



Numerical investigation of time fractional nonlinear Klein-Gordon equation

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Abstract

In the current paper, we investigate the numerical solution of time fractional nonlinear Klein-Gordon equation. The time fractional derivative which has been described in the Caputo sense, is approximated by a scheme of order $O(\tau^{3-\alpha})$, $1 \leq \alpha \leq 2$, and the fully discrete scheme is obtained using finite volume method with piecewise linear basis functions. We prove that the proposed method is stable and convergent with order $O(\tau^{3-\alpha} + h^2)$. To show the accuracy, superiority and efficiency of the proposed method, we present two examples and compare the results with some methods developed in literature.

Keywords. Caputo time-fractional derivative, Finite difference scheme, Finite volume method, Stability, Convergence.

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1. INTRODUCTION

Fractional derivatives are a helpful tool for describing some physical difficulties since they offer more accurate representations of real-world problems than integer-order derivatives. Fractional calculus is a highly significant branch of mathematics and plays a severe and critical role in describing complex dynamical behaviour in various application fields [10, 14, 18, 23].

Oscar Klein and Walter Gordon introduced the Klein-Gordon (KG) equation in 1927 as a fascinating differential equation that it describes relativistic electrons. Schrodinger initially considered the KG equation as a quantum wave equation in his research for an equation defining de Broglie waves. The KG equation is a partial differential equation (PDE) that describes a variety of physical phenomena such as DNA dynamics, ferroelectric and ferromagnetic domain walls, dislocations and Josephson junctions. It has attracted much attention in studying solitons [6, 27, 28], condensed matter physics [3], elementary particle behavior, dispersive wave phenomena, dislocations propagation in crystals, nonlinear optics, relativistic physics, and quantum field theory [31, 32].

Fractional KG equation in a natural way emerges within the framework of relativistic quantum mechanics from the expression of relativistic energy [8]. Our attention in this paper is to focus on the nonlinear KG equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + bu + cu^\beta = a \frac{\partial^2 u}{\partial x^2} + g(x, t), \quad (x, t) \in (0, L) \times (0, T], \quad (1.1)$$

with the following initial and boundary conditions

$$\begin{cases} u(x, 0) = r(x), \quad \frac{\partial}{\partial t} u(x, 0) = \psi(x), \\ u(0, t) = p_1(t), \quad u(L, t) = p_2(t), \end{cases}$$

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where for $1 < \alpha < 2$, $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ is defined as

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} (t-s)^{1-\alpha} ds, \quad (1.2)$$

in which $r(x)$, $p_1(t)$ and $p_2(t)$ are known functions, $g(x,t)$ is the source term and $\beta = 2, 3$.

Recently, several methods have been investigated to study the numerical solution of fractional KG equation, for example, homotopy analysis method [13], high-order difference scheme [16], Modified double Laplace decomposition method [26], Chebyshev polynomials-based collocation method [29], the Adomian decomposition method [12], fractional reduced differential transform method [1], implicit RBF meshless approach [4], variational iteration method (VIM) [19, 21], Legendre spectral method [15] and finite difference methods and the predictor-corrector method [22]. Nagy in [20] has solved this problem using Sinc-Chebyshev collocation method (SCM) without stability and convergence analysis. Ganji in [7] proposed the Clique polynomial method with an error bound.

The main aim of this paper is to propose a new numerical method to solve the time fractional nonlinear KG equation using Finite volume (FV) method with piecewise linear basis function. We prove the unconditional stability and convergence of proposed method and compare the obtained numerical results with other ones developed in literature. We present the applicability and superiority of suggested scheme in compare with other existing numerical methods. The FV approach is distinguished from the finite difference (FD), finite element (FE) and other approaches by the natural way in which it handles fractional partial differential equations written in conservative form. Most of the previous methods used FD schemes, due to its simplicity and easy of manipulation but makes it difficult to generalise the numerical method to deal with irregular shaped solution domains. Under the same conditions, FV is more accurate than the FD in spite of the same theoretical order, and accuracy of method is only dependent to the degree of the chosen approximation polynomials and the control volumes which have the flexibility to handle complicated domains. The FE method is also widely used for its geometric flexibility but it has more computational cost than the FD method under the same conditions. The FV method uses a volume integral formulation of the differential equations. In addition to the primary partition on the domain, we introduce a dual partition as control volumes to discretize the equation which have flexibility to handle the complicated domains and keep the conservation law of energy and mass. The outline of this article is as follows. In Section 2, we introduce the derivation of new method for the solution of Eq. (1.1). This scheme is based on approximating the time derivative of the mentioned equation by a scheme of order $O(\tau^{3-\alpha})$, $1 \leq \alpha \leq 2$, and spatial derivative with a finite volume method. In Sections 3, we prove the unconditional stability and convergence properties of the applied method. In Section 4, we report the numerical experiments for solving time fractional Eq. (1.1) with the method developed in this article for two examples. Finally, concluding remarks are drawn in Section 5.

2. DISCRETE SCHEME

Let $t_n = n\tau$ for $n = 0, 1, 2, \dots, N$, in which $\tau = \frac{T}{N}$ is the step size of time variable, $x_j = jh$ for $j = 0, 1, \dots, M$, where $h = \frac{L}{M}$ is the space step size and $x_{j-1/2} = \frac{x_j + x_{j-1}}{2}$, $j = 1, 2, \dots, M$ be the mid-point of the interval $[x_{j-1}, x_j]$. Introduce the following notations:

$$u(x, t_n) = u^n, \quad u^{n+1/2} = \frac{1}{2}(u^{n+1} + u^n), \quad \delta_t u^{n+1/2} = \frac{1}{\tau}(u^{n+1} - u^n),$$

$$u(x_j, t_n) = u_j^n, \quad \delta_x u_{j-1/2}^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad \delta_x^2 u_j^n = \frac{1}{h}(\delta_x u_{j+1/2}^n - \delta_x u_{j-1/2}^n).$$

Lemma 2.1. [30] Suppose $g(t) \in C^2[0, t_n]$, $1 < \alpha \leq 2$ then

$$\left| \int_0^{t_n} \frac{dg(s)}{ds} (t_n - s)^{1-\alpha} ds - \frac{1}{\tau} \left[a_0 g(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) g(t_k) - a_{n-1} g(t_0) \right] \right|$$

$$\leq \frac{1}{2-\alpha} \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \leq s \leq t_n} \left| \frac{d^2 g(s)}{ds^2} \right| \tau^{3-\alpha}, \quad (2.1)$$



where

$$a_l = \int_{t_l}^{t_{l+1}} \frac{dt}{t^{\alpha-1}} = \frac{1}{2-\alpha} [(t_{l+1})^{2-\alpha} - (t_l)^{2-\alpha}] = \frac{\tau^{2-\alpha}}{2-\alpha} [(l+1)^{2-\alpha} - l^{2-\alpha}], \quad l \geq 0. \quad (2.2)$$

