



Advancing the solution of time-fractional convection-diffusion-reaction equations with application to Mobile-Immobile system

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Abstract

This paper introduces a novel numerical scheme that integrates cubic B-spline basis functions with the Crank-Nicolson finite difference method to solve the time-fractional convection-diffusion-reaction equation (TFCDRE). The temporal discretization is performed using the $L1$ approximation, while CBS basis functions are employed for spatial discretization. A von Neumann stability analysis establishes the unconditional stability of the proposed scheme for the corresponding constant-coefficient problem. The convergence analysis predicts second-order accuracy in space and $(2 - \beta)$ -order accuracy in time. The accuracy and effectiveness of the method are demonstrated through several benchmark examples involving both constant and variable coefficients. Numerical results show excellent agreement with exact solutions and confirm the theoretical convergence rates through maximum and Euclidean error norms. Comprehensive graphical investigations, including two-dimensional, three-dimensional, contour, and error plots, further validate the stability and accuracy of the proposed approach. To illustrate its practical applicability, the method is applied to a fractional mobile-immobile transport model, where numerical simulations, parameter studies, and convergence analyses demonstrate its capability to accurately capture anomalous transport dynamics. All computations are carried out using Wolfram Mathematica 14.3. The proposed CBS framework provides an accurate, stable, and computationally efficient tool for solving TFCDREs arising in transport phenomena, environmental sciences, and related applied fields.

Keywords. Time fractional convection-diffusion-reaction equation, Caputo fractional derivative, Crank-Nicolson finite difference formulation, Cubic B-spline functions, Stability and convergence.

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1. INTRODUCTION

In recent decades, fractional calculus (FC) has evolved into a powerful analytical framework for describing complex physical and engineering processes characterized by memory and hereditary effects. Owing to its intrinsic nonlocal nature, FC has been widely employed to model and obtain approximate solutions for diverse phenomena in physics, electrochemistry, electromagnetism, diffusion, and general transport theory [46, 61]. At the heart of FC lies the study of fractional differential equations (FDEs), which extend classical differential equations by incorporating derivatives of non-integer order, thereby offering a more flexible and accurate description of anomalous and multiscale dynamics. Several formulations of fractional differentiation and integration have been introduced in the literature, including the Riemann-Liouville, Caputo, Hadamard, Grünwald-Letnikov, Erdélyi-Kober, Sonin-Letnikov, Riesz-Feller, Caputo-Fabrizio, and Atangana-Baleanu operators, as well as more recent local and conformable derivatives [58]. These definitions differ in their mathematical structures and analytical properties, each possessing distinct advantages and limitations depending on the intended application. Among them, the Riemann-Liouville and Caputo formulations

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are the most widely adopted in both theoretical investigations and numerical modelling. FDEs can be further classified as linear or nonlinear, as well as ordinary or partial fractional differential equations, reflecting their broad applicability.

FPDEs have been successfully applied to describe systems in convection and diffusion dynamics, fluid mechanics, groundwater flow and geohydrology [38, 45]. Other applications include inventory and transport systems, fractal networks, astrophysics, cybernetics, signal processing, fractal media, population dynamics, and geometric analysis [6, 62], among many others. Despite their wide applicability, problems involving fractional derivatives are inherently complex and analytically intractable, primarily due to the nonlocal and memory-dependent nature of fractional operators. Consequently, obtaining exact closed-form solutions for FPDEs is rarely feasible [12]. This challenge has stimulated the development of robust numerical algorithms as practical alternatives for analyzing and simulating such systems under various initial and boundary conditions.

In pursuit of greater computational efficiency and accuracy, considerable attention has been devoted to constructing advanced numerical schemes for the time-fractional convection–diffusion–reaction equation (TFCDRE) with variable coefficients. This class of equations plays a pivotal role in modelling numerous transport and reactive processes across engineering, physics, and environmental sciences. Typical applications include semiconductor device modelling, groundwater contamination prediction, biological system analysis, air pollutant dispersion, oil reservoir flow, food preservation processes, and chemical reaction kinetics [11, 27, 49, 50]. In this study, we consider the TFCDRE, expressed

$$\frac{\partial^\beta y(r, t)}{\partial t^\beta} + w_1(r) \frac{\partial y(r, t)}{\partial r} - w_2(r) \frac{\partial^2 y(r, t)}{\partial r^2} + w_3(r) y(r, t) = p(r, t), \quad t \in [0, 1], r \in [a, b], \beta \in (0, 1), \quad (1.1)$$

subject to the initial and boundary conditions

$$y(r, 0) = g_1(r), \quad a \leq r \leq b, \quad (1.2)$$

$$y(a, t) = g_2(t), \quad y(b, t) = g_3(t), \quad t \geq 0. \quad (1.3)$$

Here, the coefficients w_1, w_2 and w_3 are functions depending on r and t , with $w_1(r) \geq 0$, $w_2(r) \geq a_0 > 0$, where a_0 being a constant, and $p(r, t)$ denotes the source term. The functions $g_1(r), g_2(t), g_3(t)$ specify the initial and boundary conditions, while $\frac{\partial^\beta y(r, t)}{\partial t^\beta}$ represents the Caputo's fractional derivative (CFD) described as [46]:

$${}^c D_t^\beta y(r, t) = \frac{1}{\Gamma(n - \beta)} \int_0^t \frac{\partial^n y(r, s)}{\partial s^n} (t - s)^{n - \beta - 1} ds, \quad n - 1 < \beta \leq n, \quad n \in \mathbb{N}, \quad t > 0, \quad (1.4)$$

where Γ represent the Euler's Gamma function. As with most FPDEs, exact analytical solutions of TFCDREs are generally unavailable due to the nonlocal nature and variable coefficients of the governing operator. Consequently, robust numerical methods have been developed to obtain approximate solutions under given boundary and initial conditions. For example, a weak Galerkin finite element method (FEM) was proposed for solving time-fractional (TF) convection-diffusion problems [59]. A two-level fourth-order numerical scheme was presented for solving TFCDREs under suitable initial and boundary conditions [40].

This review paper [25] focuses on the numerical solution of the TFPDE using various discretization techniques. The time-delayed FPDEs were investigated using an extended finite difference formulation [13]. Some modern numerical and meshless approaches have been incorporated and discussed by scholars. For example, Local RBF-FD method was adopted in [48] in solving the two-dimensional modified anomalous sub-diffusion equation based on the Riemann-Liouville fractional integral. Meshless collocation techniques utilized for solving 2D fractional Klein-Kramers dynamics equation on irregular domains in [4]. The unsteady diffusion-convection problems of anisotropic functionally graded materials were investigated using the meshless local Petrov-Galerkin method [1], and the fourth order exponential time differencing Runge-Kutta schemes [28]. The Chebyshev collocation method [53] have been utilised to solve TFCDREs. Other recent numerical schemes are the finite block method adopted for nonlinear time-fractional partial integro-differential equations (TFPIDEs [2, 22]; the Lagrange polynomials used for solving a distributed-order TFPIDE on non-rectangular domains [23]. A combination of finite-difference, spectral, and finite-block methods for both square and non-rectangular regions was adopted by [24] to solve TFPIDEs with time delay on rectangular and non-rectangular domains. The distributed-order space-time fractional PDEs were discussed using the finite element method [21].



Despite the success of these numerical approaches, many are constrained by the high computational cost associated with mesh generation and refinement. To mitigate this challenge, B-spline-based techniques have attracted increasing attention due to their flexibility, accuracy, and ease of implementation. A B-spline is a mathematically defined piecewise function characterized by a set of control points, a polynomial degree, and a knot vector. The shape and smoothness of the resulting curve or surface are determined by the configuration of the knot vector, which consists of an ordered sequence of parameter values. The degree of a B-spline specifies the maximum polynomial degree employed within each segment of the piecewise-defined function. Several successful applications of B-spline approximations have been reported in the literature. For instance, cubic trigonometric B-splines (CTBS) were employed to solve the TF Burgers' equation [63], while CBS were used for solving the coupled TF Schrödinger equation [39] and TF coupled viscous Burgers equation [20]. Extended cubic B-splines (ExCBS) were applied to derive approximate solutions for the TF Allen-Cahn equation in the context of mathematical physics [19]. Quintic B-spline basis functions were employed to obtain the numerical solution of the TF diffusion equation [51]. A redefined CBS technique was utilized for solving the TF Allen-Cahn equation [30]. The ExCBS basis functions were employed to obtain the numerical solution of the TF advection-diffusion equation with the Atangana-Baleanu fractional derivative [60]. In another related study, a TF Navier-Stokes equation with the Caputo derivative was investigated using CBS functions [41].

A comparative numerical study was presented for the TF Cattaneo equation involving the Caputo-Fabrizio derivative, where CBS, CTBS, and ExCBS methods were compared in terms of accuracy and computational performance [64]. Moreso, a quadratic B-spline collocation method was developed to obtain the numerical solution of the space-fractional diffusion equation [5], while numerical solutions for fractional coupled Burgers' equations with an exponential kernel were derived using ExCBS functions [9]. Further developments and applications of B-spline-based schemes can be found in [42, 43, 54, 56], along with the references cited therein. Meanwhile, the numerical solution of the TFC-DRE (1.1) presents significant challenges due to the intricate coupling of several competing physical mechanisms. The presence of the CFD introduces a nonlocal memory effect, meaning that the solution at any time depends on its entire temporal history. This nonlocality leads to weak regularity near the initial time and complicates both stability and convergence analysis [31, 37]. At the same time, convection-dominated regimes may generate steep gradients and spurious numerical oscillations, while the diffusion term imposes stiffness constraints [7, 34]. The reaction component can further amplify or dampen the solution, increasing sensitivity to discretization errors. These challenges are compounded by spatially variable coefficients, which produce non-uniform solution behaviour and make stability control more difficult.

To effectively address these issues, the proposed CBS-Crank-Nicolson scheme combines smooth, locally supported spatial approximation with an implicit time integration strategy. The CBS basis functions provide enhanced accuracy and reduce oscillatory behaviour, particularly in convection-dominated regions, while the Crank-Nicolson method ensures second-order accuracy and improved stability without restrictive time-step conditions. Consequently, the proposed method provides a stable and accurate framework for handling the combined effects of memory, advection, diffusion, and reaction in time-fractional models, overcoming many of the limitations observed in conventional numerical approaches. Special attention is devoted to the stability, convergence, accuracy, and computational efficiency of the proposed scheme, as well as the application to mobile-immobile transport systems. The remainder of this paper is structured as follows: Section 2 presents the basis function of CBS while Section 3 established the methodology. The scheme's stability was focused on Section 4, whereas Section 5 highlighted the convergence. Section 6 provides the numerical examples and their interpretations, followed by an application study in Section 7, and the concluding remarks in Section 8.

2. BASIS FUNCTIONS

Suppose $N \in \mathbb{Z}^+$, the spatial domain $[a, b]$, be partitioned into equal subintervals of length $h = r_{j+1} - r_j = \frac{b-a}{N}$, such that $a = r_0 < r_1 < \dots < r_N = b$ and $r_j = r_0 + jh$, $j = 0, 1, \dots, N$. Suppose also that $Y(r, t)$ and $y(r, t)$ are the approximate and exact solutions, then the CBS approximate solution to Equation (1.3) can be expressed as [35]:

$$\underline{Y}(r, t) = \sum_{j=-1}^{N+1} Q_j(t) Z_j(r), \quad (2.1)$$



where $Q_j(t)$, the unknown coefficients, are to be determined at each time level and $Z_j(r)$, the CBS functions represented as

$$Z_j(r) = E_1 \begin{cases} (r - r_{j-2})^3, & \text{if } r \in [r_{j-2}, r_{j-1}), \\ E_2 + 3h(r - r_{j-1})^2 - 3(r - r_{j-1})^3, & \text{if } r \in [r_{j-1}, r_j), \\ E_3 + 3h(r_{j+1} - r)^2 - 3(r_{j+1} - r)^3, & \text{if } r \in [r_j, r_{j+1}), \\ (r_{j+2} - r)^3, & \text{if } r \in [r_{j+1}, r_{j+2}), \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

where $E_1 = \frac{1}{6h^3}$, $E_2 = h^3 + 3h^2(r - r_{j-1})$, and $E_3 = h^3 + 3h^2(r_{j+1} - r)$. The CBS basis functions defined in Eq. (2.2) are derived from the classical B-spline formulation based on the Cox-de Boor recursion relations, originally introduced by [14] and further developed by [16].

Remark 2.1. The inclusion of indices $j = -1$ and $j = N + 1$ in Eq. (2.1) introduces two additional coefficients, $Q_{-1}(t)$ and $Q_{N+1}(t)$, which arise naturally from the support of CBS basis functions. However, these coefficients are not independent unknowns. In accordance with standard CBS collocation procedures, they are eliminated using the prescribed boundary conditions (1.3). Specifically by substituting the boundary values $y(a, t) = g_2(t)$ and $y(b, t) = g_3(t)$ into the spline approximation (2.1), two additional linear relations are obtained, which allow $Q_{-1}(t)$ and $Q_{N+1}(t)$ to be expressed in terms of the interior coefficients $Q_0(t), Q_1(t), \dots, Q_N(t)$. This procedure effectively reduces the total number of unknowns and ensures that the resulting algebraic system is closed and uniquely solvable.

