



Burger equation analysis utilizing the fractal fractional derivative

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Abstract

This paper conducts an analytical investigation of Burger's equation by incorporating fractal dimensions and non-integer orders of differentiation into the conventional framework of fractional calculus. As a case study, the Caputo-Fabrizio operator, the fractal fractional derivatives with Caputo derivative, and the ABC are studied. The notion of a fractal medium is seen in numerous practical applications. This paper develops multiple strategies for the Schnakenberg model by utilizing freshly introduced numerical methods. This study develops a solution to the Burger problem using several fractal-fractional derivative kernels. The stability study of the fractal fractional derivative is also examined.

Keywords. Sustainable Development Goals, Burgers equation, Fractal fractional derivative, Anomalous diffusion, Nonlinear transport phenomena, Sustainability-oriented mathematical modeling.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. INTRODUCTION

The Burgers' equation is a crucial non-linear partial differential equation (PDE) extensively utilized in fluid mechanics to represent phenomena such as shock wave generation and turbulence. Nonetheless, the conventional integer-order derivatives employed in the equation may inadequately represent the intricate behavior of fluids in specific circumstances. Burger's equation is a fundamental nonlinear partial differential equation utilized in various domains, including fluid dynamics, acoustics, and traffic flow. Fractal fractional derivatives provide an innovative mathematical instrument to describe complicated systems that display self-similarity and irregularity across several scales. This equation, mostly employed in fluid dynamics, is a mathematical description of the diffusion wave equation. Numerous physical phenomena exist, such as one-dimensional sound waves in a viscous shock wave within a viscous medium, magnetohydrodynamic waves in a medium with limited electrical conductivity, turbulence, and waves in a fluid contained within viscous elastic tubes. This is the context in which fractal fractional derivatives are applicable.

Received: 05 December 2025; Accepted: 22 October 2026.

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Fractal fractional calculus provides a broader framework than conventional calculus. It presents the notion of fractal dimension, facilitating a more refined characterization of processes exhibiting memory effects and self-similar structures. Fractal derivatives can encapsulate the historical context of a process, which is essential for comprehending fluid dynamics in porous media or intricate geometries. Conventional diffusion models may fail to adequately depict real-world occurrences. Interjection Fractal derivatives can represent processes with diffusion rates that diverge from conventional power-law behaviour. The fractal dimension signifies the geometric intricacy of the material. Expanding the utilisation of fractal fractional derivatives beyond Burger's equation to further nonlinear systems in fluid dynamics, plasma physics, biology, and finance. Examining the impact of fractal dimensions on the dynamics and stability of complex systems might yield significant insights into their behaviour. This study's consequences extend across various fields, such as turbulence modelling, materials science, and complex system dynamics, providing improved forecasting skills and sophisticated control tactics for systems exhibiting fractal-like properties. This research advances theoretical frameworks and practical applications in nonlinear dynamics by integrating fractal geometry and fractional calculus. The analysis of the Burgers equation using fractal fractional derivatives supports the Sustainable Development Goals by enabling accurate modeling of nonlinear, memory-dependent transport processes relevant to sustainability challenges.

Many mathematical modelling disciplines have been greatly enriched by fractional differential equation (FDE) theory in recent decades. Unlike classical or integer order differential concerns, scholars have studied FDEs for their relevance and realism. Complex fractional models use Caputo and Riemann-Liouville fractional drivers more than others. Researchers used Hadamard and Caputo-Hadamard fractional operators to study various models. Caputo and Fabrizio introduced the Caputo-Fabrizio operator, a nonlinear fractional operator without a kernel, [see, [3]], and Nieto [see, [13]] investigated many of the operator's features that year.

The system is challenging to solve analytically due to the incorporation of fractional derivatives. Consequently, numerical solutions are necessary. Numerical simulations comprise the preponderance of the research conducted on fractional PDEs. Some of the most recent papers on fractional PDEs can be found in [11, 12]. Following Momani [18], numerous other academics conducted research on the fractional Burgers equation. The Adomian decomposition approach was employed by Chen and An [5] to identify numerical solutions to the coupled fractional Burgers problem. Legendre's spectral-collocation approach was employed by Bhrawy et al. [2] to numerically approximate the fractional Burgers equation. In order to approximate the solution of the space and time fractional Burgess equations, Inc [see, [15]] implemented a variational iteration method. The generalised differential transform method was employed by Liu and Hou [see, [12]] to solve the fractional coupled Burgers equation. The literature and research papers on the analytical study of Burger's equation using fractal fractional derivatives disclose a burgeoning area of research at the intersection of fractal geometry, fractional calculus, and nonlinear dynamics [7–9].

Shafiq [20] discussed the application of fractional calculus to Burger's equation, exploring different types of fractional derivatives and their implications. Malyk [14] focused on analytical methods for solving Burger's equation using fractional derivatives in the time domain, providing detailed derivations and solutions. Sugimoto [21] discussed the application of fractal calculus, including fractal fractional derivatives, to Burger's equation and similar nonlinear equations, highlighting theoretical advancements and applications [see, for example, [4, 6, 10, 16, 17, 19, 23]]. Al though, Ahmad [1] focused on numerical methods this paper provides insights into the application of fractional derivatives to Burger's equation, which complements analytical approaches. This paper explores the use of local fractional derivatives, a type of fractal fractional derivative, to solve Burger's equation analytically, demonstrating its applicability in modeling complex systems. Among the various non-singular fractional operators, the Caputo–Fabrizio derivative has gained increasing attention due to its exponential kernel and finite-memory behavior. In the context of the Burgers equation, this operator has substantial advantages over classical and power-law-based fractional derivatives, particularly in terms of numerical stability, physical realism for viscous flows, and consistency with the classical model. Motivated by these qualities, the present study combines the Caputo–Fabrizio operator alongside the Atangana–Baleanu derivative to analyze both short- and long-memory effects in fractal media [see, [21, 22]].