Taking the integration of (1.1) over a control volume $[x_{j-1/2}, x_{j+1/2}]$ for $j = 1, 2, \dots, M-1$, gives

$$\begin{aligned} & \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial^\alpha u(x, t_{n+1/2})}{\partial t^\alpha} dx + \int_{x_{j-1/2}}^{x_{j+1/2}} bu(x, t_{n+1/2}) dx + \int_{x_{j-1/2}}^{x_{j+1/2}} cf(u(x, t_{n+1/2})) dx \\ &= \int_{x_{j-1/2}}^{x_{j+1/2}} a \frac{\partial^2 u(x, t_{n+1/2})}{\partial x^2} dx + \int_{x_{j-1/2}}^{x_{j+1/2}} g(x, t_{n+1/2}) dx, \end{aligned} \quad (2.3)$$

where $f(u) = u^\beta$. We assume that f is a continuous function and satisfies the Lipschitz condition

$$|f - \tilde{f}| \leq L_1 |u - \tilde{u}|, \quad (2.4)$$

in which L_1 is the Lipschitz constant and $\tilde{f} = f(\tilde{u}(x, t))$. Now, let T_h be a uniform partition of the space interval $[0, L]$ with $I_j = ([x_{j-1}, x_j])_{j=1,2,\dots,M}$ and define the space V_h as the set of continuous and piecewise-linear functions on the spatial partition T_h . Suppose the nodal basis functions $\phi_i(x)$, $i = 1, 2, \dots, M-1$, be as

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{Otherwise,} \end{cases}$$

then $u_h(x, t_n) \in V_h$ can be expressed as

$$u_h(x, t_n) = \sum_{i=1}^{M-1} u_i^n \phi_i(x). \quad (2.5)$$

Note that if $u(x) \in C^2[0, L]$, then it can be shown [9]

$$u(x) = \sum_{i=1}^{M-1} u_i(t) \phi_i(x) + \mathcal{O}(h^2).$$

Using the Taylor series expansion, the nonlinear reaction term can be approximated in the following form

$$f(u(x, t_{n+1/2})) = f\left(\frac{3}{2}u(x, t_n) - \frac{1}{2}u(x, t_{n-1})\right) + \mathcal{O}(\tau^2).$$

The corresponding finite volume scheme using (2.5) can be written as

$$\begin{aligned} & \sum_{i=1}^{M-1} \frac{\partial^\alpha u_i(t_{n+1/2})}{\partial t^\alpha} \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_i(x) dx + b \sum_{i=1}^{M-1} u_i^{n+1/2} \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_i(x) dx \\ &+ c \int_{x_{j-1/2}}^{x_{j+1/2}} f\left(\sum_{i=1}^{M-1} \left(\frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right) \phi_i(x)\right) dx = a \sum_{i=1}^{M-1} u_i^{n+1/2} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial^2 \phi_i(x)}{\partial x^2} dx \\ &+ \frac{1}{2} \int_{x_{j-1/2}}^{x_{j+1/2}} (g(x, t_n) + g(x, t_{n+1})) dx + \mathcal{O}(h^3 + h\tau^2), \quad j = 1, \dots, M-1. \end{aligned} \quad (2.6)$$

By direct calculations, it can be concluded that [2]

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \phi_i(x) dx = \begin{cases} \frac{h}{8}, & |j-i|=1, \\ \frac{3h}{4}, & j=i, \\ 0, & \text{Otherwise,} \end{cases}$$



$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial^2 \phi_i(x)}{\partial x^2} dx = \begin{cases} \frac{1}{h}, & |j-i|=1, \\ -\frac{2}{h}, & j=i, \\ 0, & \text{Otherwise.} \end{cases}$$

Now the finite volume scheme (2.6) can be rewritten as

$$\begin{aligned} & \frac{h}{8} \left(\frac{\partial^\alpha u_{j-1}(t_{n+1/2})}{\partial t^\alpha} + 6 \frac{\partial^\alpha u_j(t_{n+1/2})}{\partial t^\alpha} + \frac{\partial^\alpha u_{j+1}(t_{n+1/2})}{\partial t^\alpha} \right) + \frac{bh}{8} (u_{j-1}^{n+1/2} + 6u_j^{n+1/2} + u_{j+1}^{n+1/2}) \\ & + c \int_{x_{j-1/2}}^{x_{j+1/2}} f \left(\sum_{i=1}^{M-1} \left(\frac{3}{2} u_i^n - \frac{1}{2} u_i^{n-1} \right) \phi_i(x) \right) dx = \frac{a}{h} (u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}) \\ & + G_j^{n+1/2} + \mathcal{O}(h^3 + h\tau^2), \quad j = 1, \dots, M-1, \end{aligned} \quad (2.7)$$

where

$$G_j^{n+1/2} = \frac{1}{2} \int_{x_{j-1/2}}^{x_{j+1/2}} (g(x, t_n) + g(x, t_{n+1})) dx.$$

The last term on the left-hand side of (2.7) can be approximated using the mid-point quadrature rule as follows

$$\begin{aligned} & \int_{x_{j-1/2}}^{x_{j+1/2}} f \left(\sum_{i=1}^{M-1} \left(\frac{3}{2} u_i^n - \frac{1}{2} u_i^{n-1} \right) \phi_i(x) \right) dx \\ & = hf \left(\sum_{i=1}^{M-1} \left(\frac{3}{2} u_i^n - \frac{1}{2} u_i^{n-1} \right) \phi_i(x_j) \right) + \mathcal{O}(h^3) \\ & = hf \left(\frac{3}{2} u_j^n - \frac{1}{2} u_j^{n-1} \right) + \mathcal{O}(h^3), \quad j = 1, \dots, M-1. \end{aligned} \quad (2.8)$$

Therefore we have

$$\begin{aligned} & \frac{h}{8} \left(\frac{\partial^\alpha u_{j-1}(t_{n+1/2})}{\partial t^\alpha} + 6 \frac{\partial^\alpha u_j(t_{n+1/2})}{\partial t^\alpha} + \frac{\partial^\alpha u_{j+1}(t_{n+1/2})}{\partial t^\alpha} \right) + \frac{bh}{8} (u_{j-1}^{n+1/2} + 6u_j^{n+1/2} + u_{j+1}^{n+1/2}) \\ & + chf \left(\frac{3}{2} u_j^n - \frac{1}{2} u_j^{n-1} \right) = \frac{a}{h} (u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}) \\ & + G_j^{n+1/2} + \mathcal{O}(h^3 + h\tau^2), \end{aligned} \quad (2.9)$$

