du | \mathcal{F}_t^{\bar{N}}\right), \quad (2.3)$$

and if $\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+} (k(s, z))^2 1_{(0, \lambda(s)]}(z) dz ds\right) < \infty$ for given σ -field $\mathcal{G}$ and $\mathcal{G}$ -measurable intensity λ

$$\mathbb{E}\left(\exp\left\{iu \int_0^t \int_{\mathbb{R}_+} k(s, z) N(ds)\right\} | \mathcal{G}\right) = \exp\left\{\int_0^t \int_{\mathbb{R}_+} (e^{iuk(s, z)} - 1) 1_{(0, \lambda(s)]}(z) dz ds\right\}. \quad (2.4)$$

We refer the reader to Chapter 5 of [10] for more details. Let us first derive some properties of the stochastic intensity mentioned in (2.2). Denote $\tilde{N}(ds, dz) = \bar{N}(dt, dz) - dt dz$.

Lemma 2.1. *For every $p \geq 2$, and every $T < \infty$,*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} e^{p\beta t} |\lambda(t)|^p\right) < \infty, \quad \text{and} \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{p\lambda(t)}\right) < \infty.$$

Proof. We will prove it in [Appendix A](#). □



3. HAWKES-DRIVEN STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, we introduce the model, state the assumptions, and consider the properties of the solution in lemmas that we require in the main results.

Let $(\Omega, \mathcal{F}, P)$ be a Wiener-Poisson space with a risk neutral probability P . We assume that the underlying asset price $S = (S_t)_{t \in [0, T]}$ with the jump stochastic intensity process $\lambda = (\lambda(t))_{t \in [0, T]}$ of Hawkes process N_t can be governed by the following system of SDEs:

$$\begin{cases} dS_t &= S_0 + \int_0^t (\mu - (e^{J_s} - 1)\lambda(s))S_s ds + \int_0^t \sigma S_s dW_s + \int_0^t (e^{J_s} - 1)S_s N(ds) \\ d\lambda(t) &= \beta(\lambda_0 - \lambda(t))dt + \alpha dN_t, \end{cases} \quad (3.1)$$

where $(W_t)_{t \in [0, T]}$ is a Brownian motion, N is independent of W_t , μ denotes the riskless interest rate, J is a measurable function, σ is a positive constant. We call this process S_t a Hawkes-driven process. Assume the following conditions throughout the paper.

Condition H1:

- Assume that for every $p \geq 1$, we have $\int_0^T (e^{J_s} - 1)^{2p} ds < \infty$.
- The function J is an non-negative strictly increasing function with $J_0 = 0$.

The assumption that J is non-negative and strictly increasing with $J_0 = 0$ ensures that jump sizes are positive and grow with time, which is consistent with empirical observations of asset price jumps (e.g., larger jumps occur during periods of high volatility). Non-decreasing paths allow the business time to be replaced by an intrinsic trading clock. The boundedness of $\int_0^T (e^{J_s} - 1)^{2p} ds$ ensures that the jump component has finite moments of all orders, which is necessary for the Malliavin calculus framework. Economically, this condition rules out infinite-activity jumps and ensures that the self-exciting feedback mechanism remains stable.

The stability condition $\alpha < \beta$ guarantees that the Hawkes process is non-explosive, meaning that the intensity does not grow without bound after a jump, which is a standard requirement in financial applications. Mancini et al. [37] have fitted the JPM jumps to three types of Hawkes models, using maximum likelihood estimation, and obtained calibrated parameters.

From (2.3), the solution to the stochastic differential (3.1) is as follows: see [4].

$$\begin{aligned} S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s + \int_0^t \int_{\mathbb{R}_+} (e^{J_s} - 1)S_s 1_{(0, \lambda(s)]}(z) [\bar{N}(ds, dz) - dz ds] \\ &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s + \int_0^t \int_{\mathbb{R}_+} (e^{J_s} - 1)S_s 1_{(0, \lambda(s)]}(z) \tilde{N}(ds, dz). \end{aligned} \quad (3.2)$$

According to the Itô formula in [42], the solution of the stochastic differential equation (3.2) is in the form:

$$\begin{aligned} S_t &= S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t + \int_0^t \int_{\mathbb{R}_+} \{ \ln[1 + (e^{J_s} - 1)1_{(0, \lambda(s)]}(z)] - (e^{J_s} - 1)1_{(0, \lambda(s)]}(z) \} dz ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_+} \ln[1 + (e^{J_s} - 1)1_{(0, \lambda(s)]}(z)] \tilde{N}(ds, dz) \right\}. \end{aligned}$$

Since $1_{(0, \lambda(s)]}(z) = 1$ if and only if $z \in (0, \lambda(s)]$, so

$$\begin{aligned} S_t &= S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t + \int_0^t (J_s - e^{J_s} + 1)\lambda(s) ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_+} J_s 1_{(0, \lambda(s)]}(z) \tilde{N}(ds, dz) \right\} =: S_0 e^{X_t}. \end{aligned} \quad (3.3)$$

Let Y_t be the solution of the following SDE

$$dY_t = Y_t (e^{J_t} - 1) 1_{(0, \lambda(s)]}(z) \tilde{N}(ds, dz), \quad Y_0 = 1. \quad (3.4)$$



From the Itô formula, we have

$$\begin{aligned} Y_t &= \exp \left\{ \int_0^t (J_s - e^{J_s} + 1) \lambda(s) ds + \int_0^t \int_{\mathbb{R}_+} J_s 1_{(0, \lambda(s)]}(z) \tilde{N}(ds, dz) \right\} \\ &= \exp \left\{ - \int_0^t (e^{J_s} - 1) \lambda(s) ds + \int_0^t \int_{\mathbb{R}_+} J_s 1_{(0, \lambda(s)]}(z) \bar{N}(ds, dz) \right\}. \end{aligned}$$

With a similar proof of Theorem 3.1 in [46] and Section 5.1.1 of [38], the equation (3.4) has a unique solution.

3.1. Properties of the solution. Here, we establish some properties of the solution S_t that are needed in the main results.

Lemma 3.1. *Let X_t and Y_t defined in (3.3) and (3.4). Under Condition H1, for every $p \geq 2$,*

$$\mathbb{E}(Y_t^p) < \infty, \quad \text{and} \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{pX_t}\right) < \infty.$$

Proof. We will prove it in Appendix B. □

Lemma 3.2. *Let Y_t defined in (3.4). Under Condition H1, for every $p \geq 1$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}(S_t^{-p}) < \infty.$$

Proof. We will prove it in Appendix B. □

In the last part of this subsection, note that getting the partial derivatives of S_t with respect to S_0 shows that the stochastic flow of S_t exists and it is

$$\frac{\partial S_t}{\partial S_0} = \frac{S_t}{S_0} = \exp\{X_t\} = Y_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}.$$

As a consequence of Lemma 2.1, this flow is in L^p -space for every $p \geq 2$.

3.2. Weak derivatives of the solution. Now, we calculate the Malliavin derivative of the solution S_t of a Hawkes SDE, in accordance with the definitions and lemmas mentioned in Appendix F, in which an essential difference in stochastic intensity appears. Based on Equation (3.3) and Proposition F.1, Malliavin derivative of $D_{u,z}^{\bar{N}} S_t$ for every $u < t$ is as follows

$$D_{u,z}^{\bar{N}} S_t = D_{u,z}^{\bar{N}} e^{X_t} = S_0 e^{X_t + D_{u,z}^{\bar{N}} X_t} - S_0 e^{X_t} = S_0 e^{X_t} (e^{D_{u,z}^{\bar{N}} X_t} - 1) = S_t (e^{D_{u,z}^{\bar{N}} X_t} - 1),$$

using Equations (F.2) and (F.3) in Appendix F, which follows from Theorem 3.1 of [49], we therefore have

$$D_{u,z}^{\bar{N}} X_t = J_u 1_{(0, \lambda(u)]}(z) + \int_u^t k_{(u,z)}(s) N_{(u,z)}(ds) - \int_u^t (e^{J_s} - 1) D_{u,z}^{\bar{N}} \lambda(s) ds. \quad (3.5)$$

Additionally, the Malliavin derivative of λ is as follows

$$D_{u,z}^{\bar{N}} \lambda(t) = \alpha e^{-\beta(t-u)} 1_{(0, \lambda(u)]}(z) + \int_u^t \alpha e^{-\beta(t-s)} \text{sign}(D_{u,z}^{\bar{N}} \lambda(s)) N_{(u,z)}(ds), \quad (3.6)$$

in which, for every $s > u$, $k_{(u,z)}(s)$ is as in (F.4) and $N_{(u,z)}(ds)$ as in (F.5).

The presence of $D_{u,z}^{\bar{N}} \lambda(s)$ in the Malliavin derivative of X_t (see (3.5)) reflects the self-exciting nature of the Hawkes process: a perturbation in the jump times alters not only the current asset price, but also the future jump intensity through the feedback mechanism. This coupling, which is absent in standard jump-diffusion models, complicates the invertibility proof- as the Malliavin derivative of the intensity appears in the numerator of the weight- and necessitates the new exponential bounds developed in Lemma 3.6.

Lemma 3.3. *For every $2 \leq p \leq 4$, if $e^{4\alpha} < 1 + 4\beta$, then*

$$\mathbb{E}\left(\sup_{0 \leq u \leq t \leq T} e^{p D_{u,z}^{\bar{N}} \lambda(t)}\right) < \infty.$$



Proof. See [Appendix B](#). □

In the rest of the section, we will show that the inverse of $D_{u,z}^{\overline{N}} S_t$ exists and belongs to $L^p(\Omega)$ for every $p \geq 2$. Substitute (3.6) into (3.5), and use the Fubini theorem to derive

$$\begin{aligned} D_{u,z}^{\overline{N}} X_t &= J_u 1_{(0,\lambda(u)]}(z) + \int_u^t k_{(u,z)}(s) N_{(u,z)}(ds) - \int_u^t (e^{J_s} - 1) D_{u,z}^{\overline{N}} \lambda(s) ds \\ &= J_u 1_{(0,\lambda(u)]}(z) + \int_u^t J_s \text{sign}(D_{u,z}^{\overline{N}} \lambda(s)) N_{(u,z)}(ds) - \alpha 1_{(0,\lambda(u)]}(z) \int_u^t (e^{J_s} - 1) e^{-\beta(s-u)} ds \\ &\quad - \int_u^t \int_u^s \alpha (e^{J_s} - 1) e^{-\beta(s-l)} \text{sign}(D_{u,z}^{\overline{N}} \lambda(l)) N_{(u,z)}(dl) ds \\ &= \left(J_u - \alpha \int_u^t (e^{J_s} - 1) e^{-\beta(s-u)} ds \right) 1_{(0,\lambda(u)]}(z) + \int_u^t J_s \text{sign}(D_{u,z}^{\overline{N}} \lambda(s)) N_{(u,z)}(ds) \\ &\quad - \alpha \int_u^t \left(\int_l^t (e^{J_s} - 1) e^{-\beta(s-l)} ds \right) \text{sign}(D_{u,z}^{\overline{N}} \lambda(l)) N_{(u,z)}(dl), \end{aligned}$$

Define $F(v) := J_v - \int_v^t (e^{J_s} - 1) \alpha e^{-\beta(s-v)} ds$, for every $0 \leq v \leq t$, and rewrite $D_{u,z}^{\overline{N}} X_t$ in the following form.

$$D_{u,z}^{\overline{N}} X_t = F(u) 1_{(0,\lambda(u)]}(z) + \int_u^t F(l) \text{sign}(D_{u,z}^{\overline{N}} \lambda(l)) N_{(u,z)}(dl). \quad (3.7)$$

Now, we will consider some properties of the function $F(\cdot)$ that we need in the following.

Define $G(v) = e^{J_v} - 1 - \beta \int_v^t (e^{J_s} - 1) ds$ for every $0 \leq v \leq t$. Due to the strictly increasing property of the function $J_\cdot$, its positivity, and the facts that $G(0) < 0$ and $G(t) > 0$, we deduce that $G(\cdot)$ is a strictly increasing function with exactly one root, called v_0 . It is obvious that for every $v > v_0$, $G(v) > 0$.

As a consequence, we show a similar result for the function $F(\cdot)$ in the following lemma.

Proposition 3.4. *Given the above conditions, the $D_{u,z}^{\overline{N}} X_t$ mentioned in equation (3.7) is integrable.*

Proof. See [Appendix G](#). □

Lemma 3.5. *The function $F : [0, T] \rightarrow \mathbb{R}$ is an increasing function and there exists some $v_1 \in [v_0, t]$ such that the function F is positive on $[v_1, t]$.*

Proof. Take the derivative of F and result for every $v > v_0$

$$F'(v) = J'_v - \beta e^{\beta v} \int_v^t \alpha e^{-\beta s} (e^{J_s} - 1) ds + \alpha (e^{J_v} - 1) \geq J'_v + \alpha G(v) > 0.$$

It shows that $F(\cdot)$ is increasing on $[v_0, t]$. If $F(v_1) \geq 0$, then we put $v_1 := v_0$ and the proof is completed. If $F(v_0) < 0$, the fact that $F(t) > 0$ and increasing property of F deduce the existence some $v_1 \in [v_0, t]$ such that for every $v > v_1$, $F(v) > 0$. □

Lemma 3.6. *There exists some constant C_0 such that for any $r \geq 0$, and every $u > v_1$,*

$$\mathbb{E} \left(\exp \left\{ -r \int_u^t F(l) \text{sign}(D_{u,z}^{\overline{N}} \lambda(l)) N_{(u,z)}(dl) \right\} \right) \leq \mathbb{E} \left(\exp \left\{ TC_0 \alpha r 1_{[0,\lambda(u)]}(z) 1_{\{F(u) \leq \frac{1}{r}\}} J_u \right\} \right).$$

Proof. See [Appendix G](#). □

This lemma is necessary to ensure that the Malliavin weight is well-defined and non-degenerate.

Lemma 3.7. *For any $p \geq 1$, and $(u, z) \in [0, T] \times \mathbb{R}_+$ with $u > v_1$ such that $P(\lambda(u) > z) > 0$,*

$$\mathbb{E} \left(\frac{1_{(0,\lambda(u)]}(z)}{|D_{u,z}^{\overline{N}} X_t|^p} \right) < \infty.$$



Proof. Our proof is motivated by the proof of Theorem 1.2 in [46]. From the definition of the Gamma function and Lemma 3.6, we have

$$\begin{aligned}
\mathbb{E}\left(\frac{1_{(0,\lambda(u)]}(z)}{(D_{u,z}^{\bar{N}}X_t)^p}\right) &= \frac{1}{\Gamma(p)}\mathbb{E}\left(\int_0^\infty r^{p-1}1_{(0,\lambda(u)]}(z)\exp\left\{-rF(u)1_{(0,\lambda(u)]}(z)ds\right.\right. \\
&\quad \left.\left.-r\int_u^t F(l)\text{sign}(D_{u,z}^{\bar{N}}\lambda(l))N_{(u,z)}(dl)\right\}dr\right) \\
&= \frac{1}{\Gamma(p)}\left(\int_0^\infty r^{p-1}\mathbb{E}^{\frac{1}{2}}\left(1_{(0,\lambda(u)]}(z)e^{-2rF(u)1_{(0,\lambda(u)]}(z)}\right)\times\mathbb{E}^{\frac{1}{2}}\left(\exp\left\{2TC_0\alpha r1_{[0,\lambda(u)]}(z)1_{\{F(u)\leq\frac{1}{r}\}}J_u\right\}dr\right)\right) \\
&\leq \frac{1}{\Gamma(p)}\left(\int_0^\infty r^{p-1}P(\lambda(u)>z)e^{-rF(u)}e^{TC_0\alpha r1_{\{F(u)\leq\frac{1}{r}\}}J_u}dr\right) \\
&\leq P(\lambda(u)>z)\frac{1}{\Gamma(p)}\int_0^\infty r^{p-1}e^{-rF(u)}[e^{TC_0\alpha\frac{J_u}{F(u)}}]dr = P(\lambda(u)>z)\frac{1}{F(u)^p}[e^{TC_0\alpha\frac{J_u}{F(u)}}] < \infty.
\end{aligned}$$

□

4. DELTA OF OPTIONS DERIVED BY THE SOLUTION OF HAWKES SDES

In this section, we find the Wiener-Poisson weight for calculating the delta of the following two types of options: European options and Asian options, when the underlying assets are driven by Hawkes processes. First, from (2.3), we can write that

$$\delta^N(u) = \int_0^T u(t)(N(dt) - \lambda(t)dt) = \int_0^T \int_{\mathbb{R}_+} u(t)1_{(0,\lambda(t)]}(z)(\bar{N}(dt, dz) - dzdt).$$

By placing $\nu(dz) = dzdt$ and $N_0 = \bar{N}$ in Appendix F and Proposition F.2 with $u(t, z) = u(t)1_{(0,\lambda(t)]}(z)$, we obtain the duality relation as follows.

$$\begin{aligned}
\mathbb{E}(F\delta^N(u)) &= \mathbb{E}\left(F\int_0^T \int_{\mathbb{R}_+} u(t)1_{(0,\lambda(t)]}(z)(\bar{N}(dt, dz) - dzdt)\right) \\
&= \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} D_{t,z}^{\bar{N}}Fu(t)1_{(0,\lambda(t)]}(z)dzdt\right) = \mathbb{E}\left(\langle D_{t,z}^{\bar{N}}F, u(t)1_{(0,\lambda(t)]}(z) \rangle_{L^2([0,T]\times\mathbb{R}_0)}\right).
\end{aligned}$$

In the Poisson approach, from Proposition F.2, the sensitivity analysis is represented in the following lemma.