The CBS functions maintain substantial geometric properties like local support, convex hull property, geometric invariability, partition of unity, symmetry, and non-negativity. Additionally, the splines $\{Z_{-1}, Z_0, \dots, Z_{N+1}\}$ have been arranged in that form that they can be engaged as basis over the assumed domain $[a, b]$. The approximate solution $Y(r_j, t^n) = Y_j^n$ of the relation (2.1) and (2.2) together with its first and second derivatives can be expressed as [14, 16]:

$$\begin{cases} Y_j^n = Y(r_j, t^n) = a_1 Q_{j-1}^n + a_2 Q_j^n + a_1 Q_{j+1}^n, \\ (Y_r)_j^n = Y_r(r_j, t^n) = -b_1 Q_{j-1}^n + b_2 Q_j^n + b_1 Q_{j+1}^n, \\ (Y_{rr})_j^n = Y_{rr}(r_j, t^n) = c_1 Q_{j-1}^n + c_2 Q_j^n + c_1 Q_{j+1}^n, \end{cases} \quad (2.3)$$

where $a_1 = \frac{1}{6}$, $a_2 = \frac{2}{3}$, $b_1 = \frac{1}{2h}$, $b_2 = 0$, $c_1 = \frac{1}{h^2}$, and $c_2 = -\frac{2}{h^2}$.

3. DESCRIPTION OF NUMERICAL SCHEME

Presume the time domain $[0, T]$ be partitioned equally in M subintervals of length $\Delta t = \frac{T}{M}$, using the knots $0 = t_0 < t_1 < \dots < t_M = T$, where $t_n = t_0 + n\Delta t$ and $n = 0, 1, \dots, M$. The CFD of $\beta \in (0, 1)$ involved in TFCDRE (1.1) is discretized at $t = t_{n+1}$ as [20]:

$$\frac{\partial^\beta y(r, t_{n+1})}{\partial t^\beta} = \frac{1}{\Gamma(2-\beta)} \sum_{z=0}^n u_z \frac{y(r, t_{n+1-z}) - y(r, t_{n-z})}{\Delta t^\beta} + Q_{\Delta t}^{n+1}, \quad (3.1)$$

where $u_z = (z+1)^{1-\beta} - z^{1-\beta}$. The $L1$ approximation used in this work is based on the piecewise linear interpolation of the CFD and is one of the most widely used numerical schemes in fractional calculus. It was originally developed and rigorously analyzed by [57] and [32]. Moreover, the truncation error $Q_{\Delta t}^{n+1}$, confirmed in [32] and expressed as

$$|Q_{\Delta t}^{n+1}| \leq D_p \Delta t^{2-\beta}, \quad (3.2)$$

where D_p is a constant. At this point, we can verify that [30]:

- $u_z > 0$, $z = 0, 1, 2, \dots, n$,
- $1 = u_0 > u_1 > u_2 > \dots > u_z$, $u_z \rightarrow 0$, as $z \rightarrow \infty$,
- $\sum_{z=0}^n (u_z - u_{z+1}) + u_{z+1} = (1 - u_1) + \sum_{z=1}^{n-1} (u_z - u_{z+1}) + u_n = 1$.



Let $y^{n+1}(r) = y(r, t_{n+1})$, $\beta_0 = \frac{1}{\Delta t^\beta \Gamma(2-\beta)}$, then employing (3.1) and θ -weighted scheme, Eq. (1.1) attains the form

$$\begin{aligned} \beta_0 y^{n+1} + \theta [w_1(r) y_r^{n+1} - w_2(r) y_{rr}^{n+1} + w_3(r) y^{n+1}] &= \beta_0 y^n - (1-\theta) [w_1(r) y_r^n \\ - w_2(r) y_{rr}^n + w_3(r) y^n] - \beta_0 \sum_{z=1}^n u_z [y^{n+1-z} - y^{n-z}] p^{n+1}, \quad (n = 0, 1, 2, \dots, N). \end{aligned} \quad (3.3)$$

The system (3.3) becomes implicit for $\theta = 1$, explicit for $\theta = 0$, and Crank-Nicolson for $\theta = 0.5$. Using (2.3) in (3.3), we attain

$$\begin{aligned} m_1 Q_{j-1}^{n+1} + m_2 Q_j^{n+1} + m_3 Q_{j+1}^{n+1} &= \beta_0 [a_1 Q_{j-1}^n + a_2 Q_j^n + a_1 Q_{j+1}^n] - \beta_0 \sum_{z=1}^n u_z \left(a_1 [Q_{j-1}^{n+1-z} - Q_{j-1}^{n-z}] \right. \\ &\quad \left. + a_2 [Q_j^{n+1-z} - Q_j^{n-z}] + a_1 [Q_{j+1}^{n+1-z} - Q_{j+1}^{n-z}] \right) \\ &\quad - (1-\theta) [m_4 Q_{j-1}^n + m_5 Q_j^n + m_6 Q_{j+1}^n] + p_j^{n+1}, \quad j = 0, 1, 2, \dots, N, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} m_1 &= \beta_0 a_1 - \theta w_1(r) b_1 - \theta w_2(r) c_1 + \theta w_3(r) a_1, \quad m_2 = \beta_0 a_2 + \theta w_1(r) b_2 - \theta w_2(r) c_2 + \theta w_3(r) a_2, \\ m_3 &= \beta_0 a_1 + \theta w_1(r) b_1 - \theta w_2(r) c_1 + \theta w_2(r) a_1, \quad m_4 = (1-\theta) (-w_1(r) b_1 - w_2(r) c_1 + w_3(r) a_1), \\ m_5 &= (1-\theta) (w_1(r) b_2 - w_2(r) c_2 + w_3(r) a_2), \quad m_6 = (1-\theta) (w_1(r) b_1 - w_2(r) c_1 + w_3(r) a_1). \end{aligned}$$

The expression (3.4) possesses $(N+1)$ linear equations in $(N+3)$ unknowns $(Q_{-1}, Q_0, Q_1, \dots, Q_{N+1})^T$ can be determined. Two additional equations are needed from the boundary conditions (1.3) to secure a unique solution. That is

$$a_1 Q_{-1}^{n+1} + a_2 Q_0^{n+1} + a_1 Q_1^{n+1} = g_2(t^{n+1}), \quad (3.5)$$

$$a_1 Q_{N-1}^{n+1} + a_2 Q_N^{n+1} + a_1 Q_{N+1}^{n+1} = g_3(t^{n+1}). \quad (3.6)$$

The system then becomes

$$A Q^{n+1} = -D Q^n + B (\beta_0 u_n Q^0 + \beta_0 \sum_{z=0}^{n-1} (u_z - u_{z+1}) Q^{n-z}) + P, \quad (3.7)$$

where

$$\begin{aligned} A &= \begin{bmatrix} a_1 & a_2 & a_1 & \cdots & \cdots & 0 \\ m_1 & m_2 & m_3 & \cdots & \cdots & 0 \\ & m_1 & m_2 & m_3 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & m_1 & m_2 & u_3 & \\ \vdots & \cdots & & m_1 & m_2 & m_3 \\ 0 & \cdots & & a_1 & a_2 & a_1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ m_4 & m_5 & m_6 & \cdots & \cdots & 0 \\ & m_4 & m_5 & m_6 & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ \vdots & \cdots & m_4 & m_5 & m_6 & \\ \vdots & \cdots & & m_4 & m_5 & m_6 \\ 0 & \cdots & & 0 & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ a_1 & a_2 & a_1 & \cdots & \cdots & 0 \\ & a_1 & a_2 & a_1 & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_1 & a_2 & a_1 \\ 0 & \cdots & \cdots & a_1 & a_2 & a_1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad Q^{n+1} = \begin{bmatrix} Q_{-1}^{n+1} \\ Q_0^{n+1} \\ Q_1^{n+1} \\ \vdots \\ Q_{N-1}^{n+1} \\ Q_N^{n+1} \\ Q_{N+1}^{n+1} \end{bmatrix}, \quad P = \begin{bmatrix} g_2^{n+1} \\ p_0^{n+1} \\ p_1^{n+1} \\ \vdots \\ p_{N-1}^{n+1} \\ p_N^{n+1} \\ g_3^{n+1} \end{bmatrix}. \end{aligned}$$

Consequently, we acquire from the system $(N+3) \times (N+3)$ order of linear equations that can be determined uniquely. Before commencing the iteration on the considered scheme, from the initial conditions, two equations are secured

$$\begin{cases} (Y_r)_j^0 = g_1'(r_j), & j = 0, N, \\ (Y)_j^0 = g_1(r_j), & j = 0, 1, \dots, N. \end{cases} \quad (3.8)$$

The CBS approximation leads to an $(N+1)$ -dimensional system for the initial coefficients Q^0 , which cannot be uniquely determined using only the initial condition $y(r, 0) = g_1(r)$. To close this system, two additional relations are required. Following standard practice in B-spline collocation methods, these are obtained by imposing derivative conditions at the boundary nodes, namely $(Y_r)_j^0 = g_1'(r_j)$ for $j = 0, N$. This approach is widely used to construct the initial coefficient vector by combining function and derivative values at the boundaries. Thus, we obtain $(N+3) \times (N+3)$ system order of linear equations which can be represented in a matrix form as:

$$A_1 Q^0 = B_1. \quad (3.9)$$

$$\begin{bmatrix} -b_1 & b_2 & b_1 & \cdots & \cdots & 0 \\ a_1 & a_2 & a_1 & \cdots & \cdots & 0 \\ 0 & a_1 & a_2 & a_1 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & \cdots & & a_1 & a_2 & a_1 & 0 \\ 0 & \cdots & \cdots & & a_1 & a_2 & a_1 \\ 0 & \cdots & \cdots & & -b_1 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} Q_{-1}^0 \\ Q_0^0 \\ Q_1^0 \\ \vdots \\ Q_{N-1}^0 \\ Q_N^0 \\ Q_{N+1}^0 \end{bmatrix} = \begin{bmatrix} g_1'(r_0) \\ g_1(r_0) \\ g_1(r_1) \\ \vdots \\ g_1(r_{N-1}) \\ g_1(r_N) \\ g_1'(r_N) \end{bmatrix}.$$

3.1. Algorithm.

- Formulate the TFCBRE model.
- Discretize the governing equation using the CTD and the CBS basis functions.
- Impose the boundary conditions to ensure a unique numerical solution.
- Assemble the resulting linear system of dimension $(N+3) \times (N+3)$.
- Solve the resulting tridiagonal system using the Thomas algorithm.
- At each time step, compute the numerical solution using Wolfram Mathematica 14.3 on an HP laptop with 16GB of RAM and an Intel Core i7-7500 CPU operating at 2.70 GHz.

4. STABILITY ANALYSIS

Remark 4.1. The governing equation considered in this work is presented in a general form with variable coefficients. However, since the classical von Neumann Fourier stability analysis is rigorously applicable only to linear constant-coefficient problems on uniform grids, the stability analysis in this section is carried out for the constant-coefficient version of Eq. (1.1). Accordingly, the convection coefficient $w_1(r)$, the diffusion coefficient $w_2(r)$, and reaction coefficient $w_3(r)$ are assumed to be constants throughout this section.

Then to investigate the stability properties of the proposed numerical scheme, we employed the classical von Neumann Fourier stability analysis for the constant-coefficient form of Eq. (1.1), explicitly written as:

$$\frac{\partial^2 y(r, t)}{\partial t^2} + w_1 \frac{\partial y(r, t)}{\partial r} - w_2 \frac{\partial^2 y(r, t)}{\partial r^2} + w_3 y(r, t) = p(r, t), \quad t \in [0, 1],$$

where w_1, w_2 , and w_3 are constants. Let φ_j^n represent the Fourier growth factor and $\tilde{\varphi}_j^n$ its estimation value. The error

$$\Pi_j^n = \varphi_j^n - \tilde{\varphi}_j^n, \quad n = 0, 1, \dots, M; \quad j = 1, 2, \dots, N-1, \quad (4.1)$$

where $\Pi^n = [\Pi_1^n, \Pi_2^n, \dots, \Pi_{N-1}^n]^T$. Without loss of generality, we shall consider the stability analysis of the proposed scheme for a force-free case ($p_j^{n+1} = 0$). Applying Eqs. (3.3) and (4.1), we obtain

$$\beta_0 \Pi_j^{n+1} + \theta (w_1 (\Pi_r)_j^{n+1} - w_2 (\Pi_{rr})_j^{n+1} + w_3 \Pi_j^{n+1}) = u_n \beta_0 \Pi_j^0$$



$$-(1-\theta)(w_1(\Pi_r)_j^n - w_2(\Pi_{rr})_j^n + w_3\Pi_j^n) - \beta_0 \sum_{z=0}^{n-1} (u_z - u_{z+1})\Pi_j^{n-z}.$$

So, the error equation becomes

$$\begin{aligned} & (\beta_0 a_1 - \theta w_1 b_1 - \theta w_2 c_1 + \theta w_3 a_1)\Pi_{j-1}^{j+1} + (\beta_0 a_2 - \theta w_2 c_2 + \theta w_3)\Pi_j^{n+1} \\ & + (\beta_0 a_1 + \theta w_1 b_1 - \theta w_2 c_1 + \theta w_3 a_1)\Pi_{j+1}^{n+1} = \beta_0 u_n (a_1 \Pi_{j-1}^0 + a_2 \Pi_j^0 + a_1 \Pi_{j+1}^0) \\ & + \beta_0 \sum_{z=0}^{n-1} (u_z - u_{z+1}) [a_1 \Pi_{j-1}^{n-z} + a_2 \Pi_j^{n-z} + a_1 \Pi_{j+1}^{n-z}] - (1-\theta)w_1 (-b_1 \Pi_{j-1}^n + b_2 \Pi_j^n + b_1 \Pi_{j+1}^n) \\ & + (1-\theta)w_2 (c_1 \Pi_{j-1}^n + c_2 \Pi_j^n + c_1 \Pi_{j+1}^n) - (1-\theta)w_3 (a_1 \Pi_{j-1}^n + a_2 \Pi_j^n + a_1 \Pi_{j+1}^n). \end{aligned} \quad (4.2)$$