Using Lemma 2.1, we can obtain the following fully discrete difference scheme

$$\begin{aligned} & \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t u_{j-1}^{n+1/2}, \psi_{j-1}) + \frac{6h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t u_j^{n+1/2}, \psi_j) + \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t u_{j+1}^{n+1/2}, \psi_{j+1}) \\ & + \frac{bh}{8} (u_{j-1}^{n+1/2} + 6u_j^{n+1/2} + u_{j+1}^{n+1/2}) + chf \left(\frac{3}{2} u_j^n - \frac{1}{2} u_j^{n-1} \right) = ha\delta_x^2 u_j^{n+1/2} \\ & + G_j^{n+1/2} + R_j^n, \quad j = 1, 2, \dots, M-1, n = 0, 1, \dots, N-1, \end{aligned} \quad (2.10)$$

where

$$\mathcal{L}(u_i^{n+1/2}, \psi_i) = a_0 u_i^{n+1/2} - \sum_{k=1}^n (a_{n-k} - a_{n-k+1}) u_i^{k-1/2} - a_n \psi_i,$$

and

$$|R_j^n| \leq C(h^3 + h\tau^{3-\alpha}).$$

Finally if U^n be the approximation of u^n and by omitting the small term $R_j^{n+1/2}$, (2.10) can be rewritten in the following matrix-vector form

$$\frac{h}{\tau\Gamma(2-\alpha)} \mathcal{A} \mathcal{L}(\delta_t U^{n+1/2}, \psi) + bhAU^{n+1/2} + chF^{n+1/2} = haBU^{n+1/2} + G^{n+1/2}, \quad (2.11)$$



where $U^{n+1/2} = \frac{U^n + U^{n+1}}{2}$, $U^n = (U_1^n, U_2^n, \dots, U_{M-1}^n)^T$, $\psi = (\psi_1, \psi_2, \dots, \psi_{M-1})^T$, A is a tridiagonal matrix with entries, $\frac{6}{8}$ on the diagonal and $\frac{1}{8}$ on the sub- and superdiagonal, B is a tridiagonal matrix with entries, $-\frac{2}{h^2}$ on the diagonal and $\frac{1}{h^2}$ on the sub- and superdiagonal,

$$\begin{aligned}\mathcal{L}(\delta_t U^{n+1/2}, \psi) &= (\mathcal{L}(\delta_t U_1^{n+1/2}, \psi_1), \dots, \mathcal{L}(\delta_t U_{M-1}^{n+1/2}, \psi_{M-1}))^T, \\ F^{n+1/2} &= (f(\frac{3}{2}U_1^n - \frac{1}{2}U_1^{n-1}), \dots, f(\frac{3}{2}U_{M-1}^n - \frac{1}{2}U_{M-1}^{n-1}))^T.\end{aligned}$$

3. STABILITY AND CONVERGENCE ANALYSIS

In this section, we will analyse the stability and convergence of the proposed finite volume method. Let $u = (u_0, u_1, \dots, u_M)^T$, then we define the following inner product and norms

$$(u, v) = h \sum_{i=0}^M u_i v_i, \quad \|u\| = \sqrt{(u, u)},$$

$$|u|_1 = \sqrt{h \sum_{i=1}^M (\delta_x u_{i-1/2})^2}, \quad \|u\|_* = \sqrt{\frac{b}{4} \|u\|^2 + \frac{a}{2} |u|_1^2}.$$

Now we give some lemmas which will be used for the analyse of proposed method.

Lemma 3.1. *A symmetric positive definite matrix A admits the Cholesky factorization $A = HH^T$, where H is a lower triangular matrix with positive diagonal entries and for any vectors X, Y we have $(AX, Y) = (HH^T X, Y) = (H^T X, H^T Y)$.*

Lemma 3.2. [5] *For any $G = G_1, G_2, G_3, \dots$ and Q, we have*

$$\sum_{n=1}^N \left[a_0 G_n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) G_k - a_{n-1} Q \right] G_n \geq \tau \frac{t_N^{1-\alpha}}{2} \sum_{n=1}^N G_n^2 - \frac{t_N^{2-\alpha}}{2(2-\alpha)} Q^2, N = 1, 2, 3, \dots$$

Lemma 3.3. [11] *For any symmetric matrix G, the property of Rayleigh-Ritz ratio is*

$$\lambda_{\min}(G) \leq \frac{(GX, X)}{(X, X)} \leq \lambda_{\max}(G),$$

where $\lambda_{\min}, \lambda_{\max}$ denote the smallest and largest eigenvalue of G, respectively.

Remark 3.1. Using Lemmas 3.1 and 3.3 we can write

$$\|H^T e^{m+1}\|^2 = (H^T e^{m+1}, H^T e^{m+1}) = (Ae^{m+1}, e^{m+1}),$$

so we have

$$\lambda_{\min}(A) \|e^{m+1}\|^2 \leq \|H^T e^{m+1}\|^2 \leq \lambda_{\max}(A) \|e^{m+1}\|^2.$$

Also

$$\|H^{-1} e^n\|^2 = (H^{-1} e^n, H^{-1} e^n) = ((H^{-1})^T H^{-1} e^n, e^n) = ((H^T)^{-1} H^{-1} e^n, e^n) = (A^{-1} e^n, e^n),$$

so we have

$$\lambda_{\min}(A^{-1}) \|e^n\|^2 \leq \|H^{-1} e^n\|^2 \leq \lambda_{\max}(A^{-1}) \|e^n\|^2.$$

Remark 3.2. Note that A in (2.11) is a symmetric positive definite matrix and $\lambda_{\min}(A) = \frac{1}{2}$ and $\lambda_{\max}(A) = 1$.

Lemma 3.4. [17, 24] (discrete Gronwall inequality) *Let φ_n be a nonnegative sequence and the sequence v_n satisfies*

$$\begin{cases} v_0 \leq c_0 \\ v_n \leq c_0 + \sum_{k=0}^{n-1} d_k + \sum_{k=0}^{n-1} \varphi_k v_k, \quad n \geq 1, \end{cases}$$



If $c_0 \geq 0$ and $d_0 \geq 0$, we have

$$v_n \leq \left(c_0 + \sum_{k=0}^{n-1} d_k \right) \exp \left(\sum_{k=0}^{n-1} \varphi_k \right).$$

Let $\bar{U}^n$ be the solution of

$$\begin{aligned} & \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \bar{U}_{j-1}^{n+1/2}, \psi_{j-1}) + \frac{6h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \bar{U}_j^{n+1/2}, \psi_j) + \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \bar{U}_{j+1}^{n+1/2}, \psi_{j+1}) \\ & + \frac{bh}{8} (\bar{U}_{j-1}^{n+1/2} + 6\bar{U}_j^{n+1/2} + \bar{U}_{j+1}^{n+1/2}) + chf \left(\frac{3}{2} \bar{U}_j^n - \frac{1}{2} \bar{U}_j^{n-1} \right) \\ & = ha\delta_x^2 \bar{U}_j^{n+1/2} + G_j^{n+1/2}, \quad j = 1, 2, \dots, M-1, \end{aligned} \quad (3.1)$$

where $\bar{U}_0^n = U_0^n$, $\bar{U}_M^n = U_M^n$, $\bar{U}_j^0 = r_j + \varphi_j$ and $\left(\frac{\partial \bar{U}}{\partial t} \right)_j^0 = \psi_j + \kappa_j$, in which φ_j and κ_j are small perturbation of r_j and ψ_j for $j = 1, 2, \dots, M-1$, respectively. Then we have the following stability result.