Lemma 4.1. *If there is some function $u(.,.)$ such that the function $v(.,.)$, defined as $v(t, z) := 1_{(0,\lambda(t)]}(z)u(t, z)$, belongs to the domain of $\delta^{\bar{N}}$ and*

$$\mathbb{E}\left(f'(S_T)\frac{\partial S_T}{\partial S_0}\right) = \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} 1_{(0,\lambda(t)]}(z)u(t, z)\{f(S_T + D_{t,z}^{\bar{N}}S_T) - f(S_T)\}dzdt\right), \quad (4.1)$$

then $\Delta^{\bar{N}} := \frac{\partial}{\partial S_0}\mathbb{E}(f(S_T)) = \mathbb{E}(f(S_T)\delta^{\bar{N}}(v))$.

Proof. According to Propositions F.1 and F.2, we derive

$$\begin{aligned}
\frac{\partial}{\partial S_0}\mathbb{E}(f(S_T)) &= \mathbb{E}\left(f'(S_T)\frac{\partial S_T}{\partial S_0}\right) \\
&= \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} 1_{(0,\lambda(t)]}(z)u(t, z)\{f(S_T + D_{t,z}^{\bar{N}}S_T) - f(S_T)\}dzdt\right) \\
&= \mathbb{E}\left(\left\langle 1_{(0,\lambda(.)]}(.)u(.,.), D^{\bar{N}}(f(S_T)) \right\rangle_{L^2([0,T]\times\mathbb{R}_0)}\right) = \mathbb{E}(f(S_T)\delta^{\bar{N}}(v)).
\end{aligned}$$

□



As an application, we may calculate the delta in the Wiener-Poisson space for the following examples desired in [29], when f is considered as a payoff function.

4.1. Delta Greeks for European options. Consider the payoff function of a European option as $f(S_T) = (S_T - K)^+$, for a strike price K . Then from Equation (1.42) in [39], or applying (F.1) with $u = \frac{h(\cdot)S_T}{\int_0^T h(s)D_s^W S_T ds}$ and $F = f(S_T)$, for every $h \in L^2[0, T]$ such that $\int_0^T D_s^W S_T ds \neq 0$, we deduce

$$\begin{aligned}\Delta^W &= \frac{\partial}{\partial S_0} \mathbb{E}(f(S_T)1_{\bar{N}=0}) = \mathbb{E}(1_{\bar{N}=0} f'(S_T) \frac{\partial S_T}{\partial S_0}) = \frac{1}{S_0} \mathbb{E}(1_{\bar{N}=0} f'(S_T) S_T) \\ &= \frac{1}{S_0} \mathbb{E}\left(1_{\bar{N}=0} f(S_T) \delta^W\left(\frac{h(\cdot)S_T}{\int_0^T h(s)D_s^W S_T ds}\right)\right).\end{aligned}$$

Setting $h \equiv 1$ and by using $D_s^W S_T = \sigma S_T$, we result

$$\Delta^W = \frac{1}{S_0 T \sigma} \mathbb{E}\left(1_{\bar{N}=0} f(S_T) W_T\right). \quad (4.2)$$

Define $G_{t,z} = D_{t,z}^{\bar{N}} S_T + S_T - K = S_T e^{D_{t,z}^{\bar{N}} X_T} - K$. According to Lemma 3.5 and Equation (3.7), we know if $S_T > K$, then $S_T e^{D_{t,z}^{\bar{N}} X_T} > K$ on $t \in [v_1, T]$, and $S_T e^{D_{t,z}^{\bar{N}} X_T} < K$ an happen when $t \in [0, v_1]$. Denote $B := \{(t, z) \in [0, T] \times \mathbb{R}_+; S_T e^{D_{t,z}^{\bar{N}} X_T} > K\}$, and set $\mathcal{B}_1 := \int \int_B 1_{(0, \lambda(s)]}(z_0) dz_0 ds$, and $\mathcal{B}_2 := \int \int_{B^c} 1_{(0, \lambda(s)]}(z_0) dz_0 ds$. We know that $\mathcal{B}_1 + \mathcal{B}_2 = \int_0^T \lambda(t) dt$. Now, we can define the function u_1 as follows.

$$u_1(t, z) = \begin{cases} \frac{H_K(S_T)(1 + \frac{K}{S_T})}{2S_0 \mathcal{B}_1 (\exp\{D_{t,z}^{\bar{N}} X_T\} - 1)} & (t, z) \in B \cap ([v_1, T] \times \mathbb{R}_0), \\ -\frac{H_K(S_T)}{2S_0 \mathcal{B}_2} & (t, z) \in B^c \cap ([0, v_1] \times \mathbb{R}_0), \end{cases} \quad (4.3)$$

where $H_y(x) = 1_{x>y}$ is the Heaviside function and 1_A is the indicator function of the set A . Substituting (4.3) in Equation (4.1), it justifies that u_1 is well defined, then Lemma 4.1 can apply and

$$\Delta^{\bar{N}} = \mathbb{E}\left(f(S_T) \delta^{\bar{N}}(v_1^S)\right), \quad \text{for} \quad v_1^S(t, z) := 1_{(0, \lambda(t)]}(z) u_1(t, z), \quad (4.4)$$

if v_1^S belongs to $\text{Dom}(\delta^{\bar{N}})$. This will be shown in the following lemma and will be proved in Appendix C. To sake of notation, denote $u^B(t, z) := 1_{(0, \lambda(t)]}(z) 1_{B \cap ([v_1, T] \times \mathbb{R}_0)} \mathcal{H}_1(t, z)$ and $u^{B^c}(t, z) := 1_{(0, \lambda(t)]}(z) 1_{B^c \cap ([0, v_1] \times \mathbb{R}_0)} \mathcal{H}_2(t, z)$, where

$$\mathcal{H}_1(t, z) := \frac{(1 + \frac{K}{S_T}) H_K(S_T)}{\mathcal{B}_1 (\exp\{D_{t,z}^{\bar{N}} X_T\} - 1)}, \quad \mathcal{H}_2(t, z) := \frac{H_K(S_T)}{\mathcal{B}_2}.$$

Lemma 4.2. Two processes u^B and u^{B^c} belong to $\text{Dom}(\delta^{\bar{N}})$.

Proof. We will prove the claim in Appendix C. □

4.2. Delta Greeks for Simple Asian options. Consider a simple Asian options with the payoff function $F = f(Y_T) = (Y_T - K)^+$, where

$$Y_T = \frac{1}{T} \int_0^T S_t dt.$$

Recall Section 2.2 in [39], put $h(t) = S_t$, and use the relation

$\delta^W(Gu^a) = G\delta^W(u^a) - \int_0^T D_t^W G u_t^a dt$, with $u^a(\cdot) = S$ and $G = \frac{1}{Y_T}$, to derive

$$H := \delta^W\left(\frac{Y_T S}{\int_0^T S_u D_u^W Y_T du}\right) = \delta^W\left(\frac{Y_T S}{\frac{\sigma}{T} \int_0^T S_u \int_u^T S_t dt du}\right) = \delta^W\left(\frac{Y_T S}{\frac{\sigma}{2T} (\int_0^T S_u du)^2}\right)$$



$$= 2\delta^W\left(\frac{S}{T\sigma Y_T}\right) = \frac{2}{T\sigma Y_T}\delta^W(S) + \frac{2}{T\sigma}\int_0^T \frac{D_t^W Y_T}{Y_T^2} S_t dt = \frac{2}{T\sigma Y_T}\delta^W(S) + T.$$

Then, applying (F.1) with $u = 1_{\bar{N}=0} \frac{h(\cdot)Y_T}{\int_0^T h(s)D_s^W Y_T ds}$ and $F = f(Y_T)$, we obtain

$$\begin{aligned}\Delta^W &= \frac{\partial}{\partial S_0} \mathbb{E}(1_{\bar{N}=0} f(Y_T)) = \mathbb{E}(1_{\bar{N}=0} f'(Y_T) \frac{\partial Y_T}{\partial S_0}) = \frac{1}{S_0} \mathbb{E}(1_{\bar{N}=0} f'(Y_T) Y_T) \\ &= \frac{1}{S_0} \mathbb{E}\left(f(Y_T) \delta^W\left(1_{\bar{N}=0} \frac{Y_T h(\cdot)}{\int_0^T h(u) D_u^W Y_T du}\right)\right) = \frac{1}{S_0} \mathbb{E}\left(f(Y_T) 1_{\bar{N}=0} \left[\frac{2}{T\sigma Y_T} \delta^W(S) + T\right]\right).\end{aligned}\quad (4.5)$$

Lemma 4.3. *If there is some function $u(\cdot, \cdot)$ such that the function $v(\cdot, \cdot)$, defined as $v(t, z) = 1_{(0, \lambda(t)]}(z) u(t, z)$, belongs to the domain of $\delta^{\bar{N}}$ and*

$$\mathbb{E}(f'(Y_T) \frac{\partial Y_T}{\partial S_0}) = \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} 1_{(0, \lambda(t)]}(z) u(t, z) \{f(Y_T + D_{t,z}^{\bar{N}} Y_T) - f(Y_T)\} dz dt\right), \quad (4.6)$$

then $\Delta^{\bar{N}} := \frac{\partial}{\partial S_0} \mathbb{E}(f(Y_T)) = \mathbb{E}(f(Y_T) \delta^{\bar{N}}(v))$.

Proof. The proof is similar to Lemma 4.1. □

Denote $C_Y := \{(t, z) \in [0, T] \times \mathbb{R}_+; D_{t,z}^{\bar{N}} Y_T + Y_T - K > 0\}$, and set $\mathcal{C}_1 := \int \int_{C_Y} 1_{(0, \lambda(s)]}(z_0) dz_0 ds$, and $\mathcal{C}_2 := \int \int_{(C_Y)^c} 1_{(0, \lambda(s)]}(z_0) dz_0 ds$. We know that $\mathcal{C}_1 + \mathcal{C}_2 = \int_0^T \lambda(t) dt$. Since

$$f(Y_T + D_{t,z}^{\bar{N}} Y_T) - f(Y_T) = 1_{C_Y}(t, z) \frac{1}{T} \int_t^T S_u (e^{D_{t,z}^{\bar{N}} X_u} - 1) du + 1_{(C_Y)^c}(t, z) (K - Y_T),$$

we can define the function u_2 as follows. For the sake of notation, denote

$$\mathcal{H}_3(t, z) := \frac{T(K + Y_T) H_K(Y_T)}{\mathcal{C}_1 \int_t^T S_u (\exp\{D_{t,z}^{\bar{N}} X_u\} - 1) du}, \quad \mathcal{H}_4(t, z) := \frac{H_K(Y_T)}{\mathcal{C}_2},$$

and define

$$u_2(t, z) = \begin{cases} \frac{1}{2S_0} \mathcal{H}_3(t, z) & (t, z) \in C_Y \cap ([v_1, T] \times \mathbb{R}_0), \\ -\frac{1}{2S_0} \mathcal{H}_4(t, z) & (t, z) \in (C_Y)^c \cap ([0, v_1] \times \mathbb{R}_0). \end{cases} \quad (4.7)$$

Define $v_2(t, z) := 1_{(0, \lambda(t)]}(z) u_2(t, z)$. Clearly, the condition (4.6) of Lemma 4.3 holds for this definition of u_2 , and therefore if $v_2 \in \text{Dom}(\delta^{\bar{N}})$,

$$\Delta^{\bar{N}} = \mathbb{E}(f(Y_T) \delta^{\bar{N}}(v_2)). \quad (4.8)$$

To show that v_2 belongs $\text{Dom}(\delta^{\bar{N}})$, denote $v^C(t, z) := 1_{(0, \lambda(t)]}(z) 1_{C_Y \cap ([v_1, T] \times \mathbb{R}_0)} \mathcal{H}_3(t, z)$ and $v^{C^c}(t, z) := 1_{(0, \lambda(t)]}(z) 1_{(C_Y)^c \cap ([0, v_1] \times \mathbb{R}_0)}$.

Lemma 4.4. *Two processes v^C and v^{C^c} belong to $\text{Dom}(\delta^{\bar{N}})$.*

Proof. We will prove the claim in Appendix D. □

5. THE CONVERGENCE OF THE NUMERICAL SCHEME

There are several numerical schemes to approximate asset dynamics for special issues in finance, such as [45], [44], and [1]. The stability of numerical schemes is widely studied, for example in [18]. For further details on the Monte Carlo implementation of Asian options, the reader is referred to [2].

In the numerical experiments, the convergence of the numerical scheme used to discretize the processes and compute the delta in the numerical examples is essential. So we first justify the convergence of our method to the exact solution.



Using [8], consider equidistant partition of the interval $[0, T]$: $t_i = t_i(n) = \frac{iT}{n}, i = 0, 1, 2, \dots, n$, for any $n \in \mathbb{N}$, and define the discretized process of X , corresponds to the given partition, in the following form

$$\begin{aligned} X_{t_j}^n &= X_0 + \left(\mu - \frac{\sigma^2}{2}\right)t_j + \sigma W_{t_j} + \int_0^{t_j} \int_{\mathbb{R}_+} (1 - e^{J_s^n}) 1_{(0, \lambda^n(s)]}(z) dz ds \\ &\quad + \int_0^{t_j} \int_{\mathbb{R}_+} J_s^n N^n(ds, dz) = X_0 + \left(\mu - \frac{\sigma^2}{2}\right)t_j + \sigma W_{t_j} \\ &\quad + \int_0^{t_j} \int_{\mathbb{R}_+} (1 - e^{J_s^n}) 1_{(0, \lambda^n(s)]}(z) dz ds + \int_0^{t_j} \int_{\mathbb{R}_+} J_s^n 1_{(0, \lambda^n(s)]}(z) \bar{N}(ds, dz), \end{aligned}$$

where for every $s \in [t_i, t_{i+1}]$, we have $J_s^n = J_{t_i}$, and

$$\lambda^n(s) = \lambda^n(t_i) + \int_{t_i}^s \beta(\lambda_0 - \lambda^n(t_i)) du + \alpha \int_{t_i}^s \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) \bar{N}(du, dz). \quad (5.1)$$

Here, $N^n(dt, dz)$ is a Poisson process associated to $\bar{N}$ with stochastic intensity λ^n . In this sense, $N - N^n$ is a Poisson process with stochastic intensity $\lambda - \lambda^n$. Finally, we define $S_{t_j}^n = \exp\{X_{t_j}^n\}$.

