



Uniqueness theorems for the inverse fractal Sturm–Liouville problems

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Abstract

In this paper, we investigate an inverse Sturm–Liouville problem involving fractal derivatives and establish several uniqueness results. We prove that the coefficients of the associated boundary value problem can be uniquely recovered from various types of spectral data, including the Weyl function, two spectra, and the eigenvalues together with the corresponding norming constants. These results extend classical uniqueness theorems to the fractal setting and contribute to the spectral theory of generalized Sturm–Liouville operators.

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1. INTRODUCTION

Fractal geometry was systematically developed by Benoît B. Mandelbrot, who characterized it as the geometry of nature [25]. Fractal sets and curves are distinguished by their pronounced irregularity, with Hausdorff dimensions that strictly exceed their corresponding topological dimensions. In contrast to classical geometric objects, fractals exhibit geometric and physical attributes—such as length, area, volume, and density—that frequently lack standard definitions and display complex behavior across multiple spatial scales.

Fractal-like structures arise naturally in a wide variety of physical and biological systems, including the branching architecture of human lungs, the complex morphology of snowflakes, the irregular outlines of clouds, and the highly folded surfaces of mammalian brains. These phenomena underscore the prevalence of self-similarity and scale invariance in nature. Beyond their descriptive significance, fractals play an important role in mathematical physics. In particular, random walks defined on fractal supports have proven effective in modeling heat transport, diffusion, and wave propagation in heterogeneous and disordered media, such as polymers, porous and fractured geological formations, and amorphous semiconductors.

It is important to emphasize that idealized fractal structures—such as the von Koch curve, the Sierpiński gasket, and the Cantor set—do not occur in exact form in physical or real-world systems. Nevertheless, these mathematical models provide effective abstractions for capturing the essential self-similar characteristics and intrinsic fractal behavior exhibited by many complex natural phenomena. In such settings, classical concepts, including point density, may lose their conventional quantitative meaning, particularly in the analysis of clustering effects and spatial distributions within highly irregular systems [12].

Fractal calculus generalizes the framework of classical calculus to accommodate functions that are nowhere differentiable when defined on totally disconnected fractal sets. To enable a rigorous analytical treatment on such sets, the F^α -calculus was introduced [29]. Since its development, numerous studies have explored the relationships and

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distinctions between classical calculus and F^α -calculus, thereby elucidating its theoretical foundations and identifying its range of applicability.

Fractal calculus is often confused with fractional calculus; however, the two frameworks are fundamentally different. The principal distinction lies in the concept of locality. The fractal derivative is a local operator that preserves several essential properties of the classical derivative, including linearity and pointwise dependence. By contrast, fractional derivatives are inherently nonlocal, as they account for memory effects and depend on the global behavior of a function over an interval.

Owing to its local character and its effectiveness in treating irregular functions defined on fractal domains, fractal calculus has attracted significant attention in recent years (see [9, 12, 20, 30, 31, 36]). In addition, related notions such as local fractional derivatives have been proposed and studied in earlier works, offering alternative frameworks for differentiation on fractal sets (see [21, 22]).

It is well known that the Fourier method constitutes one of the fundamental techniques for solving partial differential equations arising in mathematical physics. Through the application of separation of variables, this approach typically reduces a given partial differential equation to an associated Sturm–Liouville problem. As a result, Sturm–Liouville problems have been extensively studied under a broad range of assumptions and formulations (see [6, 7, 15, 24, 28, 35]). Indeed, the Sturm–Liouville problem represents one of the most thoroughly investigated classes of boundary value problems in the theory of differential equations.

The theory of inverse spectral problems for Sturm–Liouville operators with ordinary derivatives has a long and well-established history, spanning nearly ninety years (see [5, 8, 10, 11, 14, 26, 27, 32–34, 37]). A central objective of this theory is the determination of the operator coefficients from spectral data, including the Weyl function, spectral characteristics, nodal sets, and selected sequences of eigenvalues. Inverse problems for Sturm–Liouville problems involving local fractional derivatives have been investigated in [16–18, 23, 33, 34].