Theorem 3.1. *The finite volume scheme (2.11) is unconditionally stable.*

Proof. Let $\epsilon_i^n = U_i^n - \bar{U}_i^n$, $i = 1, 2, \dots, M-1$ and $\delta_t \epsilon^n = (\delta_t \epsilon_1^n, \dots, \delta_t \epsilon_{M-1}^n)^T$. Subtracting (2.11) from (3.1), gives

$$\begin{aligned} & \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \epsilon_{j-1}^{n+1/2}, \kappa_{j-1}) + \frac{6h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \epsilon_j^{n+1/2}, \kappa_j) + \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \epsilon_{j+1}^{n+1/2}, \kappa_{j+1}) \\ & + \frac{bh}{8} (\epsilon_{j-1}^{n+1/2} + 6\epsilon_j^{n+1/2} + \epsilon_{j+1}^{n+1/2}) \\ & + ch \left(f \left(\frac{3}{2} U_j^n - \frac{1}{2} U_j^{n-1} \right) - f \left(\frac{3}{2} \bar{U}_j^n - \frac{1}{2} \bar{U}_j^{n-1} \right) \right) = ha\delta_x^2 \epsilon_j^{n+1/2}. \end{aligned} \quad (3.2)$$

Multiplying (3.2) by $\tau \delta_t \epsilon_j^{n+1/2}$ and summing up for j from 0 to M and for n from 1 to $m-1$, result

$$\begin{aligned} & \sum_{j=0}^M \sum_{n=1}^{m-1} \frac{h}{8\Gamma(2-\alpha)} [\mathcal{L}(\delta_t \epsilon_{j-1}^{n+1/2}, \kappa_{j-1}) + 6\mathcal{L}(\delta_t \epsilon_j^{n+1/2}, \kappa_j) + \mathcal{L}(\delta_t \epsilon_{j+1}^{n+1/2}, \kappa_{j+1})] \delta_t \epsilon_j^{n+1/2} \\ & + \sum_{j=0}^M \sum_{n=1}^{m-1} \frac{bh\tau}{8} (\epsilon_{j-1}^{n+1/2} + 6\epsilon_j^{n+1/2} + \epsilon_{j+1}^{n+1/2}) \delta_t \epsilon_j^{n+1/2} \\ & + \sum_{j=0}^M \sum_{n=1}^{m-1} ch\tau \left(f \left(\frac{3}{2} U_j^n - \frac{1}{2} U_j^{n-1} \right) - f \left(\frac{3}{2} \bar{U}_j^n - \frac{1}{2} \bar{U}_j^{n-1} \right) \right) \delta_t \epsilon_j^{n+1/2} \\ & = \sum_{j=0}^M \sum_{n=1}^{m-1} ha\tau \delta_x^2 \epsilon(x_j, t_{n+1/2}) \delta_t \epsilon_j^{n+1/2}. \end{aligned} \quad (3.3)$$

Regarding to $\delta_t \epsilon_0^{n+1/2} = \delta_t \epsilon_M^{n+1/2} = 0$, we can write

$$\begin{aligned} & \sum_{j=0}^M \sum_{n=1}^{m-1} \frac{h}{8\Gamma(2-\alpha)} \left[\mathcal{L}(\delta_t \epsilon_{j-1}^{n+1/2}, \kappa_{j-1}) + 6\mathcal{L}(\delta_t \epsilon_j^{n+1/2}, \kappa_j) + \mathcal{L}(\delta_t \epsilon_{j+1}^{n+1/2}, \kappa_{j+1}) \right] \delta_t \epsilon_j^{n+1/2} \\ & = \sum_{n=1}^{m-1} \frac{1}{\Gamma(2-\alpha)} \left(A\mathcal{L}(\delta_t \epsilon^{n+1/2}, \kappa), \delta_t \epsilon^{n+1/2} \right). \end{aligned} \quad (3.4)$$



Now using Lemmas 3.1 and 3.2, we have

$$\begin{aligned} \frac{1}{\Gamma(2-\alpha)} \sum_{n=1}^{m-1} (A\mathcal{L}(\delta_t \epsilon^{n+1/2}, \kappa), \delta_t \epsilon^{n+1/2}) &= \frac{1}{\Gamma(2-\alpha)} \sum_{n=1}^{m-1} (\mathcal{L}(H^T \delta_t \epsilon^{n+1/2}, \kappa), H^T \delta_t \epsilon^{n+1/2}) \\ &\geq \frac{1}{\Gamma(2-\alpha)} \left[\tau \frac{t_{m-1}^{1-\alpha}}{2} \sum_{n=1}^{m-1} \|H^T \delta_t \epsilon^{n+1/2}\|^2 - \frac{t_{m-1}^{2-\alpha}}{2(2-\alpha)} \|H^T \kappa\|^2 \right]. \end{aligned} \quad (3.5)$$

Also, we can write

$$\begin{aligned} \tau a h \sum_{j=1}^{M-1} \sum_{n=1}^{m-1} \delta_x^2 \epsilon_j^{n+1/2} \delta_t \epsilon_j^{n+1/2} &= -\tau a \sum_{n=1}^{m-1} h \sum_{j=1}^{M-1} (\delta_x \epsilon_{j-1/2}^{n+1/2}) \delta_t (\delta_x \epsilon_{j-1/2}^{n+1/2}) \\ &= \tau a \sum_{j=1}^M \frac{h}{2\tau} \sum_{n=1}^{m-1} \left[(\delta_x \epsilon_{j-1/2}^{n+1} + \delta_x \epsilon_{j-1/2}^n) (\delta_x \epsilon_{j-1/2}^{n+1} - \delta_x \epsilon_{j-1/2}^n) \right] \\ &= -\frac{a}{2} \sum_{n=1}^{m-1} \left[h \sum_{j=1}^M (\delta_x \epsilon_{j-1/2}^{n+1})^2 - h \sum_{j=1}^M (\delta_x \epsilon_{j-1/2}^n)^2 \right] = -\frac{a}{2} \sum_{n=1}^{m-1} (|\epsilon^{n+1}|_1^2 - |\epsilon^n|_1^2) n = -\frac{a}{2} (|\epsilon^m|_1^2 - |\epsilon^1|_1^2). \end{aligned} \quad (3.6)$$

In addition

$$\begin{aligned} \sum_{j=0}^M \sum_{n=1}^{m-1} \frac{bh\tau}{8} (\epsilon_{j-1}^{n+1/2} + 6\epsilon_j^{n+1/2} + \epsilon_{j+1}^{n+1/2}) \delta_t \epsilon_j^{n+1/2} &= b\tau \sum_{n=1}^{m-1} (A\epsilon^{n+1/2}, \delta_t \epsilon^{n+1/2}) \\ &= \sum_{n=1}^m b\tau (H^T \epsilon^{n+1/2}, H^T \delta_t \epsilon^{n+1/2}) = \sum_{n=1}^{m-1} b\tau (H^T (\frac{\epsilon^{n+1} + \epsilon^n}{2}), H^T (\frac{\epsilon^{n+1} - \epsilon^n}{\tau})) \\ &= \frac{b}{2} \sum_{n=1}^{m-1} \|H^T \epsilon^{n+1}\|^2 - \|H^T \epsilon^n\|^2 = \frac{b}{2} (\|H^T \epsilon^m\|^2 - \|H^T \epsilon^1\|^2). \end{aligned} \quad (3.7)$$

