

Certain Simpson's and Newton's type inequalities in Quantum Multiplicative Calculus

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Abstract

This paper investigates the application of quantum multiplicative calculus to establish novel variants of Simpson's and Newton's type inequalities for quantum multiplicative differentiable functions. The motivation for employing multiplicative calculus lies in its ability to model discrete structure, proportional changes and exponential growth more effectively than classical calculus, making it a powerful framework for analyzing phenomena governed by discrete and multiplicative dynamics. The study derives two fundamental integral identities, which form the foundation for these new inequalities. Utilizing concepts from multiplicative convexity theory and Hölder's inequality, sharper and more generalized bounds are obtained. The proposed inequalities provide refined error estimates for quadrature rules, enhancing the precision of numerical integration methods. Numerical and graphical analyses are presented to validate the theoretical findings and demonstrate their practical utility. These results contribute to a deeper understanding of inequalities in multiplicative and quantum calculus, with implications for future research in integral inequalities, optimization, and numerical analysis.

Keywords. Q-calculus, Multiplicative calculus, Simpson's inequality.

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1. INTRODUCTION

Mathematical inequalities are a tool for examining the variations in magnitude or value between two or more quantities and represent their relationship. In contrast to equations that indicate equality, inequalities demonstrate conditions where a particular quantity is greater than or less than another. These interactions can involve functions, variables, or other complex mathematical components instead of solely integers. Generally, several values that fulfill the criteria represent the solutions to inequalities. Inequalities are crucial in various fields, including optimization, economics, finance, and biology, as the foundation for analyses and conclusions concerning complex problems. Through studying the theory of inequalities, mathematicians develop approaches that are useful across different mathematical domains for analysis and solutions. Considering everything, mathematical inequalities operate as powerful instruments to perform analytical and quantitative evaluations of various values or quantities. In mathematical analysis, inequalities are essential for assessing and comparing functions, establishing integral bounds, and modeling constraints in real-life scenarios. In specific, integral inequalities have numerous advantages in a broad spectrum of scientific domains and are often used in both applied and pure mathematics. Simpson's integral inequalities, a well-known result in numerical mathematics, is one such example.

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Simpson's method is a renowned tool for numerically integrating and the approximation of definite integrals. This technique is named after Thomas Simpson (1710-1761), even though an identical method was utilized by Johannes Kepler around a century ago. Therefore, the method is often called Kepler's rule. Simpson's method is one specific variant of the Newton-Cotes quadrature methodology, referred to as the three-point rule, signifying that estimations generated via a three-step quadratic kernel are frequently labelled as Newton-type results.

1. Simpson's 1/3 rule (Simpson's quadrature formula):

$$\int_{\nu}^{\mu} \Upsilon(x) dx \approx \frac{1}{6} \left[\Upsilon(\nu) + 4\Upsilon\left(\frac{\nu + \mu}{2}\right) + \Upsilon(\mu) \right].$$

2. Simpson's 3/8 rule (Newton-Cotes quadrature formula):

$$\int_{\nu}^{\mu} \Upsilon(x) dx \approx \frac{1}{8} \left[\Upsilon(\nu) + 3\Upsilon\left(\frac{2\nu + \mu}{3}\right) + 3\Upsilon\left(\frac{\nu + 2\mu}{3}\right) + \Upsilon(\mu) \right].$$

Several researchers have turned their attention to the Simpson-type inequalities for a range of functions in the past few years. Some studies have examined Simpson- and Newton-type inequalities for a convexity context, as the convexity framework serves as a reliable and versatile technique for tackling various kinds of problems in the fields of applied and pure mathematics. Dragomir et al. [17] established distinct Simpson-type inequalities and analyzed their importance in quadratic formulations for numerical integration. Alomari et al. [4] proposed Simpson-type inequalities for s -convex functions, whereas Sarikaya et al. [43] explored different variants of these inequalities in the context of s -convexity. Newton-type inequalities for harmonic and p -harmonic convex functions are provided by Noor et al. [36, 37]. Chen et al. [16] introduced Simpson-type inequalities for s -convex functions within the setting of fractional integrals. Iftikhar et al. [27] identified new Newton-type inequalities for functions governed by generalized convex local fractional derivatives. Moumen et al. [32] explored Simpson-type inequalities for multiplicatively convex functions by applying fractional integrals, whereas Soontharanon et al. [46] suggested a more general framework for quantum Simpson and Newton-type inequalities. Gao et al. [22] contributed via the development of Newton-type inequalities rooted in convexity features. A multiplicative fractional variant of Simpson's and Newton-type inequalities applicable to differentiable multiplicative convex functions are presented by Ali et al. [5]. Mateen et al. [31] examined the attributes of Newton-Cotes type inequalities involving multiplicative generalized convex functions and gave computational analysis. Simpson and Ostrowski- type inequalities for generalized convex functions multiplicative calculus are formulated by Zhan et al. [51].

Frequently called q -calculus, quantum calculus is an extension of calculus that excludes the idea of limits. This method is particularly suitable for systems where traditional methods are inefficient since it offers a discrete alternative for continuous calculus. The q -integral and the q -derivative were introduced by Jackson[28], who formally formalized the q -calculus, which Leonhard Euler initially developed in the early 18th century. Let Υ be a real valued function; the q -derivative of $\Upsilon(x)$ is defined as:

$$D_q \Upsilon(x) = \frac{\Upsilon(x) - \Upsilon(qx)}{x - qx}, \quad (1.1)$$

with $0 < q < 1$. The q -derivative can also be called the Jackson derivative. The following series expansion defines the classic Jackson integral of a real valued function Υ :

$$\int_0^x \Upsilon(x) d_q x = (1 - q)x \sum_{k=0}^{\infty} q^k \Upsilon(q^k x). \quad (1.2)$$

Tariboon et.al. [47] gave the following definition of q -derivative and q -integral for the left quantum operator: Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a continuous function, and $q \in (0, 1)$, the q_ν -derivative of Υ at $x \in [\nu, \mu]$ is defined as:

$${}_\nu D_q \Upsilon(x) = \frac{\Upsilon(\nu + q(x - \nu)) - \Upsilon(x)}{(q - 1)(x - \nu)}. \quad (1.3)$$



Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a continuous function, and $q \in (0, 1)$, then q_ν -definite integral of Υ at $\varkappa \in [\nu, \mu]$ is defined as:

$$\int_{\nu}^{\varkappa} \Upsilon(\varkappa) {}_{\nu}d_q \varkappa = (q-1)(\nu - \varkappa) \sum_{k=0}^{\infty} q^k \Upsilon(\nu + (\varkappa - \nu)q^k), \quad \varkappa \in [\nu, \mu]. \quad (1.4)$$

Similarly, Bermudo et. al. [12] gave the concept of right q -derivative and integral as:

$${}^{\mu}D_q \Upsilon(\varkappa) = \frac{\Upsilon(\mu + q(\varkappa - \mu)) - \Upsilon(\varkappa)}{(q-1)(\varkappa - \mu)}. \quad (1.5)$$

the q^μ -definite integral of Υ at $\varkappa \in [\nu, \mu]$ is defined as:

$$\int_{\varkappa}^{\mu} \Upsilon(\varkappa) {}^{\mu}d_q \varkappa = (q-1)(\varkappa - \mu) \sum_{k=0}^{\infty} q^k \Upsilon(\mu + (\varkappa - \mu)q^k), \quad \varkappa \in [\nu, \mu]. \quad (1.6)$$

Remark 1.1. If we choose $\nu = 0$ in Eq. (1.4), then we get the classical Jackson q -integral, defined in Eq. (1.2).

$$\int_0^1 \Upsilon(\varkappa) d_q \varkappa = (1-q) \sum_{k=0}^{\infty} q^k \Upsilon(q^k),$$

similarly, if $\mu = 1$ and $\varkappa = \nu = 0$ in Eq. (1.6), then

$$\int_0^1 \Upsilon(\varkappa) {}^1d_q \varkappa = (1-q) \sum_{k=0}^{\infty} q^k \Upsilon(1 - q^k).$$

Unlike conventional calculus, which relies primarily on limits, q -calculus considers variations between subsequent values in a way established by a parameter q . The q -calculus converts again to conventional calculus when q approaches 1. The q -calculus has recently been extensively utilized in numerous fields, such as mathematics and physics [2, 20, 44, 45]. Recently, much research has applied q -calculus to established error-bound results of different integral inequalities [3, 6, 7, 14, 23–25, 38, 39].

Grossman and Katz established the first-ever non-Newtonian calculus framework, which they termed geometric calculus, between 1967 and 1970. In the years afterwards, they created an entire series of non-Newtonian calculi, collectively, which profoundly changed the classical calculus that Newton and Leibniz initially proposed in the 17th century. This class, referred to as multiplicative calculus, substitutes the product and ratio over the spectrum of real positive values with the standard sum and difference operations, respectively. The system proves particularly helpful for addressing exponentially changing functions. It is significant to keep in mind that Bashirov et al. [13] established the entire mathematical foundation for multiplicative calculus. The concern of deciding if it is appropriate to develop a tool that has less versatility arises since applications of multiplicative calculus are narrower compared to Newtonian and Leibnizian conventional calculus. This paradox can be compared to the justification for mathematicians' utilization of polar coordinates in spite of a more applicable representation of points on a plane presented through rectangular coordinates. The authors suggest that because exponential decay and growth occur frequently in domains like finance, biology, and economics, multiplicative calculus becomes especially effective in these fields. The detailed overview of multiplicative calculus can be seen in recently published articles [8–11, 15, 18, 21, 29, 30, 33–35, 40–42, 50].