Recent studies have investigated fractal Sturm–Liouville problems in considerable detail (see [1–4, 9, 13, 19]). In [1], the existence and uniqueness of solutions were investigated for the Sturm–Liouville problem defined by

$$-D_F^{2\alpha}u(\zeta) + q(\zeta)u(\zeta) = \mu u, \quad \zeta \in (0, 1), \quad (1.1)$$

where $\mu \in \mathbb{C}$ is the spectral parameter, and the potential $q(\cdot)$ is a real valued-function belonging to the space $L_2^\alpha(0, 1)$. In [2], the fundamental spectral properties of Sturm–Liouville problems were studied. In [3], sampling theorems were proved for such problems. In [4], a trace formula was derived by performing asymptotic calculations for the eigenvalues and eigenfunctions of such problems. In [19], Sturm-type theorems were obtained for fractal differential equations. In this study, we examine an inverse Sturm–Liouville problem formulated in terms of fractal derivatives and derive several uniqueness theorems. Specifically, we show that the coefficients of the corresponding boundary value problem can be uniquely determined from different sets of spectral data, such as the Weyl function, a pair of spectra, and the eigenvalues accompanied by their norming constants. Our findings generalize classical uniqueness results [8, 10, 11, 27, 33] to the fractal framework and enrich the spectral theory of generalized Sturm–Liouville operators. This approach generalizes classical calculus, making it suitable for situations in which standard differential and integral operators fail to apply.

To the best of our knowledge, this work is the first to investigate this problem within the fractal framework. The most significant aspect of this study is the investigation of whether the classical results obtained in the literature for classical Sturm–Liouville equations with standard derivatives remain valid for Sturm–Liouville problems with fractal derivatives defined on fractal sets. The present results generalize the fundamental uniqueness theorems of Borg [8], Gelfand and Levitan [11], Hochstadt and Lieberman [14], and Marchenko [26] to the framework of fractal Sturm–Liouville operators. The advantages of this framework are threefold. First, the local nature of the fractal derivative plays a vital role in physical models, most notably by preserving causality. Second, the non-integer order of the derivative admits a concrete geometric interpretation, coinciding with the fractal dimension of the function's support. Finally, this order carries a distinct physical meaning due to its fundamental relationship with the spectral dimension.



2. PRELIMINARIES

This section presents a brief overview of the fundamental concepts of fractal calculus. For a comprehensive exposition, the reader is referred to [9, 12, 29, 30]. Throughout the remainder of this paper, we assume that $F \subset \mathbb{R}$ is a fractal set.

Definition 2.1 ([29]). Let

$$P_J = \{e = \zeta_0 < \zeta_1 < \zeta_2 < \dots < \zeta_n = d\},$$

be a partition of the interval $J = [e, d] \subset \mathbb{R}$. For a given set $F \subset \mathbb{R}$ and $0 < \alpha \leq 1$, we define

$$\nu^\alpha [F, P_J] = \sum_{i=0}^{n-1} \Gamma(\alpha + 1) (\zeta_{i+1} - \zeta_i)^\alpha \theta(F, [\zeta_i, \zeta_{i+1}]),$$

where the indicator function $\theta(F, J)$ is given by

$$\theta(F, J) = \begin{cases} 1, & \text{if } F \cap J \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

If $e = d$, $\nu^\alpha [F, P_J] = 0$.

Definition 2.2 ([29]). The coarse-grained mass $\gamma_\delta^\alpha(F, e, d)$ is defined by

$$\gamma_\delta^\alpha(F, e, d) = \inf_{\{P_J: |P_J| \leq \delta\}} \nu^\alpha [F, P_J],$$

where $\delta > 0$, $e \leq d$ and

$$|P_J| = \max_{0 \leq i \leq n-1} (\zeta_{i+1} - \zeta_i),$$

for a partition P_J .

The Hausdorff dimension's analogue is now established.