Now, the initial and boundary conditions can be written as

$$\Pi_j^0 = g_1(r_j), \quad j = 1, 2, \dots, N, \quad (4.3)$$

and

$$\Pi_0^n = g_2(t_n), \quad \Pi_N^n = g_3(t_n), \quad n = 0, 1, 2, \dots, M. \quad (4.4)$$

Define the mesh function as:

$$\Pi^n = \begin{cases} \Pi_j^n, & r_j - \frac{h}{2} < r \leq r_j + \frac{h}{2}, \quad j = 1, 2, \dots, N-1, \\ 0, & a \leq r \leq a + \frac{h}{2}, \text{ or } b - \frac{h}{2} \leq r \leq b. \end{cases} \quad (4.5)$$

where Π_j^n represents the numerical solution at time step n and spatial grid point j . The Fourier series expansion for $\Pi^n(r)$ can be written as:

$$\Pi^n(r) = \sum_{m=-\infty}^{\infty} \delta^n(m) e^{\frac{i2\pi mr}{(b-a)}}, \quad n = 0, 1, 2, \dots, M, \quad (4.6)$$

where

$$\delta^n(m) = \frac{1}{(b-a)} \int_a^b \Pi^n(r) e^{\frac{-i2\pi mr}{(b-a)}} dr, \quad (4.7)$$

and

$$\Pi^n = [\Pi_1^n, \Pi_2^n, \dots, \Pi_{N-1}^n]^T. \quad (4.8)$$

Using the norm $\|\cdot\|_2$, we secure

$$\begin{aligned} \|\Pi^n\|_2 &= \left(\sum_{i=1}^{N-1} h |\Pi_i^n|^2 \right)^{\frac{1}{2}} \\ &= \left(\int_a^{a+\frac{h}{2}} |\Pi^n|^2 dr + \sum_{i=1}^{N-1} \int_{r_i-\frac{h}{2}}^{r_i+\frac{h}{2}} |\Pi^n|^2 dr + \int_{b-\frac{h}{2}}^b |\Pi^n|^2 dr \right)^{\frac{1}{2}} \\ &= \left(\int_a^b |\Pi^n|^2 dr \right)^{\frac{1}{2}}. \end{aligned}$$

Applying Parseval's identity [47], we achieve

$$\int_a^b |\Pi^n(r)|^2 dr = \sum_{m=-\infty}^{\infty} |\delta^n(m)|^2.$$



Thus, we attain

$$\|\Pi^n\|_2^2 = \sum_{m=-\infty}^{\infty} |\delta^n(m)|^2. \quad (4.9)$$

Assume that Eq. (4.2)-(4.4) have solution in Fourier series form as [19]:

$$\Pi_j^n = \delta^n e^{i\mu h j}, \quad (4.10)$$

where $\mu = \frac{2\pi m}{b-a}$, $i = \sqrt{-1}$, δ , h , μ are the Fourier coefficient, the element size, and the mode number respectively. Using Eq. (4.10) in Eq. (4.2), we secure

$$\begin{aligned} & (\beta_0 a_1 - \theta w_1 b_1 - \theta w_2 c_1 + \theta w_3 a_1) \delta^{n+1} e^{i\mu h(j-1)} + (\beta_0 a_2 + \theta w_1 b_2 - \theta w_2 c_2 + \theta w_3 a_2) \delta^{n+1} e^{i\mu h j} \\ & + (\beta_0 a_1 + \theta w_1 b_1 - \theta w_2 c_1 + \theta w_3 a_1) \delta^{n+1} e^{i\mu h(j+1)} = \beta_0 u_n [a_1 e^{i\mu h(j-1)} + a_2 e^{i\mu h j} + a_1 e^{i\mu h(j+1)}] \delta^0 \\ & + \beta_0 \sum_{z=0}^{n-1} (u_z - u_{z+1}) [a_1 e^{i\mu h(j-1)} + a_2 e^{i\mu h j} + a_1 e^{i\mu h(j+1)}] \delta^{n-z} - (1-\theta) w_1 [-b_1 e^{i\mu h(j-1)} + b_1 e^{i\mu h(j+1)}] \delta^n \\ & + (1-\theta) w_2 [c_1 e^{i\mu h(j-1)} + c_2 e^{i\mu h j} + c_1 e^{i\mu h(j+1)}] \delta^n - (1-\theta) w_3 [a_1 e^{i\mu h(j-1)} + a_2 e^{i\mu h j} + a_1 e^{i\mu h(j+1)}] \delta^n. \end{aligned}$$

Divide the above expression throughout by $e^{i\mu h}$, apply the concept $e^{i\mu h} + e^{-i\mu h} = 2 \cos(\alpha h)$, and repositioning the terms, we attain

$$\begin{aligned} & [(2\beta_0 a_1 - 2\theta w_2 c_1 + 2\theta w_3 a_1) \cos(\mu h) + \beta_0 a_2 - \theta w_2 c_2 + \theta w_3 a_2 + (2i\theta w_1 b_1) \sin(\mu h)] \delta^{n+1} \\ & = \beta_0 u_n [a_2 + 2a_1 \cos(\mu h)] \delta^0 - (1-\theta) [2i w_1 b_1 \sin(\mu h) - w_2 (c_2 + 2c_1 \cos(\mu h)) \\ & + w_3 (a_2 + 2a_1 \cos(\mu h))] \delta^n + \beta_0 (a_2 + 2a_1 \cos(\mu h)) \sum_{z=0}^{n-1} (u_z - u_{z+1}) \delta^{n-z}. \end{aligned}$$

After applying $\cos(\mu h) = 1 - 2 \sin^2(\frac{\mu h}{2})$ in the above expression, we obtain

$$\begin{aligned} & [(2\beta_0 a_1 - 2\theta w_2 c_1 + 2\theta w_3 a_1) + \beta_0 a_2 - \theta w_2 c_2 + \theta w_3 a_2] \delta^{n+1} = \beta_0 u_n [a_2 + 2a_1] \delta^0 \\ & - (1-\theta) [-w_2 (c_2 + 2c_1) + w_3 (a_2 + 2a_1)] \delta^n + \beta_0 (a_2 + 2a_1) \sum_{z=0}^{n-1} (u_z - u_{z+1}) \delta^{n-z}. \end{aligned}$$

By replacing the values of a_1, a_2, b_1, c_1 and c_2 in the above equation and after further simplification, we have

$$\delta^{n+1} = \frac{u_n \beta_0}{\Psi} \delta^0 - \frac{\Psi_1}{\Psi} \delta^n + \frac{\beta_0}{\Psi} \sum_{z=0}^{n-1} (u_z - u_{z+1}) \delta^{n-z}, \quad (4.11)$$

where

$$\begin{aligned} \Psi_1 &= \frac{(1-\theta) [3i h w_1 \sin(\mu h) + 12 w_2 \sin^2(\frac{\mu h}{2}) + h^2 w_3 (3 - 2 \sin^2(\frac{\mu h}{2}))]}{(3 - 2 \sin^2(\frac{\mu h}{2})) h^2}, \\ \Psi &= (\beta_0 + \theta w_3) + \frac{12 \theta w_2 \sin^2 + 3 \theta i h w_1 \sin(\mu h) + (\frac{\rho h}{2})}{h^2 (3 - 2 \sin^2(\frac{\mu h}{2}))}. \end{aligned}$$

For every complex number $f_1 + i f_2 \geq f_3 + i f_4$ it holds if and only if $f_1 \geq f_3$ or $f_2 \geq f_4$ [15]. Therefore, from the foregoing, $\Psi \geq \alpha_0$ and $\Psi_1 > 0$.

Proposition 4.2. Presume that δ^n is a solution of (4.11), then we gain

$$|\delta^n| \leq |\delta^0|, \quad n = 0, 1, 2, \dots, M. \quad (4.12)$$



Proof. Mathematical induction is employed to validate the inequality (4.12). For $n = 0$, expression (4.11) becomes

$$\begin{aligned}\delta^1 &= \frac{u_0\beta_0}{\Psi}\delta^0 - \frac{\Psi_1}{\Psi}\delta^0 \leq \frac{u_0\beta_0}{\Psi}\delta^0, \\ \delta^1 &\leq \delta^0, \quad \Psi \geq \beta_0, \\ |\delta^{n+1}| &\leq |\delta^0|.\end{aligned}$$

Let $n = 0, 1, 2, \dots, N-1$, $|\delta^n| \leq |\delta^0|$ is true, certainly from expression (4.11), we gain

$$\begin{aligned}\delta^{n+1} &= \frac{u_n\beta_0}{\Psi}\delta^0 - \frac{\Psi_1}{\Psi}\delta^n + \frac{\beta_0}{\Psi} \sum_{z=0}^{n-1} (u_z - u_{z+1})\delta^{n-z}, \\ |\delta^{n+1}| &\leq \frac{u_n\beta_0}{\Psi} |\delta^0| + \frac{\beta_0}{\Psi} \sum_{z=0}^{n-1} (u_z - u_{z+1}) |\delta^{n-z}| \\ &\leq \frac{u_n\beta_0}{\Psi} |\delta^0| + \frac{\beta_0}{\Psi} \sum_{z=0}^{n-1} (u_z - u_{z+1}) |\delta^0| \\ &\leq \frac{1}{\Psi} \beta_0 \left[\sum_{z=0}^{n-1} (u_z - u_{z+1}) + u_n \right] |\delta^0| \\ &= \frac{1}{\Psi} \beta_0 [(1 - u_1) + (u_1 - u_2) + \dots + (u_{n-1} - u_n) + u_n] |\delta^0|, \\ |\delta^{n+1}| &= \frac{1}{\Psi} \beta_0 \delta^0 \leq |\delta^0|.\end{aligned}$$

□

Theorem 4.3. *The implicit collocation technique given in Eq. (3.4) is unconditionally stable for the constant-coefficient problem.*

Proof. Employing Eq. (3.4) and Proposition 4.2, we have

$$\|\Pi^n\|_2 \leq \|\Pi^0\|_2, \quad n = 0, 1, 2, \dots, M,$$

which signifies that the suggested method is unconditionally stable. □

Remark 4.4. The above stability result is established under the assumption of constant diffusion, convection, and reaction coefficients. For variable-coefficient TFCDREs, the stability analysis may require alternative approaches, such as energy estimates or matrix stability techniques. Nevertheless, the present analysis provides valuable insight into the stability characteristics of the proposed numerical scheme and is consistent with the assumptions underlying the classical von Neumann methodology.

5. CONVERGENCE ANALYSIS

We employed the procedure used in [29] in considering the convergence analysis of the suggested method.

The convergence analysis is based on the local truncation error of the $L1$ approximation for the CFD and the approximation properties of the CBS basis functions. The $L1$ discretization contributes a temporal error of order ($O(\Delta t^{2-\beta})$), while the CBS collocation approximation yields second-order spatial accuracy. Assuming sufficient regularity of the exact solution and stability of the discrete scheme, the global error satisfies the estimate. Now, we brought forward some important theorems and lemma [17, 26].

Theorem 5.1. [17, 26] *Supposing $y(r, t) \in C^4[a, b]$, $p \in C^2[a, b]$ respectively, and $\Phi = \{a = r_0, r_1, \dots, r_n = b\}$ be the equally partitioning of $[a, b]$ of length h such that $r_j = a + jh$, $j = 0, 1, \dots, N$. Again, let $\hat{Y}(r, t)$ be the CBS approximation of Eq. (1.1) at points $r_j \in \Phi$, then there exists a constant G_j independent of h , we attain*

$$\|D^j(y(r, t) - \hat{Y}(r, t))\|_\infty \leq G_j h^{4-j}, \quad j = 0, 1, 2. \quad (5.1)$$



Lemma 5.2. [8] The CBS basis $\{Z_{-1}, Z_0, \dots, Z_{N+1}\}$ expressed in Eq. (2.2) satisfy the inequality

$$\sum_{j=-1}^{N+1} |Z_j(r)| \leq \frac{5}{3}, \quad 0 \leq r \leq 1. \quad (5.2)$$

Theorem 5.3. The numerically approximated spline solution $Y(r, t)$ for the exact solution $y(r, t)$ of TFCDRE (1.1)–(1.3) exists. In addition, if $p \in C^2[0, 1]$, we secure

$$\|y(r, t) - Y(r, t)\|_\infty \leq Gh^2, \quad \forall t \geq 0, \quad (5.3)$$

where h is satisfactorily small and G is a positive constant and not depending on h .