Let

$$\tilde{F}^{n+1/2} = \left(f\left(\frac{3}{2}\bar{U}_1^n - \frac{1}{2}\bar{U}_1^{n-1}\right), \dots, f\left(\frac{3}{2}\bar{U}_{M-1}^n - \frac{1}{2}\bar{U}_{M-1}^{n-1}\right) \right)^T,$$

then we have

$$\begin{aligned} \sum_{j=1}^{M-1} \sum_{n=1}^{m-1} ch\tau \left(f\left(\frac{3}{2}U_j^n - \frac{1}{2}U_j^{n-1}\right) - f\left(\frac{3}{2}\bar{U}_j^n - \frac{1}{2}\bar{U}_j^{n-1}\right) \right) \delta_t \epsilon^{n+1/2} \\ = \sum_{n=1}^{m-1} c\tau \left(HH^{-1}(F^{n+1/2} - \tilde{F}^{n+1/2}), \delta_t \epsilon^{n+1/2} \right) = \sum_{n=1}^m c\tau \left(H^{-1}(F^{n+1/2} - \tilde{F}^{n+1/2}), H^T \delta_t \epsilon^{n+1/2} \right) \\ \leq \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \left(\frac{|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \|H^{-1}(F^{n+1/2} - \tilde{F}^{n+1/2})\|^2 + \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{t_{m-1}^{1-\alpha}}{|c|\Gamma(2-\alpha)} \|H^T \delta_t \epsilon^{n+1/2}\|^2 \right) \\ \leq \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \hat{C}L_1 (\|H^{-1}\epsilon^n\|^2 + \|H^{-1}\epsilon^{n-1}\|^2) + \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{t_{m-1}^{1-\alpha}}{|c|\Gamma(2-\alpha)} \|H^T \delta_t \epsilon^{n+1/2}\|^2, \end{aligned} \quad (3.8)$$



where $\hat{C}$ is a positive constant and L_1 is the Lipschitz constant. Substituting (3.5)-(3.8) into (3.3), give

$$\begin{aligned} & \tau \frac{t_{m-1}^{1-\alpha}}{2\Gamma(2-\alpha)} \sum_{n=1}^{m-1} \|H^T \delta_t \epsilon^{n+1/2}\|^2 - \frac{t_{m-1}^{2-\alpha}}{2\Gamma(3-\alpha)} \|H^T \kappa\|^2 + \frac{b}{2} (\|H^T \epsilon^m\|^2 - \|H^T \epsilon^1\|^2) \\ & \leq \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \hat{C} L_1 (\|H^{-1} \epsilon^n\|^2 + \|H^{-1} \epsilon^{n-1}\|^2) + \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{t_{m-1}^{1-\alpha}}{|c|\Gamma(2-\alpha)} \|H^T \delta_t \epsilon^{n+1/2}\|^2 \\ & \quad - \frac{a}{2} (|\epsilon^m|_1^2 - |\epsilon^1|_1^2), \end{aligned} \quad (3.9)$$

or

$$\begin{aligned} & \frac{b}{2} \|H^T \epsilon^m\|^2 + \frac{a}{2} |\epsilon^m|_1^2 \leq \frac{a}{2} |\epsilon^1|_1^2 + \frac{b}{2} \|H^T \epsilon^1\|^2 \\ & \quad + \frac{|c|\tau}{2} \sum_{n=0}^{m-1} \frac{|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \hat{C} L_1 2 \|H^{-1} \epsilon^n\|^2 + \frac{t_{m-1}^{2-\alpha}}{2\Gamma(3-\alpha)} \|H^T \kappa\|^2. \end{aligned} \quad (3.10)$$

Using Remark 3.1, we have

$$\frac{b}{4} \|\epsilon^m\|^2 + \frac{a}{2} |\epsilon^m|_1^2 \leq \frac{a}{2} |\epsilon^1|_1^2 + \frac{b}{2} \|\epsilon^1\|^2 + \frac{|c|\tau}{2} \sum_{n=0}^{m-1} \frac{|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} 4\hat{C} L_1 \|\epsilon^n\|^2 + \frac{t_{m-1}^{2-\alpha}}{2\Gamma(3-\alpha)} \|H^T \kappa\|^2, \quad (3.11)$$

or

$$\|\epsilon^m\|_*^2 \leq \|\epsilon^1\|_*^2 + 2\tau \sum_{n=0}^{m-1} \frac{c^2 \Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \hat{C} L_1 \|\epsilon^n\|_*^2 + \frac{t_{m-1}^{2-\alpha}}{2\Gamma(3-\alpha)} \|H^T \kappa\|^2.$$

Finally Lemma 3.4, results

$$\|\epsilon^m\|_*^2 \leq \frac{t_{m-1}^{2-\alpha}}{2\Gamma(3-\alpha)} \|H^T \kappa\|^2 \exp(1 + 2t_{m-1}^{\alpha-1} t_m \hat{C} L_1 c^2 \Gamma(2-\alpha)), \quad (3.12)$$

which completes the proof. $\square$

Theorem 3.2. *The finite volume scheme (2.11) is convergent.*

Proof. Let u^n be the exact solution of (1.1) and U^n be the solution of finite volume scheme (2.11). Let $\varepsilon_j^n = u_j^n - U_j^n$, $j = 1, 2, \dots, M-1$ and $\delta_t \epsilon^n = (\delta_t \varepsilon_1^n, \dots, \delta_t \varepsilon_{M-1}^n)^T$. Subtracting (2.11) from (2.10), we have the following error equation

$$\begin{aligned} & \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \varepsilon_{j-1}^{n+1/2}, 0) + \frac{6h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \varepsilon_j^{n+1/2}, 0) + \frac{h}{8\tau\Gamma(2-\alpha)} \mathcal{L}(\delta_t \varepsilon_{j+1}^{n+1/2}, 0) \\ & + \frac{bh}{8} (\varepsilon_{j-1}^{n+1/2} + 6\varepsilon_j^{n+1/2} + \varepsilon_{j+1}^{n+1/2}) \\ & + ch \left(f\left(\frac{3}{2}u_j^n - \frac{1}{2}u_j^{n-1}\right) - f\left(\frac{3}{2}U_j^n - \frac{1}{2}U_j^{n-1}\right) \right) \\ & = ha\delta_x^2 \varepsilon_j^{n+1/2} + R_j^{n+1/2}, \quad j = 1, 2, \dots, M-1. \end{aligned} \quad (3.13)$$

Multiplying (3.13) by $\tau \delta_t \varepsilon_j^{n+1/2}$ and summing up for j from 0 to M and for n from 1 to $m-1$ and with similar calculation and using

$$\begin{aligned} \sum_{j=0}^M \sum_{n=1}^{m-1} \tau R_j^{n+1/2} \delta_t \varepsilon_j^{n+1/2} &= \sum_{n=1}^{m-1} \frac{\tau}{h} (H H^{-1} R^{n+1/2}, \delta_t \varepsilon^{n+1/2}) = \frac{\tau}{h} \sum_{n=1}^{m-1} (H^{-1} R^{n+1/2}, H^T \delta_t \varepsilon^{n+1/2}) \\ &\leq \sum_{n=1}^{m-1} \frac{\tau}{2h} \left[\frac{ht_{m-1}^{1-\alpha}}{2\Gamma(2-\alpha)} \|H^T \delta_t \varepsilon^{n+1/2}\|^2 + \frac{2\Gamma(2-\alpha)}{ht_{m-1}^{1-\alpha}} \|H^{-1} R^{n+1/2}\|^2 \right], \end{aligned} \quad (3.14)$$