A major milestone in the field of mathematics is the rise of q -multiplicative calculus (quantum multiplicative calculus), establishing an entirely novel structure to analyze complex systems that show both discrete and multiplicative behaviors. This novel idea targets an essential dimension, as traditional approaches often fail to address phenomena that span both discrete and continuous domains. Quantum multiplicative calculus unifies the fundamental concepts of q -calculus with multiplicative calculus, permitting better modeling of systems in fields such as biology, economics, and quantum physics, where discrete scaling and proportional growth are significant. Besides strengthening our knowledge of classical inequalities, the approach delivers better tools along with improved bounds for the analysis of dynamic systems. This novel approach has an extensive list of potential applications, improving estimates in economics and natural sciences and also strengthening the understanding of complex mathematical structures. The newly developed approach answers a major gap, phenomena that involve both discrete and continuous domains. Unfortunately, we



have found limited literature in this potential area. As an example, Yener et al. [48] established some basic aspects of the q -multiplicative calculus. The results of the q -multiplicative Laplace transform are stated by Yilmazre et al. [49]. Mubashir et al. [26] outlined the properties of the right q -multiplicative operator q^μ and established some fresh midpoint type inequalities.

In this study, we combine the ideas of q -calculus and multiplicative calculus in order to develop some new Simpsion's and Newton's type inequalities, motivated by a very small amount of published literature for q -multiplicative calculus. More accurate modeling of systems in areas including quantum physics, economics, and biology, among others where both proportional growth and discrete scaling are of critical importance, is made doable by q -multiplicative calculus. Despite enhancing our knowledge of classical inequalities, this approach enables better tools and more refined bounds for dynamic system analysis.

We constitute this paper as follows: In Section 2, we provide an overview of q -multiplicative calculus. In Section 3, we prove two equalities, necessary to present our main results. In Section 4, we derive novel Simpson's and Newton's type inequalities in q -multiplicative calculus, while Section 5 focuses on numerical and graphical behaviours of the results proven in section 4. Finally, Section 6 sums up all the study with concluding remarks and future research directions.

2. q -MULTIPLICATIVE CALCULUS

This section includes some preliminaries and theorems related to q -multiplicative calculus that will enable us to prove our main results. Before presenting preliminaries in q -multiplicative calculus, let us define some key definitions related to q -calculus and convexity theory.

Definition 2.1. [19] A function $\Upsilon : k \subset \mathbb{R} \rightarrow [0, \infty)$ is said to be multiplicatively convex if

$$\Upsilon(tx + (1-t)y) \leq [\Upsilon(x)]^t [\Upsilon(y)]^{1-t}, \quad (2.1)$$

for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.2. [19] Let Υ be a convex function defined on the real interval $K \subset \mathbb{R}$. If $\varkappa_1, \varkappa_2, \varkappa_3, \dots, \varkappa_n \in K$ and $\mu_1, \mu_2, \mu_3, \dots, \mu_n \geq 0$, then the Jensen's inequality is stated as:

$$\Upsilon \left(\sum_{i=1}^n \lambda_i \varkappa_i \right) \leq \sum_{i=1}^n \lambda_i \Upsilon(\varkappa_i). \quad (2.2)$$

Definition 2.3. A natural number in q -calculus for $0 < q < 1$, denoted by $[\alpha]_q$ is defined by:

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q} = 1 + q + q^2 + \dots + q^{\alpha-1}, \quad (2.3)$$

The factorial of this natural number is defined by:

$$[\alpha]_q! = \begin{cases} 1, & \alpha = 0, \\ [\alpha]_q [\alpha - 1]_q [\alpha - 2]_q \dots [1]_q, & \alpha \in N. \end{cases} \quad (2.4)$$

The q -exponential function $e_q^\varkappa$ based on its classical exponential function e^x is defined by:

$$e_q^\varkappa = \sum_{k=0}^{\infty} \frac{\varkappa^k}{[k]_q!}. \quad (2.5)$$

2.1. Q -Multiplicative Integral.

Definition 2.4. Let $\Upsilon(\varkappa) > 0$ and $\ln \Upsilon$ be integrable on $[\nu, \mu]$, then the multiplicative integral of Υ on $[\nu, \mu]$ exists and is determined by the following expression:

$$\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{d\varkappa} = e^{\int_{\nu}^{\mu} \ln \Upsilon(\varkappa) d\varkappa}. \quad (2.6)$$



Definition 2.5. [49] Let $\Upsilon(\mathcal{X})$ be a bounded positive function on $0 < \nu < \mu$. The q -multiplicative integral can be stated as

$$\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X}))^{d_q \mathcal{X}} = e_q^{\int_{\mathcal{X}}^{\mu} \ln \Upsilon(\mathcal{X}) d_q \mathcal{X}}, \quad q \in (0, 1). \quad (2.7)$$

Definition 2.6. [26] Let $\Upsilon(\mathcal{X})$ be a bounded positive function on $0 < \nu < \mu$. The right q -multiplicative integral can be stated as

$$\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X}))^{\mu d_q \mathcal{X}} = e_q^{\int_{\mathcal{X}}^{\mu} \ln \Upsilon(\mathcal{X})^{\mu} d_q \mathcal{X}}, \quad q \in (0, 1). \quad (2.8)$$

By using (1.4), we have

$$\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X}))^{\mu d_q \mathcal{X}} = e_q^{(1-q)(\mu-\mathcal{X}) \sum_{\mu=0}^{\infty} q^{\mu} \ln \Upsilon(q^{\mu} \mathcal{X} + (1-q^{\mu})\mu)}, \quad q \in (0, 1). \quad (2.9)$$

Theorem 2.7. [26] Suppose that Υ and g are q^* -integrable on $[\nu, \mu]$. Then, we can establish the following properties:

- (i) $\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X})^k)^{\mu d_q \mathcal{X}} = \int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X})^{\mu d_q \mathcal{X}})^k, \quad k \in \mathbb{R};$
- (ii) $\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X})g(\mathcal{X}))^{\mu d_q \mathcal{X}} = \int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X}))^{\mu d_q \mathcal{X}} \int_{\mathcal{X}}^{\mu} (g(\mathcal{X}))^{\mu d_q \mathcal{X}};$
- (iii) $\int_{\mathcal{X}}^{\mu} \left(\frac{\Upsilon(\mathcal{X})}{g(\mathcal{X})} \right)^{\mu d_q \mathcal{X}} = \frac{\int_{\mathcal{X}}^{\mu} (\Upsilon(\mathcal{X}))^{\mu d_q \mathcal{X}}}{\int_{\mathcal{X}}^{\mu} (g(\mathcal{X}))^{\mu d_q \mathcal{X}}}.$

Theorem 2.8 (Fundamental Theorem of q -Multiplicative calculus). [26] Let $\Upsilon(\mathcal{X})$ be a differentiable positive and continuous on a segment $0 \leq \nu \leq \mu$, if ${}^{\mu}D_q^* \mathbb{F}(\mathcal{X}) = \Upsilon(\mathcal{X})$, then

$$\int_{\nu}^{\mu} (\Upsilon(\mathcal{X}))^{\mu d_q \mathcal{X}} = \frac{\mathbb{F}(\mu)}{\mathbb{F}(\nu)}. \quad (2.10)$$

Corollary 2.9. [26] Let $\Upsilon(\mathcal{X})$ be q^* -differentiable and ${}^{\mu}D_q^*[\Upsilon(\mathcal{X})]$ be q^* -integrable on $[\nu, \mu]$,

$$\int_{\nu}^{\mu} [{}^{\mu}D_q^* \Upsilon(\mathcal{X})]^{\mu d_q \mathcal{X}} = \frac{\Upsilon(\mu)}{\Upsilon(\nu)}. \quad (2.11)$$

Theorem 2.10. [26] Let $\Upsilon(\mathcal{X})$ be a q^* -differentiable and $g(\mathcal{X})$ be q -differentiable, they are continuous on $0 \leq \nu < \mu$, then

$$\int_{\nu}^{\mu} \left[({}^{\mu}D_q^* \Upsilon(\mathcal{X}))^{g(\mathcal{X})} \right]^{\mu d_q \mathcal{X}} = \frac{\Upsilon(\mu)^{g(\mu)}}{\Upsilon(\nu)^{g(\nu)}} \frac{1}{\int_{\nu}^{\mu} (\Upsilon((q\mathcal{X}) + (1-q)\nu)^{\mu} D_q g(\mathcal{X}))^{\mu d_q \mathcal{X}}}. \quad (2.12)$$

2.2. Q -Multiplicative Derivative.

Definition 2.11. [13] The multiplicative derivative of a function $\Upsilon(\mathcal{X})$, denoted by $\frac{d^*}{dx} \Upsilon(\mathcal{X})$ or $\Upsilon^*(\mathcal{X})$ can be defined as :

$$\Upsilon^*(\mathcal{X}) = \lim_{\Delta \mathcal{X} \rightarrow 0} \left(\frac{\Upsilon(\mathcal{X} + \Delta \mathcal{X})}{\Upsilon(\mathcal{X})} \right)^{\frac{1}{\Delta \mathcal{X}}}, \quad \Upsilon(\mathcal{X}) > 0. \quad (2.13)$$

If $\Upsilon(\mathcal{X}) > 0$ and $\Upsilon(\mathcal{X})$ is differentiable at $\mathcal{X}$, then $\Upsilon^*(\mathcal{X})$ exists, and its relation between ordinary derivative $\Upsilon'(\mathcal{X})$ is as follows:

$$\Upsilon^*(\mathcal{X}) = e^{(\ln \Upsilon(\mathcal{X}))'} = e^{\frac{\Upsilon'(\mathcal{X})}{\Upsilon(\mathcal{X})}},$$

$\Upsilon^{*(k)}(\mathcal{X})$ is k^{th} order multiplicative derivative and $\Upsilon^{*(0)}(\mathcal{X}) = \Upsilon(\mathcal{X})$.

Now, we present a new definition of q -multiplicative derivative by taking into consideration the basic concept of q -calculus.