Proof. Let $\hat{Y}(r, t) = \sum_{j=-1}^{N+1} Q_j^n(t) Z_j(r)$ be the approximated spline solution for $Y(r, t)$. Utilizing triangular inequality, we arrive at the following

$$\|y(r, t) - Y(r, t)\|_\infty \leq \|y(r, t) - \hat{Y}(r, t)\|_\infty + \|\hat{Y}(r, t) - Y(r, t)\|_\infty. \quad (5.4)$$

By Theorem (5.1), we gain

$$\|D^j(y(r, t) - \hat{Y}(r, t))\|_\infty \leq G_j h^{4-j}, \quad j = 0, 1, 2. \quad (5.5)$$

Employing the inequality (5.5), we gain

$$\|y(r, t) - Y(r, t)\|_\infty \leq G_0 h^4 + \|\hat{Y}(r, t) - Y(r, t)\|_\infty. \quad (5.6)$$

The collocation conditions are

$$Ly(r_j, t) = LY(r_j, t) = p(r_j, t), \quad j = 0 : 1 : N.$$

Presume

$$L\hat{Y}(r_j, t) = \hat{p}(r_j, t), \quad j = 0 : 1 : N.$$

Therefore, the difference equations $L(\hat{Y}(r_j, t) - Y(r_j, t))$ of the propounded technique at any time level n , can be expressed as

$$\begin{aligned} & ([\beta_0 + \theta w_3(r)]a_1 - \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j-1}^{n+1} + ([\beta_0 + \theta w_3(r)]a_2 - \theta w_2(r)c_2)\xi_j^{n+1} \\ & + ([\beta_0 + \theta w_3(r)]a_1 + \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j+1}^{n+1} = \beta_0(a_1\xi_{j-1}^n + a_2\xi_j^n + a_1\xi_{j+1}^n) \\ & - \beta_0 \sum_{z=1}^n u_l \{a_1[\xi_{j-1}^{n+1-z} - \xi_{j-1}^{n-z}] + a_2[\xi_j^{n+1-z} - \xi_j^{n-z}] + a_1[\xi_{j+1}^{n+1-z} - \xi_{j+1}^{n-z}]\} \\ & - (1 - \theta)w_1(r)(-b_1\xi_{j-1}^n + b_1\xi_{j+1}^n) + (1 - \theta)w_2(r)(c_1\xi_{j-1}^n + c_2\xi_j^n + c_1\xi_{j+1}^n) \\ & - (1 - \theta)w_3(r)(a_1\xi_{j-1}^n + a_2\xi_j^n + a_1\xi_{j+1}^n) + p_j^{n+1}. \end{aligned} \quad (5.7)$$

The boundary conditions can be written as:

$$a_1\xi_{j-1}^{n+1} + a_2\xi_j^{n+1} + a_1\xi_{j+1}^{n+1} = 0, \quad j = 0, N,$$

where

$$\xi_j^n = D_j^n - d_j^n, \quad j = -1, 0, 1, \dots, N+1, \quad (5.8)$$

and

$$V_j^n = h^2[p_j^n - \hat{p}_j^n], \quad j = 0, 1, 2, \dots, N.$$

From inequality (5.5), we receive

$$|V_j^n| = h^2|p_j^n - \hat{p}_j^n| \leq Gh^4, \quad j = 0, 1, 2, \dots, N. \quad (5.9)$$

Consider $V^n = \max\{|V_j^n|; 0 \leq j \leq N\}$, $\hat{X}_j^n = |\xi_j^n|$ and $\hat{X}^n = \max\{|\hat{X}_j^n|, 0 \leq j \leq N\}$.

When $n = 0$, Eq. (5.7) becomes

$$\begin{aligned} & ([\beta_0 + \theta w_3(r)]a_1 - \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j-1}^1 + ([\beta_0 + \theta w_3(r)]a_2 - \theta w_2(r)c_2)\xi_j^1 \\ & + ([\beta_0 + \theta w_3(r)]a_1 + \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j+1}^1 = \beta_0(a_1\xi_{j-1}^0 + a_2\xi_j^0 + a_1\xi_{j+1}^0) \end{aligned}$$



$$- (1 - \theta) [(-w_1(r)b_1 - w_2(r)c_1 + w_3(r)a_1)\xi_{j-1}^0 + (w_2(r)c_2 + w_3(r)a_2)\xi_j^0 + (w_1(r)b_1 - w_1(r)c_1 + w_3(r)a_1)\xi_{j+1}^0] + \frac{1}{h^2}V_j^1, \quad j = 0, 1, 2, \dots, N.$$

Applying the initial condition, $X^0 = 0$:

$$([\beta_0 + \theta w_3(r)]a_2 - \theta w_2(r)c_2)\xi_j^1 = -((\beta_0 + w_3(r))a_1 - \theta w_2(r)c_1)(\xi_{j-1}^1 + \xi_{j+1}^1) + \theta w_1(r)b_1(\xi_{j-1}^1 - \xi_{j+1}^1) + \frac{1}{h^2}V_j^1.$$

Taking norms of ξ_j^n and V_j^n with adequately small h , we gain

$$\hat{X}_j^1 \leq \frac{3Gh^4}{(\beta_0 + w_3(r))h^2 + 12\theta w_2(r) - 3\theta w_1(r)h}, \quad j = 0, 1, 2, \dots, N.$$

and engaging boundary conditions, we conclude that

$$\hat{X}^1 \leq G_1 h^2, \quad (5.10)$$

where G_1 does not aligned on h .

Mathematical induction on n is employed. Supposing $\hat{X}_j^k \leq G_e h^2$ is true for $e = 1, 2, \dots, n$, and $G = \max\{G_e : 0 \leq e \leq n\}$, then Eq. (5.7) becomes

$$\begin{aligned} & ([\beta_0 + \theta w_3(r)]a_1 - \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j-1}^{n+1} + ([\beta_0 + \theta w_3(r)]a_2 - \theta w_2(r)c_2)\xi_j^{n+1} \\ & + ([\beta_0 + \theta w_3(r)]a_1 + \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j+1}^{n+1} = \beta_0(a_1\xi_{j-1}^n + a_2\xi_j^n + a_1\xi_{j+1}^n) \\ & - \beta_0 \sum_{z=1}^n u_z \{a_1[\xi_{j-1}^{n+1-z} - \xi_{j-1}^{n-z}] + a_2[\xi_j^{n+1-z} - \xi_j^{n-z}] + a_1[\xi_{j+1}^{n+1-z} - \xi_{j+1}^{n-z}]\} \\ & - (1 - \theta) [(-w_1(r)b_1 - w_2(r)c_1 + w_3(r)a_1)\xi_{j-1}^n + (w_2(r)c_2 + w_3(r)a_2)\xi_j^n \\ & + (w_1(r)b_1 - w_2(r)c_1 + w_3(r)a_1)\xi_{j+1}^n] + \frac{1}{h^2}V_j^n. \end{aligned}$$

After reorganizing the terms, we attain

$$\begin{aligned} & ([\beta_0 + \theta w_3(r)]a_1 - \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j-1}^{n+1} + ([\beta_0 + \theta w_3(r)]a_2 - \theta w_2(r)c_2)\xi_j^{n+1} \\ & + ([\beta_0 + \theta w_3(r)]a_1 + \theta w_1(r)b_1 - \theta w_2(r)c_1)\xi_{j+1}^{n+1} = \beta_0[(u_0 - u_1)(a_1\xi_{j-1}^n + a_2\xi_j^n + a_1\xi_{j+1}^n) \\ & + (u_1 - u_2)(a_1\xi_{j-1}^{n-1} + a_2\xi_j^{n-1} + a_1\xi_{j+1}^{n-1}) + \dots + (u_{n-1} - u_n)(a_1\xi_{j-1}^1 + a_2\xi_j^1 + a_1\xi_{j+1}^1) \\ & + u_n(a_1\xi_{j-1}^0 + a_2\xi_j^0 + a_1\xi_{j+1}^0)] - (1 - \theta) [(-w_1(r)b_1 - w_2(r)c_1 + w_3(r)a_1)\xi_{j-1}^n \\ & + (w_2(r)c_2 + w_3(r)a_2)\xi_j^n + (w_1(r)b_1 - w_2(r)c_1 + w_3(r)a_1)\xi_{j+1}^n] + \frac{1}{h^2}V_j^{n+1}. \end{aligned}$$

Taking absolute values of ξ_j^{n+1} and V_j^{n+1} , we get

$$\hat{X}_j^{n+1} \leq \frac{3h^2 \left(\beta_0 \sum_{z=0}^{n-1} (u_z - u_{z+1})Gh^2 + Gh^2 - (1 - \theta)G_n h^2 \right)}{(\beta_0 + w_3(r))h^2 + 12\theta w_2(r) - 3\theta w_1(r)h}. \quad (5.11)$$

From the end conditions

$$\hat{X}_j^{n+1} \leq Gh^2, \quad j = -1, N + 1.$$

Definitely, for all values of n , we obtain

$$\hat{X}_j^{n+1} \leq Gh^2. \quad (5.12)$$

Now, from the above inequality

$$\hat{Y}(r, t) - Y(r, t) = \sum_{j=-1}^{N+1} (d_j(t) - D_j(t))Z_j(r).$$



Hence, from Lemma (5.2) and inequality (5.12), we attain

$$\|\hat{Y}(r, t) - Y(r, t)\|_\infty \leq \frac{5}{3}Gh^2. \quad (5.13)$$

Applying (5.13), the triangular inequality (5.3) becomes

$$\|y(r, t) - Y(r, t)\|_\infty \leq \|y(r, t) - \hat{Y}(r, t)\|_\infty + \|\hat{Y}(r, t) - Y(r, t)\|_\infty. \quad (5.14)$$

Employing the inequality (5.1), (5.12), and (5.14), we gain

$$\|y(r, t) - \hat{Y}(r, t)\|_\infty + \|\hat{Y}(r, t) - Y(r, t)\|_\infty \leq G_0h^4 + \frac{5}{3}Gh^2 = \tilde{G}h^2,$$

where $\tilde{G} = G_0h^2 + \frac{5}{3}G$. Therefore, there exists a constant $\tilde{G}$ independent of h .

$$\|y(r, t) - \hat{Y}(r, t)\|_\infty \leq \tilde{G}h^2.$$

□

Applying the preceding theorem and relation (3.2), it can be concluded that the proposed numerical is anticipated to attain unconditional convergence. Hence,

$$\|y(r, t) - Y(r, t)\|_\infty \leq \tilde{G}h^2 + D_p\Delta t^{2-\beta},$$

where $\beta \in (0, 1]$, $\tilde{G}$, and D_p a constant. Consequently, the error analysis indicates that the proposed scheme is expected to achieve convergence of order $O(h^2 + \Delta t^{2-\beta})$ which is confirmed by the numerical experiments.

6. EXPOSITORY EXAMPLES AND DISCUSSIONS

In this section, we present three numerical experiments to demonstrate the computational efficiency and simplicity of our scheme by employing the error norms $L_\infty(N_j)$, $L_2(N_j)$: that is

$$L_2(N_j) = \sqrt{h \sum_{j=0}^N |Y(r_j, t) - y(r_j, t)|^2}, \quad L_\infty(N_j) = \max_{0 \leq j \leq N} |Y(r_j, t) - y(r_j, t)|,$$

and the experimental order of convergence (EOC) is computed using $EOC = \frac{\log(L_\infty(N_j)/L_\infty(2N_j))}{\log(2)}$, where $(L_\infty(N_j))$ and $(L_\infty(2N_j))$ denote the maximum errors obtained on two successive meshes with (N_j) and $(2N_j)$ spatial subdivisions, respectively. All numerical simulations, error computations, convergence orders, and graphical visualizations reported in this study were generated using Mathematica 14.3. The tabulated results were obtained directly from the corresponding computational outputs.

Example 6.1. Consider the following TFCDRE given by

$$\frac{\partial^\beta y(r, t)}{\partial t^\beta} + \frac{1}{(1+r)} \frac{\partial y(r, t)}{\partial r} - \frac{\partial^2 y(r, t)}{\partial r^2} - (1+2r)^2 y(r, t) = p(r, t), \quad r \in [0, 1], \quad t \in [0, 1], \quad (6.1)$$

where the exact solution, as derived in [61], is $y(r, t) = (1 + t^{3+\alpha})r(2-r)$. The initial and boundary conditions are prescribed as $g_1(r) = r(2-r)$, $g_2(t) = 0$, and $g_3(t) = (1 + t^{3+\beta})$. The corresponding source term is

$$p(r, t) = \frac{1}{6}\Gamma(4+\beta)t^3r(2-r) + (1+t^{3+\beta})\left(\frac{4}{1+r} - r(2-r)(1+2r)^2\right).$$

The approximate piecewise polynomial solution is obtained from the general CBS (2.1) representation,

$$Y(r, t) = Q_{j-1}^n Z_{j-1}(r) + Q_j^n Z_j(r) + Q_{j+1}^n Z_{j+1}(r),$$

where $Z_j(r)$ are the CBS basis functions and Q_j^n the time-dependent coefficients evaluated at each time level. Tables 1-4 and Figures 1-4 summarize the results for this example under various discretization parameters. For instance, Eq. (6.2) represents the explicit piecewise polynomial expressions of $Y(r, t)$ when $N = 100$, $\beta = 0.25$, $\Delta t = 0.01$, and time $t = 1$. Table 1 compares the approximate and exact solutions for $N = 100$, and $\beta = 0.25, 0.50$, revealing excellent



numerical-analytical agreement with negligible absolute error. Tables 2 and 3 confirm the accuracy and stability of the method at different fractional orders and time levels. In Table 4, the approximate solutions and error norms are presented for varying mesh densities N when $\alpha = 0.45$, $\Delta t = 0.0125$, and $t = 1$. Figure 1 shows the 2D comparison of the approximate and exact solutions when $N = 100$, $\Delta t = 0.0125$, and $\beta = 0.35$. Figure 2 illustrates both 2D and 3D absolute error distributions for $\beta = 0.55$, $N = 100$, and $\Delta t = 0.01$, while Figure 3 provides a 3D visualization of the approximate versus exact solutions at $N = 70$, $\beta = 0.65$, $\Delta t = 0.02$, and $t = 1$. In Figure 4 displays contour plots of the exact and numerical solutions and their corresponding error fields for $N = 32$, $\alpha = 0.7$, $\Delta t = 0.01$, and $t = 1$. The results consistently validate the accuracy, efficiency, and robustness of the proposed numerical scheme in solving the TFCBRE.