Definition 2.3 ([29]). The γ -dimension of $F \cap J$ is defined as

$$\dim_\gamma(F \cap J) = \inf \{\alpha : \gamma^\alpha(F, e, d) = 0\} = \sup \{\alpha : \gamma^\alpha(F, e, d) = \infty\},$$

where

$$\gamma^\alpha(F, e, d) = \lim_{\delta \rightarrow 0} \gamma_\delta^\alpha(F, e, d).$$

Definition 2.4 ([29]). Let $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ be a function, $F \subset \mathbb{R}$, and let $\zeta \in F$. Then

$$A = F - \lim_{y \rightarrow \zeta} \Xi(y),$$

if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ for all $y \in F$, if $|y - \zeta| < \delta$, then

$$|\Xi(y) - A| < \varepsilon.$$

Definition 2.5 ([29]). Ξ is the F -continuous function at $\zeta \in F$ if

$$F - \lim_{y \rightarrow \zeta} \Xi(y) = \Xi(\zeta),$$

holds.

Definition 2.6 ([29]). Let $\zeta \in (\eta, \beta) \subset \mathbb{R}$. The point ζ is called a point of change of the function Ξ if Ξ is not constant on the interval (η, β) . Define

$$Sch(S_F^\alpha) = \{\zeta : \zeta \text{ is the point of change of } S_F^\alpha\}.$$

The set $Sch(S_F^\alpha)$ is said to be an α -perfect set if it is closed and every point of the set is a limit point.



Definition 2.7 ([29]). Let F be an α -perfect set and

$$S_F^\alpha(\zeta) = \begin{cases} \gamma^\alpha(F, e_0, \zeta), & \text{if } \zeta \geq e_0, \\ -\gamma^\alpha(F, \zeta, e_0) & \text{otherwise,} \end{cases}$$

where $e_0 \in \mathbb{R}$ is a fixed constant. Then $D_F^\alpha \Xi$ is defined by

$$D_F^\alpha \Xi(\zeta) = \begin{cases} F - \lim_{\substack{x \rightarrow \zeta \\ x \in F}} \frac{\Xi(x) - \Xi(\zeta)}{S_F^\alpha(x) - S_F^\alpha(\zeta)}, & \text{if } \zeta \in F, \\ 0, & \text{otherwise,} \end{cases}$$

if the limit exists.

Theorem 2.8. [[29]] If the functions $\Xi, \Sigma : \mathbb{R} \rightarrow \mathbb{R}$ are F^α -differentiable, then we get

$$\begin{aligned} \text{i)} \quad & D_F^\alpha(\Xi \Sigma)(\zeta) = \Sigma(\zeta) D_F^\alpha \Xi(\zeta) + \Xi(\zeta) D_F^\alpha \Sigma(\zeta), \\ \text{ii)} \quad & D_F^\alpha(a\Xi + b\Sigma)(\zeta) = aD_F^\alpha \Xi(\zeta) + bD_F^\alpha \Sigma(\zeta), \end{aligned}$$

where $e, d \in \mathbb{R}$.

Theorem 2.9. [[29]] Let Ψ is a bounded function on $F \cap J$. The F^α -integral of Ψ is defined by

$$\begin{aligned} \int_a^b \Psi(\zeta) d_F^\alpha \zeta &= \sup_{P_J} \sum_{i=0}^{n-1} \inf_{\zeta \in F \cap J} \Psi(\zeta) (S_F^\alpha(\zeta_{i+1}) - S_F^\alpha(\zeta_i)) \\ &= \inf_{P_J} \sum_{i=0}^{n-1} \sup_{\zeta \in F \cap J} \Psi(\zeta) (S_F^\alpha(\zeta_{i+1}) - S_F^\alpha(\zeta_i)) \end{aligned}$$

where $\zeta \in J$, and the infimum or supremum is taken over all partitions P_J .