2. PRELIMINARIES

In this section, we briefly review the fundamental concepts and operators of fractal–fractional calculus that are required throughout this study. The discussion is limited to those definitions and properties that are directly employed



in the formulation of the Burgers' equation, the construction of numerical schemes, and the subsequent stability analysis. This approach ensures mathematical completeness while avoiding unnecessary repetition.

Fractal-fractional derivatives combine the effects of fractal geometry and fractional-order differentiation, thereby providing an effective framework for modeling physical processes characterized by geometric irregularity, memory effects, and nonlocal behavior. Let $\alpha \in (0, 1)$ denote the fractional order and $\beta \in (0, 1)$ represent the fractal dimension. The presence of the fractal dimension β enables the modeling of transport phenomena occurring in heterogeneous and geometrically irregular media. The aforementioned operators constitute the mathematical foundation of the fractal-fractional Burgers' equation investigated in this work. Each operator incorporates a distinct memory kernel, namely power-law, exponential, or Mittag-Leffler type, thereby allowing a comparative analysis of their influence on solution behavior, numerical performance, and stability characteristics. These definitions will be employed throughout Sections 3–5 in the development of numerical schemes and the corresponding stability analysis.

Definition 2.1. Let $f \in C^1([0, T])$. The Caputo-type fractal-fractional derivative of order α and fractal dimension β is defined by

$${}^{CFD}_t D_t^{\alpha, \beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau^\beta, \quad 0 < \alpha, \beta < 1, \quad (2.1)$$

where $d\tau^\beta$ denotes the fractal measure and $\Gamma(\cdot)$ represents the Gamma function. This operator preserves physically meaningful initial conditions and is particularly suitable for describing systems governed by power-law memory effects.

Definition 2.2. The Caputo-Fabrizio fractal-fractional derivative of order α and fractal dimension β is given by

$${}^{CFD}_t D_t^{\alpha, \beta} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \frac{d}{d\tau} f(\tau) d\tau^\beta, \quad (2.2)$$

where $M(\alpha)$ is a normalization function satisfying

$$M(0) = M(1) = 1.$$

The non-singular exponential kernel eliminates the singularity present in classical fractional operators and is therefore well suited for processes exhibiting finite memory effects and enhanced numerical stability.

Definition 2.3. The Atangana-Baleanu fractal-fractional derivative in the Caputo sense with Mittag-Leffler kernel is defined as

$${}^{ABD}_t D_t^{\alpha, \beta} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right) \frac{d}{d\tau} f(\tau) d\tau^\beta, \quad (2.3)$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function and $B(\alpha)$ is a normalization constant. Owing to its non-singular and non-local kernel, this operator provides an effective framework for modeling anomalous diffusion and complex transport processes characterized by long-range memory effects.

It is worth noting that when $\beta = 1$, the above operators reduce to their corresponding classical fractional derivatives. Furthermore, as $\alpha \rightarrow 1$, they converge to the standard first-order derivative. The inclusion of the fractal dimension β allows the incorporation of geometric irregularities and heterogeneous structures into the mathematical model, thereby extending the applicability of fractional calculus to a broader class of real-world phenomena.

3. NUMERICAL RESOLUTION OF THE BURGERS' EQUATION UTILIZING CAPUTO FRACTAL-FRACTIONAL DERIVATIVE

3.1. Problem formulation. In this subsection, we formulate the fractal-fractional Burgers' equation under consideration. Specifically, we investigate the one-dimensional Burgers' equation incorporating a Caputo-type fractal-fractional time derivative, which accounts for both memory effects and the geometric complexity of the underlying medium.

$${}^CD_t^{\alpha, \beta} u(x, t) + u(x, t) \frac{\partial u(x, t)}{\partial x} = \nu \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in [a, b], \quad t > 0. \quad (3.1)$$

subject to the initial condition

$$u(x, 0) = u_0(x), \quad (3.2)$$



where $\nu > 0$ denotes the kinematic viscosity coefficient. The parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ represent the fractional order and fractal dimension, respectively.

The incorporation of the Caputo fractal–fractional derivative introduces memory effects associated with the geometric complexity of the underlying medium. Specifically, the fractional order α governs the temporal nonlocality and hereditary characteristics of the system, whereas the fractal dimension β characterizes the irregular geometric structure of the medium. Consequently, the proposed model extends the classical Burgers' equation and provides an effective framework for describing transport phenomena in porous, heterogeneous, and fractal-like media. To construct the numerical scheme, the spatial and temporal domains are discretized as

$$x_i = a + i\Delta x, \quad i = 0, 1, \dots, N, \quad (3.3)$$

and

$$t_n = n\Delta t, \quad n = 0, 1, \dots, M, \quad (3.4)$$

where Δx and Δt denote the spatial and temporal step sizes, respectively.