$$Y(r, t) = \begin{cases} 1.62448 \times 10^{-14} + 4.00007r - 1.99996r^2 - 0.0000876017r^3, & r \in [0.00, 0.01), \\ 9.71785 \times 10^{-13} + 4.00007r - 1.99996r^2 - 0.0000885572r^3, & r \in [0.01, 0.02), \\ 8.83957 \times 10^{-12} + 4.00007r - 1.99996r^2 - 0.0000895407r^3, & r \in [0.02, 0.03), \\ 3.61422 \times 10^{-11} + 4.00007r - 1.99996r^2 - 0.0000905519r^3, & r \in [0.03, 0.04), \\ 1.02627 \times 10^{-10} + 4.00007r - 1.99996r^2 - 0.0000915907r^3, & r \in [0.04, 0.05), \\ 2.35866 \times 10^{-10} + 4.00007r - 1.99996r^2 - 0.0000926567r^3, & r \in [0.05, 0.06), \\ \vdots & \vdots \\ 7.27527 \times 10^{-7} + 4.00006r - 1.99994r^2 - 0.000134306r^3, & r \in [0.49, 0.50), \\ 6.61585 \times 10^{-7} + 4.00007r - 1.99994r^2 - 0.000133779r^3, & r \in [0.50, 0.51), \\ 5.71366 \times 10^{-7} + 4.00007r - 1.99994r^2 - 0.000133099r^3, & r \in [0.51, 0.52), \\ 4.53358 \times 10^{-7} + 4.00007r - 1.99994r^2 - 0.000132259r^3, & r \in [0.52, 0.53), \\ \vdots & \vdots \\ -0.00017336 + 4.00068r - 2.00067r^2 + 0.000165002r^3, & r \in [0.97, 0.98), \\ -0.000185645 + 4.00072r - 2.00071r^2 + 0.000178054r^3, & r \in [0.98, 0.99), \\ -0.0001985 + 4.00076r - 2.00075r^2 + 0.000191303r^3, & r \in [0.99, 1.00). \end{cases} \quad (6.2)$$

TABLE 1. The comparison of approximate and exact solutions for Example 6.1 at $N = 100$, $\Delta t = 0.01$ and $t = 1$.

r	Exact	Approximate Solution		Error	
		$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.25$	$\beta = 0.50$
0.1	0.38	0.3800075043	0.3800531259	$7.50426447 \times 10^{-6}$	$5.31259784 \times 10^{-5}$
0.2	0.72	0.7200151724	0.7201074152	$1.51723772 \times 10^{-5}$	$1.07415192 \times 10^{-4}$
0.3	1.02	1.0200223794	1.0201584606	$2.23794189 \times 10^{-5}$	$1.58460558 \times 10^{-4}$
0.4	1.28	1.2800284234	1.2802013151	$2.84233708 \times 10^{-5}$	$2.01315082 \times 10^{-4}$
0.5	1.50	1.5000325348	1.5002305478	$3.25347991 \times 10^{-5}$	$2.30547793 \times 10^{-4}$
0.6	1.68	1.6800339125	1.6802404812	$3.39125157 \times 10^{-5}$	$2.40481179 \times 10^{-4}$
0.7	1.82	1.8200317939	1.8202256731	$3.17938984 \times 10^{-5}$	$2.25673131 \times 10^{-4}$
0.8	1.92	1.9200255655	1.9201816866	$2.55655309 \times 10^{-5}$	$1.81686576 \times 10^{-4}$
0.9	1.98	1.9800149133	1.9801061455	$1.49132846 \times 10^{-5}$	$1.06145494 \times 10^{-4}$

Example 6.2. Consider the following TFCBRE

$$\frac{\partial^\beta y(r, t)}{\partial t^\beta} + \frac{\partial y(r, t)}{\partial r} - \frac{\partial^2 y(r, t)}{\partial r^2} + y(r, t) = p(r, t), \quad (r, t) \in [0, 1] \times [0, 1], \quad (6.3)$$



TABLE 2. Absolute errors of Example 6.1 for different values of β when $\Delta t = 0.01$, $N = 100$, and $t = 1$.

r	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.1	1.47585×10^{-6}	1.15197×10^{-5}	5.31260×10^{-5}	2.11448×10^{-4}	7.71749×10^{-4}
0.2	2.98388×10^{-6}	2.32910×10^{-5}	1.07415×10^{-4}	4.48753×10^{-4}	1.56054×10^{-3}
0.3	4.40100×10^{-6}	3.43553×10^{-5}	1.58461×10^{-4}	6.30824×10^{-4}	2.30309×10^{-3}
0.4	5.58886×10^{-6}	4.36357×10^{-5}	2.01315×10^{-4}	6.30824×10^{-4}	2.92852×10^{-3}
0.5	6.39595×10^{-6}	4.99516×10^{-5}	2.30548×10^{-4}	9.18666×10^{-4}	3.35863×10^{-3}
0.6	6.66477×10^{-6}	5.20730×10^{-5}	2.40481×10^{-4}	9.59064×10^{-4}	3.51069×10^{-3}
0.7	6.24584×10^{-6}	4.88276×10^{-5}	2.25673×10^{-4}	9.01038×10^{-4}	3.30379×10^{-3}
0.8	5.01965×10^{-6}	3.92704×10^{-5}	1.81687×10^{-4}	7.26475×10^{-4}	2.66938×10^{-3}
0.9	2.92622×10^{-6}	2.09055×10^{-4}	1.06145×10^{-4}	4.25191×10^{-4}	1.56642×10^{-3}

TABLE 3. Absolute errors for Example 6.1 when $\Delta t = 0.002$, $N = 100$, and $\beta = 0.45$ at different time levels.

r	$t = 0.1$	$t = 0.5$	$t = 0.9$	$t = 1.0$
0.1	7.72920×10^{-8}	1.06726×10^{-6}	2.68336×10^{-6}	3.16046×10^{-6}
0.2	1.56301×10^{-7}	2.15798×10^{-6}	5.42546×10^{-6}	6.39007×10^{-6}
0.3	2.30858×10^{-7}	2.15798×10^{-6}	8.00374×10^{-6}	9.42638×10^{-6}
0.4	2.94114×10^{-7}	4.04759×10^{-6}	1.01683×10^{-5}	1.19747×10^{-5}
0.5	3.38437×10^{-7}	4.63951×10^{-6}	1.16450×10^{-5}	1.37118×10^{-5}
0.6	3.55528×10^{-7}	4.84581×10^{-6}	1.21469×10^{-5}	1.42999×10^{-5}
0.7	3.36871×10^{-7}	4.55557×10^{-6}	1.13991×10^{-5}	1.34160×10^{-5}
0.8	2.74616×10^{-7}	3.67612×10^{-6}	9.17756×10^{-6}	1.07976×10^{-5}
0.9	1.62957×10^{-7}	2.15388×10^{-6}	5.36194×10^{-6}	6.30567×10^{-6}

TABLE 4. Exact and approximate solution with error norms when $\beta = 0.45$, $\Delta t = 0.0125$, and $t = 1$ for Example 6.1.

r	Exact	Approximate Solution			
		$N = 20$	$N = 40$	$N = 80$	$N = 120$
0.1	0.38	0.38005177	0.38005178	0.38005179	0.38005179
0.2	0.72	0.72010465	0.72010469	0.72010471	0.72010471
0.3	1.02	1.02015436	1.02015444	1.02015446	1.02015447
0.4	1.28	1.28019606	1.28021962	1.28019622	1.28019623
0.5	1.50	1.50022446	1.50022464	1.50022469	1.50022470
0.6	1.68	1.68023404	1.68023427	1.68023433	1.68023435
0.7	1.82	1.82021952	1.82021979	1.82021985	1.82021987
0.8	1.92	1.92017664	1.92017689	1.92017695	1.92017697
0.9	1.98	1.98010313	1.98010330	1.98010334	1.98010335
L_∞		2.34038×10^{-4}	2.34273×10^{-4}	2.34343×10^{-4}	2.34392×10^{-4}
L_2		7.36345×10^{-4}	1.04235×10^{-3}	1.47446×10^{-3}	1.80592×10^{-3}

where the exact analytical solution is $y(r, t) = t^2 \sin(2\pi r)$ [31]. The initial and boundary conditions are specified as $y(r, 0) = 0$, $y(0, t) = y(1, t) = 0$, while the corresponding source term is given by

$$p(r, t) = 2\pi t^2 \cos(2\pi r) + (4\pi^2 + 1)t^2 \sin(2\pi r) + \frac{2t^{2-\beta}}{\Gamma(3-\beta)} \sin(2\pi r).$$



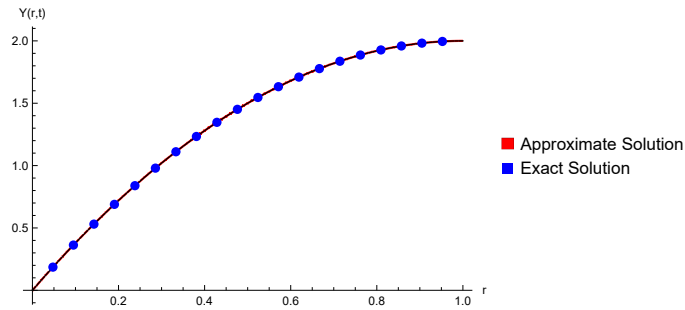


FIGURE 1. Approximate and exact solutions representation for Example 6.1 when $\beta = 0.35$, $N = 100$, $\Delta t = 0.0125$, and time $t = 1$.

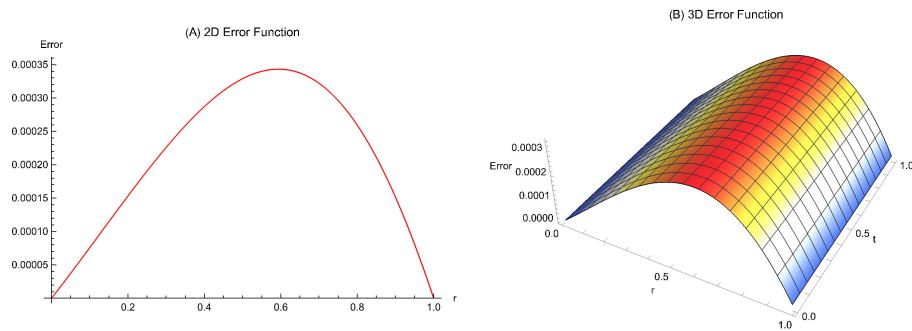


FIGURE 2. 2D and 3D error profiles when $\beta = 0.55$, $N = 100$, and $t = 1$ for Example 6.1.

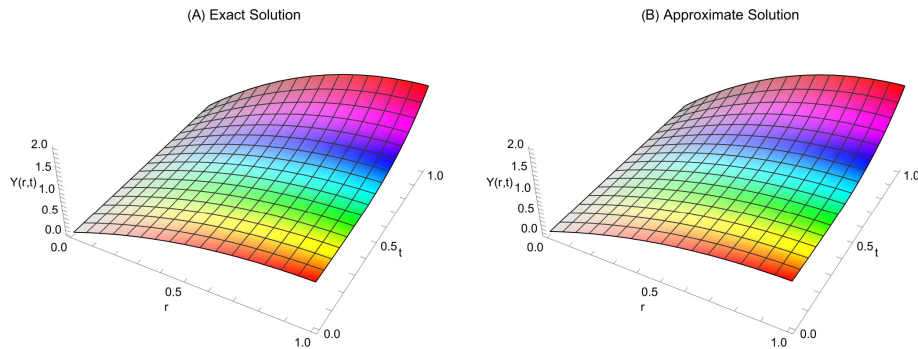


FIGURE 3. 3D approximate and exact solutions profiles when $\beta = 0.65$, $N = 70$, $\Delta t = 0.02$ at $t = 1$, and $r \in [0, 1]$ for Example 6.1.

The piecewise polynomial approximations for this problem is presented in Eq. (6.4), and the numerical experiments performed for parameters $\beta = 0.35$, $N = 100$, $\Delta t = 0.0125$, and time $t = 1$. Table 5 summarizes the computed error norms, EOC, and CPU execution time for $\Delta t = 0.002$ at $t = 1$. For validation, the results were benchmarked against those reported in [31]. The proposed CBS-Crank-Nicolson method exhibits better accuracy and consistency with the theoretical convergence rate, confirming its high precision and computational efficiency. Table 6 reports the L_∞ and L_2 error norms at multiple time levels, demonstrating that the numerical error decreases systematically with mesh

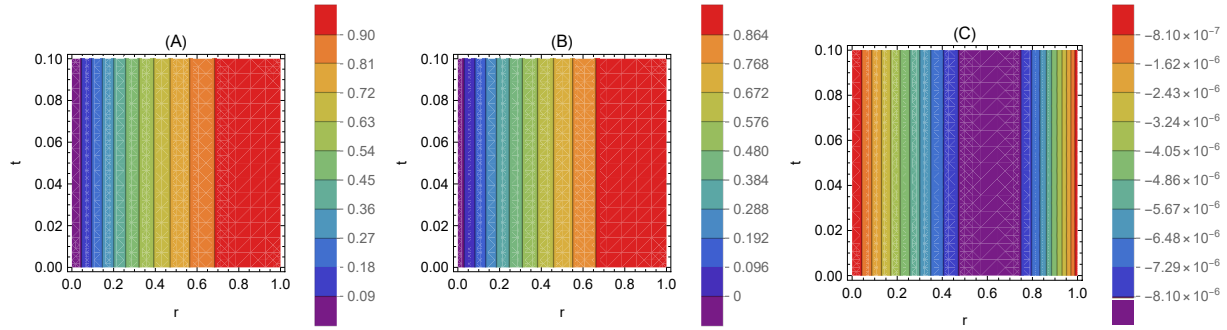


FIGURE 4. The contour plots of (A) the exact solution, (B) approximate solutions, and (C) error for $\alpha = 0.7$, $N = 32$, $\Delta t = 0.01$ at time $t = 1$ for Example 6.1.

refinement and fractional order. Table 7 highlights the absolute error at selected spatial locations r for $\beta = 0.3, 0.7$, $\Delta t = 0.0125$ at time $t = 1$. The results show excellent pointwise agreement and minimal deviation across the spatial domain.