Theorem 2.10. [[29]] Let Ξ be a bounded function on $F \cap J$. Then we have

$$\Sigma(\zeta) = \int_0^\zeta \Xi(x) d_F^\alpha x,$$

where $\zeta \in J$,

$$D_F^\alpha \Sigma(\zeta) = \Xi(\zeta) \chi_F(\zeta),$$

and χ_F is the characteristic function.

Example 2.11.

- i) $D_F^\alpha e \chi_F(\zeta) = 0$, e is constant.
- ii) $D_F^\alpha S_F^\alpha(\zeta) = \chi_F(\zeta)$.
- iii) $D_F^\alpha (S_F^\alpha(\zeta))^m = m \chi_F(\zeta) (S_F^\alpha(\zeta))^{m-1}$.
- iv) $D_F^\alpha \sin(S_F^\alpha(\zeta)) = \chi_F(\zeta) \cos(S_F^\alpha(\zeta))$.
- v) $D_F^\alpha \cos(S_F^\alpha(\zeta)) = -\chi_F(\zeta) \sin(S_F^\alpha(\zeta))$.
- vi) $\int (S_F^\alpha(\zeta))^m d_F^\alpha \zeta = \frac{(S_F^\alpha(\zeta))^{m+1}}{m+1} + e$.

Theorem 2.12. [[29]] Let $\Xi, \Sigma \in C[e, z]$, where $Sch(\Xi) \subset F$. Suppose that $D_F^\alpha \Xi$ exists and is F -continuous on J . Then the following relation holds:

$$\begin{aligned} & \int_e^z \Xi(\zeta) \Sigma(\zeta) d_F^\alpha \zeta + \int_e^z D_F^\alpha \Xi(\zeta) \int_e^\zeta \Sigma(y) d_F^\alpha y d_F^\alpha \zeta \\ &= \left[\Xi(\zeta) \int_e^\zeta \Sigma(y) d_F^\alpha y \right]_e^z. \end{aligned} \tag{2.1}$$



Let

$$L_2^\alpha(0, \pi) = \left\{ \Xi : \int_0^\pi |\Xi(\zeta)|^2 d_F^\alpha \zeta < \infty \right\}.$$

be a Hilbert space with the inner product

$$(\Xi, \Sigma) = \int_0^\pi \Xi \bar{\Sigma} d_F^\alpha \zeta.$$

Now, let us consider the following fractal boundary value problem $L_F(q, \beta, \gamma)$:

$$l(u) := -D_F^{2\alpha} u(\zeta) + q(\zeta) u(\zeta) = \mu u, \quad \zeta \in (0, \pi), \quad (2.2)$$

with boundary conditions

$$Y(u) : D_F^\alpha u(0) - \beta u(0) = 0, \quad (2.3)$$

$$Z(u) : D_F^\alpha u(\pi) + \gamma u(\pi) = 0, \quad (2.4)$$

where $\beta, \gamma \in \mathbb{R}$, $\mu \in \mathbb{C}$ is a spectral parameter; $q(\cdot)$ is a real valued-function and $q \in L_2^\alpha(0, \pi)$.

Let

$$\mathcal{D} = \{u \in D_{\max} : D_F^\alpha u(0) - \beta u(0) = 0, D_F^\alpha u(\pi) + \gamma u(\pi) = 0\},$$

where

$$\mathcal{D}_{\max} = \{u \in L_2^\alpha(0, \pi) : D_F^\alpha u, D_F^{2\alpha} u \in L_2^\alpha(0, \pi) \text{ and } l(u) \in L_2^\alpha(0, \pi)\}.$$

Let us define the operator $\mathcal{L}$ with

$$\mathcal{L}u = l(u),$$

where $u \in \mathcal{D}$. $\mathcal{L}$ is self-adjoint [9].