Definition 2.12. [49] Let $\Upsilon(\varkappa) > 0$ and $0 < q < 1$. The q -analogue of the multiplicative derivative, the q -multiplicative derivative is defined by:

$$D_q^* \Upsilon(\varkappa) = \left(\frac{\Upsilon(q\varkappa)}{\Upsilon(\varkappa)} \right)^{\frac{1}{(q-1)\varkappa}}. \quad (2.14)$$

Definition 2.13. [26] Let $\Upsilon(\varkappa) > 0$ and $0 < q < 1$. The ${}^\mu q$ -analogue of the multiplicative derivative, the right q -multiplicative derivative is defined by:

$${}^\mu D_q^* \Upsilon(\varkappa) = \left(\frac{\Upsilon(q\varkappa + (1-q)\mu)}{\Upsilon(\varkappa)} \right)^{\frac{1}{(q-1)(\varkappa-\mu)}}. \quad (2.15)$$

Also, if in Eq. (2.15), $q \rightarrow 1^-$ and for $\mu = 0$, we obtain as follows:

$$\lim_{q \rightarrow 1^-} {}^0 D_q^* \Upsilon(\varkappa) = \frac{d^*}{d\varkappa^*} \Upsilon(\varkappa).$$

The μ th order q -multiplicative derivative can be denoted by ${}^\mu D_q^{*(\mu)} \Upsilon(\varkappa)$ for $\mu \in W$.

Theorem 2.14. [49] Let $\Upsilon(\varkappa)$ be a q -differentiable positive function and $0 < q < 1$. Then, we have:

$$D_q^* \Upsilon(\varkappa) = e_q^{D_q(\ln \Upsilon(\varkappa))}. \quad (2.16)$$

Theorem 2.15. [26] Let $\Upsilon(\varkappa)$ be a q -differentiable positive function and $0 < q < 1$. Then, we have :

$${}^\mu D_q^* \Upsilon(\varkappa) = e_q^{{}^\mu D_q(\ln \Upsilon(\varkappa))}. \quad (2.17)$$

In [1], the authors provided the correct formulation of Definition 2.5 and Theorem 2.14, resolving the inconsistency in the inverse relationship between the q -exponential and logarithmic functions. Moreover, they pointed out that the natural exponential function e (rather than the q -exponential e_q) should be used in the corresponding expressions. In particular, they proposed the following definition:

Definition 2.16. If we assume that Υ is a positive and bounded function on $0 < \mu < \nu$, then for $\tau \in I$,

$$\int_\mu^\nu (\Upsilon(\varkappa))^{d_q \varkappa} = \exp \left(\int_\mu^\nu \ln(\Upsilon(\varkappa)) d_q \varkappa \right) = \exp \left(\int_0^\nu \ln(\Upsilon(\varkappa)) d_q \varkappa - \int_0^\mu \ln(\Upsilon(\varkappa)) d_q \varkappa \right)$$

is called a q^* - or q -multiplicative integral of Υ over I .

Proposition 2.17. If we assume a positive and q -differentiable function Υ , then the following equality holds:

$$D_q^* \Upsilon(\varkappa) = \exp(D_q \ln(\Upsilon(\varkappa))).$$

A positive function $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}^+$ is said to be quantum multiplicative differentiable (or q^* -differentiable) on (ν, μ) if its q -multiplicative derivative $D_q^* \Upsilon(\varkappa)$ exists at every point $\varkappa \in (\nu, \mu)$.

Corollary 2.18. [26] Let $\Upsilon(\varkappa)$ be a positive function and $q \in (0, 1)$. The n th order q -multiplicative derivative of $\Upsilon(\varkappa)$ is stated by:

$${}^\mu D_q^{*(n)} [\Upsilon(\varkappa)] = e^{{}^\mu D_q^{(n)} [\ln \Upsilon(\varkappa)]}, \quad n \in W. \quad (2.18)$$

The following properties of q^* -derivative exist:

Theorem 2.19. [26] Let Υ and g be q^* -differentiable functions and κ be a constant. Then, we have:

- (i) ${}^\mu D_q^* [\kappa \Upsilon(\varkappa)] = {}^\mu D_q^* [\Upsilon(\varkappa)]$;
- (ii) ${}^\mu D_q^* [\Upsilon(\varkappa) g(\varkappa)] = {}^\mu D_q^* [\Upsilon(\varkappa)] {}^\mu D_q^* [g(\varkappa)]$;
- (iii) ${}^\mu D_q^* \left[\frac{\Upsilon(\varkappa)}{g(\varkappa)} \right] = \frac{{}^\mu D_q^* \Upsilon(\varkappa)}{{}^\mu D_q^* g(\varkappa)}$;
- (iv) ${}^\mu D_q^* [\Upsilon \circ g(\varkappa)] = {}^\mu D_{q,g}^* [\Upsilon(g(\varkappa))] {}^\mu D_{q,g} g(\varkappa)$;
- (v) ${}^\mu D_q^* [\Upsilon(\varkappa)^{g(\varkappa)}] = [{}^\mu D_q^* \Upsilon(\varkappa)]^{g(q\varkappa + (1-q)\mu)} \Upsilon(\varkappa)^{{}^\mu D_{q,g} g(\varkappa)}$,



where ${}^{\mu}D_{q,g}^{*}[\Upsilon(g(\varkappa))] = \left[\frac{\Upsilon(g(q\varkappa+(1-q)\mu))}{\Upsilon(g(\varkappa))} \right]^{\frac{1}{g(q\varkappa+(1-q)\mu)-g(\varkappa)}}$.

Theorem 2.20. [26] Let Υ and g be q^{*} -differentiable functions. Then, for any integer $\mu \geq 0$

$${}^{\mu}D_q^{*(\mu)} \left[\Upsilon(\varkappa)^{g(\varkappa)} \right] = e^{\sum_{k=0}^{\mu} \left[\begin{matrix} \mu \\ k \end{matrix} \right]_q} {}^{\mu}D_q^k \left[\ln \Upsilon(q^{\mu-k}\varkappa + (1-q^{\mu-k})\mu) \right] {}^{\mu}D_q^{\mu-k}[g(\varkappa)] \quad (2.19)$$

2.3. Hermite–Hadamard Inequalities in q -multiplicative calculus.

Theorem 2.21. [26] Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a multiplicative convex function on (ν, μ) and $0 < q < 1$. Then, we have

$$\Upsilon \left(\frac{\nu + q\mu}{[2]_q} \right) \leq \left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\nu d_q \varkappa} \right)^{\frac{1}{\mu-\nu}} \leq (\Upsilon(\nu)\Upsilon(\mu))^{\frac{1}{[2]_q}}. \quad (2.20)$$

Remark 2.22. If we take $q \rightarrow 1^{-}$ in Theorem 2.21, we deduce the following inequality:

$$\Upsilon \left(\frac{\nu + \mu}{2} \right) \leq \left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{dx} \right)^{\frac{1}{\mu-\nu}} \leq \sqrt{\Upsilon(\nu)\Upsilon(\mu)}, \quad (2.21)$$

which is presented in [8, 9].

Theorem 2.23. [26] Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a multiplicative convex function on (ν, μ) and $0 < q < 1$. Then, we have

$$\Upsilon \left(\frac{q\nu + \mu}{[2]_q} \right) \leq \left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\nu d_q \varkappa} \right)^{\frac{1}{\mu-\nu}} \leq (\Upsilon(\nu)^q \Upsilon(\mu))^{\frac{1}{[2]_q}}. \quad (2.22)$$

By multiplying the respective terms in Theorem 2.21 and Theorem 2.23 yields the following corollary:

Corollary 2.24. [26] Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a multiplicative convex function and $q \in (0, 1)$. Then we have

$$\Upsilon \left(\frac{q\nu + \mu}{[2]_q} \right) \Upsilon \left(\frac{\nu + q\mu}{[2]_q} \right) \leq \left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\nu d_q \varkappa} \int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\mu d_q \varkappa} \right)^{\frac{1}{\mu-\nu}} \leq \Upsilon(\nu)\Upsilon(\mu). \quad (2.23)$$

One can change the lower bound in Eq. (2.23) to the square of the lower bound in Eq. (2.21).

Corollary 2.25. [26] For any multiplicative convex function $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ and $q \in (0, 1)$,

$$\left(\Upsilon \left(\frac{\nu + \mu}{2} \right) \right)^2 \leq \left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\nu d_q \varkappa} \int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\mu d_q \varkappa} \right)^{\frac{1}{\mu-\nu}} \leq \Upsilon(\nu)\Upsilon(\mu). \quad (2.24)$$

3. QUANTUM MULTIPLICATIVE INTEGRAL INEQUALITIES

This section presents two quantum multiplicative integral equalities necessary for attaining our main results.

Lemma 3.1. Let $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a q^{*} -differentiable function on (ν, μ) and $0 < q < 1$. If Υ and $\ln \Upsilon$ are continuous on $[\nu, \mu]$, and that the q^{*} -integral of ${}^{\mu}D_q^{*}\Upsilon$ exists, then the following identity holds:

$$\frac{\left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\nu d_q \varkappa} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} = \left(\int_0^1 \left(({}^{\mu}D_q^{*}\Upsilon(t\nu + (1-t)\mu))^{w(t)} \right)^{d_q t} \right)^{(\mu-\nu)}, \quad (3.1)$$

where

$$w(t) = \begin{cases} qt - \frac{1}{6}, & 0 \leq t < \frac{1}{2}, \\ qt - \frac{5}{6}, & \frac{1}{2} < t \leq 1. \end{cases}$$



Proof. To prove this Lemma, we can write as:

$$\int_0^1 \left(({}^\mu D_q^* \Upsilon(t\nu + (1-t)\mu))^{w(t)} \right)^{d_q t} = e^{\int_0^1 (w(t)(\ln \Upsilon)_q'(t\nu + (1-t)\mu)) d_q t}. \quad (3.2)$$

Here we used ${}^\mu D_q \Upsilon(\mathcal{X}) = (\ln \Upsilon)_q'(\mathcal{X})$, The definition of $w(t)$ and the primary features of the quantum integral make it obvious that

$$\begin{aligned} \int_0^1 w(t)(\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t &= \frac{2}{3} \int_0^{\frac{1}{2}} (\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t + \int_0^1 qt(\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t \\ &\quad - \frac{5}{6} \int_0^1 (\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t. \end{aligned} \quad (3.3)$$

By definition, we have

$$(\ln \Upsilon)_q'(t\nu + (1-t)\mu) = \frac{\ln \Upsilon(qt\nu - (1-qt)\mu) - \ln \Upsilon(t\nu + (1-t)\mu)}{(1-q)(\mu - \nu)t},$$