Figure 5 depicts the temporal evolution of the approximate and exact solutions for $\beta = 0.25$, $N = 32$, and $\Delta t = 0.004$, highlighting the accuracy and stability of the numerical predictions over time. Figure 6 presents both 2D and 3D error distributions for $N = 100$, $\beta = 0.3$, and $\Delta t = 0.0125$, revealing smooth, uniformly small error profiles. Figure 7 provides a 3D visualization of the approximate versus exact solutions at $N = 80$, $\beta = 0.95$, $\Delta t = 0.01$ at $t = 1$. Finally, Figure 8, illustrates the contour plots of the exact and approximate solutions alongside their corresponding error surfaces for $N = 40$, $\alpha = 0.75$, $\Delta t = 0.01$, and $t = 1$.

$$Y(r, t) = \begin{cases} -1.578598 \times 10^{-13} + 6.28135r - 0.000917423r^2 - 41.31572r^3, & r \in [0.00, 0.01), \\ -1.631705 \times 10^{-7} + 6.281399r - 0.00581253r^2 - 41.15255r^3, & r \in [0.01, 0.02), \\ -2.767816 \times 10^{-6} + 6.28179r - 0.02534737r^2 - 40.82697r^3, & r \in [0.02, 0.03), \\ -1.590888 \times 10^{-5} + 6.283104r - 0.06915091r^2 - 40.34027r^3, & r \in [0.03, 0.04), \\ -5.724716 \times 10^{-5} + 6.286205r - 0.1466602r^2 - 39.69436r^3, & r \in [0.04, 0.05), \\ -1.575679 \times 10^{-4} + 6.292224r - 0.2670451r^2 - 38.89179r^3, & r \in [0.05, 0.06), \\ \vdots & \vdots \\ -2.023692 + 24.70472r - 61.9729r^2 + 41.31583r^3, & r \in [0.49, 0.50), \\ -2.023678 + 24.70463r - 61.97273r^2 + 41.31572r^3, & r \in [0.50, 0.51), \\ -2.002033 + 24.57731r - 61.72308r^2 + 41.15254r^3, & r \in [0.51, 0.52), \\ -1.956254 + 24.3132r - 61.21517r^2 + 40.82696r^3, & r \in [0.52, 0.53), \\ \vdots & \vdots \\ 34.56935 - 116.248r + 122.5062r^2 - 40.8276r^3, & r \in [0.97, 0.98), \\ 34.87557 - 117.1854r + 123.4628r^2 - 41.15295r^3, & r \in [0.98, 0.99), \\ 35.03367 - 117.6645r + 123.9467r^2 - 41.31589r^3, & r \in [0.99, 1.00). \end{cases} \quad (6.4)$$

Example 6.3. Consider the following TFCDRE:

$$\frac{\partial^\beta y(r, t)}{\partial t^\beta} + \frac{\partial y(r, t)}{\partial r} - \frac{\partial^2 y(r, t)}{\partial r^2} + y(r, t) = p(r, t), \quad (r, t) \in (0, 1) \times (0, 1], \quad (6.5)$$



TABLE 5. Error norms when $\Delta t = 0.002$ and $t = 1$ of Example 6.2.

β	N	Method in [31]	Proposed Method		Order	CPU
		L_2	L_2	$\bar{L}_\infty$		
0.4	4	4.62×10^{-1}	2.41334×10^{-1}	1.87145×10^{-1}	...	5.7312
	8	2.38×10^{-1}	9.46134×10^{-2}	5.18147×10^{-2}	1.8527	9.1475
	16	1.22×10^{-1}	3.41748×10^{-2}	1.32355×10^{-2}	1.9689	13.377
	32	6.20×10^{-2}	1.21434×10^{-2}	3.34907×10^{-3}	1.9826	32.155
	64	3.13×10^{-2}	4.29687×10^{-3}	8.38618×10^{-4}	1.9977	76.631
	128	1.57×10^{-2}	1.51697×10^{-3}	2.09491×10^{-4}	2.0011	304.14
0.6	4	4.55×10^{-1}	2.40174×10^{-1}	1.85946×10^{-1}	...	5.4195
	8	2.34×10^{-1}	9.40981×10^{-2}	5.14458×10^{-2}	1.8538	6.4385
	16	1.20×10^{-1}	3.39784×10^{-2}	1.31370×10^{-2}	1.9694	10.082
	32	6.08×10^{-2}	1.20675×10^{-2}	3.32157×10^{-3}	1.9837	21.928
	64	3.07×10^{-2}	4.26223×10^{-3}	8.30313×10^{-4}	2.0001	78.421
	128	1.54×10^{-2}	1.49376×10^{-3}	2.05873×10^{-4}	2.0119	300.84
0.8	4	4.48×10^{-1}	2.39013×10^{-1}	1.84726×10^{-1}	...	4.1787
	8	2.30×10^{-1}	9.35720×10^{-2}	5.10647×10^{-2}	1.8549	6.0344
	16	1.18×10^{-1}	3.37639×10^{-2}	1.30298×10^{-2}	1.9705	10.266
	32	5.97×10^{-2}	1.19649×10^{-2}	3.28615×10^{-3}	1.9873	21.413
	64	3.01×10^{-2}	4.18942×10^{-3}	8.14362×10^{-4}	2.0127	78.638
	128	1.51×10^{-2}	1.41645×10^{-3}	1.94668×10^{-4}	2.0647	272.96

TABLE 6. Error norms when $N = 100$, $\Delta t = 0.00625$, and $r \in [0, 1]$ of Example 6.2.

t	L_∞		L_2	
	$\beta = 0.35$	$\beta = 0.75$	$\beta = 0.35$	$\beta = 0.75$
0.2	1.13739×10^{-5}	2.10179×10^{-5}	7.34212×10^{-5}	1.32667×10^{-5}
0.4	5.21856×10^{-5}	1.76031×10^{-5}	3.35266×10^{-4}	1.19870×10^{-4}
0.6	1.20915×10^{-4}	8.49299×10^{-5}	7.75247×10^{-4}	5.52922×10^{-4}
0.8	2.17676×10^{-4}	1.80823×10^{-4}	1.39389×10^{-3}	1.16753×10^{-3}
1.0	3.42550×10^{-4}	3.05311×10^{-4}	2.19158×10^{-3}	1.96354×10^{-3}

TABLE 7. Absolute error for different values of β where $N = 80$, $\Delta t = 0.0125$ at $t = 1$ of Example 6.2.

r	Exact	Approximate Solution		Error	
	Solutions	$\beta = 0.3$	$\beta = 0.7$	$\beta = 0.3$	$\beta = 0.7$
0.1	0.5877852523	0.5875002372	0.5875335708	$2.85015074 \times 10^{-4}$	$2.51681449 \times 10^{-4}$
0.2	0.9510565163	0.9505693080	0.9506267034	$4.87208272 \times 10^{-4}$	$4.29812906 \times 10^{-4}$
0.3	0.9510565163	0.9505288064	0.9505919460	$5.27709868 \times 10^{-4}$	$4.64570324 \times 10^{-4}$
0.4	0.5877852523	0.5873954212	0.5874438964	$3.89831114 \times 10^{-4}$	$3.41355880 \times 10^{-4}$
0.5	0	-0.0001254537	-0.0001063859	$1.25453740 \times 10^{-4}$	$1.06385883 \times 10^{-4}$
0.6	-0.5877852523	-0.5876204922	-0.5876343164	$1.64760129 \times 10^{-4}$	$1.50935923 \times 10^{-4}$
0.7	-0.9510565163	-0.9506867397	-0.9507243918	$3.69776569 \times 10^{-4}$	$3.32124529 \times 10^{-4}$
0.8	-0.9510565163	-0.9506459725	-0.9506893474	$4.10543757 \times 10^{-4}$	$3.67168944 \times 10^{-4}$
0.9	-0.5877852523	-0.5875151420	-0.5875440643	$2.70110306 \times 10^{-4}$	$2.41187955 \times 10^{-4}$

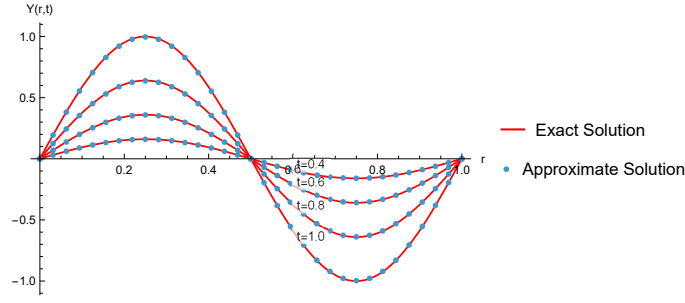


FIGURE 5. Comparison plot of approximate and exact solutions with $\beta = 0.25$, $\Delta t = 0.004$ for Example 6.2 at different time levels.

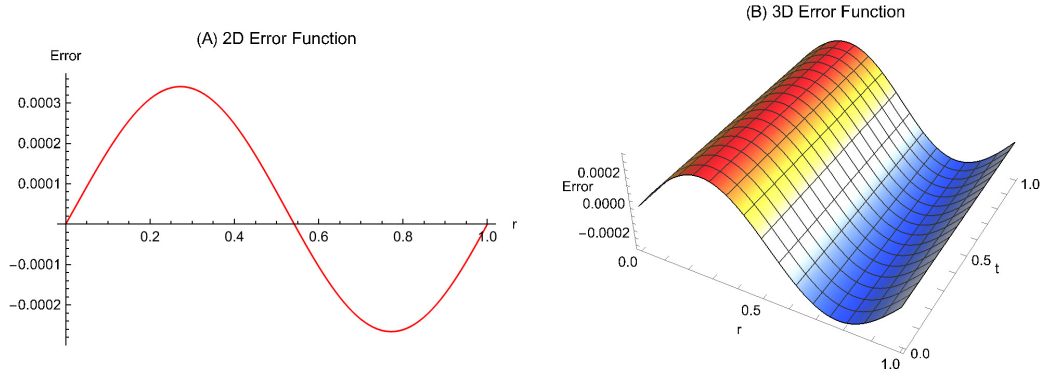


FIGURE 6. 2D and 3D Error profiles when $N = 100$, $\beta = 0.3$ at $T = 1$ of Example 6.2.

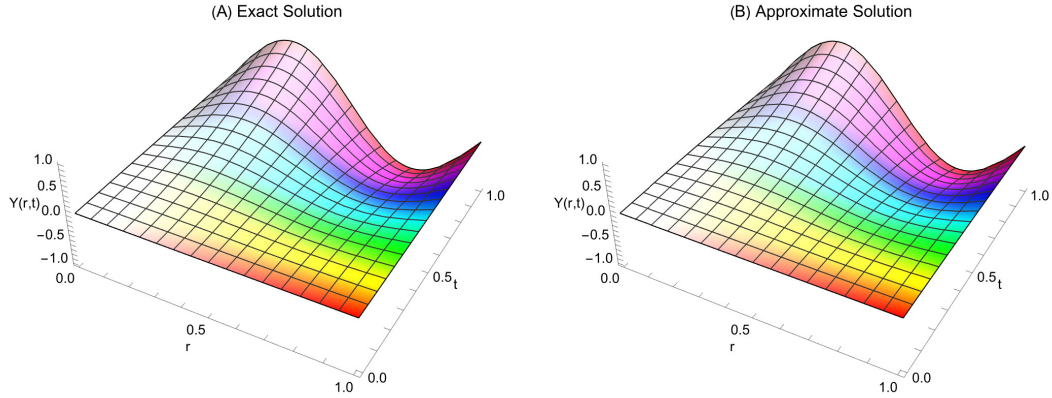


FIGURE 7. 3D approximate and exact solutions representation when $\beta = 0.90$, $N = 70$, $\Delta t = 0.01$, $T = 1$, and $r \in [0, 1]$ for Example 6.2.

where the exact analytical solution is $p(r, t) = (t^3 + t^\beta) \sin(2\pi r)$. The corresponding initial and boundary conditions are $y(r, 0) = 0$, $y(0, t) = y(1, t) = 0$. The source term is expressed as

$$p(r, t) = \left(\frac{6t^{3-\beta}}{\Gamma(4-\alpha)} + \Gamma(\beta+1) \right) \sin(2\pi r) + 2\pi(t^3 + t^\beta) \cos(2\pi r) + (t^3 + t^\beta)(4\pi^2 + 1) \sin(2\pi r).$$

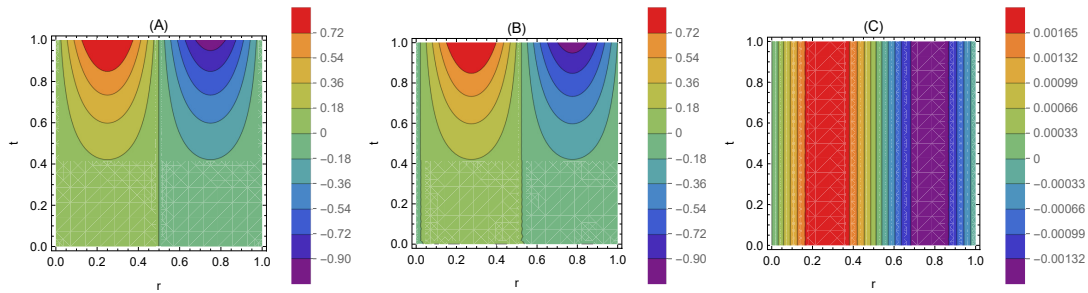


FIGURE 8. The contour plots of (A) the exact solution, (B) approximate solutions, and (C) error for $\alpha = 0.75$, $N = 40$, $\Delta t = 0.01$ at time $T = 1$ for Example 6.2.