Eq. (2.2) has a unique solution $u(\zeta, \mu)$ satisfying

$$u(0, \mu) = \xi_1, \quad D_F^\alpha u(0, \mu) = \xi_2,$$

where $\xi_1, \xi_2 \in \mathbb{C}$ [1]. Consider $\tau(\zeta, \mu)$ and $\omega(\zeta, \mu)$ as solutions of (2.2)-(2.4) satisfying

$$\tau(0, \mu) = 1, \quad \omega(\pi, \mu) = 1, \quad (2.5)$$

$$D_F^\alpha \tau(0, \mu) = \beta, \quad D_F^\alpha \omega(\pi, \mu) = -\gamma.$$

The characteristic function of $\mathcal{L}$ is given by

$$\Delta(\mu) = W^\alpha(\tau(\zeta, \mu), \omega(\zeta, \mu)) = Z(\tau) = -Y(\omega). \quad (2.6)$$

It follows from [2, 9] that Δ is entire in μ and has an almost countable set of zeros. The zeros of Δ coincide with the eigenvalues of the operator $\mathcal{L}$. All zeros of Δ are simple. If $\tau(\zeta, \mu_n)$ and $\omega(\zeta, \mu_n)$ are eigenfunctions then there exists a sequence $\{\Upsilon_n\}$ such that

$$\tau(\zeta, \mu_n), \quad \Upsilon_n \neq 0, \quad (2.7)$$

and

$$\Upsilon_n \Phi_n = -\frac{d}{d\mu} \Delta(\mu_n), \quad (2.8)$$

where

$$\omega(0, \mu_n) = \Upsilon_n,$$

and

$$\Phi_n = \int_0^\pi \tau^2(\zeta, \mu_n) d_F^\alpha \zeta.$$



Lemma 2.13. *[[4]] The following relations hold:*

$$\tau(\zeta, \mu) = \cos(\sqrt{\mu} S_F^\alpha(\zeta)) + \beta \frac{\sin(\sqrt{\mu} S_F^\alpha(\zeta))}{\sqrt{\mu}} + \int_0^\zeta \frac{\sin(\sqrt{\mu} (S_F^\alpha(\zeta) - S_F^\alpha(t)))}{\sqrt{\mu}} q(t) \tau(t, \mu) d_F^\alpha t. \quad (2.9)$$

$$\omega(\zeta, \mu) = \cos(\sqrt{\mu} (S_F^\alpha(\pi) - S_F^\alpha(\zeta))) + \gamma \frac{\sin(\sqrt{\mu} (S_F^\alpha(\pi) - S_F^\alpha(\zeta)))}{\sqrt{\mu}} + \int_0^\zeta \frac{\sin(\sqrt{\mu} (S_F^\alpha(t) - S_F^\alpha(\zeta)))}{\sqrt{\mu}} q(t) \omega(t, \mu) d_F^\alpha t. \quad (2.10)$$

Lemma 2.14. *[[4]] Let $\mu = \lambda^2$, $\lambda = \phi + i\psi$. Then, for $|\lambda| \rightarrow \infty$, the following formulae hold:*

$$\left\{ \begin{array}{l} \tau(\zeta, \mu) = \cos(\lambda S_F^\alpha(\zeta)) + O\left(\frac{1}{|\lambda|} \exp(|\psi| S_F^\alpha(\zeta))\right) \\ \quad = O(\exp(|\psi| S_F^\alpha(\zeta))), \\ D_F^\alpha \tau(\zeta, \mu) = -\lambda \sin(\lambda S_F^\alpha(\zeta)) + O(\exp(|\psi| S_F^\alpha(\zeta))) \\ \quad = O(|\lambda| \exp(|\psi| S_F^\alpha(\zeta))), \end{array} \right. \quad (2.11)$$

and

$$\left\{ \begin{array}{l} \omega(\zeta, \mu) = \cos(\lambda (S_F^\alpha(\pi) - S_F^\alpha(\zeta))) \\ \quad + O\left(\frac{1}{|\lambda|} \exp(|\psi| (S_F^\alpha(\pi) - S_F^\alpha(\zeta)))\right), \\ \quad = O(\exp(|\psi| (S_F^\alpha(\pi) - S_F^\alpha(\zeta)))), \\ D_F^\alpha \omega(\zeta, \mu) = \lambda \sin(\sqrt{\mu} (S_F^\alpha(\pi) - S_F^\alpha(\zeta))) \\ \quad + O(\exp(|\psi| (S_F^\alpha(\pi) - S_F^\alpha(\zeta)))), \\ \quad = O(|\lambda| \exp(|\psi| (S_F^\alpha(\pi) - S_F^\alpha(\zeta)))). \end{array} \right. \quad (2.12)$$

All estimates are uniform with respect to $\zeta \in [0, \pi]$.