In this sense, for every t_{i+1} ,

$$\lambda^n(t_{i+1}) = \lambda^n(t_i) + \beta \Delta t (\lambda_0 - \lambda^n(t_i)) + \alpha \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) \bar{N}(dt, dz).$$

Take the expectation and result

$$\begin{aligned} \mathbb{E}(\lambda^n(t_{i+1})) &= \mathbb{E}(\lambda^n(t_i)) + \beta \Delta t (\lambda_0 - \mathbb{E}(\lambda^n(t_i))) + \alpha \mathbb{E}(\lambda^n(t_i)) \Delta t \\ &= \beta \Delta t \lambda_0 + [1 + (\alpha - \beta) \Delta t] \mathbb{E}(\lambda^n(t_i)) \leq \beta \Delta t \lambda_0 + \mathbb{E}(\lambda^n(t_i)). \end{aligned}$$

Hence, for every $i = 1, \dots, n$,

$$\sup_{1 \leq i \leq n} \mathbb{E}(\lambda^n(t_i)) \leq \sup_{1 \leq i \leq n} \lambda_0 (1 + (i-1) \beta \Delta t) < \lambda_0 (1 + \beta). \quad (5.2)$$

In addition, taking the expectation from (5.1) and then applying (5.2) deduce for every $s \in [t_i, t_{i+1}]$

$$\begin{aligned} \mathbb{E}(|\lambda^n(s) - \lambda^n(t_i)|) &= \alpha \mathbb{E} \left(\int_{t_i}^s \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) dz dt \right) \leq \alpha \Delta t \sup_{1 \leq i \leq n} \mathbb{E}(\lambda^n(t_i)) \\ &\leq \alpha \lambda_0 (1 + \beta) \Delta t, \end{aligned} \quad (5.3)$$

and for every $p \geq 2$, from Burkholder-Davis-Gundy inequality,

$$\begin{aligned} \mathbb{E}(|\lambda^n(s) - \lambda^n(t_i)|^p) &\leq \alpha^p \left[\mathbb{E} \left(\int_{t_i}^s \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) dz dt \right) + \mathbb{E} \left(\int_{t_i}^s \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) dz dt \right)^{\frac{p}{2}} \right] \\ &\leq \alpha^p \Delta t \sup_{1 \leq i \leq n} \mathbb{E}(\lambda^n(t_i)) + \alpha^p (\Delta t)^{\frac{p}{2}-1} \mathbb{E} \left(\int_{t_i}^s \int_{\mathbb{R}_+} 1_{(0, \lambda^n(t_i)]}(z) dz dt \right) \\ &\leq \alpha^p \lambda_0 (1 + \beta) \left(\Delta t + (\Delta t)^{\frac{p}{2}} \right) \leq \alpha^p \lambda_0 (1 + \beta) \Delta t. \end{aligned} \quad (5.4)$$

Lemma 5.1. For every $p \geq 2$, there exists some constant $c_{1,p}$ such that for every $i = 1, \dots, n$

$$\sup_{t_i \leq t \leq t_{i+1}} \mathbb{E} \left(\lambda(t) - \lambda^n(t) \right)^p \leq c_{1,p} n^{-1}.$$

Proof. See [Appendix H](#). □

We prove the convergence of X_t^n and S_t^n in the following lemma.



Lemma 5.2. *Assume that the function J is a Lipschitz continuous function. Then, there exists some constant $c_2 > 0$ such that*

$$\mathbb{E}(X_T - X_T^n)^{2p} \leq c_2 n^{-\frac{p}{2}}, \quad \mathbb{E}(S_T - S_T^n)^{2p} \leq c_2 n^{-\frac{p}{2}}. \quad (5.5)$$

Proof. See [Appendix H](#). □

6. NUMERICAL EXAMPLE

In this section, we provide numerical results for the prices and the delta Greeks of European and Asian call options under the Hawkes dynamics.

The program was implemented on a computer system equipped with an Intel(R) Core i7-9700 K CPU and 64 GB of memory.

Consider the SDE (3.1) with the parameters $\sigma = 0.10$, $\alpha = 0.30$, $\beta = 0.80$, $\gamma = 0.20$, $\mu = 0.05$, $T = 1$, $\lambda_0 = 1$, $S_0 = 5$. Figure 1 illustrates a simulated sample path of the Hawkes process, displaying both the intensity $\lambda(t)$ over time and the corresponding event times (jump arrival times). The figure clearly demonstrates the self-exciting behaviour of the process: each event induces an immediate upward jump in the intensity, which then decays exponentially back to the baseline. This visualisation effectively captures the clustering phenomenon inherent in the Hawkes dynamics, showing how past events increase the likelihood of future events. The European call option and the simple Asian call option

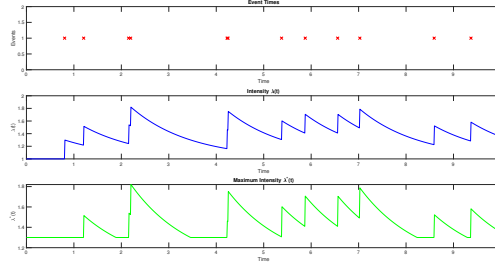


FIGURE 1. Simulated sample path of the Hawkes process over the time horizon $T = 10$.

with the expiration date T and the strike price K are defined as follows.

$$f(S_T) = \max(S_T - K, 0), \quad f(S_T, Y_T) = \max\left(\frac{1}{T} \int_0^T S_t dt - K, 0\right).$$

The averaging mechanism in the definition of $Y_T = \frac{1}{T} \int_0^T S_t dt$ smooths out extreme price movements caused by the Hawkes jump process. Consequently, the terminal payoff distribution of the Asian option has lower variability than that of the corresponding European option. Under the same model parameters and Hawkes dynamics, the Asian option price is strictly lower than that of its European counterpart.

Figure 2 shows the price of European and Asian call options with $h = 0.010$, the number of simulated paths 1000, and $K = S_0 \times u$ which $u = 0.05, 0.10, 0.15, \dots, 1.50, 1.55, 1.60$. As predicted, the European call price is higher than the Asian call price at every strike K , due to the volatility-reducing effect of averaging. The widening gap at lower strike prices K (i.e., in-the-money options) arises from the convexity of the payoff function, which affects European options more strongly. Also, the averaging in Asian options mitigates the effect of large upward jumps. This behavior is consistent with standard results in exotic option pricing and is amplified under the self-exciting Hawkes dynamics due to the heavier tails of the return distribution.

To investigate the impact of the Hawkes parameters on the asset and option prices, we perform a sensitivity analysis with respect to the self-excitation strength α and the decay rate β . Figure 3 presents the behaviour of the terminal asset price S_T as a function of α and β . The asset price increases monotonically with α , reflecting the amplifying effect of stronger jump clustering on the expected terminal price. In contrast, the asset price remains almost constant across



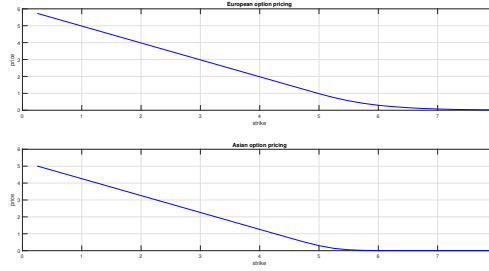
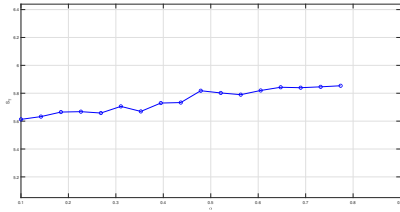
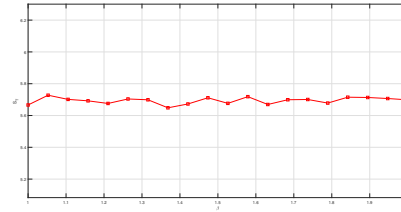


FIGURE 2. European and Asian call option prices versus strike price K . European call prices are consistently higher than Asian call prices for all strike levels. The path averaging in the Asian payoff reduces the overall volatility of the underlying process. The gap between the two curves widens as K decreases (i.e., as the option moves further in the money), reflecting the stronger convexity and path-dependence of the European payoff compared to the averaged Asian payoff.



(A) Asset price S_T vs. α .



(B) Asset price S_T vs. β .

FIGURE 3. Sensitivity of the terminal asset price S_T to the Hawkes parameters with $S_0 = 5$ and $J_t = \frac{\gamma t}{1+t}$, $t \geq 0$.

the entire range of β , indicating a negligible sensitivity to the decay rate. Figures 4 and 5 display the European and Asian option prices, respectively. The European option price exhibits a mild increasing trend with respect to α , while it remains stable with respect to β , consistent with its dependence on the terminal asset price. Similarly, the Asian option price shows a moderate increase with α , followed by a saturation effect for higher values, due to the averaging mechanism inherent in the Asian payoff. Meanwhile, it remains nearly constant across the β range, confirming that the decay rate has a limited impact on path-dependent options. Overall, the sensitivity analysis reveals that the self-excitation strength α plays a more significant role in determining asset and option prices. At the same time, the decay rate β has a secondary effect, particularly for Asian options. To calculate the delta, we present the exact delta expressions for the European and simple Asian call options, respectively, as follows.

$$\Delta_0 = \mathbb{E}[H_K(S_T) \frac{S_T}{S_0}], \quad \text{and} \quad \Delta_0 = \mathbb{E}[H_K(Y_T) \frac{Y_T}{S_0}].$$

In [36], semi-analytical valuation formulas are derived for foreign equity options under a Hawkes jump-diffusion model, using the Fourier transform method. They compute the Greeks and optimal hedging strategies under the mean-variance criterion for options with clustered jumps and cross-market jump propagation. Callegaro et al. [14] study a quadratic hedging problem in an affine Heston model with self-exciting jumps of Hawkes type. They apply semi-static strategies that combine dynamic trading in the underlying asset with static positions in a basket of European options, reducing the hedging error compared with purely dynamic strategies. Gonzato and Sgarra [23] compute two optimal hedging strategies on a Hawkes-type process based on futures trading: one minimizing variance, and a second strategy that also accounts for third-order moment contributions to reflect investor risk attitudes.

Our Malliavin-based Wiener-Poisson (WP) method offers distinct advantages over these approaches. Unlike Fourier

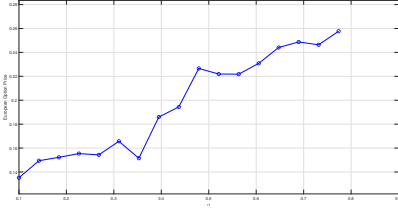
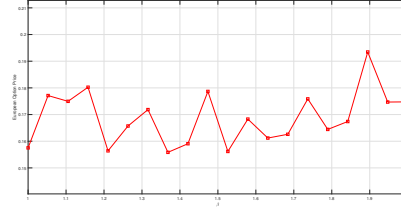
(A) European option price vs. α .(B) European option price vs. β .

FIGURE 4. Sensitivity of the European call option price to the Hawkes parameters with $S_0 = 5$, $K = 6$ and $J_t = \frac{\gamma t}{1+t}$, $t \geq 0$.

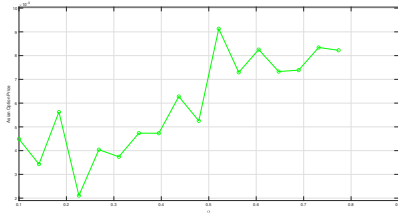
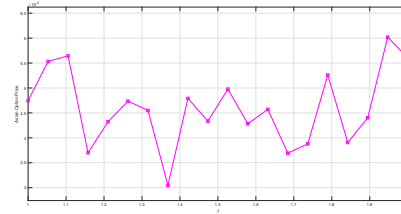
(A) Asian option price vs. α .(B) Asian option price vs. β .

FIGURE 5. Sensitivity of the Asian call option price to the Hawkes parameters with $S_0 = 5$, $K = 6$ and $J_t = \frac{\gamma t}{1+t}$, $t \geq 0$.

transform methods that rely on semi-analytical approximations, our approach provides an exact probabilistic representation of the delta through Malliavin weights. Compared to variance-optimal and semi-static strategies, which require solving complex optimization problems and often rely on affine model structures, our method directly computes the delta with a single simulation per path, avoiding the need for perturbed simulations.

The method WM (Wiener-Malliavin weight) is based on (4.2) and (4.5) for European and Asian options, respectively, and the method PM (Poisson-Malliavin weight) is based on (4.4) and (4.8) for European and Asian options, respectively. The WP (Wiener-Poisson weight) method combines two methods: WM and PM. Whereas the symmetric finite difference approach for the European call option gives

$$\Delta = \frac{\partial}{\partial S_0} \mathbb{E}[\max(S_T - K, 0)] = \frac{\mathbb{E}[f(S_T)|S_0 + h] - \mathbb{E}[f(S_T)|S_0 - h]}{2h},$$

the symmetric finite difference approach for a simple Asian call option gives

$$\Delta = \frac{\partial}{\partial S_0} \mathbb{E}[\max(Y_T - K, 0)] = \frac{\mathbb{E}[f(Y_T)|S_0 + h] - \mathbb{E}[f(Y_T)|S_0 - h]}{2h},$$

Figures 6 and 7 show the behavior of the terms Δ , Δ^W , Δ^N and $\Delta^W/2 + \Delta^N/2$. The jumps are modeled using Hawkes processes. The execution time of the program code in the Malliavin method WP and the finite-difference method in the European call option are 0.6345×10^5 and 1.2530×10^5 , respectively. In the simple Asian call option, the values are 1.0688×10^5 and 1.2621×10^5 . This efficiency gain arises because the WP method requires only one simulation per path; meanwhile, finite differences require two perturbed simulations (with $S_0 + h$ and $S_0 - h$) to compute the delta. The Malliavin estimator converges robustly, avoiding the bias-variance trade-off that inherently constrains the finite difference method, which requires careful tuning of the perturbation step size h . Additionally, a CPU time comparison is included, confirming the computational efficiency of our method. This reduction in computational cost is particularly



significant for large-scale portfolio applications and real-time risk management, where frequent rebalancing is required.

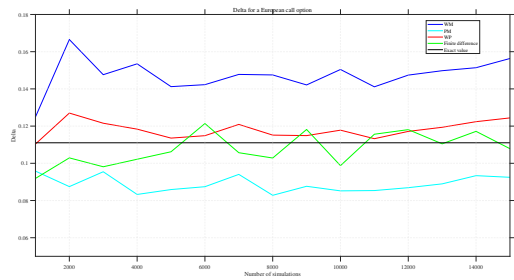


FIGURE 6. The delta of the European call option under Hawkes process dynamics.

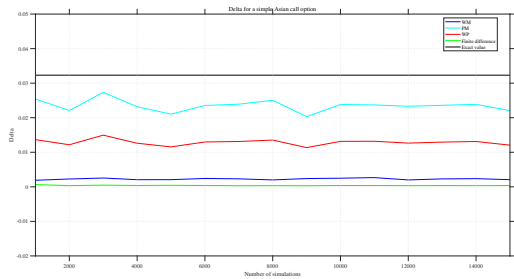


FIGURE 7. The delta of the simple Asian call option under Hawkes process dynamics.

The mean squared errors for the computation of the delta of the European call option and the simple Asian call option are presented in Tables 2 and 3, respectively. The numerical results presented above carry important

TABLE 2. The mean squared errors of the four methods for the European call option.

The Method	MSE of the European call option
WM	0.0014
PM	0.0005
WP	0.0001
<i>Symmetric Finite Difference</i>	0.0001

TABLE 3. The mean squared errors of the four methods for the simple Asian call option.

The Method	MSE of the simple Asian option
WM	0.0009
PM	0.0001
WP	0.0004
<i>Symmetric Finite Difference</i>	0.0010

implications for risk management under Hawkes jump dynamics. The MSE comparisons (Tables 2 and 3) demonstrate



that the Wiener-Poisson (WP) method achieves an MSE of 0.0001 for European options, matching the accuracy of the symmetric finite difference approach, while requiring only half the computational time. For Asian options, the WP method achieves an MSE of 0.0004, significantly lower than the finite-difference MSE of 0.0010, again in half the time. These results indicate that the WP method not only delivers superior accuracy for path-dependent exotic options but also substantially reduces operational costs, enabling more frequent portfolio rebalancing and faster responses to changing market conditions.