we can obtain

$$\begin{aligned}
& \tau \frac{t_{m-1}^{1-\alpha}}{2\Gamma(2-\alpha)} \sum_{n=1}^{m-1} \|H^T \delta_t \varepsilon^{n+1/2}\|^2 + \frac{b}{2} (\|H^T \varepsilon^m\|^2 - \|H^T \varepsilon^1\|^2) \\
& \leq \frac{|c|\tau}{2} \sum_{n=1}^{m-1} \frac{2|c|\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \hat{C}L_1 (\|H^{-1} \varepsilon^n\|^2 + \|H^{-1} \varepsilon^{n-1}\|^2) + \frac{\tau}{4} \sum_{n=1}^{m-1} \frac{t_{m-1}^{1-\alpha}}{\Gamma(2-\alpha)} \|H^T \delta_t \varepsilon^{n+1/2}\|^2 \\
& - \frac{a}{2} (\|\varepsilon^m\|_1^2 - \|\varepsilon^1\|_1^2) + \sum_{n=1}^{m-1} \frac{\tau t_{m-1}^{1-\alpha}}{4\Gamma(2-\alpha)} \|H^T \delta_t \varepsilon^{n+1/2}\|^2 + \frac{\Gamma(2-\alpha)}{h t_{m-1}^{1-\alpha}} \|H^{-1} R^{n+1/2}\|^2,
\end{aligned} \tag{3.15}$$

so we have

$$\|\varepsilon^m\|_*^2 \leq \frac{a}{2} \|\varepsilon^1\|_1^2 + \frac{b}{2} \|\varepsilon^1\|^2 + \frac{2\tau\Gamma(2-\alpha)}{h^2 t_{m-1}^{1-\alpha}} \sum_{n=1}^{m-1} \|R^{n+1/2}\|^2 + \hat{C}L_1 \frac{4c^2\tau\Gamma(2-\alpha)}{t_{m-1}^{1-\alpha}} \sum_{n=1}^{m-1} \|\varepsilon^n\|_*^2.$$

Using Lemma 3.4, we obtain

$$\|\varepsilon^m\|_*^2 \leq \frac{2\tau\Gamma(2-\alpha)}{h^2 t_{m-1}^{1-\alpha}} \sum_{n=1}^{m-1} \|R^{n+1/2}\|^2 \exp(1 + 4t_{m-1}^\alpha \hat{C}L_1 c^2 \Gamma(2-\alpha)). \tag{3.16}$$

Regarding to

$$\|R^{n+1/2}\|^2 = \sum_{j=0}^M h |R_j^{n+1/2}|^2 \leq CL(h\tau^{3-\alpha} + h^3)^2,$$

we can write

$$\|\varepsilon^m\|_*^2 \leq \frac{\bar{C}}{h^2} (h\tau^{3-\alpha} + h^3)^2, \tag{3.17}$$

where $\bar{C} = 2LCT^\alpha \Gamma(2-\alpha) \exp(1 + 4T^\alpha \hat{C}L_1 c^2 \Gamma(2-\alpha))$. Finally we have

$$\|\varepsilon^m\|_* \leq \sqrt{\bar{C}} (\tau^{3-\alpha} + h^2), \tag{3.18}$$

which completes the proof. $\square$

4. NUMERICAL RESULTS

In current section, we investigate the accuracy and stability of the proposed method for different values of h , Δt , and α using some examples. We obtain the computational order of the proposed method as

$$Order = \frac{\log(\varepsilon_1/\varepsilon_2)}{\log(h_1/h_2)},$$

in which ε_1 and ε_2 are errors correspond to the step sizes h_1 and h_2 , respectively. We performed our computations using Matlab R2017a software on a Intel(R) Core(TM) i3 CPU M 330 @ 2.13GHz processor with 4 GB of memory.

Example 4.1. We consider the nonlinear Klein-Gordon equation (1.1) with $a = 1$, $b = 0$, $c = 1$, $\beta = 2$, $0 \leq x \leq 1$, $0 < t \leq 1$ with the source term

$$g(x, t) = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} - \alpha)} (1-x)^{\frac{5}{2}} t^{\frac{3}{2}-\alpha} - \frac{15}{4} (1-x)^{\frac{1}{2}} t^{\frac{3}{2}} + (1-x)^5 t^3.$$

The initial and boundary conditions are

$$\begin{aligned}
u(x, 0) &= 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \\
u(0, t) &= t^{\frac{3}{2}}, \quad u(1, t) = 0,
\end{aligned}$$

and the exact solution is given by

$$u(x, t) = (1-x)^{\frac{5}{2}} t^{\frac{3}{2}}.$$



In Tables 1,2,3 we compare the results of the proposed method with the methods developed in [19–21]. As we see the present method is more accurate. We present the error and computational order in spatial and time directions with different values of τ , h and α in Tables 4,5. This tables show that the computational orders are in good agreement with theoretical ones.

TABLE 1. Comparison of errors with $\alpha = 1.9$ for Example 4.1.

x	SCM [20]		FVM		
	$n = m = 3$	$n = m = 9$	$h = \tau = 0.1$	$h = \tau = 0.01$	$h = 0.01, \tau = 0.001$
0.1	1.2328×10^{-2}	4.3675×10^{-4}	3.9471×10^{-4}	3.2193×10^{-5}	1.7376×10^{-6}
0.2	1.0041×10^{-2}	9.8359×10^{-5}	9.0236×10^{-4}	5.1850×10^{-5}	4.9721×10^{-7}
0.3	6.1374×10^{-3}	4.8897×10^{-4}	1.4625×10^{-3}	5.3646×10^{-5}	9.7947×10^{-6}
0.4	3.1044×10^{-3}	7.6534×10^{-4}	2.0215×10^{-3}	3.1669×10^{-5}	2.8627×10^{-5}
0.5	9.9389×10^{-4}	9.3043×10^{-4}	2.5114×10^{-3}	2.2958×10^{-5}	5.9616×10^{-5}
0.6	2.8807×10^{-4}	9.4248×10^{-4}	2.8310×10^{-3}	1.2355×10^{-4}	1.0631×10^{-4}
0.7	2.0332×10^{-3}	7.5585×10^{-4}	2.8367×10^{-3}	2.7502×10^{-4}	1.7339×10^{-4}
0.8	6.4035×10^{-3}	4.2006×10^{-4}	2.3840×10^{-3}	4.0644×10^{-4}	2.2980×10^{-4}
0.9	8.1942×10^{-3}	5.4848×10^{-5}	1.4251×10^{-3}	3.3166×10^{-4}	1.6733×10^{-5}

TABLE 2. Comparison of errors with $\alpha = 1.5$ for Example 4.1.