It requires us to solve the integral on the right side of Eq. (3.3) to conclude this proof. So, by using Eq. (1.5) for the first quantum integral in Eq. (3.3), we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} (\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t \\ &= \int_0^{\frac{1}{2}} \frac{\ln \Upsilon(qt\nu - (1-qt)\mu) - \ln \Upsilon(t\nu + (1-t)\mu)}{(1-q)(\mu - \nu)} d_q t, \\ &= \frac{1}{(\mu - \nu)} \sum_{k=0}^{\infty} \ln \Upsilon \left(\frac{1}{2} q^{k+1} \nu + \left(1 - \frac{1}{2} q^{k+1} \right) \mu \right) - \frac{1}{(\mu - \nu)} \sum_{k=0}^{\infty} \ln \Upsilon \left(\frac{1}{2} q^k \nu + \left(1 - \frac{1}{2} q^k \right) \mu \right), \\ &= \frac{1}{(\mu - \nu)} \left[\ln \Upsilon(\mu) - \ln \Upsilon \left(\frac{\nu + \mu}{2} \right) \right] = \ln \left(\frac{\Upsilon(\mu)}{\Upsilon \left(\frac{\nu + \mu}{2} \right)} \right)^{\frac{1}{(\mu - \nu)}}. \end{aligned} \quad (3.4)$$

Similarly, by statement (1.6), it is noticed that

$$\begin{aligned} &\int_0^1 qt(\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t \\ &= \frac{(1-q)(\mu - \nu)}{(\mu - \nu)^2} \sum_{k=0}^{\infty} q^k \ln \Upsilon \left(\frac{1}{2} q^k \nu + \left(1 - \frac{1}{2} q^k \right) \mu \right) - \frac{1}{(\mu - \nu)} \ln \Upsilon(\nu) \\ &= \frac{1}{(\mu - \nu)^2} \int_{\nu}^{\mu} \ln \Upsilon(\mathcal{X})^{\mu} d_q \mathcal{X} - \ln \Upsilon(\nu)^{\frac{1}{(\mu - \nu)}}. \end{aligned} \quad (3.5)$$

Eventually, we possess

$$\begin{aligned} \int_0^1 (\ln \Upsilon)_q'(t\nu + (1-t)\mu) d_q t &= \frac{\ln \Upsilon(\mu) - \ln \Upsilon(\nu)}{(\mu - \nu)}, \\ &= \ln \left(\frac{\Upsilon(\mu)}{\Upsilon(\nu)} \right)^{\frac{1}{(\mu - \nu)}}. \end{aligned} \quad (3.6)$$

By using Eq. (3.3)-(3.6) in Eq. (3.2), we have

$$\begin{aligned} &\int_0^1 \left(({}^\mu D_q^* \Upsilon(t\nu + (1-t)\mu))^{w(t)} \right)^{d_q t} \\ &= e^{\frac{2}{3} \ln \left(\frac{\Upsilon(\mu)}{\Upsilon \left(\frac{\nu + \mu}{2} \right)} \right)^{\frac{1}{(\mu - \nu)}} + \frac{1}{(\mu - \nu)^2} \int_{\nu}^{\mu} \ln \Upsilon(\mathcal{X})^{\mu} d_q \mathcal{X} - \ln \Upsilon(\nu)^{\frac{1}{(\mu - \nu)}} - \frac{5}{6} \ln \left(\frac{\Upsilon(\mu)}{\Upsilon(\nu)} \right)^{\frac{1}{(\mu - \nu)}}}, \end{aligned} \quad (3.7)$$



$$= e^{\ln\left(\frac{\Upsilon(\mu)}{\Upsilon(\frac{\nu+\mu}{2})}\right)^{\frac{2}{3(\mu-\nu)}} + \left(\int_{\nu}^{\mu} \ln \Upsilon(\varkappa)^{\mu} d_q \varkappa\right)^{\frac{1}{(\mu-\nu)^2}} - \ln \Upsilon(\nu)^{\frac{1}{(\mu-\nu)}} - \ln\left(\frac{\Upsilon(\mu)}{\Upsilon(\nu)}\right)^{\frac{5}{6(\mu-\nu)}}}.$$

By multiplying either side of Eq. (3.7) by $(\mu - \nu)$ while utilizing the characteristics of $\ln$ and q -multiplicative integral, we acquire the identity needed. This concludes the proof. $\square$

In the next lemma, we provide the identity which is necessary to deduce novel quantum multiplicative integral inequalities of Newton type.

Lemma 3.2. Assume that $\Upsilon : [\nu, \mu] \rightarrow \mathbb{R}$ be a q^* -differentiable function on (ν, μ) and $0 < q < 1$. Also suppose that Υ and $\ln \Upsilon$ are continuous on $[\nu, \mu]$, and that the q^* -integral of ${}^{\mu}D_q^* \Upsilon$ exists, then the following identity holds:

$$\frac{\left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\mu} d_q \varkappa\right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{2\nu+\mu}{3}\right)\right)^3 \left(\Upsilon\left(\frac{\nu+2\mu}{3}\right)\right)^3 \Upsilon(\mu)\right]^{\frac{1}{8}}} = \left(\int_0^1 \left({}^{\mu}D_q^* \Upsilon(t\nu + (1-t)\mu)\right)^{w(t)} d_q t\right)^{(\mu-\nu)}, \quad (3.8)$$

where

$$w(t) = \begin{cases} qt - \frac{1}{8}, & 0 \leq t < \frac{1}{3}, \\ qt - \frac{1}{2}, & \frac{1}{3} \leq t < \frac{2}{3}, \\ qt - \frac{7}{8}, & \frac{2}{3} \leq t \leq 1. \end{cases}$$

Proof. We know that

$$\int_0^1 \left({}^{\mu}D_q^* \Upsilon(t\nu + (1-t)\mu)\right)^{w(t)} d_q t = e^{\int_0^1 (w(t)(\ln \Upsilon)'_q(t\nu + (1-t)\mu)) d_q t}. \quad (3.9)$$

considering the notion of $w(t)$ and the basic features of the quantum integral, it implies that

$$\begin{aligned} \int_0^1 w(t)(\ln \Upsilon)'_q(t\nu + (1-t)\mu) d_q t &= \frac{3}{8} \left[\int_0^{\frac{1}{3}} (\ln \Upsilon)'_q(t\nu + (1-t)\mu) d_q t + \int_0^{\frac{2}{3}} (\ln \Upsilon)'_q(t\nu + (1-t)\mu) d_q t \right] \\ &\quad + \int_0^1 \left(qt - \frac{7}{8}\right) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) d_q t. \end{aligned} \quad (3.10)$$

The required equality can be easily established if the remaining part of this proof proceed in the same approach as the proof of Lemma 3.1. $\square$

4. SIMSPON'S TYPE INEQUALITIES FOR QUANTUM MULTIPLICATIVE INTEGRALS

Theorem 4.1. Suppose that the condition stated in Lemma 3.1 hold. If $|{}^{\mu}D_q^* \Upsilon|$ is multiplicatively convex and integrable on $[\nu, \mu]$, then we can state

$$\left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\varkappa))^{\mu} d_q \varkappa\right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right)\right)^4 \Upsilon(\mu)\right]^{\frac{1}{6}}} \right| \leq \left[|{}^{\mu}D_q^* \Upsilon(\nu)|^{(\Omega_1(q)+\Omega_3(q))} \times |{}^{\mu}D_q^* \Upsilon(\mu)|^{(\Omega_2(q)+\Omega_4(q))} \right]^{(\mu-\nu)}, \quad (4.1)$$

Where $\Omega_i, i = 1, 2, 3, 4$ are defined by

$$\begin{aligned} \Omega_1(q) &= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| t d_q t = \begin{cases} \frac{-2q^2-2q+1}{24[2]_q[3]_q}, & 0 \leq q < \frac{1}{3}, \\ \frac{18q^2+18q-7}{216[2]_q[3]_q}, & \frac{1}{3} \leq q < 1, \end{cases} \\ \Omega_2(q) &= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| (1-t) d_q t = \begin{cases} \frac{-4q^3+1}{24[2]_q[3]_q}, & 0 \leq q < \frac{1}{3}, \\ \frac{36q^3+12q^2+12q+1}{216[2]_q[3]_q}, & \frac{1}{3} \leq q < 1, \end{cases} \end{aligned}$$



$$\Omega_3(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| t d_q t = \begin{cases} \frac{-6q^2 - 6q + 15}{24[2]_q[3]_q}, & 0 \leq q < \frac{5}{6}, \\ \frac{18q^2 + 18q + 25}{216[2]_q[3]_q}, & \frac{5}{6} \leq q < 1, \end{cases}$$

and

$$\Omega_4(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| (1-t) d_q t = \begin{cases} \frac{-8q^3 + 8q^2 + 8q - 5}{24[2]_q[3]_q}, & 0 \leq q < \frac{5}{6}, \\ \frac{12q^2 + 12q + 5}{216[2]_q[3]_q}, & \frac{5}{6} \leq q < 1. \end{cases}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned} \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{X}))^{\mu} d_q \mathcal{X} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}}, &= \left(\int_0^1 \left((\mu D_q^* \Upsilon(t\nu + (1-t)\mu))^{w(t)} \right)^{d_q t} \right)^{(\mu-\nu)}, \\ &= e^{(\mu-\nu) \int_0^{\frac{1}{2}} \left((qt - \frac{1}{6}) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{1}{2}}^1 \left((qt - \frac{5}{6}) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right) d_q t}. \end{aligned}$$