Finally, the convergence analysis confirms that the WP method yields stable delta estimates with a modest number of simulated paths, ensuring scalability for large portfolios and reducing estimator variance—a critical feature for avoiding costly hedging errors. Overall, the WP method offers a compelling combination of accuracy, computational efficiency, and valuable parameter sensitivity insights, making it a practical and economically advantageous tool for risk management under Hawkes jump-diffusion models.

7. CONCLUSION

The main purpose of this article is to compute the delta of exotic options, like Asian options, on underlying assets driven by Hawkes jump-diffusion models, using Malliavin calculus. This research is based on the relationship between the integral operators with respect to the Hawkes process N and those with respect to the associated Poisson process $\bar{N}$. This relationship helps us to control the exponential moments of the solutions of Hawkes SDEs and of the stochastic intensity process λ_t . It also leads to a new duality relation between the Skorokhod integral of a process with respect to N and the Malliavin derivative of that with respect to $\bar{N}$. As an application, we establish a new Wiener–Poisson weight to compute the delta for both European and Asian options and compare the results with the exact solution and the finite differences using numerical simulations and the Monte Carlo method. The convergence of the numerical method in delta computation is verified in a separate section.

ACKNOWLEDGMENT

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APPENDIX A. PROOF OF LEMMA 2.1

Proof of Lemma 2.1: First we can deduce from the definition of $\lambda(t)$ and its stability, $\alpha < \beta$ that for every $0 \leq t \leq T$,

$$\mathbb{E}\left(\int_0^t \lambda(s) ds\right) = \frac{\lambda_0}{\alpha - \beta} \left(\alpha e^{(\alpha - \beta)t} - \beta\right) < \lambda_0 \frac{\beta}{\beta - \alpha}. \quad (\text{A.1})$$

For each integer $R \geq 1$, define the stopping time $\tau_R = \inf\{t \geq 0 : \lambda(t) \geq R\} \wedge T$, and let

$$\mathcal{S}_{\tau_R}(t_1) := \mathbb{E}\left(\sup_{0 \leq s \leq t_1} e^{p\beta(s \wedge \tau_R)} |\lambda(s \wedge \tau_R)|^p\right).$$

From (2.3), Burkholder–Davis–Gundy inequality for jump processes, (A.1) and the facts that $ab \leq \frac{1}{2}(a^2 + b^2)$ and $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \in \mathbb{R}$, for every $0 \leq t_1 \leq T$, there exists some constant C_p such that

$$\begin{aligned} \mathcal{S}_{\tau_R}(t_1) &= \mathbb{E}\left(\sup_{0 \leq s \leq t_1} |e^{\beta(s \wedge \tau_R)} \lambda_0 + \alpha \int_0^{s \wedge \tau_R} \int_{\mathbb{R}_+} e^{\beta r} 1_{(0, \lambda(r)]}(z) \bar{N}(dz, dr)|^p\right) \\ &= 2^{p-1} \mathbb{E}\left(e^{p\beta T} \lambda_0^p + \alpha^p \left| \int_0^{t_1} \int_{\mathbb{R}_+} 1_{r \leq \tau_R} e^{\beta r} 1_{(0, \lambda(r)]}(z) dz dr \right|^p\right. \\ &\quad \left. + \alpha^p \sup_{0 \leq s \leq t_1} \left| \int_0^{s \wedge \tau_R} \int_{\mathbb{R}_+} e^{\beta r} 1_{(0, \lambda(r)]}(z) \tilde{N}(dz, dr) \right|^p\right) \\ &\leq 2^{p-1} e^{p\beta T} \lambda_0^p + 2^{p-1} \alpha^p \mathbb{E}\left(\int_0^{t_1} 1_{r \leq \tau_R} e^{\beta r} \lambda(r) dr\right)^p \end{aligned}$$



$$\begin{aligned}
& + C_p \mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}_+} 1_{r \leq \tau_R} e^{2\beta r} 1_{(0, \lambda(r)]}^2(z) dz dr \right)^{\frac{p}{2}} + C_p \mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}_+} 1_{r \leq \tau_R} e^{p\beta r} 1_{(0, \lambda(r)]}^p(z) dz dr \right) \\
& \leq 2^{p-1} e^{p\beta T} \lambda_0^p + (2T)^{p-1} \alpha^p \mathbb{E} \left(\int_0^{t_1} e^{p\beta(r \wedge \tau_R)} |\lambda(r \wedge \tau_R)|^p dr \right) \\
& + C_p T^{\frac{p}{2}-1} \mathbb{E} \left(\int_0^{t_1} e^{p\beta(r \wedge \tau_R)} \lambda^{\frac{p}{2}}(r \wedge \tau_R) dr \right) + C_p e^{p\beta T} \mathbb{E} \left(\int_0^{t_1} \lambda(r) dr \right) \\
& \leq 2^{p-1} e^{p\beta T} \lambda_0^p + (2T)^{p-1} \alpha^p \mathbb{E} \left(\int_0^{t_1} \sup_{0 \leq s \leq r} \{e^{p\beta(s \wedge \tau_R)} \lambda^p(s \wedge \tau_R)\} dr \right) + C_p e^{p\beta T} \frac{\lambda_0 \beta}{\beta - \alpha} \\
& + \frac{1}{2} C_p T^{\frac{p}{2}-1} \mathbb{E} \left(\int_0^{t_1} \sup_{0 \leq s \leq r} \{e^{p\beta(s \wedge \tau_R)} (1 + \lambda^p(s \wedge \tau_R))\} dr \right).
\end{aligned}$$

The Grönwall inequality deduces that $\mathcal{S}_{\tau_R}(t_1) \leq A$, where A is a finite constant depending on p , T , and the model parameters. Taking the limit $R \rightarrow \infty$ and using the monotone convergence theorem completes the proof of the first part.

To establish the exponential integrability of $\lambda(t)$, we apply Itô's formula for jump processes to the function $f(x) = e^{px}$ with $p \geq 1$.

$$e^{p\lambda(t)} = e^{p\lambda(0)} + \int_0^t e^{p\lambda(s)} p\beta(\lambda_0 - \lambda(s)) ds + \int_0^t \int_{\mathbb{R}_+} e^{p\lambda(s)} (p\alpha + e^\alpha - 1) 1_{(0, \lambda(s)]}(z) \bar{N}(dz, ds). \quad (\text{A.2})$$

Taking expectation for every $p \geq \frac{e^\alpha - 1}{\beta - \alpha}$ and applying (2.4), we result for every $0 \leq t \leq T$

$$\begin{aligned}
\mathbb{E}(e^{p\lambda(t \wedge \tau_R)}) & = e^{p\lambda(0)} + p\beta\lambda_0 \mathbb{E} \left(\int_0^{t \wedge \tau_R} e^{p\lambda(s)} ds \right) + (p\alpha - p\beta + e^\alpha - 1) \mathbb{E} \left(\int_0^{t \wedge \tau_R} e^{p\lambda(s)} \lambda(s) ds \right) \\
& \leq e^{p\lambda(0)} + p\beta\lambda_0 \int_0^t \mathbb{E}(e^{p\lambda(s \wedge \tau_R)}) ds.
\end{aligned}$$

The Grönwall inequality results $\mathbb{E}(e^{p\lambda(t \wedge \tau_R)}) < e^{p\lambda(0)} e^{p\beta\lambda_0 t}$. So when $R \rightarrow \infty$, it shows that for every $p \geq 1$,

$$\sup_{0 \leq t \leq T} \mathbb{E}(e^{p\lambda(t)}) < \infty. \quad (\text{A.3})$$

In addition, taking expectation from the supremum of (A.2), and then applying (2.4), and Burkholder-Davis-Gundy inequality we deduce for every $0 \leq t_1 \leq T$

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq s \leq t_1} e^{p\lambda(s \wedge \tau_R)} \right) & \leq e^{p\lambda_0} + p\beta\lambda_0 \int_0^{t_1} \mathbb{E} \left(\sup_{0 \leq r \leq s} e^{p\lambda(r \wedge \tau_R)} \right) ds \\
& + (p\alpha + e^\alpha - 1)^2 \mathbb{E} \left(\left| \int_0^{t \wedge \tau_R} e^{2p\lambda(s)} \lambda(s) ds \right|^{\frac{1}{2}} \right) + (p\alpha + e^\alpha - 1) \mathbb{E} \left(\int_0^{t_1} e^{p\lambda(s \wedge \tau_R)} \lambda(s \wedge \tau_R) ds \right) \\
& \leq e^{p\lambda(0)} + p\beta\lambda_0 \int_0^{t_1} \mathbb{E} \left(\sup_{0 \leq r \leq s} e^{p\lambda(r \wedge \tau_R)} \right) ds \\
& + \frac{1}{2} (p\alpha + e^\alpha - 1)^2 \int_0^{t_1} \mathbb{E} \left(e^{2p\lambda(s \wedge \tau_R)} \lambda(s \wedge \tau_R) \right) ds + \frac{1}{2} (p\alpha + e^\alpha - 1)^2 \\
& + (p\alpha + e^\alpha - 1) \int_0^{t_1} \mathbb{E} \left(\frac{1}{2} e^{2p\lambda(s \wedge \tau_R)} + \frac{1}{2} \lambda^2(s \wedge \tau_R) \right) ds.
\end{aligned}$$

where we used since $\mathbb{E} \left(e^{2p\lambda(s \wedge \tau_R)} \lambda(s \wedge \tau_R) \right) \leq \mathbb{E} \left(\frac{1}{2} e^{4p\lambda(s \wedge \tau_R)} + \frac{1}{2} \lambda^2(s \wedge \tau_R) \right)$, in the last inequality. From (A.3) and the moment bounds from Lemma 2.1, all terms on the right-hand side are finite. Finally, applying the Grönwall inequality and then taking $R \rightarrow \infty$, the proof is completed.



APPENDIX B. PROOF OF LEMMAS 3.1, 3.2, AND 3.3

Proof of Lemma 3.1: By adding and subtracting the term $\frac{1}{2} \int_0^t (e^{2pJ_s} - 1) \lambda(s) ds$ to the exponent of the expression formula of Y_t and then applying the Cauchy-Schwarz inequality, to reach the martingale process $\exp \left\{ - \int_0^t (e^{2pJ_s} - 1) \lambda(s) ds + \int_0^t \int_{\mathbb{R}_+} 2pJ_s 1_{(0, \lambda(s)]}(z) \overline{N}(ds, dz) \right\}$. This result

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E}(Y_t^p) &= \sup_{0 \leq t \leq T} \mathbb{E} \left(\exp \left\{ - \int_0^t p(e^{J_s} - 1) \lambda(s) ds + \int_0^t \int_{\mathbb{R}_+} pJ_s 1_{(0, \lambda(s)]}(z) \overline{N}(ds, dz) \right\} \right) \\ &\leq \sup_{0 \leq t \leq T} \mathbb{E} \left(\exp \left\{ - \int_0^t (e^{2pJ_s} - 1) \lambda(s) ds + \int_0^t \int_{\mathbb{R}_+} 2pJ_s 1_{(0, \lambda(s)]}(z) \overline{N}(ds, dz) \right\} \right)^{\frac{1}{2}} \\ &\quad \sup_{0 \leq t \leq T} \mathbb{E} \left(\exp \left\{ \int_0^t (e^{2pJ_s} - 1) - 2p(e^{J_s} - 1) \lambda(s) ds \right\} \right)^{\frac{1}{2}} \\ &\leq 1 \times \sup_{0 \leq t \leq T} \mathbb{E} \left(\exp \left\{ \int_0^t e^{2pJ_s} \lambda(s) ds \right\} \right)^{\frac{1}{2}} \leq \sup_{0 \leq t \leq T} \mathbb{E} \left(\exp \left\{ \sup_{0 \leq s \leq T} \lambda(s) \int_0^t e^{2pJ_r} dr \right\} \right)^{\frac{1}{2}} < \infty, \end{aligned}$$

where we used from Lemma 2.1 and **Condition H1** in the last inequality.

To prove the second part, the definition of X_t and its relation to Y_t show that it is sufficient to prove the claim just for Y_t . To estimate the supremum of the jump martingale, we apply the Burkholder–Davis–Gundy inequality ([47], Lemma 2.3) to Y_t . This inequality bounds the p -th moment of the supremum by the sum of the L^p -norm of the jump measure and the $p/2$ -norm of its quadratic variation. So, there exists some constant C_p such that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s|^p \right) &\leq Y_0^p + C_p \mathbb{E} \left(\int_0^t \int_{\mathbb{R}_+} Y_s^2 (e^{J_s} - 1)^2 1_{(0, \lambda(s)]}^2(z) dz ds \right)^{\frac{p}{2}} \\ &\quad + C_p \mathbb{E} \left(\int_0^t \int_{\mathbb{R}_+} Y_s^p (e^{J_s} - 1)^p 1_{(0, \lambda(s)]}^p(z) dz ds \right) \\ &\leq Y_0^p + C_p T^p \int_0^t \mathbb{E} \left(Y_s^p (e^{J_s} - 1)^p \lambda^{\frac{p}{2}}(s) \right) ds + C_p \int_0^t \mathbb{E} \left(Y_s^p (e^{J_s} - 1)^p \lambda(s) \right) ds \\ &=: Y_0^p + C_p \int_0^t \left(T^p I_1(s) + I_2(s) \right) ds. \end{aligned} \tag{B.1}$$

Applying Jensen's inequality, there exists some constant c that

$$\begin{aligned} \int_0^t (T^p I_1(s) + I_2(s)) ds &\leq 2T^{p+1} \sup_{0 \leq s \leq t} \mathbb{E} \left(Y_s^{2p} \right) + 4T^p \mathbb{E} \left(\int_0^t (e^{J_s} - 1)^{4p} ds \right) \\ &\quad + 4T^{p+1} \sup_{0 \leq s \leq t} \mathbb{E} \left(\lambda^{2p}(s) \right) + 2T \sup_{0 \leq s \leq t} \mathbb{E} \left(Y_s^{2p} \right) + 4 \mathbb{E} \left(\int_0^t (e^{J_s} - 1)^{4p} ds \right) + 4T \sup_{0 \leq s \leq t} \mathbb{E} \left(\lambda^4(s) \right). \end{aligned} \tag{B.2}$$

Substitute (B.2) into (B.1), and use the first part of Lemma 2.1 and **Condition H1** to complete the proof.

Proof of Lemma 3.2: We note that assumptions (5.95), (5.96), and (5.98) of Lemma 5.19 in [38] for the coefficients of the SDE (3.2) are satisfied. We know

$$S_t^{-p} = S_0^{-p} \exp \left\{ -p \left(\mu - \frac{1}{2} \sigma^2 \right) t - p \sigma W_t + p \int_0^t (e^{J_s} - 1) \lambda(s) ds - p \int_0^t \int_{\mathbb{R}_0} J_s 1_{(0, \lambda(t)]}(z) \overline{N}(ds, dz) \right\}.$$

Similar to the proof of Lemma 3.1, by adding and subtracting the term $\frac{1}{2} \int_0^t (e^{-2pJ_s} - 1) \lambda(s) ds$ to the exponent and then applying the Cauchy-Schwarz inequality, we derive the martingale process $\exp \left\{ - \int_0^t (e^{-2pJ_s} - 1) \lambda(s) ds - \right.$



$\int_0^t \int_{\mathbb{R}_+} 2pJ_s 1_{(0, \lambda(s)]}(z) \overline{N}(ds, dz) \Big\}$. This fact, **Condition H1**, and Lemma 2.1 conclude

$$\begin{aligned} \mathbb{E}(S_t^{-p}) &\leq S_0^{-p} \mathbb{E}^{\frac{1}{2}} \left(\exp \left\{ -2p(\mu - \tfrac{1}{2}\sigma^2)t - 2p\sigma W_t + \int_0^t [2p(e^{J_s} - 1) + e^{-2pJ_s} - 1]\lambda(s)ds \right\} \right) \\ &\quad \times \mathbb{E}^{\frac{1}{2}} \left(\exp \left\{ - \int_0^t (e^{-2pJ_s} - 1)\lambda(s)ds - \int_0^t \int_{\mathbb{R}_+} 2pJ_s 1_{(0, \lambda(s)]}(z) \overline{N}(ds, dz) \right\} \right) \\ &\leq S_0^{-p} \mathbb{E}^{\frac{1}{2}} \left(\exp \left\{ -2p(\mu - \tfrac{1}{2}\sigma^2)t - 2p\sigma W_t + \int_0^t [2p(e^{J_s} - 1)]\lambda(s)ds \right\} \right) \times 1 \\ &\leq \frac{1}{2} S_0^{-p} \left[\mathbb{E} \left(\exp \left\{ -4p((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t) \right\} \right) + \mathbb{E} \left(\exp \left\{ 4p \sup_{0 \leq s \leq t} \lambda(s) \int_0^T (e^{J_s} - 1)ds \right\} \right) \right]^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

where we used the fact $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, for every $x, y \in \mathbb{R}$.

