



Modelling Ponzi schemes as a diffusive process: A novel PDE framework and analysis

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Abstract

The paper discusses the development of a novel Partial Differential Equation (PDE) framework to model and analyse the dynamics of Ponzi schemes, focusing on investor capital distribution and sustainability conditions. While PDEs have been a foundation to modern quantitative finance, their application in the context of the specific problem of Ponzi schemes have been limited. The existing literature is deficient in spatio-temporal (partial derivatives of cashflows and time), distribution-based model that can describe the continuous evolution of the investor capital distribution over time. It also does not incorporate advection (the aging of investments), diffusion (the spread of investment sizes), and source/sink terms (new investments/withdrawals) in a single cohesive framework. Our proposed PDE-based model is designed to describe the bubble and bust cycle of the schemes through the extension of the aggregate view of most differential equation-based models to a continuous setting and provides the analytical tractability that is often missing in complex ABMs (agent-based models). By drawing on the mathematical traditions of other PDE-based models, we offer a novel and rigorous tool for understanding the life cycle of this type of financial fraud – Ponzi scheme.

Keywords. Ponzi schemes, PDE, Financial investment, Financial fraud, Cashflow, Recruitment rate, Withdrawal rate.

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1. INTRODUCTION

Ponzi schemes, named after the infamous Charles Ponzi [14], represent a perennial and devastating form of financial fraud that is mathematically designed to fail. Unlike legitimate investments that generate realistic returns from profit-earning investments, these schemes sustain themselves by using capital from new investors to pay fabricated returns to earlier participants, thereby creating a liability structure that grows exponentially against a static or non-existent asset base [4]. This fundamental structure creates an inherent, mathematical inevitability of collapse, as the liability to investors grows exponentially while the underlying asset base remains fictitious and depleted by successive withdrawals of matured investments. Despite increased regulatory vigilance and public awareness, sophisticated Ponzi schemes, such as the one perpetrated by Bernard Madoff [8], continue to emerge, causing catastrophic financial losses and eroding public confidence in financial systems [15].

While the economic and legal characteristics of these schemes are well-documented, their mathematical formalization has often relied on simplified models that treat investor populations as aggregate, homogeneous groups [5]. Such existing models often rely on deterministic, continuous-time frameworks or simplistic assumptions about investor behaviour which are frequently based on ordinary differential equations (ODEs) or discrete stochastic processes, which capture the core instability but lack the resolution to describe the internal structure and distributional dynamics of the scheme, such as the evolution of investor capital over time and the wave-like propagation of financial distress. The persistence of these schemes underscores the critical need for robust, quantitative models that can elucidate their internal dynamics, predict their lifespan, and inform early detection mechanisms. While the fraudulent nature of Ponzi schemes is well-understood in legal and economic circles, a precise mathematical formalization of their growth, sustainability, and critical point of failure remains an active area of research.

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The application of Partial Differential Equations (PDEs) offers a powerful, yet largely unexplored, framework for analysing such systems. Inspired by their use in quantitative finance (e.g., the Black-Scholes equation) and physical sciences, PDE models can describe how the mass of investors and their capital evolve across a continuous state space, incorporating variables such as the time since investment, cashflows, and the rate of capital withdrawal/recruitment. This approach provides a granular view of the scheme’s internal mechanics, revealing dynamics that are obscured in lumped-parameter models.

In this paper, we develop a novel PDE-based model of a Ponzi scheme. Our primary contribution is the formulation and analysis of a diffusion PDE that governs the time evolution of the investor capital density function. This model allows us to:

- Derive a fundamental sustainability condition analogous to those found in population dynamics [3], which links the scheme’s longevity to parameters such as the rate of new investor recruitment, promised returns, and capital withdrawal rates.
- Characterize the intrinsic blow-up behaviour of the solution, providing a rigorous mathematical proof of the scheme’s inevitable collapse under a wide range of conditions.
- Identify critical thresholds and phase transitions between slow decay and rapid implosion, offering new insights into critical point of failure for forensic finance and regulatory oversight.