For convenience, we introduce the notation

$$u(x_i, t_n) = u_i^n. \quad (3.5)$$

The initial condition is given by

$$u(x, 0) = f(x), \quad (3.6)$$

or equivalently,

$$u_i^0 = f(x_i), \quad i = 0, 1, \dots, N. \quad (3.7)$$

Throughout this study, $\nu > 0$ denotes the constant kinematic viscosity coefficient.

$${}^{FFP}D_t^{\alpha, \beta} u(x, t) = \nu \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) \frac{\partial u(x, t)}{\partial x}. \quad (3.8)$$

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t u(x, \tau) (t-\tau)^{-\alpha} d\tau = \nu u_{xx}(x, t) - u(x, t) u_x(x, t). \quad (3.9)$$

Using

$$\frac{d}{d\tau^\beta} = \frac{1}{\beta t^{\beta-1}} \frac{d}{dt},$$

we obtain

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x, \tau) (t-\tau)^{-\alpha} d\tau \frac{1}{\beta t^{\beta-1}} = \nu u_{xx}(x, t) - u(x, t) u_x(x, t). \quad (3.10)$$

Integrating both sides with respect to time gives

$$u(x, t) - u(x, 0) = \frac{1}{\Gamma(\alpha)} \int_0^t \beta \tau^{\beta-1} \left(\nu u_{xx}(x, \tau) - u(x, \tau) u_x(x, \tau) \right) (t-\tau)^{\alpha-1} d\tau. \quad (3.11)$$

For convenience's, we allowed

$$F(x, \tau) = \beta \tau^{\beta-1} (\nu u_{xx} - u u_x). \quad (3.12)$$

Now we have,

$$u(x, t) - u(x, 0) = \frac{1}{\Gamma(\alpha)} \int_0^t F(x, \tau) (t-\tau)^{\alpha-1} d\tau. \quad (3.13)$$

The following method is used to discretise the equation above

$$u_i^{n+1} - u_i^0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(x_i, \tau) (t_{n+1} - \tau)^{\alpha-1} d\tau. \quad (3.14)$$



$$u_i^{n+1} = u_i^0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(x_i, \tau) (t_{n+1} - \tau)^{\alpha-1} d\tau. \quad (3.15)$$

Using Lagrange's polynomial in two steps, we get

$$p_j(\tau) = \frac{\tau - t_{j-1}}{t_j - t_{j-1}}, F(x_i, t_j) - \frac{\tau - t_j}{t_j - t_{j-1}}, F(x_i, t_{j-1}) \quad (3.16)$$

$$u(x_i, t_{n+1}) = u(x_i, 0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} p_j(\tau) (t_{n+1} - \tau)^{\alpha-1} d\tau, \quad (3.17)$$

$$\begin{aligned} u(x_i, t_{n+1}) &= u(x_i, 0) + \sum_{j=0}^n \frac{h^\alpha F(x_i, t_j)}{\Gamma(\alpha+2)} \left[(n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+1+2\alpha) \right] \\ &\quad - \sum_{j=0}^n \frac{h^\alpha F(x_i, t_{j-1})}{\Gamma(\alpha+2)} \left[(n+1-j)^{\alpha+1} (n-j+2+\alpha) - (n-j)^\alpha (n-j+1+\alpha) \right], \end{aligned} \quad (3.18)$$

where

$$F(x_i, t_j) = \beta t_j^{\beta-1} \left(d \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + a - u_i^j + (u_i^j)^2 v_i^j \right). \quad (3.19)$$

Now with the help of above information we obtain

$$\begin{aligned} u(x_i, t_{n+1}) &= u(x_i, 0) + \sum_{j=0}^n \frac{h^\alpha \beta t_j^{\beta-1}}{\Gamma(\alpha+2)} \left(d \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta x} + a - u_i^j + (u_i^j)^2 v_i^j \right) \\ &\quad \times \left[(n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+1+2\alpha) \right] \\ &\quad - \sum_{j=0}^n \frac{d h^\alpha \beta t_j^{\beta-1}}{\Gamma(\alpha+2)} \left(\frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{(\Delta x)^2} + \frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{2\Delta x} + a - u_i^{j-1} + (u_i^{j-1})^2 v_i^{j-1} \right) \\ &\quad \times \left[(n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right]. \end{aligned} \quad (3.20)$$

We simplify the equation above by breaking it down into smaller components, and let

$$\begin{aligned} A_{n,j}^{\alpha,\beta} &= \frac{h^\alpha \beta t_j^{\beta-1} d K_{n,j}^\alpha}{\Gamma(\alpha+2)(\Delta x)^2}, \\ B_{n,j}^{\alpha,\beta} &= \frac{h^\alpha \beta t_j^{\beta-1} K_{n,j}^\alpha}{2\Gamma(\alpha+2)\Delta x}, \\ C_{n,j}^{\alpha,\beta} &= \frac{h^\alpha \beta t_j^{\beta-1} d K_{n,j}^\alpha}{\Gamma(\alpha+2)}, \\ D_{n,j}^{\alpha,\beta} &= \frac{h^\alpha \beta t_j^{\beta-1} K_{n,j}^\alpha}{\Gamma(\alpha+2)}. \end{aligned}$$

For the second component, the same expression applies. Therefore, we separate to get

$$\begin{aligned} u_i^{n+1} &= u_i^0 + \sum_{j=0}^n \left[A_{n,j}^{\alpha,\beta} (u_{i+1}^j - 2u_i^j + u_{i-1}^j) + B_{n,j}^{\alpha,\beta} (u_{i+1}^j - u_{i-1}^j) + C_{n,j}^{\alpha,\beta} - D_{n,j}^{\alpha,\beta} u_i^j (1 - u_i^j v_i^j) \right] \\ &\quad - \sum_{j=0}^n \left[A_{n,j-1}^{\alpha,\beta} (u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + B_{n,j-1}^{\alpha,\beta} (u_{i+1}^{j-1} - u_{i-1}^{j-1}) + C_{n,j-1}^{\alpha,\beta} - D_{n,j-1}^{\alpha,\beta} u_i^{j-1} (1 - u_i^{j-1} v_i^{j-1}) \right]. \end{aligned} \quad (3.21)$$



the numerical results for the solution $(u(x,t))$ corresponding to different values of fractional order α and fractal dimension β obtained from the Caputo fractal-fractional derivative for the suggested system equation, which (3.1) are shown in the following.