Proof of Lemma 3.3: According to Equation (3.6), and apply the Itô formula for the function $f(z) = e^{pz}$, $p \geq 1$,

$$d \left(e^{pD_{u,z}^{\overline{N}} \lambda(t)} \right) = -p\beta D_{u,z}^{\overline{N}} \lambda(t) e^{pD_{u,z}^{\overline{N}} \lambda(t)} dt + e^{pD_{u,z}^{\overline{N}} \lambda(t)} \left(e^{p\alpha \operatorname{sign}(D_{u,z}^{\overline{N}} \lambda(t))} - 1 \right) N_{(u,z)}(dt).$$

Therefore, the integral form is as follows.

$$\begin{aligned} e^{pD_{u,z}^{\overline{N}} \lambda(t)} &= e^{p\alpha 1_{(0, \lambda(u)]}(z)} - p\beta \int_u^t D_{u,z}^{\overline{N}} \lambda_s e^{pD_{u,z}^{\overline{N}} \lambda(s)} ds \\ &\quad + \int_u^t e^{pD_{u,z}^{\overline{N}} \lambda(s)} \left(e^{p\alpha \operatorname{sign}(D_{u,z}^{\overline{N}} \lambda(s))} - 1 \right) N_{(u,z)}(ds), \end{aligned} \quad (\text{B.3})$$

Using the fact that $1_{(0, \lambda(u)]}(z) \leq |\lambda(u)|/z$ for $z > 0$, to estimate $\mathbb{E}(e^{p\alpha 1_{(0, \lambda(u)]}(z)}) \leq 1 + (e^{p\alpha} - 1) \frac{\mathbb{E}(|\lambda(u)|^2)}{z^2}$, and result

$$\begin{aligned} \mathbb{E} \left(e^{pD_{u,z}^{\overline{N}} \lambda(t)} \right) &= \mathbb{E} \left(e^{p\alpha 1_{(0, \lambda(u)]}(z)} \right) - p\beta \int_u^t \mathbb{E} \left(D_{u,z}^{\overline{N}} \lambda(s) e^{pD_{u,z}^{\overline{N}} \lambda(s)} \right) ds \\ &\quad + \int_u^t \mathbb{E} \left(D_{u,z}^{\overline{N}} \lambda(s) e^{pD_{u,z}^{\overline{N}} \lambda(s)} \left(e^{p\alpha \operatorname{sign}(D_{u,z}^{\overline{N}} \lambda(s))} - 1 \right) \right) ds \\ &\leq \mathbb{E} \left(e^{p\alpha 1_{(0, \lambda(u)]}(z)} \right) + (e^{p\alpha} - 1 - p\beta) \int_u^t \mathbb{E} \left(D_{u,z}^{\overline{N}} \lambda_s e^{pD_{u,z}^{\overline{N}} \lambda(s)} \right) ds \\ &\leq 1 + (e^{p\alpha} - 1) \frac{\mathbb{E}(|\lambda(u)|^2)}{z^2} + (e^{p\alpha} - 1 - p\beta) \int_u^t \mathbb{E} \left(D_{u,z}^{\overline{N}} \lambda_s e^{pD_{u,z}^{\overline{N}} \lambda(s)} \right) ds. \end{aligned}$$

Consequently, since $e^{4\alpha} < 1 + 4\beta$, we result

$$\sup_{0 \leq t \leq T} \mathbb{E}(e^{4D_{u,z}^{\overline{N}} \lambda(t)}) < \infty. \quad (\text{B.4})$$

In addition, taking the expectation of the supremum of (B.3), and using $\operatorname{sign}(D_{u,z}^{\overline{N}} \lambda(s)) > 0$, and the Burkholder–Davis–Gundy inequality, we derive

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq u \leq t \leq T} e^{pD_{u,z}^{\overline{N}} \lambda(t)} \right) &\leq \max\{e^{p\alpha}, 1\} + (e^{p\alpha} - 1) \mathbb{E} \left(\left| \int_u^T e^{2D_{u,z}^{\overline{N}} \lambda(s)} D_{u,z}^{\overline{N}} \lambda(s) ds \right|^{\frac{1}{2}} \right) \\ &\quad + (e^{p\alpha} - 1) \int_u^T \mathbb{E}(e^{pD_{u,z}^{\overline{N}} \lambda(s)} D_{u,z}^{\overline{N}} \lambda(s)) ds \\ &\leq \max\{e^{p\alpha}, 1\} + \frac{1}{2} (e^{p\alpha} - 1) \int_u^T \mathbb{E} \left(e^{2D_{u,z}^{\overline{N}} \lambda(s)} D_{u,z}^{\overline{N}} \lambda_s \right) ds + \frac{1}{2} (e^{p\alpha} - 1) \end{aligned}$$



$$+ (e^{p\alpha} - 1) \int_u^T \mathbb{E} \left(\frac{1}{2} e^{2pD_{u,z}^{\bar{N}} \lambda(s)} + \frac{1}{2} D_{u,z}^{\bar{N}} \lambda_s^2(s) \right) ds,$$

where we use the fact $e^{2pD_{u,z}^{\bar{N}} \lambda(s)} D_{u,z}^{\bar{N}} \lambda(s) \leq \frac{1}{2} e^{4pD_{u,z}^{\bar{N}} \lambda(s)} + \frac{1}{2} (D_{u,z}^{\bar{N}} \lambda(s))^2$, in the last inequality. Applying Grönwall's inequality once more completes the proof.

APPENDIX C. PROOF OF LEMMA 4.2

According to Theorem 13.15 in [41], it is sufficient to show that $u^B = 1_{(0,\lambda(t)]}(z) \mathcal{H}_1(t, z)$ and $u^{B^c} = 1_{(0,\lambda(t)]}(z) \mathcal{H}_2(t, z)$ belong to $\mathbb{D}_N^{1,2}$. We require three steps to do this. In Step one, we show that u^B and u^{B^c} belong $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$. In the sequence, we see that the variable $\mathcal{H}_1(\cdot, \cdot)$ on $\mathcal{A}_1 := \{(t, z) \in [0, T] \times \mathbb{R}_0; 1_{(0,\lambda(t)]}(z) 1_{B \cap ([v_1, T] \times \mathbb{R}_0)}(t, z) = 1\}$, and the variable $\mathcal{H}_2(\cdot, \cdot)$ on

$\mathcal{A}_2 := \{(t, z) \in [0, T] \times \mathbb{R}_0; 1_{(0,\lambda(t)]}(z) 1_{B^c \cap ([0, v_1] \times \mathbb{R}_0)}(t, z) = 1\}$ are Malliavin differentiable whose derivatives belong $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$ in the next two steps. To this end, we apply the following lemma from [40].

Lemma C.1. [40] *Given $F, G \in \mathbb{D}_N^{1,2}$ that $FG \in L^2(\Omega)$ and $(F + D^{\bar{N}} F)(G + D^{\bar{N}} G) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$, the product FG also belongs to $\mathbb{D}_N^{1,2}$ and*

$$D_{t,z}^{\bar{N}}(FG) = F D_{t,z}^{\bar{N}} G + G D_{t,z}^{\bar{N}} F + D_{t,z}^{\bar{N}} F D_{t,z}^{\bar{N}} G.$$

As a consequence, we set $F = \frac{F_1}{G_1}$ and $G = G_1$ for every $F_1, G_1 \in \mathbb{D}_N^{1,2}$, such that $\frac{F_1}{G_1} \in \mathbb{D}_N^{1,2}$ and the right hand side of the following equality belongs $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$, we deduce

$$D^{\bar{N}} \left(\frac{F_1}{G_1} \right) = \frac{G_1 D^{\bar{N}} F_1 - F_1 D^{\bar{N}} G_1}{G_1 (G_1 + D^{\bar{N}} G_1)}. \quad (\text{C.1})$$

Step one: We first note that $1 + \frac{K}{S_T} < 2$, if $S_T > K$, and the following inequality holds $\frac{1}{B_1^4} \leq \int_B \frac{1}{\lambda(s)^4} ds$. Therefore

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(|1_{(0,\lambda(t)]}(z) \mathcal{H}_1(t, z)|^2) dz dt \leq 2 \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{B_1^2 (e^{D_{t,z}^{\bar{N}} X_T} - 1)^2}) dz dt \\ & \leq \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{(D_{t,z}^{\bar{N}} X_T)^4}) dz dt + \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{B_1^4}) dz dt \\ & \leq \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{(D_{t,z}^{\bar{N}} X_T)^4}) dz dt + \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \int_B \frac{1}{\lambda(s)^4} ds) dz dt \\ & \leq \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{(D_{t,z}^{\bar{N}} X_T)^4}) dz dt + \int_0^T \int_{\mathbb{R}_+} \frac{1}{\lambda_0^4} \mathbb{E}(1_{(0,\lambda(t)]}(z)) dz dt < \infty, \end{aligned}$$

where we used Lemma 3.7 and (A.1) in the last inequality. In a similar way,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(|1_{(0,\lambda(t)]}(z) \mathcal{H}_2(t, z)|^2) dz dt \leq \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \frac{1}{B_2^2}) dz dt \\ & \leq \int_0^T \int_{\mathbb{R}_+} \mathbb{E}(1_{(0,\lambda(t)]}(z) \int_B \frac{1}{\lambda(s)^2} ds) dz dt \leq \int_0^T \int_{\mathbb{R}_+} \frac{1}{\lambda_0^2} \mathbb{E}(1_{(0,\lambda(t)]}(z)) dz dt < \infty. \end{aligned}$$

Then, u^B and u^{B^c} belong $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

Step two: We show that $1_{\mathcal{A}_1}(t, z) D^{\bar{N}}(\mathcal{H}_1(t, z)) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$. We are moving forward step by step to do this. The Malliavin derivative of the process $e^{D_{t,z}^{\bar{N}} X_T}$ is derived in Appendix E, explicitly for the rest of the reader;



nevertheless, this explicit representation is not directly needed in this proof.

(a): For every $0 \leq r \leq t$ and $z_0 \leq \lambda(r)$, from (C.1)

$$\begin{aligned} \left| D_{r,z_0}^{\bar{N}} \left[\frac{1}{e^{D_{t,z}^{\bar{N}} X_T} - 1} \right] \right| &= \left| \frac{e^{D_{t,z}^{\bar{N}} X_T} \left[e^{D_{r,z_0}^{\bar{N}} D_{t,z}^{\bar{N}} X_T} - 1 \right]}{\left(e^{D_{t,z}^{\bar{N}} X_T} - 1 \right) \left[e^{D_{r,z_0}^{\bar{N}} D_{t,z}^{\bar{N}} X_T + D_{t,z}^{\bar{N}} X_T} - 1 \right]} \right| \\ &\leq \left| \frac{e^{D_{t,z}^{\bar{N}} X_T}}{e^{D_{t,z}^{\bar{N}} X_T} - 1} \right| \leq 1 + \left| \frac{1}{D_{t,z}^{\bar{N}} X_T} \right|, \end{aligned}$$

where we used the facts $D_{t,z}^{\bar{N}} X_T > 0$ in the last inequality. Then using Lemmas 2.1 and 3.7

$$\begin{aligned} &\mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| D_{r,z_0}^{\bar{N}} \left[\frac{1}{e^{D_{t,z}^{\bar{N}} X_T} - 1} \right] \right|^2 dz_0 dr \right) \\ &\leq \mathbb{E} \left(\int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} dz_0 dr \right) + \mathbb{E} \left(\left(\frac{1_{(0,\lambda(t)]}(z)}{D_{t,z}^{\bar{N}} X_T} \right)^2 \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} dz_0 dr \right) \\ &\leq T \sup_{0 \leq r \leq T} \mathbb{E}(\lambda(r)) + \mathbb{E}^{\frac{1}{2}} \left(\left(\frac{1_{(0,\lambda(t)]}(z)}{D_{t,z}^{\bar{N}} X_T} \right)^4 \right) \mathbb{E}^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} dz_0 dr \right) < \infty. \end{aligned} \quad (\text{C.2})$$

In addition, from Lemma C.1 it follows that

$$D_{r,z_0}^{\bar{N}} \left(\frac{K}{S_T} \right) = \frac{-K D_{r,z_0}^{\bar{N}} S_T}{S_T(S_T + D_{r,z_0}^{\bar{N}} S_T)} = \frac{K S_T (e^{D_{r,z_0}^{\bar{N}} X_T} - 1)}{S_T^2 e^{D_{r,z_0}^{\bar{N}} X_T}} = \frac{K (e^{D_{r,z_0}^{\bar{N}} X_T} - 1)}{S_T e^{D_{r,z_0}^{\bar{N}} X_T}}.$$

Therefore, given $S_T > K$ and Lemma 3.2, we have that $\left| \frac{K}{S_T} \right| < 1$ and also $\frac{e^{D_{r,z_0}^{\bar{N}} X_T} - 1}{e^{D_{r,z_0}^{\bar{N}} X_T}} < 1$, so it can be concluded that

$$\left| D_{r,z_0}^{\bar{N}} \left(\frac{K}{S_T} \right) \right|^2 < 1. \quad (\text{C.3})$$

On the other hand, since $D_{r,z_0}^{\bar{N}} \left(\frac{1}{\mathcal{B}_1} \right) = \frac{-D_{r,z_0}^{\bar{N}} \mathcal{B}_1}{\mathcal{B}_1(\mathcal{B}_1 + D_{r,z_0}^{\bar{N}} \mathcal{B}_1)}$, therefore

$$\mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| D_{r,z_0}^{\bar{N}} \left(\frac{1}{\mathcal{B}_1} \right) \right|^2 dz_0 dr \right) \leq \mathbb{E} \left(\frac{\int_0^T \lambda(r) dr}{\mathcal{B}_1^4} \right) \leq \mathbb{E} \left(\frac{1}{\int_0^T \lambda_s ds} \right)^2 < \infty. \quad (\text{C.4})$$

(b): We now prove that $1_{\mathcal{A}_1}(t, z) D_{r,z_0}^{\bar{N}} \left(\left(1 + \frac{K}{S_T}\right) H_K(S_T) \right)$ and $1_{\mathcal{A}_1}(t, z) D_{r,z_0}^{\bar{N}} \left[\frac{1}{\mathcal{B}_1 (e^{D_{t,z}^{\bar{N}} X_T} - 1)} \right]$ belong to $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