The piecewise cubic B-spline polynomial solution of Example 6.3 for $\beta = 0.75$, $N = 100$, $\Delta t = 0.0125$, and $T = 1$ is presented in Eq. (6.6). Table 8 reports the approximate solutions at various spatial grid points for $\Delta t = 0.01$ and $t = 1$ under different mesh sizes N . The results indicate a rapid convergence of the numerical approximation to the analytical solution as N increases, confirming the spatial accuracy of the proposed CBS-Crank-Nicolson approach. Table 9 highlights the error norms, the convergence order, and the CPU corresponding to different fractional orders when $\Delta t = 0.00625$ and $t = 1$. Table 10, presents the maximum absolute errors corresponding to different fractional orders β at $\Delta t = 0.004$ and $t = 1$.

Figure 9 illustrates the 2D comparison between the approximate and exact solutions for $\beta = 0.45$, $N = 100$, and $\Delta t = 0.0125$. An excellent visual agreement between the two curves demonstrates the accuracy of the CBS approximation across the computational domain. Figure 10 displays both 2D and 3D error distributions for $\beta = 0.20$, $N = 100$, and $\Delta t = 0.00625$, revealing uniformly small error magnitudes throughout the domain. Finally, Figure 11 presents the contour plots of the exact and numerical solutions, along with their corresponding error surfaces, for $N = 50$, $\alpha = 0.30$, $\Delta t = 0.004$, and $t = 1$. The contour profiles confirm the smoothness, stability, and high-resolution performance of the proposed CBS-Crank-Nicolson formulation.

$$Y(r, t) = \begin{cases} -3.68931 \times 10^{-13} + 4.00374r - 1.99813r^2 - 0.004552r^3, & r \in [0.00, 0.01), \\ 4.19254 \times 10^{-11} + 4.00374r - 1.99813r^2 - 0.0045943r^3, & r \in [0.01, 0.02), \\ 3.93361 \times 10^{-10} + 4.00374r - 1.99813r^2 - 0.00463823r^3, & r \in [0.02, 0.03), \\ 1.62366 \times 10^{-9} + 4.00374r - 1.99812r^2 - 0.00468379r^3, & r \in [0.03, 0.04), \\ 4.64437 \times 10^{-9} + 4.00374r - 1.99812r^2 - 0.00473099r^3, & r \in [0.04, 0.05), \\ 1.07467 \times 10^{-8} + 4.00374r - 1.99811r^2 - 0.00477981r^3, & r \in [0.05, 0.06), \\ \vdots & \vdots \\ 0.0000477459 + 4.00329r - 1.99652r^2 - 0.00700165r^3, & r \in [0.49, 0.50), \\ 0.0000458291 + 4.0033r - 1.99654r^2 - 0.00698632r^3, & r \in [0.50, 0.51), \\ 0.0000428168 + 4.00331r - 1.99657r^2 - 0.00696361r^3, & r \in [0.51, 0.52), \\ 0.0000385349 + 4.00334r - 1.99662r^2 - 0.00693316r^3, & r \in [0.52, 0.53), \\ \vdots & \vdots \\ -0.00870253 + 4.03408r - 2.03318r^2 + 0.00780485r^3, & r \in [0.97, 0.98), \\ -0.00934169 + 4.03603r - 2.03517r^2 + 0.00848394r^3, & r \in [0.98, 0.99), \\ -0.0100124 + 4.03806r - 2.03723r^2 + 0.00917517r^3, & r \in [0.99, 1.00). \end{cases} \quad (6.6)$$

TABLE 8. A comparison of approximate solution for Example 6.3 with $\Delta t = 0.004$ and $t = 1$.

β	r	Exact	Approximate Solutions		Error	
			$N = 70$	$N = 100$	$N = 70$	$N = 100$
0.20	0.1	1.175570505	1.174814315	1.175199467	7.56189×10^{-4}	3.71037×10^{-4}
	0.2	1.902113033	1.900819976	1.901478566	1.29306×10^{-3}	6.34467×10^{-4}
	0.3	1.902113033	1.900711506	1.901425328	1.40153×10^{-3}	6.87705×10^{-4}
	0.4	1.175570505	1.174533325	1.175061551	1.03718×10^{-3}	5.08953×10^{-4}
	0.5	0	-0.000337261	-0.000165557	3.37261×10^{-4}	1.65548×10^{-4}
	0.6	-1.175570505	-1.175138832	-1.175358776	4.31673×10^{-4}	2.11729×10^{-4}
	0.7	-1.902113033	-1.901137565	-1.901634482	9.75468×10^{-4}	4.78550×10^{-4}
	0.8	-1.902113033	-1.901028442	-1.901580925	1.08459×10^{-3}	5.32107×10^{-4}
	0.9	-1.175570505	-1.174856530	-1.175220218	7.13975×10^{-4}	3.50287×10^{-4}
0.40	0.1	1.175570505	1.174820274	1.175203619	7.50231×10^{-4}	3.66885×10^{-4}
	0.2	1.902113033	1.900830460	1.901485804	1.28257×10^{-3}	6.27228×10^{-4}
	0.3	1.902113033	1.900723558	1.901433498	1.38947×10^{-3}	6.79534×10^{-4}
	0.4	1.175570505	1.174543538	1.175068208	1.02697×10^{-3}	5.02296×10^{-4}
	0.5	0	-0.000331498	-0.000162231	3.31498×10^{-4}	1.62231×10^{-4}
	0.6	-1.175570505	-1.175138392	-1.175359335	4.32113×10^{-4}	2.11169×10^{-4}
	0.7	-1.902113033	-1.901141310	-1.901637981	9.71722×10^{-4}	4.75052×10^{-4}
	0.8	-1.902113033	-1.901033725	-1.901585342	1.07931×10^{-3}	5.27690×10^{-4}
	0.9	-1.175570505	-1.174860281	-1.175223255	7.10223×10^{-4}	3.47249×10^{-4}

TABLE 9. Error norms of Example 6.3 when $\Delta t = 0.00625$ at $t = 1$.

β	N	L_2	L_∞	Order	CPU
0.3	8	1.90337×10^{-1}	1.04419×10^{-1}	...	0.66964
	16	6.87646×10^{-2}	2.66788×10^{-2}	1.9686	1.21729
	32	2.44356×10^{-2}	6.75310×10^{-3}	1.9821	2.65129
	64	8.64677×10^{-3}	1.69094×10^{-3}	1.9977	9.33684
	128	3.05299×10^{-3}	4.22607×10^{-4}	2.0004	46.7981
0.5	8	1.89463×10^{-1}	1.03804×10^{-1}	...	0.73839
	16	6.84101×10^{-2}	2.65063×10^{-2}	1.9695	1.49828
	32	2.42685×10^{-2}	2.42685×10^{-2}	1.9848	2.71032
	64	8.53058×10^{-3}	1.6659×10^{-3}	2.0071	8.74811
	128	2.93136×10^{-3}	4.05229×10^{-4}	2.0395	35.2758
0.7	8	1.88298×10^{-1}	1.02991×10^{-1}	...	0.56988
	16	6.78774×10^{-2}	2.62547×10^{-2}	1.9685	1.16202
	32	2.39376×10^{-2}	6.59216×10^{-3}	1.9938	2.94156
	64	8.21470×10^{-3}	1.60110×10^{-3}	2.0417	13.3259
	128	2.53859×10^{-3}	3.50113×10^{-4}	2.1932	45.6882

7. APPLICATION

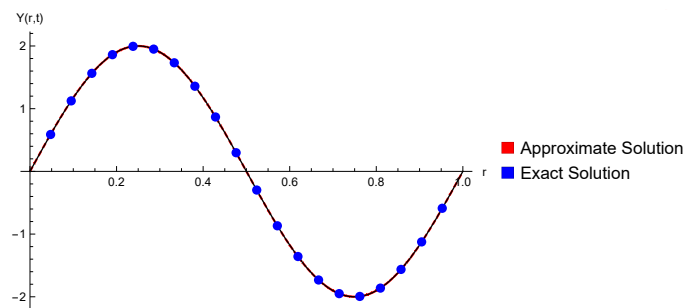
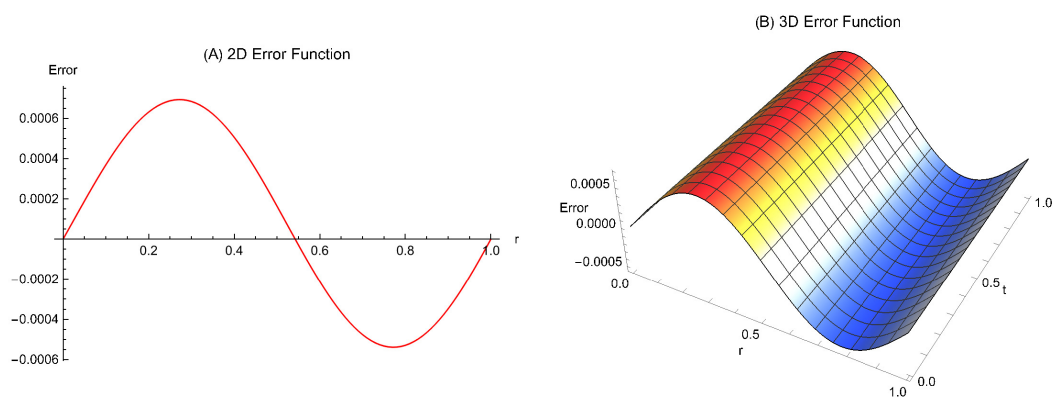
This section presents an application of the time-fractional convection-diffusion-reaction model within the framework of a mobile-immobile (MIM) system characterized by fractional dynamics. We consider the following fractional-order MIM model describing the total solute concentration:

$$\varpi_1 \frac{\partial v(r, t)}{\partial t} + \varpi_2 \frac{\partial^{\beta(r, t)} v(r, t)}{\partial t^{\beta}} + u_1 \frac{\partial v(r, t)}{\partial r} - u_2 \frac{\partial^2 v(r, t)}{\partial r^2} = g(r, t), \quad (r, t) \in \Omega = [0, L] \times [0, T], \quad (7.1)$$



TABLE 10. Maximum absolute errors of Example 6.3 for $N = 100$, $\Delta t = 0.004$ and $t = 1$.

r	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.1	3.72030×10^{-4}	3.69324×10^{-4}	3.63106×10^{-4}	3.44638×10^{-4}	2.76946×10^{-4}
0.2	6.36213×10^{-4}	6.31474×10^{-4}	6.20677×10^{-4}	5.88882×10^{-4}	4.72861×10^{-4}
0.3	6.89712×10^{-4}	6.84314×10^{-4}	6.72225×10^{-4}	6.37260×10^{-4}	5.10873×10^{-4}
0.4	5.10655×10^{-4}	5.06168×10^{-4}	4.96496×10^{-4}	4.69675×10^{-4}	3.74958×10^{-4}
0.5	1.66509×10^{-4}	1.64122×10^{-4}	1.59608×10^{-4}	1.49105×10^{-4}	1.16068×10^{-4}
0.6	1.66509×10^{-4}	2.11569×10^{-4}	2.10171×10^{-4}	2.02430×10^{-4}	1.67305×10^{-4}
0.7	4.79172×10^{-4}	4.77179×10^{-4}	4.71365×10^{-4}	4.50414×10^{-4}	3.66697×10^{-4}
0.8	5.32985×10^{-4}	5.30345×10^{-4}	5.23253×10^{-4}	4.99143×10^{-4}	4.05035×10^{-4}
0.9	3.50910×10^{-4}	3.49068×10^{-4}	3.44246×10^{-4}	3.28174×10^{-4}	2.65973×10^{-4}
L_∞	6.98881×10^{-4}	6.93514×10^{-4}	6.81414×10^{-4}	6.46178×10^{-4}	5.18350×10^{-4}
L_2	4.45982×10^{-3}	4.42964×10^{-3}	4.35847×10^{-3}	4.14148×10^{-3}	3.33559×10^{-3}

FIGURE 9. 2D graph of the approximate and exact solutions of Example 6.3 at $\beta = 0.45$, $\Delta t = 0.0125$, $t = 1$.FIGURE 10. 2D and 3D Error profiles when $N = 100$, $\beta = 0.3$ at $T = 1$ of Example 6.2.

with the initial,

$$v(r, 0) = \psi_1(r), \quad 0 \leq r \leq L,$$

(7.2)



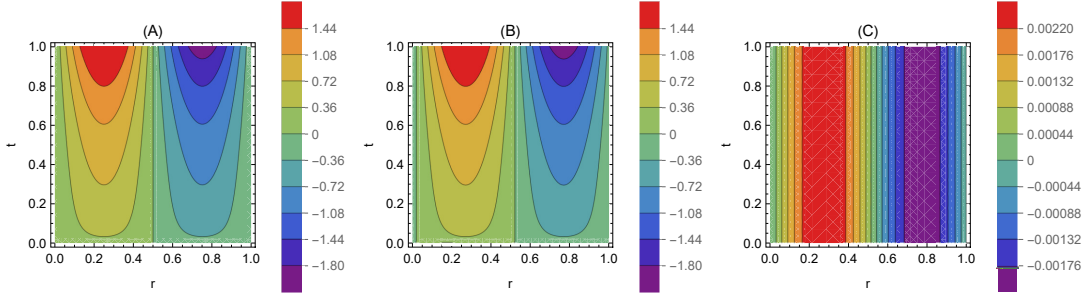


FIGURE 11. The contour plots of (A) the exact solution, (B) approximate solutions, and (C) absolute error for $\beta = 0.3$, $N = 50$, $\Delta t = 0.004$, and time $t = 1$ for Example 6.3.

and boundary conditions

$$v(0, t) = \psi_2(t), \quad v(L, t) = \psi_3(t), \quad 0 \leq t \leq T. \quad (7.3)$$

Here, $v(r, t)$ denotes the total solute concentration in the MIM phase at spatial location r and time t , describing its evolution temporally and spatially. The parameter $\varpi_1 \geq 0$ represents the coefficient weighting the classical first-order time derivative, while $\varpi_2 > 0$ stands for the fractional capacity coefficient, quantifying the influence of fractional temporal dynamics on concentration changes. The coefficient $u_1 > 0$ is the velocity parameter modeling solute transport due to advection in the mobile phase, and $u_2 > 0$ is the dispersion coefficient reflecting spreading due to diffusion within the mobile phase. The term $\frac{\partial^\beta v(r, t)}{\partial t^\beta}$ is the CFD of order $(0 < \beta < 1)$, capturing anomalous diffusion effects inherent in the MIM system. The function $g(r, t)$, acts as a source or forcing term impacting concentration at position r and time t . The boundary functions $w(0, t) = \psi_2(t)$ and $v(L, t) = \psi_3(t)$ specify concentration values at the domain boundaries over time, while $v(0, t) = \psi_1(r)$ defines the initial spatial distribution of concentration at $t = 0$.