x	SCM [20]		FVM		
	$n = m = 3$	$n = m = 9$	$h = \tau = 0.1$	$h = 0.1, \tau = 0.01$	$h = 0.001, \tau = 0.01$
0.1	1.7221×10^{-2}	8.7105×10^{-4}	1.1886×10^{-4}	9.7489×10^{-6}	1.9898×10^{-8}
0.2	1.8443×10^{-2}	6.7781×10^{-4}	8.4639×10^{-5}	1.7322×10^{-5}	2.0620×10^{-7}
0.3	1.5848×10^{-2}	6.2089×10^{-4}	2.0131×10^{-5}	2.2675×10^{-5}	5.9295×10^{-7}
0.4	1.2813×10^{-2}	5.7015×10^{-4}	1.3553×10^{-4}	2.5610×10^{-5}	1.0835×10^{-6}
0.5	9.9088×10^{-3}	5.1476×10^{-4}	2.3634×10^{-4}	2.5973×10^{-5}	1.6288×10^{-6}
0.6	7.9067×10^{-3}	4.8948×10^{-4}	3.0581×10^{-4}	2.3775×10^{-5}	2.1723×10^{-6}
0.7	8.0209×10^{-3}	5.1671×10^{-4}	3.3704×10^{-4}	1.9259×10^{-5}	2.6368×10^{-6}
0.8	1.0523×10^{-2}	5.3919×10^{-4}	3.2403×10^{-4}	1.2924×10^{-5}	2.8997×10^{-6}
0.9	1.0241×10^{-2}	6.0660×10^{-4}	2.4946×10^{-4}	5.5956×10^{-6}	2.6987×10^{-6}

Example 4.2. Consider the nonlinear Klein-Gordon equation(1.1) with $a = 1$, $b = 1$, $c = 3/2$, $\beta = 3$, $0 \leq x \leq 1$, $0 < t \leq 1$,

$$g(x, t) = \frac{\Gamma(3 + \alpha)}{2} \sin(\pi x) t^2 + (1 + \pi^2) \sin(\pi x) t^{2+\alpha} + \frac{3}{2} (\sin(\pi x) t^{2+\alpha})^3$$

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0,$$

$$u(0, t) = 0, \quad u(1, t) = 0.$$

The exact solution is given by [7]

$$u(x, t) = \sin(\pi x) t^{2+\alpha}.$$

This problem is solved in [7] with clique polynomial (CPM), in [20] using Sinc-Chebyshev collocation method (SCCM) and in [19, 21] with variational iteration method (VIM). In Tables 6,7,8 we compare the numerical results of proposed method ($h = \tau = 0.001$) with the results given in [7, 20]. It can be seen that the presented technique is more accurate than the presented methods.

In Tables 9,10, we present the computational order in time and space directions with different values of h , τ and α . These tables show that the computational orders are in good agreement with theoretical ones.



TABLE 3. Comparison of the errors with different values of α for Example 4.1.

α	(x, t)	VIM [20]	SCM [20] $m = n = 5$	FVM $h = \tau = 0.01$
1.4	(0.1, 0.1)	9.2852×10^{-3}	8.4385×10^{-4}	6.6073×10^{-5}
	(0.2, 0.2)	2.2201×10^{-2}	1.1433×10^{-3}	6.7614×10^{-5}
	(0.3, 0.3)	3.5651×10^{-2}	5.3780×10^{-4}	5.8826×10^{-5}
	(0.4, 0.4)	4.9628×10^{-2}	1.5545×10^{-4}	4.7438×10^{-5}
	(0.5, 0.5)	6.4449×10^{-2}	5.3227×10^{-4}	5.2076×10^{-5}
	(0.6, 0.6)	7.9514×10^{-2}	1.3268×10^{-3}	1.3200×10^{-5}
	(0.7, 0.7)	9.1443×10^{-2}	1.9159×10^{-3}	2.5182×10^{-6}
	(0.8, 0.8)	8.7942×10^{-2}	2.0414×10^{-3}	6.6600×10^{-7}
	(0.9, 0.9)	9.2321×10^{-4}	1.8996×10^{-3}	2.6310×10^{-7}
1.6	(0.1, 0.1)	4.1518×10^{-3}	1.1685×10^{-3}	2.4486×10^{-4}
	(0.2, 0.2)	1.0319×10^{-2}	2.5887×10^{-3}	2.9941×10^{-4}
	(0.3, 0.3)	1.7757×10^{-2}	2.8863×10^{-3}	2.8228×10^{-4}
	(0.4, 0.4)	2.6987×10^{-2}	2.3912×10^{-3}	2.4877×10^{-4}
	(0.5, 0.5)	3.8327×10^{-2}	1.7692×10^{-3}	2.0753×10^{-4}
	(0.6, 0.6)	5.0993×10^{-2}	1.4174×10^{-3}	1.3907×10^{-4}
	(0.7, 0.7)	6.1379×10^{-3}	1.4334×10^{-3}	4.5172×10^{-5}
	(0.8, 0.8)	5.6577×10^{-2}	1.6653×10^{-3}	1.4232×10^{-5}
	(0.9, 0.9)	3.8618×10^{-2}	1.7449×10^{-3}	1.9445×10^{-5}

TABLE 4. Errors and computational orders with $\tau = 0.001$ for Example 4.1.

h	$\alpha = 1.4$		$\alpha = 1.5$		$\alpha = 1.6$	
	L_∞	Order	L_∞	Order	L_∞	Order
$\frac{1}{10}$	5.5035×10^{-4}	—	5.0066×10^{-4}	—	5.0871×10^{-4}	—
$\frac{1}{20}$	1.3003×10^{-4}	1.95	1.2902×10^{-4}	1.97	1.2307×10^{-4}	2.04
$\frac{1}{40}$	3.2909×10^{-5}	1.98	3.0819×10^{-5}	2.06	2.3306×10^{-5}	2.40
$\frac{1}{80}$	7.8747×10^{-6}	2.06	5.7287×10^{-6}	2.42	5.3811×10^{-6}	2.11

TABLE 5. Errors and computational orders with $h = 0.001$ for Example 4.1.

τ	$\alpha = 1.03$		$\alpha = 1.2$		$\alpha = 1.97$	
	L_∞	Order	L_∞	Order	L_∞	Order
$\frac{1}{10}$	3.6037×10^{-4}	—	3.5375×10^{-4}	—	4.0121×10^{-3}	—
$\frac{1}{20}$	9.2268×10^{-5}	1.96	9.0989×10^{-5}	1.95	2.3598×10^{-3}	0.76
$\frac{1}{40}$	2.3376×10^{-5}	1.98	2.3225×10^{-5}	1.96	1.1583×10^{-3}	1.02
$\frac{1}{80}$	5.8634×10^{-6}	1.99	5.9299×10^{-6}	1.96	4.8099×10^{-4}	1.26
$\frac{1}{160}$	1.4503×10^{-6}	2.01	1.5174×10^{-6}	1.96	1.6506×10^{-4}	1.54

5. CONCLUSION

In this paper, we developed a finite volume method for the solution of fractional nonlinear Klein-Gordon equation on a finite domain. We obtained the semi-discrete scheme using a finite difference scheme of order $O(\tau^{3-\alpha})$, $1 \leq \alpha \leq 2$ then the fully discrete scheme is achieved using finite volume method with piecewise linear basis functions. We proved that the constructed scheme is unconditionally stable and convergent with order $O(\Delta t^{3-\alpha} + h^2)$. Finally, we



TABLE 6. Comparison of errors with different values of α for Example 4.2.