Theorem 3.1. *The proposed finite difference scheme for the Caputo fractal-fractional Burgers equation is consistent with the continuous model (3.1).*

Proof. Let $u(x, t)$ be the exact solution of (3.1) satisfying

$$u \in C^{2,1}([a, b] \times [0, T]).$$

The Caputo fractal-fractional derivative of order α and fractal dimension β is defined as

$${}^{CF}D_t^{\alpha,\beta}u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x, \tau)}{\partial \tau} d\tau^\beta. \quad (3.22)$$

Using the Grünwald-type discretization together with Newton polynomial interpolation, the temporal approximation satisfies

$${}^{CF}D_t^{\alpha,\beta}u(x_i, t_n) = \delta_t^\alpha u_i^n + O(\Delta t^\alpha), \quad (3.23)$$

where δ_t^α denotes the discrete Caputo fractal-fractional operator.

The first-order spatial derivative is approximated by the central difference formula

$$\frac{\partial u}{\partial x}(x_i, t_n) = \frac{u(x_{i+1}, t_n) - u(x_{i-1}, t_n)}{2\Delta x} + O(\Delta x^2). \quad (3.24)$$

Similarly, the second-order derivative is approximated as

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) = \frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n))}{(\Delta x)^2} + O(\Delta x^2). \quad (3.25)$$

Substituting these approximations into the numerical scheme yields the local truncation error

$$\tau_i^n = O(\Delta t^\alpha + \Delta x^2). \quad (3.26)$$

Hence,

$$\lim_{\Delta t, \Delta x \rightarrow 0} \tau_i^n = 0. \quad (3.27)$$

Therefore, the proposed finite difference method is consistent with the continuous Caputo fractal-fractional Burgers equation. $\square$

Theorem 3.2. *If the initial condition is such that*

$$\|u^0\|_\infty \leq M,$$

then the numerical solution u_i^n remains bounded for all n .

Proof. We use mathematical induction.

At $n = 0$,

$$|u_i^0| = |u_0(x_i)| \leq M. \quad (3.28)$$

Assume that for some $n \geq 0$,

$$\|u^n\|_\infty \leq M. \quad (3.29)$$

Using the numerical scheme,

$$u_i^{n+1} = u_i^n - \Delta t^\alpha u_i^n \left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right) + \nu \Delta t^\alpha \left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \right). \quad (3.30)$$

Taking absolute values and using the triangle inequality,

$$|u_i^{n+1}| \leq |u_i^n| + C_1 \Delta t^\alpha |u_i^n| + C_2 \Delta t^\alpha, \quad (3.31)$$



where C_1 and C_2 depend on ν and Δx , but are finite.

Thus,

$$\|u^{n+1}\|_\infty \leq (1 + C_1 \Delta t^\alpha) \|u^n\|_\infty + C_2 \Delta t^\alpha. \quad (3.32)$$

Applying the inductive hypothesis and iterating results in

$$\|u^n\|_\infty \leq M^*, \quad (3.33)$$

for a constant M^* independent of n . Hence, the solution is bounded. $\square$

Theorem 3.3. *The numerical scheme is stable according to Lyapunov.*

Proof. Let u_i^n and $\tilde{u}_i^n$ be two numerical solutions for significantly altered beginning data. Subtracting the two schemes yields

$$e_i^{n+1} = e_i^n - \Delta t^\alpha \left[u_i^n \left(\frac{e_{i+1}^n - e_{i-1}^n}{2\Delta x} \right) + e_i^n \left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right) \right] + \nu \Delta t^\alpha \left(\frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{(\Delta x)^2} \right). \quad (3.34)$$

Using norms and Theorem 3.2's boundedness,

$$\|e^{n+1}\|_\infty \leq (1 + C \Delta t^\alpha) \|e^n\|_\infty. \quad (3.35)$$

By discrete Grönwall's inequality,

$$\|e^n\|_\infty \leq \|e^0\|_\infty e^{CT}. \quad (3.36)$$

Thus, slight perturbations in the original data cause modest perturbations in the solution, and the scheme remains stable. $\square$

Theorem 3.4. *As $\Delta t, \Delta x$ approach zero, the numerical solution approaches the precise solution.*

Proof. Let

$$\varepsilon_i^n = u(x_i, t_n) - u_i^n$$

represent the global error.

According to the consistency result (Theorem 3.1), the error satisfies

$$\varepsilon_i^{n+1} = \varepsilon_i^n + \Delta t^\alpha \mathcal{L}(\varepsilon_i^n) + \tau_i^n, \quad (3.37)$$

where τ_i^n is the local truncation error.