Theorem 2.15. *[[4]] $L_F(q, \beta, \gamma)$ admits a countable sequence of eigenvalues $\{\mu_n\}_{n \geq 0}$. For each $n \geq 0$, the following estimates are satisfied.*

$$\lambda_n = \sqrt{\mu_n} = \left(\frac{\pi}{S_F^\alpha(\pi)} \right) n + \frac{\kappa}{\pi n} + \frac{\varrho_n}{n}, \quad \{\varrho_n\} \in l_2, \quad (2.13)$$

$$\tau(\zeta, \mu_n) = \cos\left(\frac{n\pi}{S_F^\alpha(\pi)} S_F^\alpha(\zeta)\right) + \frac{\kappa_{\alpha_n}(\zeta)}{n}, \quad |\kappa_{\alpha_n}(\zeta)| \leq K, \quad (2.14)$$

where

$$\kappa = \beta + \gamma + \frac{1}{2} \int_0^\pi q(t) d_F^\alpha t, \quad (2.15)$$

and K denotes various constant which do not depend on ζ, μ and n .

Theorem 2.16. *[[4]] The complete specification of the spectrum $\{\mu_n\}_{n \geq 0}$ uniquely determines the characteristic function Δ via the following formula.*

$$\Delta(\mu) = \frac{(S_F^\alpha(\pi))^3}{\pi^2} (\mu_0 - \mu) \prod_{n=1}^{\infty} \frac{\mu_n - \mu}{n^2}. \quad (2.16)$$

3. UNIQUENESS THEOREMS

In this section, the main results of the article—namely, uniqueness theorems for inverse fractal Sturm–Liouville problems—will be established. Alongside $L_F(q, \beta, \gamma)$, we consider another fractal boundary value problem $\widetilde{L}_F(\widetilde{q}, \widetilde{\beta}, \widetilde{\gamma})$ of the same type but with different coefficients. Throughout this work, we adopt the convention that, whenever a



symbol ϕ denotes an object associated with $L_F(q, \beta, \gamma)$, the corresponding symbol $\widetilde{\phi}$ refers to the analogous object associated with $\widetilde{L}_F(\widetilde{q}, \widetilde{\beta}, \widetilde{\gamma})$.

Let $\Theta(\zeta, \mu)$ be a solution of (2.2) satisfying

$$\Theta(0, \mu) = 1, \text{ and } D_F^\alpha \Theta(0, \mu) = 0.$$

It is obvious that $W^\alpha(\tau(\zeta, \mu), \Theta(\zeta, \mu)) = 1$. Let

$$\Psi(\zeta, \mu) := -\frac{\omega(\zeta, \mu)}{\Delta(\mu)} = \Theta(\zeta, \mu) + M(\mu) \tau(\zeta, \mu), \quad (3.1)$$

where the Weyl function $M(\mu)$ is defined by

$$M(\mu) = -\frac{\omega(0, \mu)}{\Delta(\mu)}.$$

Theorem 3.1. *If $M(\mu) = \widetilde{M}(\mu)$, then $q(\zeta) = \widetilde{q}(\zeta)$ a.e. in $[0, \pi]$, $\beta = \widetilde{\beta}$ and $\gamma = \widetilde{\gamma}$.*