Taking modulus on both sides, and using the fact that $|\mu D_q^* \Upsilon|$ is multiplicative convex on $[\nu, \mu]$, we have

$$\begin{aligned} \left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{X}))^{\mu} d_q \mathcal{X} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} \right| &\leq e^{(\mu-\nu) \int_0^{\frac{1}{2}} \left(|qt - \frac{1}{6}| \left| (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right| \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{1}{2}}^1 \left(|qt - \frac{5}{6}| \left| (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right| \right) d_q t}, \\ &\leq e^{(\mu-\nu) \left[\left| (\ln \Upsilon)'_q(\nu) \right| \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| t d_q t + \left| (\ln \Upsilon)'_q(\mu) \right| \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| (1-t) d_q t \right]} \\ &\quad \times e^{(\mu-\nu) \left[\left| (\ln \Upsilon)'_q(\nu) \right| \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| t d_q t + \left| (\ln \Upsilon)'_q(\mu) \right| \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| (1-t) d_q t \right]}, \\ &= e^{(\mu-\nu) \left[\left| (\ln \Upsilon)'_q(\nu) \right| \Omega_1(q) + \left| (\ln \Upsilon)'_q(\mu) \right| \Omega_2(q) \right]} \\ &\quad \times e^{(\mu-\nu) \left[\left| (\ln \Upsilon)'_q(\nu) \right| \Omega_3(q) + \left| (\ln \Upsilon)'_q(\mu) \right| \Omega_4(q) \right]}, \\ &= e^{(\mu-\nu) \left[\left| (\ln \Upsilon)'_q(\nu) \right| (\Omega_1(q) + \Omega_3(q)) + \left| (\ln \Upsilon)'_q(\mu) \right| (\Omega_2(q) + \Omega_4(q)) \right]}, \\ &= \left[e^{\left| (\ln \Upsilon)'_q(\nu) \right| (\Omega_1(q) + \Omega_3(q))} \times e^{\left| (\ln \Upsilon)'_q(\mu) \right| (\Omega_2(q) + \Omega_4(q))} \right]^{(\mu-\nu)}, \\ &= \left[|\mu D_q^* \Upsilon(\nu)|^{(\Omega_1(q) + \Omega_3(q))} \times |\mu D_q^* \Upsilon(\mu)|^{(\Omega_2(q) + \Omega_4(q))} \right]^{(\mu-\nu)}. \end{aligned}$$

Hence, the proof is accomplished. $\square$

Remark 4.2. If we take $q \rightarrow 1^-$ in Theorem 4.1, we deduce the following inequality:

$$\left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{X}))^{d\mathcal{X}} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} \right| \leq (\Upsilon^*(\nu) \Upsilon^*(\mu))^{\frac{5}{72}(\mu-\nu)},$$

which is presented in [15, Theorem 4.1].



Theorem 4.3. Assume that condition in Lemma 3.1 is satisfied. If $|\mu D_q^* \Upsilon|^r$ is multiplicative convex for $r > 1$ and integrable on $[\nu, \mu]$ with $\frac{1}{s} + \frac{1}{r} = 1$, then there exist the following inequality

$$\left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\kappa))^{\mu} d_q \kappa \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} \right| \leq \left[\left\{ \left(|\mu D_q^* \Upsilon(\nu)|^{\frac{r}{4[2]_q}} \right) \times \left(|\mu D_q^* \Upsilon(\mu)|^{\frac{r(2q+1)}{4[2]_q}} \right) \right\}^{\frac{1}{r} \left(2^{1-\frac{1}{s}} \right)} \right. \\ \left. \times \left\{ \left(|\mu D_q^* \Upsilon(\nu)|^{\frac{3r}{4[2]_q}} \right) \times \left(|\mu D_q^* \Upsilon(\mu)|^{\frac{r(2q-1)}{4[2]_q}} \right) \right\}^{\frac{1}{r} (5^s - 2^{s-1}) \frac{1}{s}} \right]^{\frac{1}{6}(\mu-\nu)}. \quad (4.2)$$

Proof. We revisit Lemma 3.1, utilizing Hölder's inequality and using the fact that $|\mu D_q^* \Upsilon|^r$ is multiplicative convex on $[\nu, \mu]$, we observe that

$$\left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\kappa))^{\mu} d_q \kappa \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} \right| \leq e^{(\mu-\nu) \left(\int_0^{\frac{1}{2}} |qt - \frac{1}{6}|^s d_q t \right)^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \int_0^{\frac{1}{2}} t d_q t + |(\ln \Upsilon)'_q(\mu)|^r \int_0^{\frac{1}{2}} (1-t) d_q t \right)^{\frac{1}{r}} \\ \times e^{(\mu-\nu) \left(\int_{\frac{1}{2}}^1 |qt - \frac{5}{6}|^s d_q t \right)^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \int_{\frac{1}{2}}^1 t d_q t + |(\ln \Upsilon)'_q(\mu)|^r \int_{\frac{1}{2}}^1 (1-t) d_q t \right)^{\frac{1}{r}}. \quad (4.3)$$

By using the statement (1.2), it is noticed that

$$\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right|^s d_q t = \frac{(1-q)}{2} \sum_{k=0}^{\infty} q^k \left| \frac{q^{k+1}}{2} - \frac{1}{6} \right|^s, \\ \leq \frac{(1-q)}{2} \sum_{k=0}^{\infty} q^k \left| \frac{1}{2} - \frac{1}{6} \right|^s, \\ = \frac{(1-q)}{2} \frac{1}{3^s} \frac{1}{1-q}, \\ = \frac{1}{2 \cdot 3^s}. \quad (4.4)$$

Following a similar approach for $\int_{\frac{1}{2}}^1 |qt - \frac{5}{6}|^s d_q t$, and using the calculation that follow

$$\int_0^{\frac{1}{2}} t d_q t = \frac{1}{2} (1-q) \sum_{k=0}^{\infty} \frac{1}{2} q^{2k}, \\ = \frac{1}{4} (1-q) \frac{1}{1-q^2}, \\ = \frac{1}{4[2]_q}, \\ \int_0^{\frac{1}{2}} (1-t) d_q t = \frac{2q+1}{4[2]_q}, \\ \int_{\frac{1}{2}}^1 t d_q t = \int_0^{\frac{1}{2}} t d_q t - \int_0^1 t d_q t = \frac{3}{4[2]_q},$$

and

$$\int_{\frac{1}{2}}^1 (1-t) d_q t = \int_0^{\frac{1}{2}} (1-t) d_q t - \int_0^1 (1-t) d_q t = \frac{2q-1}{4[2]_q},$$



we find that

$$\begin{aligned}
& \left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{K}))^{\mu} d_q \mathcal{K} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{\nu+\mu}{2}\right) \right)^4 \Upsilon(\mu) \right]^{\frac{1}{6}}} \right| \leq e^{(\mu-\nu)\left(\frac{1}{2 \cdot 3^s}\right)^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \frac{1}{4[2]_q} + |(\ln \Upsilon)'_q(\mu)|^r \frac{2q+1}{4[2]_q} \right)^{\frac{1}{r}} \\
& \quad \times e^{(\mu-\nu)\left(\frac{5^s-2^{s-1}}{6^s}\right)^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \frac{3}{4[2]_q} + |(\ln \Upsilon)'_q(\mu)|^r \frac{2q-1}{4[2]_q} \right)^{\frac{1}{r}}, \\
& = \left[\left(e^{|\ln \Upsilon'_q(\nu)|^r \frac{1}{4[2]_q}} \times \left(e^{|\ln \Upsilon'_q(\mu)|^r \frac{2q+1}{4[2]_q}} \right) \right)^{\frac{1}{r}} \left(2^{1-\frac{1}{s}} \right) \right. \\
& \quad \times \left. \left(e^{|\ln \Upsilon'_q(\nu)|^r \frac{3}{4[2]_q}} \times \left(e^{|\ln \Upsilon'_q(\mu)|^r \frac{2q-1}{4[2]_q}} \right) \right)^{\frac{1}{r}} (5^s - 2^{s-1})^{\frac{1}{s}} \right]^{\frac{1}{6}(\mu-\nu)}, \\
& = \left[\left(\left({}^{\mu}D_q^* \Upsilon(\nu) \right)^{\frac{r}{4[2]_q}} \times \left({}^{\mu}D_q^* \Upsilon(\mu) \right)^{\frac{r(2q+1)}{4[2]_q}} \right)^{\frac{1}{r}} \left(2^{1-\frac{1}{s}} \right) \right. \\
& \quad \times \left. \left(\left({}^{\mu}D_q^* \Upsilon(\nu) \right)^{\frac{3r}{4[2]_q}} \times \left({}^{\mu}D_q^* \Upsilon(\mu) \right)^{\frac{r(2q-1)}{4[2]_q}} \right)^{\frac{1}{r}} (5^s - 2^{s-1})^{\frac{1}{s}} \right]^{\frac{1}{6}(\mu-\nu)}.
\end{aligned}$$

which concludes the proof. $\square$

5. NEWTON'S TYPE INEQUALITIES FOR QUANTUM MULTIPLICATIVE INTEGRALS

This section includes some new Newton-type inequalities for functions whose quantum multiplicative derivatives possess a multiplicatively convex nature, developed via a three-part kernel.