x	SCM [20]		CPM [7]		FVM	
	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.7$
0.1	1.6396×10^{-3}	1.5471×10^{-3}	3.7560×10^{-4}	2.8316×10^{-4}	3.1841×10^{-6}	2.1772×10^{-5}
0.2	1.2808×10^{-4}	1.1272×10^{-4}	6.1166×10^{-4}	4.3705×10^{-4}	6.0837×10^{-6}	4.1132×10^{-5}
0.3	1.0869×10^{-3}	8.9663×10^{-4}	8.0904×10^{-4}	5.6962×10^{-4}	8.4195×10^{-6}	5.6137×10^{-5}
0.4	$.04196 \times 10^{-4}$	6.3348×10^{-4}	9.3664×10^{-4}	6.5570×10^{-4}	9.9410×10^{-6}	6.5543×10^{-5}
0.5	7.8252×10^{-4}	5.6868×10^{-4}	9.7915×10^{-4}	6.8393×10^{-4}	1.0469×10^{-5}	6.8736×10^{-5}
0.6	8.4196×10^{-4}	6.3348×10^{-4}	9.3664×10^{-4}	6.5570×10^{-4}	9.9410×10^{-6}	6.5543×10^{-5}
0.7	1.0869×10^{-3}	8.9663×10^{-4}	8.0904×10^{-4}	5.6962×10^{-4}	8.4195×10^{-6}	5.6137×10^{-5}
0.8	1.2808×10^{-3}	1.1272×10^{-3}	6.1166×10^{-4}	4.3705×10^{-4}	6.0837×10^{-6}	4.1132×10^{-5}
0.9	1.6396×10^{-3}	1.5471×10^{-3}	3.7560×10^{-4}	2.8316×10^{-4}	3.1841×10^{-6}	2.1772×10^{-5}

TABLE 7. Comparison of errors with $\alpha = 1.9$ for Example 4.2.

	SCM [20]	CPM [7]	FVM	FVM
			$h = \tau = 0.001$	$h = 0.01, \tau = 0.0001$
0.1	1.4380×10^{-3}	1.4710×10^{-4}	1.4633×10^{-4}	3.3152×10^{-5}
0.2	9.4914×10^{-4}	1.7998×10^{-4}	2.7654×10^{-4}	6.2299×10^{-5}
0.3	6.7913×10^{-4}	2.1728×10^{-4}	3.7757×10^{-4}	8.4459×10^{-5}
0.4	3.9687×10^{-4}	2.4247×10^{-4}	4.4098×10^{-4}	9.8067×10^{-5}
0.5	3.2651×10^{-4}	2.4975×10^{-4}	4.6252×10^{-4}	1.0262×10^{-4}
0.6	3.9687×10^{-4}	2.4247×10^{-4}	4.4098×10^{-4}	9.8067×10^{-5}
0.7	6.7913×10^{-4}	2.1728×10^{-4}	3.7757×10^{-4}	8.4459×10^{-5}
0.8	9.4914×10^{-4}	1.7998×10^{-4}	2.7654×10^{-4}	6.2299×10^{-5}
0.9	1.4380×10^{-3}	1.4710×10^{-4}	1.4663×10^{-4}	3.3152×10^{-5}

TABLE 8. Comparison of errors with different values of α for Example 4.2.

	(x, t)	VIM [20]	SCM[20]	CPM[7]	FVM
$\alpha = 1.4$	(0.1, 0.1)	3.9211×10^{-5}	2.3809×10^{-5}	2.5306×10^{-6}	8.3383×10^{-8}
	(0.2, 0.2)	6.1713×10^{-5}	5.2644×10^{-5}	9.4500×10^{-5}	4.7838×10^{-7}
	(0.3, 0.3)	2.1989×10^{-3}	6.0187×10^{-6}	1.8244×10^{-4}	1.1511×10^{-6}
	(0.4, 0.4)	2.5545×10^{-3}	6.6640×10^{-5}	1.1825×10^{-4}	1.8873×10^{-6}
	(0.5, 0.5)	5.3405×10^{-3}	4.0011×10^{-5}	3.4665×10^{-5}	2.4543×10^{-6}
	(0.6, 0.6)	3.1409×10^{-2}	1.5837×10^{-4}	1.1197×10^{-4}	2.6974×10^{-6}
	(0.7, 0.7)	8.0092×10^{-2}	9.1922×10^{-4}	5.5498×10^{-5}	2.5614×10^{-6}
	(0.8, 0.8)	1.3528×10^{-1}	2.9084×10^{-3}	5.4616×10^{-6}	2.0605×10^{-6}
	(0.9, 0.9)	1.4272×10^{-1}	3.8732×10^{-3}	8.5389×10^{-5}	1.2226×10^{-6}
$\alpha = 1.6$	(0.1, 0.1)	1.0402×10^{-5}	2.3809×10^{-5}	1.5542×10^{-5}	1.8184×10^{-7}
	(0.2, 0.2)	1.4424×10^{-4}	5.2644×10^{-5}	7.8870×10^{-5}	1.4778×10^{-6}
	(0.3, 0.3)	6.7151×10^{-5}	6.0187×10^{-6}	2.1215×10^{-4}	4.4925×10^{-6}
	(0.4, 0.4)	3.0493×10^{-3}	6.6640×10^{-5}	1.9004×10^{-4}	8.8153×10^{-6}
	(0.5, 0.5)	1.6350×10^{-2}	4.0011×10^{-5}	3.6281×10^{-5}	1.3197×10^{-5}
	(0.6, 0.6)	4.9599×10^{-2}	1.5837×10^{-4}	7.9247×10^{-5}	1.6113×10^{-5}
	(0.7, 0.7)	1.0675×10^{-1}	1.1922×10^{-4}	5.8820×10^{-5}	1.6366×10^{-5}
	(0.8, 0.8)	1.6942×10^{-1}	2.9084×10^{-3}	1.2402×10^{-6}	1.3466×10^{-5}
	(0.9, 0.9)	1.7521×10^{-1}	3.8732×10^{-3}	6.6997×10^{-5}	7.7082×10^{-6}

TABLE 9. Errors and computational orders with $h = 1/2000$ for Example 4.2.

τ	$\alpha = 1.2$		$\alpha = 1.4$	
	L_∞	Order	L_∞	Order
$\frac{1}{10}$	1.4943×10^{-2}	—	1.6339×10^{-2}	—
$\frac{1}{20}$	4.4383×10^{-3}	1.75	5.0012×10^{-3}	1.70
$\frac{1}{40}$	1.2206×10^{-3}	1.86	1.4567×10^{-3}	1.77
$\frac{1}{80}$	3.2297×10^{-4}	1.91	4.1671×10^{-4}	1.80
$\frac{1}{160}$	8.3843×10^{-5}	1.94	1.1923×10^{-4}	1.80

TABLE 10. Errors and computational orders with $\tau = 1/10000$ for Example 4.2.

h	$\alpha = 1.2$		$\alpha = 1.4$	
	L_∞	Order	L_∞	Order
$\frac{1}{10}$	1.121×10^{-3}	—	1.3481×10^{-3}	—
$\frac{1}{20}$	4.632×10^{-3}	1.995	5.3788×10^{-4}	1.996
$\frac{1}{40}$	3.054×10^{-4}	1.998	3.3744×10^{-4}	1.998
$\frac{1}{80}$	7.181×10^{-5}	1.999	8.4583×10^{-5}	1.996

presented the numerical results and compared them with the results of existing methods in the literature. The results show the applicability and efficiency of the proposed finite volume schemes and excellent agreement with theoretical analysis. We plan to develop the FV to solve two- and three-dimensional fractional partial differential equations with conservative equation in future works.

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