Using stability (Theorem 3.3) and adding up the time steps gives

$$\|\varepsilon^n\|_\infty \leq C \sum_{k=0}^{n-1} \|\tau^k\|_\infty. \quad (3.38)$$

Since

$$\tau_i^n = O(\Delta t^\alpha + \Delta x^2), \quad (3.39)$$

we have

$$\lim_{\Delta t, \Delta x \rightarrow 0} \|\varepsilon^n\|_\infty = 0. \quad (3.40)$$

This means that the scheme leads to the exact solution. $\square$



4. CAPUTO-FABRIZIO EXPONENTIAL LAW-BASED SOLUTION

In this section, we prove the existence, uniqueness, consistency, and convergence of a numerical solution for the fractal-fractional Burgers equation expressed in the sense of the Caputo–Fabrizio derivative. The distinctive characteristic of this technique is the employment of a non-singular exponential kernel, which provides finite memory and avoids the singular behavior associated with standard power-law kernels. The Caputo–Fabrizio operator was chosen for this section because of its non-singular exponential kernel, which avoids the numerical difficulties associated with classical fractional derivatives.” This property makes it ideal for modeling Burgers-type equations that describe dissipative and short-memory processes. In this part, we consider the proposed model using the Caputo–Fabrizio operator.

$${}_0^{FFE}D_t^{\alpha,\beta}u(x,t) = \nu u_{xx}(x,t) - u(x,t)u_x(x,t) \quad (4.1)$$

Using an exponential kernel to introduce the derivative, we have

$$\frac{M(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x,\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau = \beta t^{\beta-1} (\nu u_{xx}(x,t) - u(x,t)u_x(x,t)). \quad (4.2)$$

For ease of use, we allow

$$F(x,t,u(x,t)) = \beta t^{\beta-1} (\nu u_{xx}(x,t) - u(x,t)u_x(x,t)). \quad (4.3)$$

For ease of use, we allow

$$F(x,t,u(x,t)) = \beta t^{\beta-1} (\nu u_{xx}(x,t) - u(x,t)u_x(x,t)). \quad (4.4)$$

By employing the Atangana–Baleanu integral, we derive

$$u(x,t) - u(x,0) = \frac{1-\alpha}{M(\alpha)} F(x,t,u(x,t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(x,\tau,u(x,\tau)) d\tau. \quad (4.5)$$

The following is obtained by discretizing the foregoing at (x_i, t_{n+1}) and (x_i, t_n) to get

$$u_i^{n+1} = u_i^0 + \frac{1-\alpha}{M(\alpha)} F(x_i, t_n, u_i^n) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} F(x_i, \tau, u(x_i, \tau)) d\tau, \quad (4.6)$$

and

$$u_i^n = u_i^0 + \frac{1-\alpha}{M(\alpha)} F(x_i, t_{n-1}, u_i^{n-1}) + \frac{\alpha}{M(\alpha)} \int_0^{t_n} F(x_i, \tau, u(x_i, \tau)) d\tau. \quad (4.7)$$

Which on subtraction yields

$$u_i^{n+1} - u_i^n = \frac{1-\alpha}{M(\alpha)} [F(x_i, t_n, u_i^n) - F(x_i, t_{n-1}, u_i^{n-1})] + \frac{\alpha}{M(\alpha)} \int_{t_n}^{t_{n+1}} F(x_i, \tau, u(x_i, \tau)) d\tau. \quad (4.8)$$

Theorem 4.1 (Equivalent Integral Formulation). *Let*

$$u(x,t) \in C^2(\Omega \times [0, T]),$$

be a suitably smooth function.

The fractional–fractional Burgers equation with the Caputo–Fabrizio derivative

$${}^{CF}D_t^{\alpha,\beta}u(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha < 1, \quad (4.9)$$

is equivalent to the following integral equation, subject to suitable initial and boundary conditions:

$$u(x,t) = u(x,0) + \frac{1-\alpha}{M(\alpha)} F(x,t) + \frac{\alpha}{M(\alpha)} \int_0^t F(x,\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) d\tau. \quad (4.10)$$



where

$$F(x, t) = -u(x, t) u_x(x, t) + \nu u_{xx}(x, t). \quad (4.11)$$

and $M(\alpha)$ is the normalization constant of the Caputo–Fabrizio operator.

Proof. By definition, the Caputo–Fabrizio fractional derivative is given by

$${}^{CF}D_t^\alpha u(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t u'(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) d\tau. \quad (4.12)$$

Applying the corresponding Caputo–Fabrizio fractional integral operator to both sides of Equation (4.12), and using the fundamental identity

$${}^{CF}I_t^\alpha ({}^{CF}D_t^\alpha u(t)) = u(t) - u(0), \quad (4.13)$$

we immediately obtain

$$u(x, t) - u(x, 0) = {}^{CF}I_t^\alpha [-u u_x + \nu u_{xx}]. \quad (4.14)$$

Substituting the explicit expression of the Caputo–Fabrizio fractional integral operator yields Equation (4.10). This completes the proof. $\square$

The integral formulation demonstrates that the Caputo–Fabrizio operator provides a finite-memory phenomenon, where the influence of past states decays exponentially. This feature is physically realistic for viscous flows and dramatically enhances numerical stability.

Theorem 1 (Existence and Uniqueness of the Numerical Solution). Let $\Delta t > 0$ and $\Delta x > 0$ be fixed discretization parameters. At each time level t_{n+1} , the implicit finite difference scheme obtained from Equation (4.10) admits a unique numerical solution.

Proof. We discretize Equation (4.10) in time using the uniform mesh

$$t_n = n\Delta t, \quad n = 0, 1, \dots, N.$$

The exponential kernel is approximated using Newton polynomial interpolation, yielding the recursive scheme

$$u_i^{n+1} = u_i^n + A_\alpha F_i^{n+1} + B_\alpha \sum_{k=0}^n F_i^k e^{-\lambda(n+1-k)\Delta t}, \quad (4.15)$$

where A_α and B_α are positive constants depending on the Caputo–Fabrizio fractional operator.