Theorem 5.1. *Assume that the condition in Lemma 3.2 is satisfied. If $|{}^{\mu}D_q^* \Upsilon|$ is multiplicatively convex and integrable on $[\nu, \mu]$, then the following inequality hold*

$$\begin{aligned}
& \left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{K}))^{\mu} d_q \mathcal{K} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{2\nu+\mu}{3}\right) \right)^3 \left(\Upsilon\left(\frac{\nu+2\mu}{3}\right) \right)^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| \leq \left[|{}^{\mu}D_q^* \Upsilon(\nu)|^{(\Delta_1(q)+\Delta_3(q)+\Delta_5(q))} \right. \\
& \quad \times \left. |{}^{\mu}D_q^* \Upsilon(\mu)|^{(\Delta_2(q)+\Delta_4(q)+\Delta_6(q))} \right]^{\frac{1}{8}(\mu-\nu)}, \tag{5.1}
\end{aligned}$$

where $\Delta_i, i = 1, 2, 3, 4, 5, 6$ are defined by

$$\begin{aligned}
\Delta_1(q) &= \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| t d_q t = \begin{cases} \frac{-5q^2-5q+3}{216[2]_q[3]_q}, & 0 < q < \frac{3}{8}, \\ \frac{160q^2+160q-69}{6912[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\
\Delta_2(q) &= \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| (1-t) d_q t = \begin{cases} \frac{-15q^3-q^2-q+6}{216[2]_q[3]_q}, & 0 < q < \frac{3}{8}, \\ \frac{480q^3+248q^2+248q-3}{6912[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\
\Delta_3(q) &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| t d_q t = \begin{cases} \frac{-5q^2-5q+9}{54[2]_q[3]_q}, & 0 < q < \frac{3}{4}, \\ \frac{6q^2+6q-3}{108[2]_q[3]_q}, & \frac{3}{4} \leq q < 1, \end{cases}
\end{aligned}$$



$$\Delta_4(q) = \int_{\frac{2}{3}}^{\frac{3}{4}} \left| qt - \frac{1}{2} \right| (1-t) d_q t = \begin{cases} \frac{-9q^3+5q^2+5q}{54[2]_q[3]_q}, & 0 < q < \frac{3}{4}, \\ \frac{6q^3+3}{108[2]_q[3]_q}, & \frac{3}{4} \leq q < 1, \end{cases}$$

$$\Delta_5(q) = \int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right| t d_q t = \begin{cases} \frac{-47q^2-47q+105}{216[2]_q[3]_q}, & 0 < q < \frac{7}{8}, \\ \frac{224q^2+224q+525}{6912[2]_q[3]_q}, & \frac{7}{8} \leq q < 1, \end{cases}$$

and

$$\Delta_6(q) = \int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right| (1-t) d_q t = \begin{cases} \frac{-57q^3+53q^2+53q-42}{216[2]_q[3]_q}, & 0 < q < \frac{7}{8}, \\ \frac{-96q^3+184q^2+184q-21}{6912[2]_q[3]_q}, & \frac{7}{8} \leq q < 1. \end{cases}$$

Proof. Taking into account Lemma 3.2, we have

$$\begin{aligned} \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{K}))^{\mu} d_q \mathcal{K} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) (\Upsilon(\frac{2\nu+\mu}{3}))^3 (\Upsilon(\frac{\nu+2\mu}{3}))^3 \Upsilon(\mu) \right]^{\frac{1}{8}}}, &= \left(\int_0^1 \left(({}^{\mu}D_q^* \Upsilon(t\nu + (1-t)\mu))^{w(t)} \right)^{d_q t} \right)^{(\mu-\nu)}, \\ &= e^{(\mu-\nu) \int_0^{\frac{1}{3}} \left((qt - \frac{1}{8}) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{1}{3}}^{\frac{2}{3}} \left((qt - \frac{1}{2}) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{2}{3}}^1 \left((qt - \frac{7}{8}) (\ln \Upsilon)'_q(t\nu + (1-t)\mu) \right) d_q t}. \end{aligned}$$

Taking absolute value on both sides, and using the fact $|{}^{\mu}D_q^* \Upsilon|$ is multiplicative convex on $[\nu, \mu]$, we have

$$\begin{aligned} \left| \frac{\left(\int_{\nu}^{\mu} (\Upsilon(\mathcal{K}))^{\mu} d_q \mathcal{K} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) (\Upsilon(\frac{2m+\mu}{3}))^3 (\Upsilon(\frac{\nu+2\mu}{3}))^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| &\leq e^{(\mu-\nu) \int_0^{\frac{1}{3}} \left(|qt - \frac{1}{8}| |(\ln \Upsilon)'_q(t\nu + (1-t)\mu)| \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{1}{3}}^{\frac{2}{3}} \left(|qt - \frac{1}{2}| |(\ln \Upsilon)'_q(t\nu + (1-t)\mu)| \right) d_q t} \\ &\quad \times e^{(\mu-\nu) \int_{\frac{2}{3}}^1 \left(|qt - \frac{7}{8}| |(\ln \Upsilon)'_q(t\nu + (1-t)\mu)| \right) d_q t} \\ &\leq e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \int_0^{\frac{1}{3}} |qt - \frac{1}{8}| t d_q t + |(\ln \Upsilon)'_q(\mu)| \int_0^{\frac{1}{3}} |qt - \frac{1}{8}| (1-t) d_q t \right]} \\ &\quad \times e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \int_{\frac{1}{3}}^{\frac{2}{3}} |qt - \frac{1}{2}| t d_q t + |(\ln \Upsilon)'_q(\mu)| \int_{\frac{1}{3}}^{\frac{2}{3}} |qt - \frac{1}{2}| (1-t) d_q t \right]} \\ &\quad \times e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \int_{\frac{2}{3}}^1 |qt - \frac{7}{8}| t d_q t + |(\ln \Upsilon)'_q(\mu)| \int_{\frac{2}{3}}^1 |qt - \frac{7}{8}| (1-t) d_q t \right]} \\ &= e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \Delta_1(q) + |(\ln \Upsilon)'_q(\mu)| \Delta_2(q) \right]} \\ &\quad \times e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \Delta_3(q) + |(\ln \Upsilon)'_q(\mu)| \Delta_4(q) \right]} \\ &\quad \times e^{(\mu-\nu) \left[|(\ln \Upsilon)'_q(\nu)| \Delta_5(q) + |(\ln \Upsilon)'_q(\mu)| \Delta_6(q) \right]} \\ &= \left[e^{|(\ln \Upsilon)'_q(\nu)| (\Delta_1(q) + \Delta_3(q) + \Delta_5(q))} \right] \end{aligned}$$

$$\begin{aligned}
& \times e^{(\ln \Upsilon)'_q(\mu)(\Delta_2(q)+\Delta_4(q)+\Delta_6(q))} \Big]^{(\mu-\nu)} \\
& = \left[|{}^\mu D_q^* \Upsilon(\nu)|^{(\Delta_1(q)+\Delta_3(q)+\Delta_5(q))} \right. \\
& \times \left. |{}^\mu D_q^* \Upsilon(\mu)|^{(\Delta_2(q)+\Delta_4(q)+\Delta_6(q))} \right]^{(\mu-\nu)}.
\end{aligned}$$

Hence, the proof is accomplished. $\square$

Remark 5.2. If we take $q \rightarrow 1^-$ in Theorem 4.1, we deduce the following inequality:

$$\left| \frac{\left(\int_\nu^\mu (\Upsilon(\varkappa))^{d_q \varkappa} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{2\nu+\mu}{3}\right) \right)^3 \left(\Upsilon\left(\frac{\nu+2\mu}{3}\right) \right)^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| \leq (\Upsilon^*(\nu) \Upsilon^*(\mu))^{\frac{25}{576}(\mu-\nu)},$$

which is presented in [15, Theorem 5.1].

Theorem 5.3. Let us assume that the condition specified in Lemma 3.2 is true. If $|{}^\mu D_q^* \Upsilon|^r$ is multiplicative convex for $r > 1$ and integrable on $[\nu, \mu]$ with $\frac{1}{s} + \frac{1}{r} = 1$, then the following inequality hold

$$\begin{aligned}
\left| \frac{\left(\int_\nu^\mu (\Upsilon(\varkappa))^{d_q \varkappa} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{2\nu+\mu}{3}\right) \right)^3 \left(\Upsilon\left(\frac{\nu+2\mu}{3}\right) \right)^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| & \leq \left[\left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{r}{9[2]_q}} \right) \times \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{r(3q+2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{5^s}{3.24^s} \right)^{\frac{1}{s}}} \right. \\
& \times \left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{3r}{9[2]_q}} \right) \times \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{3qr}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{2 \cdot 3^s - 1}{3.6^s} \right)^{\frac{1}{s}}} \\
& \times \left. \left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{5r}{9[2]_q}} \right) \times \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{r(3q-2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{3 \cdot 7^s - 2}{3.8^s} \right)^{\frac{1}{s}}} \right]^{(\mu-\nu)}. \quad (5.2)
\end{aligned}$$

Proof. We consider the Lemma 3.2 again. Through employing Hölder's inequality and making use of the claim that $|{}^\mu D_q^* \Upsilon|^r$ is multiplicatively convex on $[\nu, \mu]$, we discover that

$$\begin{aligned}
\left| \frac{\left(\int_\nu^\mu (\Upsilon(\varkappa))^{d_q \varkappa} \right)^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) \left(\Upsilon\left(\frac{2\nu+\mu}{3}\right) \right)^3 \left(\Upsilon\left(\frac{\nu+2\mu}{3}\right) \right)^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| & \leq e^{\left[(\mu-\nu) \left(\int_0^{\frac{1}{3}} |qt - \frac{1}{8}|^s d_q t \right)^{\frac{1}{s}} \right.} \\
& \times \left(|(\ln \Upsilon)'_q(\nu)|^r \int_0^{\frac{1}{3}} t d_q t + |(\ln \Upsilon)'_q(\mu)|^r \int_0^{\frac{1}{3}} (1-t) d_q t \right)^{\frac{1}{r}} \Big] \\
& \times e^{\left[(\mu-\nu) \left(\int_{\frac{1}{3}}^{\frac{2}{3}} |qt - \frac{1}{2}|^s d_q t \right)^{\frac{1}{s}} \right.} \\
& \times \left(|(\ln \Upsilon)'_q(\nu)|^r \int_{\frac{1}{3}}^{\frac{2}{3}} t d_q t + |(\ln \Upsilon)'_q(\mu)|^r \int_{\frac{1}{3}}^{\frac{2}{3}} (1-t) d_q t \right)^{\frac{1}{r}} \Big] \\
& \times e^{\left[(\mu-\nu) \left(\int_{\frac{2}{3}}^1 |qt - \frac{7}{8}|^s d_q t \right)^{\frac{1}{s}} \right.} \\
& \times \left. \left(|(\ln \Upsilon)'_q(\nu)|^r \int_{\frac{2}{3}}^1 t d_q t + |(\ln \Upsilon)'_q(\mu)|^r \int_{\frac{2}{3}}^1 (1-t) d_q t \right)^{\frac{1}{r}} \right]}. \quad (5.3)
\end{aligned}$$