From Lemma C.1, the fact $|D_{r,z_0}^{\bar{N}}(H_K(S_T))| = |1_{S_T + D_{r,z_0}^{\bar{N}} S_T > K} - 1_{S_T > K}| \leq 2$, and (C.3), we obtain

$$\begin{aligned} &\mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\bar{N}} \left(\left(1 + \frac{K}{S_T}\right) H_K(S_T) \right)|^2 dz_0 dr \right) \\ &\leq \mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |H_K(S_T) D_{r,z_0}^{\bar{N}} \left(1 + \frac{K}{S_T}\right)|^2 dz_0 dr \right) \\ &+ \mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| \left(1 + \frac{K}{S_T}\right) D_{r,z_0}^{\bar{N}}(H_K(S_T)) \right|^2 dz_0 dr \right) \\ &+ \mathbb{E} \left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\bar{N}}(H_K(S_T)) D_{r,z_0}^{\bar{N}} \left(1 + \frac{K}{S_T}\right)|^2 dz_0 dr \right) \end{aligned}$$



$$< \mathbb{E}\left(1_{\mathcal{A}_1}(t, z) \int_0^T \lambda(r) dr\right) (1 + 4 + 1) < \infty,$$

and from (C.2) and (C.4)

$$\begin{aligned} & \mathbb{E}\left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}} \left[\frac{1}{\mathcal{B}_1(e^{D_{t,z}^{\overline{N}} X_T} - 1)} \right]|^2 dz_0 dr\right) \\ & \leq 2\mathbb{E}\left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}} \left(\frac{1}{\mathcal{B}_1} \right) \frac{1}{e^{D_{t,z}^{\overline{N}} X_T} - 1}|^2 dz_0 dr\right) \\ & + 2\mathbb{E}\left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| \frac{1}{\mathcal{B}_1} D_{r,z_0}^{\overline{N}} \left(\frac{1}{e^{D_{t,z}^{\overline{N}} X_T} - 1} \right) \right|^2 dz_0 dr\right) \\ & + 2\mathbb{E}\left(1_{\mathcal{A}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}} \left(\frac{1}{\mathcal{B}_1} \right) D_{r,z_0}^{\overline{N}} \left(\frac{1}{e^{D_{t,z}^{\overline{N}} X_T} - 1} \right)|^2 dz_0 dr\right) < \infty. \end{aligned}$$

(c): From Lemma C.1 and the parts (a) and (b) we conclude $D_{r,z_0}^{\overline{N}} 1_{\mathcal{A}_1}(t, z) \mathcal{H}_1(t, z) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

Step three: In a similar way, one easily show that $D_{r,z_0}^{\overline{N}} 1_{\mathcal{A}_1}(t, z) \mathcal{H}_2(t, z) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

APPENDIX D. PROOF OF LEMMA 4.4

Similarly to the proof of Lemma 4.2 in Appendix C, we again invoke Theorem 13.15 of [41] to verify that the processes v^C and v^{C^c} belong to $\text{Dom}(\delta^{\overline{N}})$. The L^2 -integrability conditions required by this theorem are satisfied thanks to the same three estimates: Lemma 3.7, Lemma 3.3, and Lemma 2.1 with **Condition H1**. We now proceed with the explicit verification of the differentiability and square-integrability of the weights.

In this part, we will show that the random variable $\mathcal{H}_3(.,.)$ on the set

$\mathcal{J}_1 := \{(t, z) \in [0, T] \times \mathbb{R}_0; 1_{(0, \lambda(t)]}(z) 1_{C_Y \cap ([v_1, T] \times \mathbb{R}_0)}(t, z) = 1\}$, and the random variable $\mathcal{H}_4(.,.)$ on the set $\mathcal{J}_2 := \{(t, z) \in [0, T] \times \mathbb{R}_0; 1_{(0, \lambda(t)]}(z) 1_{C_Y^c \cap ([0, v_1] \times \mathbb{R}_0)}(t, z) = 1\}$ are differentiable whose derivatives belong $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$ in two steps.

Step one: First we show that $1_{(0, \lambda(t)]}(z) D^{\overline{N}}(\mathcal{H}_3(t, z)) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$ in the following three steps.

(a) Using Jensen inequality for the function $x \rightarrow \frac{1}{x^2}$,

$$\begin{aligned} \mathbb{E}\left(\frac{1_{(0, \lambda(t)]}(z)}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du}\right)^2 & \leq \mathbb{E}\left(\int_t^T \frac{1_{(0, \lambda(t)]}(z)}{S_u^2(e^{D_{t,z}^{\overline{N}} X_u} - 1)^2} du\right) \\ & \leq T^2 \mathbb{E}^{\frac{1}{2}}\left(\frac{1}{S_u^4}\right) \left(\mathbb{E}\left(\frac{1_{(0, \lambda(t)]}(z)}{e^{D_{t,z}^{\overline{N}} X_u} - 1}\right)^4\right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{D.1}$$

For every $0 \leq r \leq t$ and $z_0 \leq \lambda(r)$, from (C.1)

$$\begin{aligned} & \left| D_{r,z_0}^{\overline{N}} \left(\frac{1}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du} \right) \right| \\ & = \left| \frac{D_{r,z_0}^{\overline{N}} \int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du \left(\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du + D_{r,z_0}^{\overline{N}} \int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du \right)} \right| \\ & = \left| \frac{\int_r^T D_{r,z_0}^{\overline{N}} (S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du)}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du \left(\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du + \int_r^T (D_{r,z_0}^{\overline{N}} S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du) \right)} \right| \\ & \leq \left| \frac{\int_r^T D_{r,z_0}^{\overline{N}} (S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du)}{\left(\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du \right)^2 + \int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du \int_r^T (D_{r,z_0}^{\overline{N}} S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du)} \right| \end{aligned}$$



$$\leq \left| \frac{1}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du} \right|.$$

Using Equation (D.1), we have

$$\begin{aligned} & \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| D_{r,z_0}^{\overline{N}} \left[\frac{1}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du} \right] \right|^2 dz_0 dr \right) \\ & \leq \mathbb{E} \left(\int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left(\frac{1_{(0,\lambda(t)]}(z)}{\int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du} \right)^2 dz_0 dr \right) < \infty. \end{aligned}$$

Since $D_{r,z_0}^{\overline{N}}(\frac{1}{c_1}) = \frac{-D_{r,z_0}^{\overline{N}} c_1}{c_1(c_1 + D_{r,z_0}^{\overline{N}} c_1)}$, therefore

$$\begin{aligned} \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| D_{r,z_0}^{\overline{N}} \left(\frac{1}{c_1} \right) \right|_0^2 dz_0 dr \right) & \leq \mathbb{E} \left(\frac{1}{c_1^4} \left(\int_0^T \lambda(r) dr \right)^2 \right) \\ & \leq \mathbb{E} \left(\frac{1}{\int_0^T \lambda(s) ds} \right)^2 < \infty. \end{aligned} \quad (\text{D.2})$$

Hence, according to Hölder inequality, Lemmas 3.1 and 3.3, connection with the proof of Theorem 3.1 in [49] (on page 2116), we conclude

$$\begin{aligned} & \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}}(K + Y_T)|^2 dz_0 dr \right) \\ & = \frac{1}{T} \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| \int_r^T S_u(e^{D_{r,z_0}^{\overline{N}} X_u} - 1) du \right|^2 dz_0 dr \right) \\ & \leq \frac{1}{2} \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left[\int_r^T S_u^2 du + \int_r^T (e^{D_{r,z_0}^{\overline{N}} X_u} - 1)^2 du \right] dz_0 dr \right) \\ & \leq \frac{1}{2} \mathbb{E} \left(\int_0^T \int_{\mathbb{R}_+} \lambda(r) \left[\int_r^T S_u^2 du + \int_r^T (e^{D_{r,z_0}^{\overline{N}} X_u} - 1)^2 du \right] dz_0 dr \right) < \infty. \end{aligned} \quad (\text{D.3})$$

(b) We now prove that $1_{(0,\lambda(t)]}(z) D_{r,z_0}^{\overline{N}} \left[\frac{1}{c_1 \int_t^T S_u(e^{D_{t,z}^{\overline{N}} X_u} - 1) du} \right]$ and $1_{(0,\lambda(t)]}(z) D_{r,z_0}^{\overline{N}}((S_T + Y_T)H_K(S_T))$ belong to $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

From Lemma C.1 and Equation (D.3), we can write

$$\begin{aligned} & \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}}((K + Y_T)H_K(Y_T))|^2 dz_0 dr \right) \\ & \leq \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |H_K(Y_T) D_{r,z_0}^{\overline{N}}(K + Y_T)|^2 dz_0 dr \right) \\ & + \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |(K + Y_T) D_{r,z_0}^{\overline{N}}(H_K(Y_T))|^2 dz_0 dr \right) \\ & + \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r,z_0}^{\overline{N}}(H_K(Y_T)) D_{r,z_0}^{\overline{N}}(K + Y_T)|^2 dz_0 dr \right) < \infty. \end{aligned}$$



From (C.2) and (D.2)

$$\begin{aligned}
& \mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r, z_0}^{\bar{N}} \left[\frac{1}{\mathcal{C}_1 \int_t^T S_u(e^{D_{t, z}^{\bar{N}} X_u} - 1) du} \right]|^2 dz_0 dr \right) \\
& \leq 2\mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r, z_0}^{\bar{N}} \left(\frac{1}{\mathcal{C}_1} \frac{1}{\int_t^T S_u(e^{D_{t, z}^{\bar{N}} X_u} - 1) du} \right)|^2 dz_0 dr \right) \\
& + 2\mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} \left| \frac{1}{\mathcal{C}_1} D_{r, z_0}^{\bar{N}} \left(\frac{1}{\int_t^T S_u(e^{D_{t, z}^{\bar{N}} X_u} - 1) du} \right) \right|^2 dz_0 dr \right) \\
& + 2\mathbb{E} \left(1_{\mathcal{J}_1}(t, z) \int_0^T \int_{\mathbb{R}_+} 1_{\{z_0 \leq \lambda(r)\}} |D_{r, z_0}^{\bar{N}} \left(\frac{1}{\mathcal{C}_1} \right) D_{r, z_0}^{\bar{N}} \left(\frac{1}{\int_t^T S_u(e^{D_{t, z}^{\bar{N}} X_u} - 1) du} \right)|^2 dz_0 dr \right) < \infty.
\end{aligned}$$

(c) From Lemma C.1 and the parts (a) and (b) we conclude $D_{r, z_0}^{\bar{N}} 1_{(0, \lambda(t)]}(z) \mathcal{H}_3(t, z) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

Step two: In a similar way, one can easily show that $D_{r, z_0}^{\bar{N}} 1_{(0, \lambda(t)]}(z) \mathcal{H}_4(t, z) \in L^2(\Omega \times [0, T] \times \mathbb{R}_0)$.

APPENDIX E. AN EXPRESSION FOR THE SECOND ORDER MALLIAVIN DERIVATIVE

Let $\epsilon_{(u, z)}$ denotes the Dirac measure at $(u, z) \in \mathbb{R} \times \mathbb{R}_+$. To present an expression for the derivative of $e^{D_{t, z}^{\bar{N}} X_T}$, we are motivated by [49] and derive the following results.

$$\begin{aligned}
D_{r, z_0}^{\bar{N}} \left(\int_u^t J_s N_{(u, z)}(ds) \right) &= \int_u^t \int_{\mathbb{R}_+} 1_{s \leq r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} \left(\bar{N} + \epsilon_{(r, z_0)} \right)(dv, ds) \\
&+ \int_u^t \int_{\mathbb{R}_+} 1_{s > r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} \left(\bar{N} + \epsilon_{(r, z_0)} \right)(dv, ds) \\
&- \int_u^t \int_{\mathbb{R}_+} 1_{s \leq r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)} \bar{N}(dv, ds) \\
&- \int_u^t \int_{\mathbb{R}_+} 1_{s > r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)} \bar{N}(dv, ds) \\
&= \int_u^t \int_{\mathbb{R}_+} 1_{s \leq r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)} \left(\bar{N} + \epsilon_{(r, z_0)} \right)(dv, ds) \\
&+ \int_u^t \int_{\mathbb{R}_+} 1_{s > r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} \bar{N}(dv, ds) \\
&- \int_u^t \int_{\mathbb{R}_+} 1_{s \leq r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)} \bar{N}(dv, ds) \\
&- \int_u^t \int_{\mathbb{R}_+} 1_{s > r} J_s \left(1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)} \bar{N}(dv, ds) \\
&= 1_{r \geq u} J_u \left(1_{\left(0, \varphi_r(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, r)}\right)} \right]^{(z_0)} - 1_{\left(0, \varphi_r(\bar{N})|_{(-\infty, r)}\right)} \right]^{(z_0)} \\
&+ \int_u^t \int_{\mathbb{R}_+} 1_{s > r > u} J_s A_{r, u, z, z_0}(v, s) \bar{N}(dv, ds),
\end{aligned}$$

where

$$\begin{aligned}
A_{r, u, z, z_0}(v, s) &= 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} - 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(r, z_0)})|_{(-\infty, s)}\right)} \right]^{(v)} \\
&- 1_{\left(0, \varphi_s(\bar{N} + \epsilon_{(u, z)})|_{(-\infty, s)}\right)} \right]^{(v)} + 1_{\left(0, \varphi_s(\bar{N})|_{(-\infty, s)}\right)} \right]^{(v)}.
\end{aligned}$$



In the same manner,

$$\begin{aligned} D_{r,z_0}^{\overline{N}} D_{(u,z)}^{\overline{N}} \lambda(s) &= 1_{r \geq u} \alpha e^{-\beta(t-r)} \left(1 \left(0, \varphi_r(\overline{N} + \varepsilon_{(u,z)} |_{(-\infty, r)}) \right) (z_0) - 1 \left(0, \varphi_r(\overline{N} |_{(-\infty, r)}) \right) (z_0) \right) \\ &\quad + \int_u^t \int_{\mathbb{R}_+} 1_{s > r > u} \alpha e^{-\beta(t-s)} A_{r,u,z,z_0}(v, s) \overline{N}(dv, ds). \end{aligned}$$

In addition,

$$D_{r,z_0}^{\overline{N}} (J_u 1_{(0, \lambda(u)]}(z)) = J_u \left(1_{(0, \lambda(u) + D_{r,z_0}^{\overline{N}} \lambda(u)]}(z) - 1_{(0, \lambda(u)]}(z) \right).$$

APPENDIX F. MALLIAVIN CALCULUS ON WIENER-POISSON SPACE

Let us review some concepts of Malliavin calculus on Wiener-Poisson space, as demonstrated in [40, 41]. Consider the Wiener-Poisson space $(\Omega, \mathcal{F}, P)$, and a Poisson random measure N_0 associated with a Lévy process L with the Lévy measure $\nu(\cdot)$ on a complete separable metric space $(\mathbb{R}_0, \mathcal{B})$. For a positive real number T , let $L^2([0, T] \times \mathbb{R}_0^n)$ be the space of symmetric square integrable functions on the $([0, T] \times \mathbb{R}_0^n, m \otimes \nu^{\otimes n})$, where m is an atomless measure on $[0, T]$. Denote by $\mathcal{S}$ the set of all functionals $F = \varphi(\theta_1, \theta_2, \dots, \theta_n)$ where φ is a smooth function with bounded derivatives of any order and $\theta_i = \int_0^T f_i(t) dB_t$ with $f_i \in L^2([0, T])$. Define the Wiener Malliavin derivative of F by

$$D_t^W F(w) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(\theta_1, \dots, \theta_n) f_i(t).$$

For every integer $p \geq 2$, the domain of the operator D^W , denoted by $\mathbb{D}_W^{1,p}$, is the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{n,p} = \|F\|_{L^p(\Omega)} + \|D^W F\|_{L^p([0,T]^n \times \Omega)}.$$

The Skorohod operator δ is the adjoint operator of D^W from $L^2([0, T] \times \Omega)$ to $\mathbb{D}_W^{1,2}$. The following duality relation holds between these two operators, which states that for given $F \in \mathbb{D}_W^{1,2}$ and $u \in \text{Dom}(\delta^W)$

$$\mathbb{E} \left(\langle D^W F, u \rangle_{L^2([0,T])} \right) := \mathbb{E} \left(\int_0^T (D_t^W F) u_t dt \right) = \mathbb{E} (F \delta^W(u)). \quad (\text{F.1})$$

For every adapted process u , $\delta^W(u)$ can be represented by the stochastic integral $\int_0^T u(s) dW_s$.