The resulting algebraic system at each time level can be written as

$$(I + K)\mathbf{u}^{n+1} = \mathbf{b}^n, \quad (4.16)$$

where I denotes the identity matrix and K is the coefficient matrix arising from the discretization of the diffusion term.

Since the diffusion operator produces a strictly diagonally dominant matrix, the matrix $I + K$ is nonsingular. Therefore, by the Levy–Desplanques theorem, the linear system admits a unique solution at each time level. $\square$

Theorem 4.2 (Consistency and Convergence). The proposed Caputo–Fabrizio finite difference method is consistent and convergent. Moreover, its numerical solution converges to the exact solution of Equation (4.10) as

$$\Delta t, \Delta x \rightarrow 0.$$

Proof. Using the Taylor expansion of $u(x, t)$ together with the smoothness assumption, the local truncation error satisfies

$$\tau_i^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2),$$

which establishes the consistency of the proposed finite difference scheme.

From the stability analysis presented in Section 6, the numerical scheme is unconditionally stable with respect to the discrete L^2 -norm.



By the Lax–Richtmyer Equivalence Theorem, consistency together with stability implies convergence. Hence,

$$\|u_i^n - u(x_i, t_n)\| \rightarrow 0, \quad \text{as } \Delta t, \Delta x \rightarrow 0.$$

Furthermore, as $\alpha \rightarrow 1$, the Caputo–Fabrizio fractional operator reduces to the classical first-order derivative. Consequently, the proposed numerical scheme recovers the standard implicit finite difference method for the classical Burgers equation. The Caputo–Fabrizio derivative is especially well suited for viscous and dissipative systems where historical influence quickly fades because the exponential kernel incorporated in it simulates short-range memory effects. $\square$

5. ATANGANA–BALEANU NUMERICAL SOLUTION FOR MITTAG–LEFFLER LAW

The Atangana–Baleanu derivative with Mittag–Leffler kernel, which is well known for capturing long-range memory effects, governs the fractal–fractional Burgers equation that we examine in this section. The AB derivative is appropriate for anomalous diffusion events in fractal media because, in contrast to the Caputo–Fabrizio operator, it maintains a nonlocal kernel with algebraic decay. We shall examine the generalised Mittag–Leffler kernel problem

$${}^{FFM}_0 D_t^{\alpha, \beta} u(x, t) = \nu u_{xx}(x, t) - u(x, t) u_x(x, t), \quad (5.1)$$

$$\frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x, \tau) E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right] d\tau = \beta t^{\beta-1} (\nu u_{xx}(x, t) - u(x, t) u_x(x, t)), \quad (5.2)$$

$$F(x, t, u(x, t)) = \beta t^{\beta-1} (\nu u_{xx}(x, t) - u(x, t) u_x(x, t)). \quad (5.3)$$

Using the AB integral results in

$$u(x, t) - u(x, 0) = \frac{1-\alpha}{AB(\alpha)} F(x, t, u(x, t)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t F(x, \tau, u(x, \tau)) (t-\tau)^{\alpha-1} d\tau. \quad (5.4)$$

Hence, the discrete form is

$$u_i^{n+1} = u_i^0 + \frac{1-\alpha}{AB(\alpha)} F(x_i, t_{n+1}, u_i^n) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} F(x_i, \tau, u(x_i, \tau)) d\tau. \quad (5.5)$$

Now we have

$$F(x_i, t_j, u_i^n) = \beta t_j^{\beta-1} \left(\frac{\nu_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + a - u_i^j + (u_i^j)^2 v_i^j \right). \quad (5.6)$$

Putting together the above, we obtain the following numerical scheme.

$$\begin{aligned} u_i^{n+1} = & u_i^0 + \frac{1-\alpha}{AB(\alpha)} \left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + a - u_i^j + (u_i^j)^2 v_i^j \right) \\ & + \frac{\alpha}{AB(\alpha)} \sum_{j=0}^n \left[\frac{h^\alpha}{\Gamma(\alpha+2)} \beta t_j^{\beta-1} \left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + a - u_i^j + (u_i^j)^2 v_i^j \right) \right. \\ & \times \left((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha) \right) \\ & - \frac{h^\alpha}{\Gamma(\alpha+2)} \beta t_j^{\beta-1} \left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + a - u_i^j + (u_i^j)^2 v_i^j \right) \\ & \left. \times \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right]. \end{aligned} \quad (5.7)$$

For computation, we denote

$$M_\alpha = \frac{1-\alpha}{AB(\alpha)(\Delta x)^2}, \quad N_\alpha = \frac{1-\alpha}{AB(\alpha)}, \quad (5.8)$$



$$A_{n,j}^{\alpha,\beta} = \frac{\alpha}{AB(\alpha)} \frac{h^\alpha \beta t_{j-1}^{\beta-1} \gamma_{n,j}^\alpha}{\Gamma(\alpha+2)(\Delta x)^2}, \quad B_{n,j}^{\alpha,\beta} = \frac{\alpha}{AB(\alpha)} \frac{h^\alpha \beta t_{j-1}^{\beta-1} \gamma_{n,j}^\alpha}{\Gamma(\alpha+2)}. \quad (5.9)$$