By using Eq. (1.2), it is obvious that

$$\begin{aligned}
 \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right|^s d_q t &= \frac{(1-q)}{3} \sum_{k=0}^{\infty} q^k \left| \frac{q^{k+1}}{3} - \frac{1}{8} \right|^s, \\
 &\leq \frac{(1-q)}{3} \sum_{k=0}^{\infty} q^k \left| \frac{1}{3} - \frac{1}{8} \right|^s, \\
 &= \frac{(1-q)}{3} \left(\frac{5}{24} \right)^s \frac{1}{1-q}, \\
 &= \frac{5^s}{3 \cdot 24^s}.
 \end{aligned} \tag{5.4}$$

applying a similar approach to $\int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right|^s d_q t$, $\int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right|^s d_q t$, and performing calculations just as

$$\begin{aligned}
 \int_0^{\frac{1}{3}} t d_q t &= \frac{1}{3} (1-q) \sum_{k=0}^{\infty} \frac{1}{3} q^{2k}, \\
 &= \frac{1}{9} (1-q) \frac{1}{1-q^2}, \\
 &= \frac{1}{9[2]_q},
 \end{aligned}$$

$$\begin{aligned}
 \int_0^{\frac{1}{3}} (1-t) d_q t &= \frac{3q+2}{9[2]_q}, \\
 \int_{\frac{1}{3}}^{\frac{2}{3}} t d_q t &= \int_0^{\frac{2}{3}} t d_q t - \int_0^{\frac{1}{3}} t d_q t = \frac{3}{9[2]_q}, \\
 \int_{\frac{1}{3}}^{\frac{2}{3}} (1-t) d_q t &= \int_0^{\frac{2}{3}} (1-t) d_q t - \int_0^{\frac{1}{3}} (1-t) d_q t = \frac{3q}{9[2]_q}, \\
 \int_{\frac{2}{3}}^1 t d_q t &= \int_0^1 t d_q t - \int_0^{\frac{2}{3}} t d_q t = \frac{5}{9[2]_q},
 \end{aligned}$$

and

$$\int_{\frac{2}{3}}^1 (1-t) d_q t = \int_0^1 (1-t) d_q t - \int_0^{\frac{2}{3}} (1-t) d_q t = \frac{3q-2}{9[2]_q},$$

we eventually have,

$$\begin{aligned}
 \left| \frac{(\int_{\nu}^{\mu} (\Upsilon(\mathcal{X}))^{\mu} d_q \mathcal{X})^{\frac{1}{\mu-\nu}}}{\left[\Upsilon(\nu) (\Upsilon(\frac{2\nu+\mu}{3}))^3 (\Upsilon(\frac{\nu+2\mu}{3}))^3 \Upsilon(\mu) \right]^{\frac{1}{8}}} \right| &\leq e^{(\mu-\nu)(\frac{5^s}{3 \cdot 24^s})^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \frac{1}{9[2]_q} + |(\ln \Upsilon)'_q(\mu)|^r \frac{3q+2}{9[2]_q} \right)^{\frac{1}{r}} \\
 &\times e^{(\mu-\nu)(\frac{2 \cdot 3^s - 1}{3 \cdot 6^s})^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \frac{3}{9[2]_q} + |(\ln \Upsilon)'_q(\mu)|^r \frac{3q}{9[2]_q} \right)^{\frac{1}{r}} \\
 &\times e^{(\mu-\nu)(\frac{3 \cdot 7^s - 2}{3 \cdot 8^s})^{\frac{1}{s}}} \left(|(\ln \Upsilon)'_q(\nu)|^r \frac{5}{9[2]_q} + |(\ln \Upsilon)'_q(\mu)|^r \frac{3q-2}{9[2]_q} \right)^{\frac{1}{r}}, \\
 &= \left[\left\{ \left(e^{(\ln \Upsilon)'_q(\nu) \frac{r}{9[2]_q}} \right) \times \left(e^{(\ln \Upsilon)'_q(\mu) \frac{r(3q+2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{5^s}{3 \cdot 24^s} \right)^{\frac{1}{s}}} \right. \\
 &\times \left. \left\{ \left(e^{(\ln \Upsilon)'_q(\nu) \frac{3r}{9[2]_q}} \right) \times \left(e^{(\ln \Upsilon)'_q(\mu) \frac{3qr}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{2 \cdot 3^s - 1}{3 \cdot 6^s} \right)^{\frac{1}{s}}} \right]
 \end{aligned}$$



$$\begin{aligned}
& \times \left\{ \left(e^{|\ln \Upsilon'_q(\nu)| \frac{5r}{9[2]_q}} \right) \right. \\
& \times \left. \left(e^{|\ln \Upsilon'_q(\mu)| \frac{r(3q-2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{3.7^s - 2}{3.8^s} \right)^{\frac{1}{s}}} (\mu - \nu) \\
& = \left[\left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{r}{9[2]_q}} \right) \times \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{r(3q+2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{5^s}{3.24^s} \right)^{\frac{1}{s}}} \right. \\
& \times \left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{3r}{9[2]_q}} \right) \times \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{3qr}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{2.3^s - 1}{3.6^s} \right)^{\frac{1}{s}}} \\
& \times \left\{ \left(|{}^\mu D_q^* \Upsilon(\nu)|^{\frac{5r}{9[2]_q}} \right) \right. \\
& \times \left. \left(|{}^\mu D_q^* \Upsilon(\mu)|^{\frac{r(3q-2)}{9[2]_q}} \right) \right\}^{\frac{1}{r} \left(\frac{3.7^s - 2}{3.8^s} \right)^{\frac{1}{s}}} (\mu - \nu)
\end{aligned}$$

which concludes the proof. $\square$

6. NUMERICAL ANALYSIS

In this section, we provides some numerical examples and their graphical representations in order to better understand the validity, applications, and practical relevance of newly established inequalities. Additionally, we present a comparison of quadrature approximations between classical calculus and multiplicative calculus. The approximations for classical quadrature rules are given in Table 1, while the multiplicative calculus approximations (when $q \rightarrow 1^-$) are provided in Table 2. These tables illustrate the behavior of different quadrature formulas for the function $\Upsilon(x) = e^{x^2}$ on the interval $[1, 3]$.

Example 6.1. Consider a multiplicative differentiable convex functions $\Upsilon(x) = e^{x^2}$ in Theorem 4.1 and Theorem 5.1, for all $0 < q < 1$. The numerical verification shown in (see Tables 3 and 4) together with the related graphs (see Figures 1–7) demonstrate that the left-hand side consistently remains less than the right-hand side. These tables and graphs confirm the validity of our main results.

7. CONCLUSION

In this paper, we have established some new inequalities of Simpson's and Newton's types for quantum multiplicative differentiable convex functions using the framework of quantum multiplicative calculus. Two fundamental integral identities were derived, which served as the foundation for obtaining sharper and more generalized bounds. The proposed inequalities provide refined error estimates for quadrature rules, thereby enhancing the precision of numerical integration methods in the multiplicative calculus setting. We have also included a comparative analysis of quadrature approximations between classical and multiplicative calculus. The classical quadrature approximations are presented

TABLE 1. Quadrature approximations for classical calculus, taking $\Upsilon(x) = e^{x^2}$, and $[1, 3]$.

N	Quadrature Formulae	Approximation
3	Newton's Rule	1105.865
2	Simpson's 1/3 Rule	1387.365
1	Trapezoidal Rule	4052.899
0	Midpoint Formula	109.196

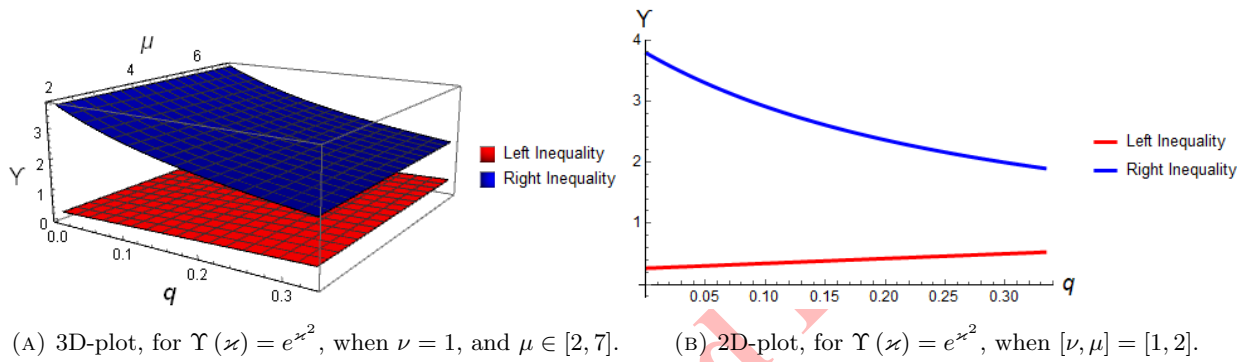
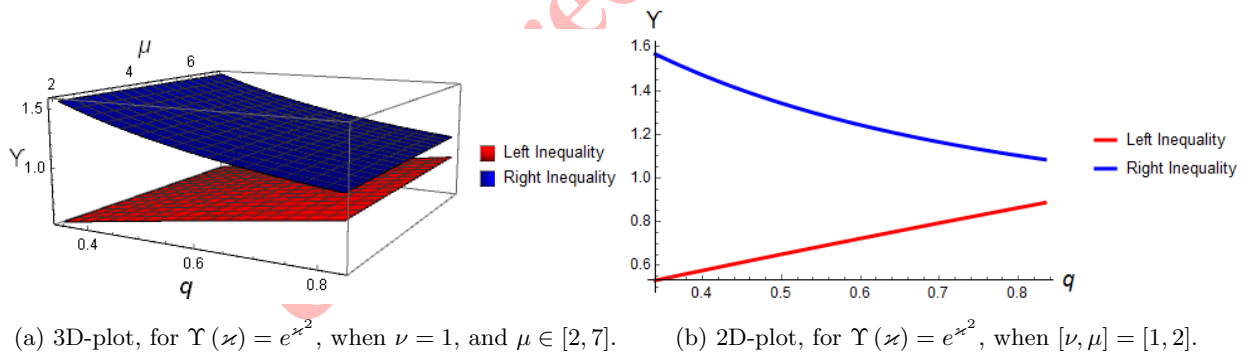
TABLE 2. Quadrature approximations for multiplicative calculus when $q \rightarrow 1^-$, taking $\Upsilon(x) = e^{x^2}$, and $[1, 3]$.