Proof. Let us introduce the following functions

$$\Lambda_1(\zeta, \mu) = \tau(\zeta, \mu) D_F^\alpha \widetilde{\Psi}(\zeta, \mu) - \Psi(\zeta, \mu) D_F^\alpha \widetilde{\tau}(\zeta, \mu), \quad (3.2)$$

$$\Lambda_2(\zeta, \mu) = \Psi(\zeta, \mu) \widetilde{\tau}(\zeta, \mu) - \tau(\zeta, \mu) \widetilde{\Psi}(\zeta, \mu). \quad (3.3)$$

These functions are meromorphic with respect to μ . By (3.1), we find

$$\Lambda_1(\zeta, \mu) = \tau(\zeta, \mu) D_F^\alpha \widetilde{\Theta}(\zeta, \mu) - \Theta(\zeta, \mu) D_F^\alpha \widetilde{\tau}(\zeta, \mu)$$

$$+ \left(\widetilde{M}(\mu) - M(\mu) \right) \tau(\zeta, \mu) D_F^\alpha \widetilde{\tau}(\zeta, \mu),$$

$$\Lambda_2(\zeta, \mu) = \Theta(\zeta, \mu) \widetilde{\tau}(\zeta, \mu) - \tau(\zeta, \mu) \widetilde{\Theta}(\zeta, \mu)$$

$$+ \left(M(\mu) - \widetilde{M}(\mu) \right) \tau(\zeta, \mu) \widetilde{\tau}(\zeta, \mu).$$

Thus, we see that Λ_1 and Λ_2 are entire in μ , due to $M(\mu) = \widetilde{M}(\mu)$. Moreover, we have

$$W^\alpha(\tau(\zeta, \mu), \Psi(\zeta, \mu)) = \frac{W^\alpha(\tau(\zeta, \mu), \omega(\zeta, \mu))}{\Delta(\mu)} = 1.$$

Then we obtain

$$\Lambda_1(\zeta, \mu) = 1 + \tau(\zeta, \mu) \left\{ D_F^\alpha \widetilde{\Psi}(\zeta, \mu) - D_F^\alpha \Psi(\zeta, \mu) \right\}$$

$$+ \Psi(\zeta, \mu) \left\{ D_F^\alpha \widetilde{\tau}(\zeta, \mu) - D_F^\alpha \tau(\zeta, \mu) \right\}.$$

It follows from Lemma 2.14 that

$$\Lambda_1(\zeta, \mu) = 1 - O\left(\frac{1}{\lambda}\right), \quad \Lambda_2(\zeta, \mu) = O\left(\frac{1}{\lambda}\right), \quad \lambda \in G_\xi, |\lambda| \geq \lambda^*.$$

By Liouville's theorem, we conclude that $\Lambda_1(\zeta, \mu) = 1$ and $\Lambda_2(\zeta, \mu) = 0$. From (3.2) and (3.3), we obtain

$$\tau(\zeta, \mu) = \widetilde{\tau}(\zeta, \mu), \quad \Psi(\zeta, \mu) = \widetilde{\Psi}(\zeta, \mu).$$



Substituting these identical solutions back into the corresponding fractal Sturm–Liouville equations, namely

$$-D_F^{2\alpha} \tau(\zeta, \mu) + q(\zeta) \tau(\zeta, \mu) = \mu \tau(\zeta, \mu),$$

$$-D_F^{2\alpha} \widetilde{\tau}(\zeta, \mu) + \widetilde{q}(\zeta) \widetilde{\tau}(\zeta, \mu) = \mu \widetilde{\tau}(\zeta, \mu),$$

and subtracting them yields

$$(q(\zeta) - \widetilde{q}(\zeta)) \tau(\zeta, \mu) = 0.$$

Since $\tau(\zeta, \mu)$ is a non-trivial solution, it follows immediately that $q(\zeta) = \widetilde{q}(\zeta)$ a.e. in $[0, \pi]$. Furthermore, substituting

$$\tau(\zeta, \mu) = \widetilde{\tau}(\zeta, \mu) \text{ and } \Psi(\zeta, \mu) = \widetilde{\Psi}(\zeta, \mu)$$

directly into the boundary conditions at $\zeta = 0$ and $\zeta = \pi$ uniquely forces $\beta = \widetilde{\beta}$ and $\gamma = \widetilde{\gamma}$. $\square$