On the other hand, given $h \in L^2([0, T] \times \mathbb{R}_0^n)$ and fixed $z \in \mathbb{R}_0^n := \mathbb{R}^n - \{0\}$, we write $h(t, \cdot, z)$ to indicate the function on $\mathbb{R}_0^{n-1}$ given by $(z_1, \dots, z_{n-1}) \rightarrow h(t, z_1, \dots, z_{n-1}, z)$. Denote by $\mathbb{D}_{N_0}^{1,2}$ the set of random variables F in $L^2(\Omega)$ with a chaotic decomposition $F = \sum_{n=0}^{\infty} I_n(h_n)$ with $h_n \in L_s^2([0, T] \times \mathbb{R}_0^n)$, satisfying

$$\sum_{n \geq 1} n n! \|h_n\|_{L^2([0,T] \times \mathbb{R}_0^n)}^2 < \infty.$$

For every $F \in \mathbb{D}_{N_0}^{1,2}$, the Malliavin derivative $D^{N_0} F$ is defined as the $L^2([0, T] \times \mathbb{R}_0)$ -valued random variable given by

$$D_{t,z}^{N_0} F = \sum_{n \geq 1} n I_{n-1}(h_n(t, \cdot, z)), \quad z \in \mathbb{R}_0.$$

The operator D^{N_0} is a closed operator from $\mathbb{D}_{N_0}^{1,2} \subset L^2(\Omega)$ into $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$ and satisfy the following rule.

Proposition F.1. [40] Let F be a random variable in $\mathbb{D}_{N_0}^{1,2}$ and let φ be a real continuous function such that $\varphi(F)$ belongs to $L^2(\Omega)$ and $\varphi(F + D^{N_0} F)$ belongs to $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$. Then $\varphi(F)$ belongs to $\mathbb{D}_{N_0}^{1,2}$ and

$$D_{t,z}^{N_0} \varphi(F) = \varphi(F + D_{t,z}^{N_0} F) - \varphi(F).$$

The following result characterizes δ^{N_0} as the adjoint operator of D^{N_0} .



Proposition F.2. [40] If $u \in \text{Dom} \delta^{N_0}$, then $\delta^{N_0}(u)$ is the unique element of $L^2(\Omega)$ such that, for all $F \in \mathbb{D}_{N_0}^{1,2}$,

$$\mathbb{E}(\langle D^{N_0} F, u \rangle_{L^2([0,T] \times \mathbb{R}_0)}) = \mathbb{E}(F \delta^{N_0}(u)).$$

Conversely, if u is a stochastic process in $L^2(\Omega \times [0, T] \times \mathbb{R}_0)$ such that, for some $G \in L^2(\Omega)$ and for all $F \in \mathbb{D}_{N_0}^{1,2}$,

$$\mathbb{E}(\langle D^{N_0} F, u \rangle_{L^2([0,T] \times \mathbb{R}_0)}) = \mathbb{E}(FG),$$

then u belongs to $\text{Dom} \delta^{N_0}$ and $\delta^{N_0}(u) = G$.

According to the proof of Theorem 3.1 in [49], the following relation holds.

$$D_{(t,z)} \left(\int_{\mathbb{R}} \int_{\mathbb{R}_+} g_1(s, v) \bar{N}(ds \times dv) \right) = g_1(t, z) + \int_{(t, \infty)} u_{(t,z)}(s) N_{(t,z)}(ds), \quad (\text{F.2})$$

$$D_{(t,z)} \left(\int_{\mathbb{R}} \int_{\mathbb{R}_+} g_1(s, v) ds dv \right) = \int_t^{+\infty} u_{(t,z)}(s) |D_{(t,z)} \lambda(s)| ds, \quad (\text{F.3})$$

with $g_1(t, \omega, z) := u(t) \mathbf{1}_{(0, \lambda(t, \omega)]}(z)$, where for every $s > u$

$$k_{(u,z)}(s) = \text{sign}(\varphi_s(\bar{N}|_{(-\infty, s)} + \epsilon_{(u,z)}) - \varphi_s(\bar{N}|_{(-\infty, s)})) J_s = \text{sign}(D_{u,z}^{\bar{N}} \lambda(s)) J_s, \quad (\text{F.4})$$

$$N_{(u,z)}(ds) = \bar{N}(ds \times (\varphi_s(\bar{N}|_{(-\infty, s)} + \epsilon_{(u,z)}) \wedge \varphi_s(\bar{N}|_{(-\infty, s)}), \varphi_s(\bar{N}|_{(-\infty, s)} + \epsilon_{(u,z)}) \vee \varphi_s(\bar{N}|_{(-\infty, s)})). \quad (\text{F.5})$$

Here, $\epsilon_{(u,z)}$ denotes the Dirac measure at $(u, z) \in \mathbb{R} \times \mathbb{R}_+$ and for ease of notation, we denoted by $a \wedge b$ and $a \vee b$ the minimum and the maximum between $a, b \in \mathbb{R}$, respectively.

APPENDIX G. PROOF OF PROPOSITION 3.4 AND LEMMA 3.6

Proof of Proposition 3.4: According to **Condition H1** and Equation (3.7), we have

$$F(v) = J_v - \int_v^t (e^{J_s} - 1) \alpha e^{-\beta(s-v)} ds, \quad 0 \leq v \leq t \leq T,$$

where the integral kernel $\alpha e^{-\beta(s-v)}$ describes the jump size and decay of the Hawkes intensity. We know that J_v is bounded on the compact interval $[0, T]$, because any monotone function on a compact interval is bounded; then $0 \leq J_v \leq J_T$. The exponential kernel is bounded and integrable, so the integral term is also bounded,

$$\left| \int_v^t (e^{J_s} - 1) \alpha e^{-\beta(s-v)} ds \right| \leq \frac{\alpha(e^{J_T} - 1)}{\beta}.$$

Therefore, the function $F(v)$ is uniformly bounded on $[0, T]$ with the explicit bound,

$$|F(v)| \leq J_T + \frac{\alpha(e^{J_T} - 1)}{\beta}.$$

Since the point process $N_{(t,z)}$ on (t, ∞) has $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -stochastic intensity

$\{|D_{t,z}^{\bar{N}} \lambda(s)|\}_{s>t}$ (see [49]), add and subtract $|D_{u,z}^{\bar{N}} \lambda(l)|$, and use equations (3.6) and (3.21) in [49] to derive

$$\begin{aligned} \mathbb{E} \left(\left| D_{u,z}^{\bar{N}} X_t \right| \right) &\leq \mathbb{E} \left(\left| F(u) \mathbf{1}_{(0, \lambda(u)]}(z) \right| \right) \\ &+ \mathbb{E} \left(\left| \int_u^t F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) (N_{(u,z)}(dl) - |D_{u,z}^{\bar{N}} \lambda(l)| + |D_{u,z}^{\bar{N}} \lambda(l)|) \right| \right) \\ &\leq J_T + \frac{\alpha(e^{J_T} - 1)}{\beta} + \mathbb{E} \left(\left| \int_u^t F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) (N_{(u,z)}(dl) - |D_{u,z}^{\bar{N}} \lambda(l)|) \right| \right) \end{aligned}$$



$$+ \mathbb{E} \left(\left| \int_u^t F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) |D_{u,z}^{\bar{N}} \lambda(l)| \right| \right) < \infty.$$

Proof of Lemma 3.6: According to the definitions (F.5) and (F.4), setting

$$\begin{aligned} \mathcal{AC}_1(s, u) &:= \varphi_s(\bar{N}|_{(-\infty, s)} + \epsilon_{(u, z)}) \wedge \varphi_s(\bar{N}|_{(-\infty, s)}), \\ \mathcal{AC}_2(s, u) &:= \varphi_s(\bar{N}|_{(-\infty, s)} + \epsilon_{(u, z)}) \vee \varphi_s(\bar{N}|_{(-\infty, s)}). \end{aligned}$$

Thanks to the proof of Theorem 3.1 in [49], stated that $N_{(u, z)}$ on (t, ∞) has $\mathcal{F}_t^{\bar{N}}$ -stochastic intensity $\{|D_{t,z} \lambda(s)|\}_{s \geq t}$, we result

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ -r \int_u^t F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) N_{(u, z)}(dl) \right\} \right) \\ &= \mathbb{E} \left(\exp \left\{ -r \int_u^t \int_{\mathbb{R}_+} F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) 1_{(\mathcal{AC}_1(l, u), \mathcal{AC}_2(l, u))}(z_0) \bar{N}(dl, dz_0) \right\} \right) \\ &\leq \mathbb{E} \left(\exp \left\{ - \int_u^t (1 - e^{-rF(s)}) |D_{u,z}^{\bar{N}} \lambda(s)| ds \right\} \right) \leq \mathbb{E} \left(\exp \left\{ -C_0 r \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} F(s) |D_{u,z}^{\bar{N}} \lambda(s)| ds \right\} \right), \quad (\text{G.1}) \end{aligned}$$

where C_0 is a constant that $1 - e^{-x} \geq C_0 x$, for every $x \in (-1, 1)$. Applying (3.6), Fubini theorem, and the fact that for every $s \in [u, t]$ we have $F(u) \leq F(s)$, will derive

$$\begin{aligned} & \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} F(s) e^{-\beta(s-u)} ds = \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} \left(J_s e^{-\beta(s-u)} - \alpha \int_s^t (e^{J_l} - 1) e^{-\beta(l-u)} dl \right) ds \\ &= \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} J_s e^{-\beta(s-u)} ds - \alpha \int_u^t \int_u^l 1_{\{F(s) \leq \frac{1}{r}\}} (e^{J_l} - 1) e^{-\beta(l-u)} ds dl \\ &\geq \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} J_s e^{-\beta(s-u)} ds - 1_{\{F(u) \leq \frac{1}{r}\}} \alpha \int_u^t (e^{J_l} - 1) e^{-\beta(l-u)} (l - u) dl \\ &\geq \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} J_s e^{-\beta(s-u)} ds - t \alpha 1_{\{F(u) \leq \frac{1}{r}\}} \int_u^t (e^{J_l} - 1) e^{-\beta(l-u)} dl \\ &= \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} J_s e^{-\beta(s-u)} ds - t 1_{\{F(u) \leq \frac{1}{r}\}} [J_u - F(u)]. \quad (\text{G.2}) \end{aligned}$$

Substituting (G.2) into (G.1) deduce that

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ -r \int_u^t F(l) \text{sign}(D_{u,z}^{\bar{N}} \lambda(l)) N_{(u, z)}(dl) \right\} \right) \\ &\leq \mathbb{E} \left(\exp \left\{ -C_0 \alpha r 1_{[0, \lambda(u)]}(z) \int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} F(s) e^{-\beta(s-u)} ds \right\} \right) \\ &\leq \mathbb{E} \left(\exp \left\{ -C_0 \alpha r 1_{[0, \lambda(u)]}(z) \left(\int_u^t 1_{\{F(s) \leq \frac{1}{r}\}} J_s e^{-\beta(s-u)} ds - t J_u + t F(u) \right) \right\} \right) \\ &\leq \mathbb{E} \left(\exp \left\{ T C_0 \alpha r 1_{[0, \lambda(u)]}(z) 1_{\{F(u) \leq \frac{1}{r}\}} J_u \right\} \right). \end{aligned}$$

APPENDIX H. PROOF OF LEMMAS 5.1 AND 5.2

Proof of Lemma 5.1: Start from the definition of the discretized intensity $\lambda^n(t)$ on the interval $[t_i, t]$, for every $i = 1, \dots, n$, and apply the Burkholder–Davis–Gundy inequality for jump processes to yield

$$\begin{aligned} \mathbb{E} \left(|\lambda(t) - \lambda^n(t)|^p \right) &\leq 2^{p-1} \mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p + \beta^p \int_{t_i}^t |\lambda(s) - \lambda^n(t_i)|^p ds \right. \\ &\quad \left. + \alpha^p \left| \int_{t_i}^t \int_{\mathbb{R}_+} \left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right) \bar{N}(ds, dz) \right|^p \right) \end{aligned}$$



$$\begin{aligned}
&\leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) + \beta^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(t_i)|^p \right) ds \right. \\
&\quad + \alpha^p \int_{t_i}^t \int_{\mathbb{R}_+} \mathbb{E} \left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right)^p dz ds \\
&\quad \left. + \alpha^p \mathbb{E} \left(\left| \int_{t_i}^t \int_{\mathbb{R}_+} \left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right)^2 dz ds \right|^{\frac{p}{2}} \right) \right]. \tag{H.1}
\end{aligned}$$

We note that $\left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right)^p \leq 2^p \left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right)^2$. Moreover, for fixed z , the integral over $\mathbb{R}_+$ of the squared difference of indicators is exactly the absolute difference of the intensities:

$$\int_{\mathbb{R}_+} \left(1_{(0, \lambda(s)]}(z) - 1_{(0, \lambda^n(t_i)]}(z) \right)^2 dz \leq |\lambda(s) - \lambda^n(t_i)|.$$

Substitute these inequalities into (H.1), apply the Cauchy-Schwarz inequality for the last term, and use the fact $xy \leq \frac{1}{2}(x^2 + y^2)$, then add and subtract the term $\lambda^n(s)$ to them and use the fact $(x + y)^p \leq 2^{p-1}(x^p + y^p)$ to derive

$$\begin{aligned}
\mathbb{E} \left(|\lambda(t) - \lambda^n(t)|^p \right) &\leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) + \beta^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(t_i)|^p \right) ds \right. \\
&\quad + 2^p \alpha^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(t_i)| \right) ds + \alpha^p (\Delta t)^{\frac{p}{2}-1} \mathbb{E} \left(\int_{t_i}^t |\lambda(s) - \lambda^n(t_i)|^{\frac{p}{2}} ds \right) \Big] \\
&\leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) + 2^p \beta^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)|^p \right) ds \right. \\
&\quad + 2^p \beta^p \int_{t_i}^t \mathbb{E} \left(|\lambda^n(s) - \lambda^n(t_i)|^p \right) ds + 2^p \alpha^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)| \right) ds \\
&\quad + 2^p \alpha^p \int_{t_i}^t \mathbb{E} \left(|\lambda^n(s) - \lambda^n(t_i)| \right) ds + \frac{1}{2} \alpha^p (\Delta t)^{\frac{p}{2}-1} 2^{p-1} \mathbb{E} \left(\int_{t_i}^t |\lambda(s) - \lambda^n(s)|^p ds \right) \\
&\quad \left. + \frac{1}{2} \alpha^p (\Delta t)^{\frac{p}{2}-1} 2^{p-1} \mathbb{E} \left(\int_{t_i}^t |\lambda^n(s) - \lambda^n(t_i)|^p ds \right) + \alpha^p (\Delta t)^{\frac{p}{2}} \right].
\end{aligned}$$

Now, substitute the upper bounds of (5.3) and (5.4) in the above inequality to obtain

$$\begin{aligned}
\mathbb{E} \left(|\lambda(t) - \lambda^n(t)|^p \right) &\leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) \right. \\
&\quad + \left(2^p \beta^p + \frac{1}{2} \alpha^p (\Delta t)^{\frac{p}{2}-1} 2^{p-1} \right) \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)|^p \right) ds \\
&\quad + 2^p \alpha^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)| \right) ds + C_{p,1} \Delta t \Big] \\
&\leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) + \left(2^p \beta^p + 2^{p-2} \alpha^p T^p \right) \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)|^p \right) ds \right. \\
&\quad \left. + \frac{2^p}{p} \alpha^p \int_{t_i}^t \mathbb{E} \left(|\lambda(s) - \lambda^n(s)|^p \right) ds + \frac{2^p}{q} \alpha^p \Delta t + C_{p,1} \Delta t \right],
\end{aligned}$$

where $C_{p,1}$ is a constant, and we apply $|\lambda(s) - \lambda^n(s)| \leq \frac{1}{p} |\lambda(s) - \lambda^n(s)|^p + \frac{1}{q}$ in the last inequality for $\frac{1}{q} + \frac{1}{p} = 1$. Applying Grönwall's inequality yields

$$\mathbb{E} \left(|\lambda(t) - \lambda^n(t)|^p \right) \leq 2^{p-1} \left[\mathbb{E} \left(|\lambda(t_i) - \lambda^n(t_i)|^p \right) + \frac{2^p}{q} \alpha^p \Delta t + C_{p,1} \Delta t \right] e^{A_{p,1} \Delta t},$$



where $A_{p,1} = 2^{p+1}\beta^p + 2^{p-2}\alpha^p T^p + \frac{2^{2p}}{p}\alpha^p$. We can continue this process recursively to derive that there exists some constant $c_{1,p}$ such that

$$\mathbb{E}\left(|\lambda(t) - \lambda^n(t)|^p\right) \leq (\lambda_0 - \lambda_0^n)e^{A_{p,1}T} + \Delta t(e^T - 1) \leq c_{1,p}\Delta t.$$