Then we get

$$\begin{aligned} u_i^{n+1} = & u_i^0 + M_\alpha \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j \right) + N_\alpha \left(a - u_i^j + \left(u_i^j \right)^2 v_i^j \right) \\ & + \sum_{j=0}^n \left[A_{n,j}^{\alpha,\beta} \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j \right) + B_{n,j}^{\alpha,\beta} \left(a - u_i^j + \left(u_i^j \right)^2 v_i^j \right) \right] \\ & - \sum_{j=0}^n \left[A_{n,j-1}^{\alpha,\beta} \left(u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1} \right) + B_{n,j-1}^{\alpha,\beta} \left(a - u_i^j + \left(u_i^j \right)^2 v_i^j \right) \right]. \end{aligned} \quad (5.10)$$

Theorem 5.1. Let $u(x, t) \in C^2(\Omega \times [0, T])$. Then the fractal-fractional Burgers equation

$${}^0FFMD_t^{\alpha,\beta} u(x, t) = \nu u_{xx}(x, t) - u(x, t) u_x(x, t), \quad 0 < \alpha < 1, \quad (5.11)$$

with the Atangana–Baleanu derivative in the Mittag–Leffler sense,

$$\frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x, \tau) E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) d\tau = \beta t^{\beta-1} (\nu u_{xx} - uu_x), \quad (5.12)$$

is equivalent to the nonlinear Volterra integral equation

$$u(x, t) - u(x, 0) = \frac{1-\alpha}{AB(\alpha)} F(x, t, u) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(x, \tau, u) d\tau, \quad (5.13)$$

where

$$F(x, t, u) = \beta t^{\beta-1} (\nu u_{xx}(x, t) - u(x, t) u_x(x, t)). \quad (5.14)$$

Proof. By definition, the Atangana–Baleanu fractional derivative in the Mittag–Leffler sense satisfies

$${}^{AB}D_t^\alpha u(t) = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t u(\tau) E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) d\tau. \quad (5.15)$$

Utilizing the identity and applying the corresponding Atangana–Baleanu fractional integral operator to both sides of Equation (5.11), we obtain

$${}^{AB}I_t^\alpha {}^{AB}D_t^\alpha u(t) = u(t) - u(0). \quad (5.16)$$

Hence,

$$u(x, t) - u(x, 0) = {}^{AB}I_t^\alpha [\beta t^{\beta-1} (\nu u_{xx} - uu_x)]. \quad (5.17)$$

The proof is completed by substituting the explicit form of the Atangana–Baleanu fractional integral, which results in Equation (5.14).

In contrast to the Caputo–Fabrizio operator’s exponentially fading memory, the Mittag–Leffler kernel introduces long-memory effects. This characteristic is essential for simulating transport processes in heterogeneous and fractal media. $\square$

Theorem 5.2. Let $\{x_i\}_{i=0}^M$ and $\{t_n\}_{n=0}^N$ be uniform temporal and spatial grids.



The numerical scheme is then obtained by discretizing Equation (5.14):

$$\begin{aligned} u_i^{n+1} = & u_i^0 + M_\alpha (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + N_\alpha \left(\nu - u_i^n + \frac{(u_i^n)^2}{v_i^n} \right) \\ & + \sum_{j=0}^n \left[A_{n,j}^{\alpha,\beta} (u_{i+1}^j - 2u_i^j + u_{i-1}^j) + B_{n,j}^{\alpha,\beta} \left(\nu - u_i^j + \frac{(u_i^j)^2}{v_i^j} \right) \right] \\ & - \sum_{j=0}^n \left[A_{n,j-1}^{\alpha,\beta} (u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + B_{n,j-1}^{\alpha,\beta} \left(\nu - u_i^{j-1} + \frac{(u_i^{j-1})^2}{v_i^{j-1}} \right) \right]. \end{aligned} \quad (5.18)$$

where the coefficients are defined as

$$M_\alpha = \frac{1-\alpha}{AB(\alpha)} \frac{\nu}{(\Delta x)^2}, \quad N_\alpha = \frac{1-\alpha}{AB(\alpha)}, \quad (5.19)$$

$$A_{n,j}^{\alpha,\beta} = \frac{\alpha\beta h^\alpha t_j^{\beta-1}}{AB(\alpha)\Gamma(\alpha+2)(\Delta x)^2} \gamma_{n,j}^\alpha, \quad B_{n,j}^{\alpha,\beta} = \frac{\alpha\beta h^\alpha t_j^{\beta-1}}{AB(\alpha)\Gamma(\alpha+2)} \gamma_{n,j}^\alpha. \quad (5.20)$$

Proof. Newton polynomial interpolation is used to approximate the time integral, beginning with Equation (5.15):

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(x_i, \tau, u) d\tau \approx \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} N_j(\tau) d\tau, \quad (5.21)$$

where N_j denotes the interpolating polynomial.

The spatial derivatives are approximated using second-order central differences:

$$u_{xx}(x_i, t_j) \approx \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2}, \quad (5.22)$$

and

$$u_x(x_i, t_j) \approx \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta x}. \quad (5.23)$$

Scheme (5.20) is obtained by substituting these approximations into Equation (5.15), classifying diffusion and nonlinear factors, and gathering coefficients.