The inclusion of the time drift term $\partial v / \partial t$ in equation (7.1) is intended to represent conventional temporal dynamics, allowing for clear differentiation of particle states. The fractional model in (7.1) constitutes a special case of the more general TFCBRE (1.1) with variable coefficients. The problem defined by equations (7.1)–(7.3) could be comfortably solvable by the proposed numerical schemes previously described in (3.3)–(3.9) and enabling numerical approximation of the fractional MIM total concentration model. However, the next example highlights the numerical simulations, parameter study, validation, and physical interpretation of the MIM transport results.

Example 7.1. Consider the following MIM model [36]:

$$\varpi_1 \frac{\partial v(r, t)}{\partial t} + \varpi_2 \frac{\partial^{\beta(r, t)} v(r, t)}{\partial t^\alpha} + u_1 \frac{\partial v(r, t)}{\partial r} - u_2 \frac{\partial^2 v(r, t)}{\partial r^2} = g(r, t), \quad (r, t) \in \Omega = [0, t] \times [0, T], \quad (7.4)$$

$$v(r, 0) = 10r^2(1 - r)^2, \quad 0 \leq r \leq 1, \quad (7.5)$$

$$v(0, t) = (1, t) = 0, \quad 0 \leq t \leq 1, \quad (7.6)$$

where $g(r, t) = g_1(r, t) + g_2(r, t) + g_3(r, t) + g_4(r, t)$, and $g_1(r, t) = 10u_1(t + 1)(2r - 6r^2 + 4r^3)$,

$$g_2(r, t) = -10u_2(t + 1)(12r^2 - 12r + 2), \quad g_3(r, t) = 10\varpi_1 r^2(1 - r)^2, \quad g_4(r, t) = \frac{10\varpi_2 r^2(1 - r)^2 t^{1-\alpha(r, t)}}{\Gamma(2 - \beta(r, t))}.$$

The exact solution Eq. (7.4) is $v(r, t) = 10r^2(t + 1)(1 - r)$. Let the parameters $\varpi_1 = u_1 = u_2 = \varpi_2 = 1$, and $\beta(r, t) = 1 - 0.5 \exp(-rt)$. Tables 11 and 12 demonstrate the accuracy and effectiveness of the proposed CBS-based scheme for the fractional MIM transport model. The numerical solutions agree closely with the exact solution, with absolute errors of order (10^{-3}) . The (L_2) and (L_∞) norms decrease consistently as the mesh is refined, confirming



the convergence of the method. The observed convergence rates are approximately second order and agree with the theoretical analysis. Furthermore, the results show that the proposed scheme accurately captures the influence of the fractional order (β), thereby providing a reliable and computationally efficient tool for simulating fractional MIM transport processes. Figures 12-14 further demonstrate the effectiveness of the proposed CBS-based numerical scheme for the fractional MIM transport model. Figure 12 shows an excellent agreement between the numerical and exact solutions at ($t = 1$), confirming the accuracy of the method. Figure 13 presents the corresponding error profiles, showing that the absolute error remains small throughout the computational domain, with the maximum near the centre. The smooth error distribution indicates the stability and robustness of the numerical approximation. Furthermore, the contour plots in Figure 14 reveal that the approximate solution closely reproduces the exact concentration field over the entire space-time domain. The associated error contour remains uniformly small, providing additional validation of the proposed scheme for simulating fractional mobile-immobile transport processes.

TABLE 11. Absolute error for different values of β where $N = 100$, $\Delta t = 0.002$ at $t = 1$ of Example 7.1.

r	Exact Solution	Approximate Solution		Absolute Error	
		$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.25$	$\beta = 0.50$
0.1	0.162	0.1623147880	0.1618368276	$3.147188000 \times 10^{-04}$	$1.631723514 \times 10^{-04}$
0.2	0.512	0.5126621556	0.5116860205	$6.621555611 \times 10^{-04}$	$3.139795398 \times 10^{-04}$
0.3	0.882	0.8829731604	0.8815394645	$9.731603708 \times 10^{-04}$	$4.605355383 \times 10^{-04}$
0.4	1.152	1.1531784010	1.1514006780	$1.178401249 \times 10^{-03}$	$5.993224909 \times 10^{-04}$
0.5	1.250	1.2512365920	1.2492898870	$1.236591695 \times 10^{-03}$	$7.101129155 \times 10^{-04}$
0.6	1.152	1.1531448580	1.1512391870	$1.144858401 \times 10^{-03}$	$7.608125985 \times 10^{-04}$
0.7	0.882	0.8829327394	0.8812796890	$9.327394472 \times 10^{-04}$	$7.203101182 \times 10^{-04}$
0.8	0.512	0.5126456447	0.5114270061	$6.456447446 \times 10^{-04}$	$5.729939476 \times 10^{-04}$
0.9	0.162	0.1623258386	0.1616738599	$3.258385774 \times 10^{-04}$	$3.261400623 \times 10^{-04}$

TABLE 12. Error norms when $\Delta t = 0.002$ and $t = 1$ of Example 7.1.

β	N	L_2	L_∞	Order	CPU
0.25	4	3.77924×10^{-01}	2.56688×10^{-01}		4.266132
	8	1.30288×10^{-01}	6.23489×10^{-02}	2.041579	7.164410
	16	4.25468×10^{-02}	1.44238×10^{-02}	2.111913	10.08044
	32	1.02137×10^{-02}	2.40585×10^{-03}	2.583833	25.59975
	64	3.31919×10^{-03}	6.54899×10^{-04}	1.877202	62.86362
0.50	4	2.60544×10^{-01}	3.83381×10^{-01}		4.603332
	8	1.35100×10^{-01}	6.47748×10^{-02}	2.008022	5.966471
	16	4.83085×10^{-02}	1.64888×10^{-02}	1.973946	9.748568
	32	1.79837×10^{-02}	4.37265×10^{-03}	1.914907	21.06923
	64	7.66138×10^{-03}	1.34917×10^{-03}	1.696436	92.81671

8. CONCLUSION

This study developed an efficient CBS collocation scheme for the numerical solution of the TFCBRE involving the CFD. The proposed framework combines the $L1$ finite difference approximation in time with CBS basis functions in space, resulting in a simple and computationally efficient numerical procedure. The theoretical analysis established a convergence rate of $O(h^2 + \Delta t^{2-\beta})$, demonstrating second-order spatial accuracy and a temporal accuracy that depends on the fractional order (β). Furthermore, the von Neumann stability analysis showed that the scheme is unconditionally



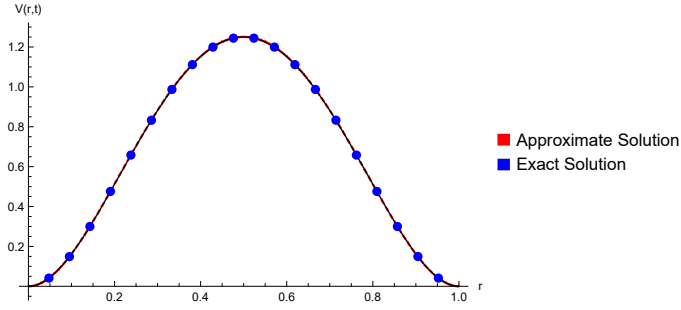


FIGURE 12. 2D graph of the approximate and exact solutions of Example 7.1 at $N = 100$, $\Delta t = 0.0125$, $t = 1$.

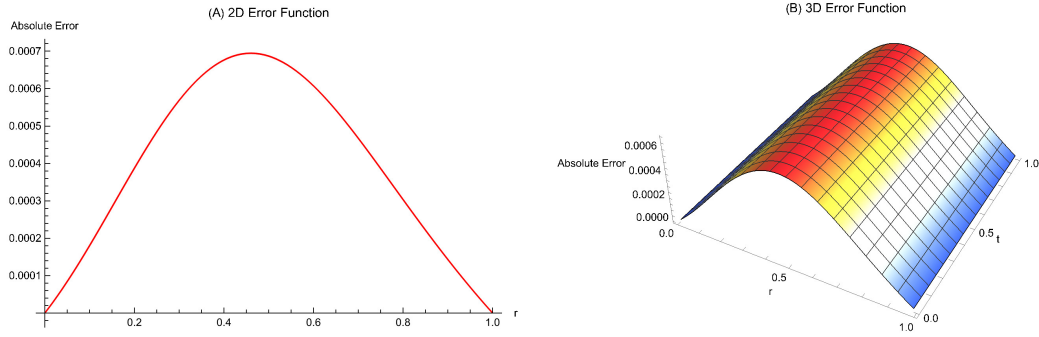


FIGURE 13. 2D and 3D Error profiles when $N = 100$, $\Delta t = 0.0125$, at $t = 1$ of Example 7.1.

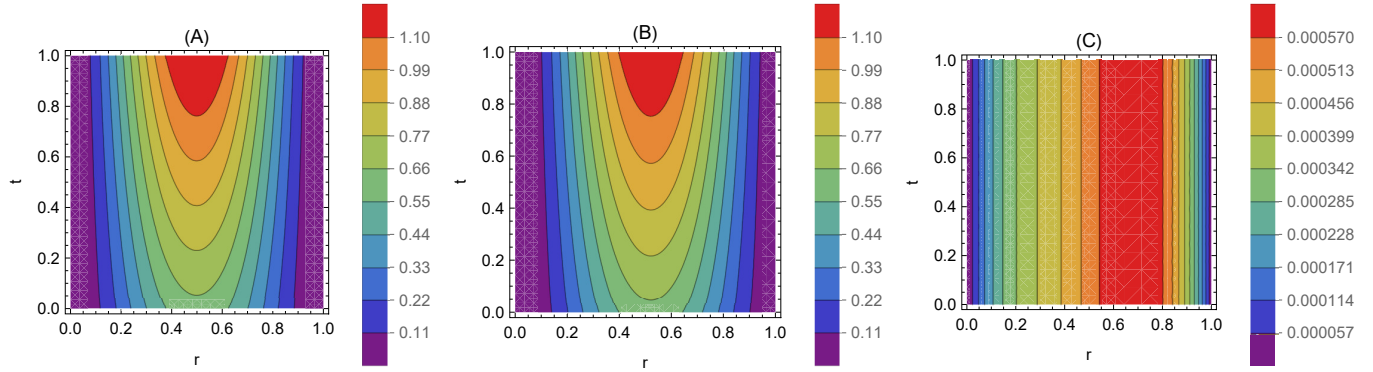


FIGURE 14. The contour plots of (A) the exact solution, (B) approximate solutions, and (C) absolute error for $N = 50$, $\Delta t = 0.004$, and time $t = 1$ for Example 7.1.

stable for the corresponding constant-coefficient model problem. The effectiveness of the proposed method was verified through several numerical examples involving both constant and variable coefficients. The numerical solutions were found to be in excellent agreement with the corresponding exact solutions, while the computed error norms confirmed the predicted convergence behaviour. Additional validation was provided through graphical investigations, including

two-dimensional, three-dimensional, and contour plots, which demonstrated the accuracy and stability of the numerical approximations over the computational domain.

To further illustrate its practical applicability, the proposed method was successfully applied to a fractional MIM transport model. The numerical results, supported by error analyses, convergence studies, parameter investigations for different fractional orders, and graphical simulations, demonstrated the capability of the scheme to accurately capture fractional transport dynamics and anomalous diffusion behaviour. These results confirm that the developed CBS approach provides a reliable computational tool for fractional transport and reactive processes. Compared with other spline-based approaches, such as classical CTBS, ExCBS, and FEM-based formulations, the proposed CBS scheme offers a relatively simple implementation, avoids the use of free tuning parameters, and produces banded algebraic systems that can be solved efficiently. However, a comprehensive quantitative comparison under identical benchmark settings, including common test problems, mesh sizes, time steps, fractional orders, and error measures, is beyond the scope of the present study and will be considered in future work. .

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AUTHOR CONTRIBUTIONS

The first draft of the manuscript was written by Anthony Anya Okeke and Nur Nadiah Abd Hamid commented on previous versions of the manuscript. All authors read and approved the final manuscript.

DECLARATIONS

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