Proof of Lemma 5.2: The following two inequalities are required to establish the convergence rate. For every $x, y \in \mathbb{R}$ and $p \in \mathbb{N}$,

$$|e^x - e^y| \leq (e^x + e^y)|x - y|, \quad (\text{H.2})$$

$$(x + y)^{2p} \leq 2^{2p-1}(x^{2p} + y^{2p}). \quad (\text{H.3})$$

From (H.3) we derive

$$\begin{aligned} \mathbb{E}\left(X_T - X_T^n\right)^{2p} &= \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} (1 - e^{J_s})1_{(0,\lambda(s)]}(z)dzds + \int_0^T \int_{\mathbb{R}_+} J_s 1_{(0,\lambda(s)]}(z)\overline{N}(ds, dz) \right. \\ &\quad \left. - \left[\int_0^T \int_{\mathbb{R}_+} (1 - e^{J_s^n})1_{(0,\lambda^n(s)]}(z)dzds + \int_0^T \int_{\mathbb{R}_+} J_s^n 1_{(0,\lambda^n(s)]}(z)\overline{N}(ds, dz)\right]\right)^{2p} \\ &\leq 4^{2p-1}\mathbb{E}\left(\int_0^T (\lambda(s) - \lambda^n(s))ds\right)^{2p} + 4^{2p-1}\mathbb{E}\left(\int_0^T (e^{J_s^n}\lambda^n(s) - e^{J_s}\lambda(s))ds\right)^{2p} \\ &\quad + 4^{2p-1}\mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} [J_s 1_{(0,\lambda(s)]}(z) - J_s^n 1_{(0,\lambda^n(s)]}(z)]\overline{N}(ds, dz)\right)^{2p} \\ &\leq 4^{2p-1}\left(T^2 \sup_{0 \leq s \leq T} \mathbb{E}\left((\lambda(s) - \lambda^n(s))^{2p}\right) + I_1 + I_2\right). \end{aligned} \quad (\text{H.4})$$

To estimate the term I_1 , from (H.2) and (H.3), and the Holder inequality, connection with Lemma 5.1 and **Condition H1**, and finally Lipschitz property of J , we result there exists some constant c_6 that

$$\begin{aligned} I_1 &\leq 2^{2p-1}\mathbb{E}\left(\int_0^T (e^{J_s^n} - e^{J_s})\lambda^n(s)ds\right)^{2p} + 2^{2p-1}\mathbb{E}\left(\int_0^T e^{J_s}(\lambda^n(s) - \lambda(s))ds\right)^{2p} \\ &\leq T^{2p-1}2^{2p-1}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \left[2^{2p-1}[(e^{J_{t_i}} - 1)^{2p} + (e^{J_s} - 1)^{2p}] + 2^{2p}\right](J_{t_i} - J_s)ds\right) \\ &\quad \times \sup_{0 \leq s \leq T} \mathbb{E}\left(\lambda^n(s)\right)^{2p} + 8^{2p-1}\mathbb{E}\left(\int_0^T (e^{J_s} - 1)^{2p}(\lambda^n(s) - \lambda(s))^{2p}ds\right) \\ &\quad + 8^{2p-1}\mathbb{E}\left(\int_0^T (\lambda^n(s) - \lambda(s))^{2p}ds\right) \leq c_6(\Delta t)^{2p}. \end{aligned}$$

Also, from (H.3) and the Burkholder-Davis-Gundy inequality ([47], Lemma 2.3) we have

$$\begin{aligned} I_2 &= \mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} J_s 1_{(0,\lambda(s)]}(z)\overline{N}(ds, dz) - \int_0^T \int_{\mathbb{R}_+} J_s 1_{(0,\lambda^n(s)]}(z)\overline{N}(ds, dz) \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}_+} J_s 1_{(0,\lambda^n(s)]}(z)\overline{N}(ds, dz) - \int_0^T \int_{\mathbb{R}_+} J_s^n 1_{(0,\lambda^n(s)]}(z)\overline{N}(ds, dz)\right)^{2p} \\ &\leq 2^{2p-1}\mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} (J_s 1_{(0,\lambda(s)]}(z) - J_s 1_{(0,\lambda^n(s)]}(z))^{2p}dzds\right) \\ &\quad + 2^{2p-1}\mathbb{E}\left(\int_0^T \int_{\mathbb{R}_+} (J_s 1_{(0,\lambda(s)]}(z) - J_s 1_{(0,\lambda^n(s)]}(z))^2dzds\right)^p \end{aligned}$$



$$\begin{aligned}
& + 2^{2p-1} \mathbb{E} \left(\int_0^T \int_{\mathbb{R}_+} (J_s 1_{(0, \lambda^n(s)]}(z) - J_s^n 1_{(0, \lambda^n(s)]}(z))^{2p} dz ds \right) \\
& + 2^{2p-1} \mathbb{E} \left(\int_0^T \int_{\mathbb{R}_+} (J_s 1_{(0, \lambda^n(s)]}(z) - J_s^n 1_{(0, \lambda^n(s)]}(z))^2 dz ds \right)^p.
\end{aligned}$$

Now, using **Condition H1**, the equation (5.2), and Lemma 5.1, there exists some constant c_7 that

$$\begin{aligned}
I_2 & \leq (2\gamma T)^{2p} \left[\sup_{0 \leq s \leq T} \mathbb{E}(|\lambda(s) - \lambda^n(s)|) \right] + (2\gamma \Delta t)^{2p} \mathbb{E} \left(\int_0^T \lambda^n(s) ds \right) \\
& + (2\gamma T)^{2p} \left[\sup_{0 \leq s \leq T} \mathbb{E}(|\lambda(s) - \lambda^n(s)|^p) \right] + (2\gamma \Delta t)^2 \mathbb{E} \left(\int_0^T \lambda^n(s) ds \right)^p \leq c_7(\Delta t),
\end{aligned}$$

Substitute the bounds of I_1 and I_2 into equation (H.4) to complete the proof.

REFERENCES

- [1] D. Ahmadian, L. V. Ballestra, and N. Karimi, *An extremely efficient numerical method for pricing options in the Black–Scholes model with jumps*, Mathematical Methods in the Applied Sciences, 44(2) (2021), 1843–1862.
- [2] D. Ahmadian, L. V. Ballestra, and F. Shokrollahi, *A Monte-Carlo approach for pricing arithmetic Asian rainbow options under the mixed fractional Brownian motion*, Chaos, Solitons & Fractals, 158 (2022), 112023.
- [3] E. Alòs and C. O. Ewald, *Malliavin differentiability of the Heston volatility and applications to option pricing*, Advances in Applied Probability, 40(1) (2008), 144–162.
- [4] E. Bacry, I. Mastromatteo, and J. F. Muzy, *Hawkes processes in finance*, Market Microstructure and Liquidity, 1(01) (2015), 1550005.
- [5] D. R. Baños, S. Ortiz-Latorre, and O. Z. Font, *Change of measure in a Heston-Hawkes stochastic volatility model*, Advances in Applied Probability, 56(4) (2024), 1370–1399.
- [6] M. P. Bavouzet and M. Messaoud, *Computation of Greeks using Malliavin’s calculus in jump-type market models*, Electronic Journal of Probability, 11 (2006), 276–300.
- [7] E. Ben-Hamou, *Application of Malliavin calculus and Wiener chaos to option pricing theory*, Thesis (Ph.D.)–London School of Economics and Political Science (United Kingdom), 2001.
- [8] V. Bezborodov, L. D. Persio, and Y. Mishura, *Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth*, Methodology and Computing in Applied Probability, 21(1) (2019), 331–366.
- [9] M. L. Bianchi and F. J. Fabozzi, *Investigating the performance of non-Gaussian stochastic intensity models in the calibration of credit default swap spreads*, Computational Economics, 46 (2015), 243–273.
- [10] P. Brémaud, *Point Process Calculus in Time and Spaces: An Introduction with Applications*, Springer Nature Switzerland, 2020.
- [11] R. Brignone and C. Sgarra, *Asian options pricing in Hawkes-type jump-diffusion models*, Annals of Finance, 16(1) (2020), 101–119.
- [12] D. Brigo and A. Alfonsi, *Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model*, Finance and Stochastics, 9(1) (2004), 29–42.
- [13] D. Brigo and N. El-Bachir, *Credit derivatives pricing with a smile-extended jump stochastic intensity model*, Available at SSRN 950208, 2006.
- [14] G. Callegaro, P. D. Tella, B. Ongarato, and C. Sgarra, *Semistatic variance-optimal hedging with self-exciting jumps*, Mathematics of Operations Research, forthcoming (2025).
- [15] Y. Chang and Y. Wang, *Option pricing under a double stochastic volatility model with stochastic interest rates and double exponential jumps with stochastic intensity*, Mathematical Problems in Engineering, 2020.
- [16] J. Chen, M. P. Clements, and A. Urquhart, *Modeling price and variance jump clustering using the marked Hawkes process*, Journal of Financial Econometrics, 22(3) (2024), 743–772.
- [17] Y. El-Khatib and N. Privault, *Computations of Greeks in a market with jumps via the Malliavin calculus*, Finance and Stochastics, 8(2) (2004), 161–179.



- [18] O. Farkhondeh Rouz and D. Ahmadian, *Stability of two classes of improved backward Euler methods for stochastic delay differential equations of neutral type*, Computational Methods for Differential Equations, 5(3) (2017), 201–213.
- [19] Y. Feng, *CVA under the Bates model with stochastic default intensity*, Journal of Mathematical Finance, 7(3) (2017), 682–698.
- [20] V. Filimonov, D. Bicchetti, N. Maystre, and D. Sornette, *Quantification of the high level of endogeneity and of structural regime shifts in commodity markets*, Journal of International Money and Finance, 42 (2014), 174–192.
- [21] R. Foschi, *Measuring discrepancies between Poisson and exponential Hawkes processes*, Methodology and Computing in Applied Probability, 23(1) (2021), 219–239.
- [22] A. Fulop, J. Li, and J. Yu, *Self-exciting jumps, learning, and asset pricing implications*, The Review of Financial Studies, 28 (2015), 876–912.
- [23] L. Gonzato and C. Sgarra, *Self-exciting jumps in the oil market: Bayesian estimation and dynamic hedging*, Energy Economics, 99 (2021), 105279.
- [24] A. G. Hawkes, *Point spectra of some mutually exciting point processes*, Journal of the Royal Statistical Society: Series B (Methodological), 33(3) (1971), 438–443.
- [25] A. G. Hawkes, *Spectra of some self-exciting and mutually exciting point processes*, Biometrika, 58(1) (1971), 83–90.
- [26] A. G. Hawkes, *Hawkes processes and their applications to finance: a review*, Quantitative Finance, 18(2) (2018), 193–198.
- [27] J. Huang, W. Zhu, and X. Ruan, *Option pricing using the fast Fourier transform under the double exponential jump model with stochastic volatility and stochastic intensity*, Journal of Computational and Applied Mathematics, 263 (2014), 152–159.
- [28] A. Hudde and L. Rüschendorf, *European and Asian Greeks for exponential Lévy processes*, Methodology and Computing in Applied Probability, 25(1) (2023), 39.
- [29] F. Huehne, *Malliavin calculus for the computation of Greeks in markets driven by pure-jump Lévy processes*, Available at SSRN 948347, 2005.
- [30] J. J. Ketelbuters and D. Hainaut, *CDS pricing with fractional Hawkes processes*, European Journal of Operational Research, 297(3) (2022), 1139–1150.
- [31] M. Kharrat and F. Bastin, *Continuation value computation using Malliavin calculus under a general volatility stochastic process for American option pricing*, Turkish Journal of Mathematics, 46(1) (2022), 71–86.
- [32] A. Khedher, *Computation of the delta in a multidimensional jump-diffusion setting with applications to stochastic volatility models*, Stochastic Analysis and Applications, 30(3) (2012), 403–425.
- [33] K. S. Leung and Y. K. Kwok, *Counterparty risk for credit default swaps: Markov chain interacting intensities model with stochastic intensity*, Asia-Pacific Financial Markets, 16 (2009), 169–181.
- [34] P. E. Lévy dit Véhel and J. Lévy Véhel, *Stochastic jump intensity models*, Journal of Risk and Decision Analysis, 7 (2018), 63–75.
- [35] W. Liu and S. P. Zhu, *Pricing variance swaps under the Hawkes jump-diffusion process*, Journal of Futures Markets, 39(6) (2019), 635–655.
- [36] Y. Ma, D. Pan, K. Shrestha, and W. Xu, *Pricing and hedging foreign equity options under Hawkes jump-diffusion processes*, Physica A: Statistical Mechanics and Its Applications, 537 (2020), 122645.
- [37] C. Mancini, R. Foschi, and F. Lilla, *Warnings about future jumps: properties of the exponential Hawkes model*, 2019.
- [38] J. L. Menaldi, *Stochastic Differential Equations with Jumps*, Mathematics Faculty Research Publications, 2008.
- [39] D. Nualart, *Malliavin Calculus and Related Topics*, 2nd ed., Springer, New York, 2010.
- [40] D. Nualart and E. Nualart, *Introduction to Malliavin Calculus*, Cambridge University Press, 2018.
- [41] G. D. Nunno, B. Øksendel, and F. Proske, *Malliavin Calculus for Lévy Processes with Applications to Finance*, Springer, 2009.
- [42] B. Øksendal and A. Sulem, *Stochastic Control of Jump Diffusions*, Springer, 2005.



- [43] M. Rambaldi, P. Pennesi, and F. Lillo, *Modeling foreign exchange market activity around macroeconomic news: Hawkes-process approach*, Physical Review E, *91*(1) (2015), 012819.
- [44] A. Rathinasamy, D. Ahmadian, and P. Nair, *Second-order balanced stochastic Runge–Kutta methods with multi-dimensional studies*, Journal of Computational and Applied Mathematics, *377* (2020), 112890.
- [45] A. Safdari-Vaighani, D. Ahmadian, and R. Javid-Jahromi, *An approximation scheme for option pricing under two-state continuous CAPM*, Computational Economics, *57*(4) (2021), 1373–1385.
- [46] Y. Song and Z. Wang, *Regularity for distribution-dependent SDEs driven by jump processes*, Stochastics and Dynamics, *22*(5) (2022).
- [47] Y. Song and X. Zhang, *Regularity of density for SDEs driven by degenerate Lévy noises*, Electronic Journal of Probability, *20*(21) (2015), 1–27.
- [48] I. M. Toke, *An introduction to Hawkes processes with applications to finance*, Lecture Notes from École Centrale Paris, BNP Paribas Chair of Quantitative Finance, 2011.
- [49] G. L. Torrisi, *Gaussian approximation of nonlinear Hawkes processes*, The Annals of Applied Probability, *26*(4) (2016), 2106–2140.
- [50] B. Z. Yang, J. Yue, M. H. Wang, and N. J. Huang, *Volatility swaps valuation under stochastic volatility with jumps and stochastic intensity*, Applied Mathematics and Computation, *355* (2019), 73–84.
- [51] B. Yilmaz, *Computation of option Greeks under hybrid stochastic volatility models via Malliavin calculus*, Modern Stochastics: Theory and Applications, *5*(2) (2018), 145–165.

Uncorrected Proof