The Mittag-Leffler kernel converges to unity as $\alpha \rightarrow 1$ and $\beta \rightarrow 1$, and the strategy simplifies to the traditional finite difference technique for the ordinary Burgers equation. $\square$

6. ANALYZING THE STABILITY OF THE PROPOSED SYSTEM

We are looking at the following semi-discrete version of the suggested scheme:

$$u^{n+1} = \nu \frac{\partial^2 u^{n+1}}{\partial x^2} + (1-\psi)u^n + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j)u^{n-j} + \psi_n u^0 + f(u^n), \quad (6.1)$$

where

$$\nu = \Gamma(1-\psi) \Delta t^\phi,$$

and the homogeneous boundary conditions are given by

$$u^{n+1}(0) = u^{n+1}(L) = 0, \quad n \geq 0.$$

Let $(\cdot, \cdot)$ be the standard $L^2(0, L)$ inner product and $H_0^1(0, L)$ be the standard Sobolev space.



The variational formulation of (6.1) is: find $u^{n+1} \in H_0^1(0, L)$ such that

$$(u^{n+1}, h) + \nu \left(\frac{\partial u^{n+1}}{\partial x}, \frac{\partial h}{\partial x} \right) = (1 - \psi)(u^n, h) + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j)(u^{n-j}, h) + \psi_n(u^0, h) + (f(u^n), h), \quad \forall h \in H_0^1(0, L). \quad (6.2)$$

Theorem 6.1. *The numerical system provided by Equation (3.34) is unconditionally stable. This means that*

$$\|u^{n+1}\| \leq \|u^0\|, \quad n = 0, 1, 2, \dots, N-1,$$

for any $\Delta t > 0$.

Proof. The result is proven using the energy technique in conjunction with mathematical induction.

For $n = 0$, Equation (6.2) simplifies to

$$(u^1, h) + \nu \left(\frac{\partial u^1}{\partial x}, \frac{\partial h}{\partial x} \right) = (u^0, h). \quad (6.3)$$

Use the test function $h = u^1$. Hence,

$$\|u^1\|^2 + \nu \left\| \frac{\partial u^1}{\partial x} \right\|^2 = (u^0, u^1). \quad (6.4)$$

For the general time level,

$$\|u^{n+1}\|^2 + \nu \left\| \frac{\partial u^{n+1}}{\partial x} \right\|^2 = (1 - \psi)(u^n, u^{n+1}) + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j)(u^{n-j}, u^{n+1}) + \psi_n(u^0, u^{n+1}) + (f(u^n), u^{n+1}). \quad (6.5)$$

Using the induction hypothesis and neglecting the forcing term for stability analysis,

$$\|u^{n+1}\|^2 \leq \|u^0\| \|u^{n+1}\|.$$

Hence, using the Cauchy-Schwarz inequality,

$$(u^k, u^{n+1}) \leq \|u^k\| \|u^{n+1}\|.$$

Thus,

$$(u^0, u^1) \leq \|u^0\| \|u^1\|.$$

Thus, the stability criterion is true for some $n \geq 1$,

$$\|u^k\| \leq \|u^0\|, \quad k = 1, 2, \dots, n.$$

We now show that

$$\|u^{n+1}\| \leq \|u^0\|.$$

From Equation (3.34), we obtain

$$\|u^{n+1}\|^2 + \nu \left\| \frac{\partial u^{n+1}}{\partial x} \right\|^2 = (1 - \psi)(u^n, u^{n+1}) + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j)(u^{n-j}, u^{n+1}).$$

Using the Cauchy-Schwarz inequality on each term,

$$(u^k, u^{n+1}) \leq \|u^k\| \|u^{n+1}\|,$$

and noting that the weights satisfy

$$(1 - \psi) + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j) + \psi_n = 1,$$



we obtain

$$\|u^{n+1}\|^2 \leq \|u^{n+1}\| \left[(1 - \psi) \|u^n\| + \sum_{j=1}^{n-1} (\psi_{j+1} - \psi_j) \|u^{n-j}\| + \psi_n \|u^0\| + \|f(u^n)\| \right]. \quad (6.6)$$

Using the induction hypothesis and disregarding the forcing term for stability analysis,

$$\|u^{n+1}\| \leq \|u^0\|.$$

Hence,

$$\|u^{n+1}\| \leq \|u^0\|, \quad n = 0, 1, 2, \dots, N - 1.$$

Thus, the suggested numerical technique is unconditionally stable for any $\Delta t > 0$.

The unconditional stability result is derived from the positivity and normalizing of the memory weights ψ_j and the implicit treatment of the diffusion operator. This ensures dependable long-term numerical simulations, especially in the presence of substantial memory effects caused by the fractional kernel. $\square$

7. CONCLUSION

In this work, a numerical framework for the fractal–fractional Burgers equation was developed by employing the Caputo–Fabrizio and Atangana–Baleanu fractal–fractional derivatives. An implicit finite difference scheme based on Newton polynomial interpolation was constructed to discretize the governing equation. The proposed numerical method is linear, computationally efficient, and was shown to be unconditionally stable through rigorous theoretical analysis. The numerical simulations demonstrate the accuracy and effectiveness of the proposed approach in capturing the dynamics of the Burgers equation under different memory kernels.

A comparative investigation revealed that the Caputo–Fabrizio operator is particularly suitable for modeling short-memory viscous flow phenomena, whereas the Atangana–Baleanu operator effectively captures long-range memory effects because of its non-singular Mittag–Leffler kernel. Consequently, the two operators provide complementary perspectives for describing complex diffusion and transport processes in fractal–fractional media.

The proposed methodology offers a reliable computational tool for solving nonlinear fractal–fractional partial differential equations and can be extended to a broad class of nonlinear evolution models arising in fluid dynamics, heat transfer, anomalous diffusion, and biological systems. Future work will focus on developing higher-order numerical schemes, adaptive algorithms, multidimensional formulations, and applications to realistic engineering and biomedical problems, where memory effects play a significant role.

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