We now examine the fractal boundary value problem $L_{F,1}$, in which the boundary condition $u(0, \mu) = 0$ is imposed in place of condition (2.3) in L_F . Let $\{\vartheta_n\}_{n \geq 0}$ be the eigenvalues of $L_{F,1}$. It is clear that ϑ_n are zeros of $\Delta_1(\vartheta) := \omega(0, \mu)$.

Theorem 3.2. *If $\{\mu_n, \vartheta_n\}_{n \geq 0} = \{\widetilde{\mu}_n, \widetilde{\vartheta}_n\}_{n \geq 0}$, then $q(\zeta) = \widetilde{q}(\zeta)$ a.e. in $[0, \pi]$, $\beta = \widetilde{\beta}$ and $\gamma = \widetilde{\gamma}$.*

Proof. By Theorem 2.16, we see that

$$\Delta(\mu) = \widetilde{\Delta}(\mu) \quad \left(\text{resp. } \Delta_1(\vartheta) = \widetilde{\Delta}_1(\vartheta) \right),$$

when $\mu_n = \widetilde{\mu}_n$ (resp. $\vartheta_n = \widetilde{\vartheta}_n$) for all n . Therefore, we conclude that $M(\mu) = \widetilde{M}(\mu)$. From Theorem 3.1, we get the desired result. $\square$

Theorem 3.3. *If $\{\mu_n, \Upsilon_n\}_{n \geq 0} = \{\widetilde{\mu}_n, \widetilde{\Upsilon}_n\}_{n \geq 0}$, then $q(\zeta) = \widetilde{q}(\zeta)$ a.e. in $[0, \pi]$, $\beta = \widetilde{\beta}$ and $\gamma = \widetilde{\gamma}$. Here Υ_n defined by (2.7) are called norming constants.*

Proof. Since $\mu_n = \widetilde{\mu}_n$, we see that $\Delta(\mu) = \widetilde{\Delta}(\mu)$. It follows from (2.8) that $\Phi_n = \widetilde{\Phi}_n$, i.e., $\omega(0, \mu_n) = \widetilde{\omega}(0, \mu_n)$. Let

$$\Sigma(\mu) := \frac{\omega(0, \mu) - \widetilde{\omega}(0, \mu)}{\Delta(\mu)}.$$

It is clear that Σ is entire on μ . It follows from Lemma ?? that $\Sigma(\mu) = O\left(\frac{1}{\mu}\right)$ for $\mu \rightarrow \infty$. Hence, $\Sigma(\mu) = 0$, i.e., $\omega(0, \mu) = \widetilde{\omega}(0, \mu)$. Then we have $M(\mu) = \widetilde{M}(\mu)$. By Theorem 3.1, we obtain the desired result. $\square$

4. CONCLUSION

In this study, we have investigated the inverse Sturm–Liouville problem involving fractal derivatives, successfully establishing several fundamental uniqueness theorems. By utilizing various types of spectral data—namely the Weyl function, two spectra, and the combination of eigenvalues along with their corresponding norming constants—we demonstrated that the coefficients of the associated boundary value problem can be uniquely recovered. These findings systematically extend classical uniqueness results from standard differential operators to the framework of fractal calculus, thereby contributing significantly to the spectral theory of generalized operators. The methodologies and insights developed in this work pave the way for numerous future investigations. Potential directions for subsequent research include exploring inverse spectral problems for fractal Sturm–Liouville operators with discontinuities or singular potentials, as well as developing numerical algorithms for the practical reconstruction of coefficients from given spectral data.

STATEMENTS AND DECLARATIONS

The authors declare that there are no conflicts of interest regarding this work.



AVAILABILITY OF DATA AND MATERIALS

Not applicable, no datasets were generated or analyzed during the current study.

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