N	Quadrature Formulae	Approximation
3	Newton's Rule	$e^{13/3} \approx 76.10$
2	Simpson's 1/3 Rule	$e^{13/3} \approx 76.10$
1	Trapezoidal Rule	$e^5 \approx 148.41$
0	Midpoint Formula	$e^8 \approx 2980.96$



TABLE 3. Numerical values of the inequality (4.1) for $\Upsilon(\varkappa) = e^{\varkappa^2}$, when $[\nu, \mu] = [1, 2]$.

For $0 < q < \frac{1}{3}$.			For $\frac{1}{3} \leq q < \frac{5}{6}$.			For $\frac{5}{6} \leq q < 1$.		
q	Left Term	Right Term	q	Left Term	Right Term	q	Left Term	Right Term
0.001	0.26439	3.78232	0.35	0.53946	1.55078	0.85	0.89873	1.49051
0.1	0.34342	2.91187	0.45	0.61458	1.40274	0.9	0.93277	1.49313
0.2	0.42307	2.36367	0.55	0.68786	1.2896	0.95	0.96652	1.49572
0.3	0.50116	1.99537	0.65	0.75952	1.20169	0.999	0.99933	1.49823
			0.75	0.82976	1.13232			
			0.83	0.88504	1.08697			

FIGURE 1. Numerical Analysis of Theorem 4.1: (a) 3D-plot, (b) 2D-plot for $0 < q < \frac{1}{3}$, computed and plotted with Mathematica.FIGURE 2. Numerical Analysis of Theorem 4.1: (a) 3D-plot, (b) 2D-plot for $\frac{1}{3} \leq q < \frac{5}{6}$, computed and plotted with Mathematica.

in Table 1, while the multiplicative counterparts (for $q \rightarrow 1^-$) are shown in Table 2. These tables demonstrate the performance of various quadrature formulas and offer valuable insights into the practical applicability of the proposed theoretical framework. The numerical results and graphical illustrations confirm the validity of the derived inequalities and highlight their usefulness across different parameter choices. Future work may focus on extending these results to other types of inequalities and error bounds within the quantum multiplicative calculus framework, as well as exploring applications in diverse fields where discrete scaling and proportional growth are essential.



(a) 3D-plot, for $\Upsilon(x) = e^{x^2}$, when $\nu = 1$, and $\mu \in [2, 7]$. (b) 2D-plot, for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

FIGURE 3. Numerical Analysis of Theorem 4.1: (a) 3D-plot, (b) 2D-plot for $\frac{5}{6} \leq q < 1$, computed and plotted with Mathematica.

TABLE 4. Numerical values of the inequality (5.1) for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

For $0 < q < \frac{3}{8}$.

q	Left Term	Right Term
0.001	0.26439	3.78232
0.1	0.34342	2.91187
0.2	0.42307	2.36367
0.3	0.50116	1.99537

For $\frac{3}{8} \leq q < \frac{3}{4}$.

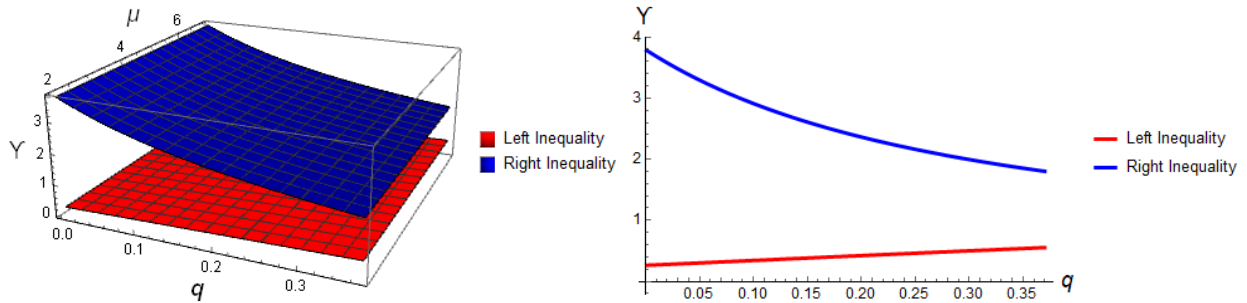
q	Left Term	Right Term
0.4	0.57727	1.74727
0.5	0.65144	1.59786
0.6	0.72388	1.47810
0.7	0.79481	1.37934
0.74	0.82280	1.34442

For $\frac{3}{4} \leq q < \frac{7}{8}$.

q	Left Term	Right Term
0.76	0.83672	1.33296
0.8	0.86440	1.31877
0.85	0.89873	1.30191

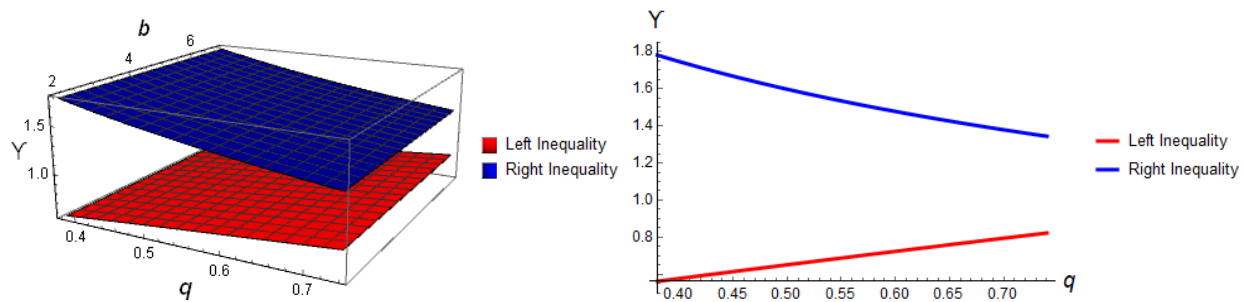
For $\frac{7}{8} \leq q < 1$.

q	Left Term	Right Term
0.9	0.93277	1.29409
0.95	0.96652	1.29574
0.999	0.99933	1.29731



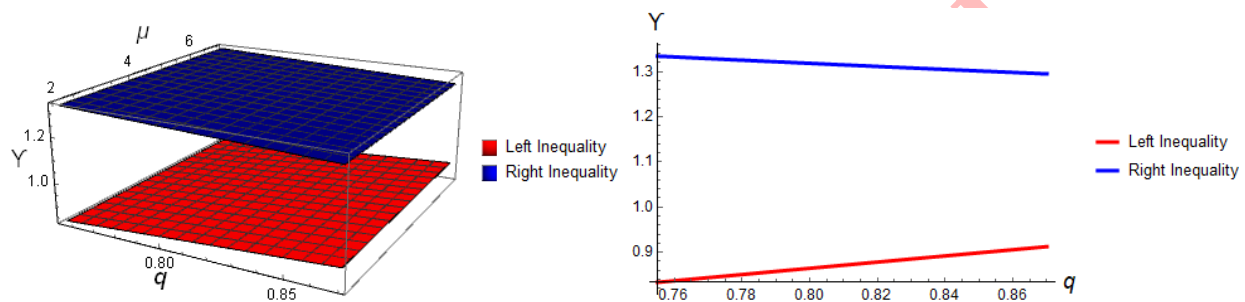
(a) 3D-plot, for $\Upsilon(x) = e^{x^2}$, when $\nu = 1$, and $\mu \in [2, 7]$. (b) 2D-plot, for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

FIGURE 4. Numerical Analysis of Theorem 5.1: (a) 3D-plot, (b) 2D-plot for $0 < q < \frac{3}{8}$, computed and plotted with Mathematica.



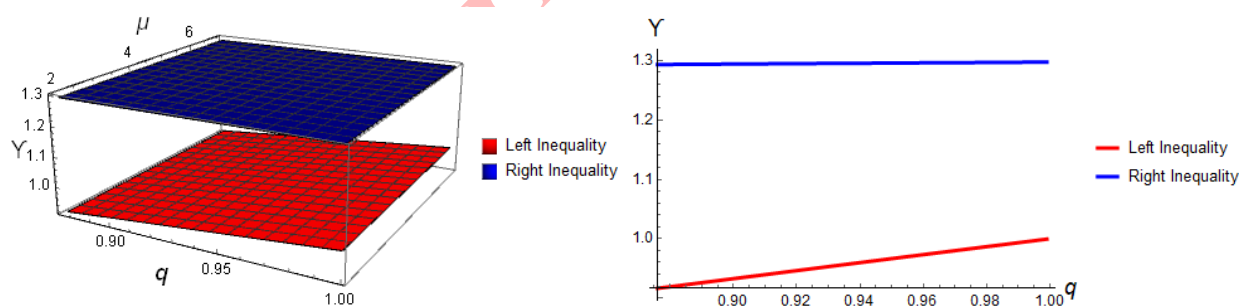
(a) 3D-plot, for $\Upsilon(x) = e^{x^2}$, when $\nu = 1$, and $\mu \in [2, 7]$. (b) 2D-plot, for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

FIGURE 5. Numerical Analysis of Theorem 5.1: (a) 3D-plot, (b) 2D-plot for $\frac{3}{8} \leq q < \frac{3}{4}$, computed and plotted with Mathematica.



(a) 3D-plot, for $\Upsilon(x) = e^{x^2}$, when $\nu = 1$, and $\mu \in [2, 7]$. (b) 2D-plot, for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

FIGURE 6. Numerical Analysis of Theorem 5.1: (a) 3D-plot, (b) 2D-plot for $\frac{3}{4} \leq q < \frac{7}{8}$, computed and plotted with Mathematica.



(a) 3D-plot, for $\Upsilon(x) = e^{x^2}$, when $\nu = 1$, and $\mu \in [2, 7]$. (b) 2D-plot, for $\Upsilon(x) = e^{x^2}$, when $[\nu, \mu] = [1, 2]$.

FIGURE 7. Numerical Analysis of Theorem 5.1: (a) 3D-plot, (b) 2D-plot for $\frac{7}{8} \leq q < 1$, computed and plotted with Mathematica.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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Uncorrected Proof