The remainder of this paper is structured as follows. Section 2 reviews the literature, positioning our PDE approach against existing modelling paradigms. Section 3 details the formulation of our model, outlining its assumptions and governing equations. This section also presents a mathematical analysis of the model, including the analytical solution, well-posedness of Model’s governing PDE, derivation of the sustainability condition, and a numerical exploration of its solutions using the 4th-order Runge-Kutta (RK4) Method in Section 4. Section 5 presents and discusses the results of our findings. Finally, Section 6 consists of conclusion and suggests directions for future research.

2. LITERATURE REVIEW

The mathematical modelling of Ponzi schemes spans economics, finance, and applied mathematics. This review categorizes the existing approaches to highlight the novelty of our PDE framework.

2.1. Economic Foundations and Aggregate Models. The foundational understanding of Ponzi schemes is rooted in their distinction from legitimate pyramid schemes and other financial bubbles. The work of Kindleberger (1978) on manias, panics, and crashes [11] provides a broad historical context, positioning Ponzi schemes as a specific type of financial fraud reliant on unsustainable exponential growth. More recently, Barlevy and Alvarez (2014) have explored the economic conditions that foster such speculative bubbles, noting that periods of low financial literacy, lack of public disclosure of credible information, and high market optimism create a fertile ground for their propagation of fraudulent financial practices [1]. From a mathematical perspective, a seminal contribution is the deterministic model of Artzrouni (2009), which uses ODEs to describe the dynamics of an investment fund that promises more than it can deliver [2]. This model effectively captures the macro-level instability but treats the investor pool as a single, homogeneous entity, lacking distributional detail.

2.2. Stochastic and Agent-Based Approaches. To incorporate randomness and heterogeneity, researchers have turned to stochastic models. Kutsenko (2025) considers a simplified Ponzi pyramid scheme with and without freedom of movement, which simulates stochastic compression with high and low turbulence [12]. Wang and Zhang (2025) in their study examines the evolution and dynamics of blockchain-based Ponzi schemes that originated from the PlusToken scam, as well as the strategic behaviour of heterogeneous investors. PlusToken was a typical blockchain-based Ponzi scheme started in 2018 and incurred over 3 million investors losing over USD 5 billion [6]. Other blockchain-based Ponzi scheme includes OneCoin, BitClub, MMM Bitcoin, CoinUp, and others. The study applies the ABM framework to analyse the evolutionary dynamics and investor performances under three distinct scenarios, i.e., the standard Ponzi scheme, the scenario involving the escape of the fraudster, and the scenario under the decrease in utility weight for invested tokens [16]. While these studies are powerful, ABMs and stochastic approaches often yield computational results rather than general analytical insights.



2.3. PDE Models in Finance and Population Dynamics. Our work in this paper is methodologically aligned with the use of PDEs in related fields. In quantitative finance, the Black-Scholes (1973) model uses a parabolic PDE to price options, modelling the evolution of a probability density [7]. In population dynamics, the McKendrick-von Foerster (1959) equation is a PDE that models the age-structured growth of a population, a structure that is highly analogous to tracking investors based on their age in the scheme [9]. It is a conservation law which states that a specific physical property of an isolated system remains constant over time, and in this case, it describes how the population density of individuals at a specific age change over time due to aging and mortality, with new individuals introduced via a birth.

Finally, while PDEs are a cornerstone of modern quantitative finance, their application to the specific problem of Ponzi schemes has been limited. The existing literature is deficient in spatio-temporal, distribution-based model that can describe the continuous evolution of the investor capital distribution. It also does not incorporate advection (the aging of investments), diffusion (the spread of investment sizes), and source/sink terms (new recruitments/withdrawals) in a single cohesive framework. Our proposed PDE model is designed to fill this gap. It extends the aggregate view of most differential equation-based models to a continuous setting and provides the analytical tractability that is often missing in complex ABMs. By drawing on the mathematical traditions of other PDE-based models, we offer a novel and rigorous tool for understanding the life cycle of this type of financial fraud – Ponzi Scheme.

3. MODEL FORMULATION, SOLUTION AND ANALYSIS

3.1. Model Formulation and Assumptions. The total capital, x in the fund at time t , given as $u(x, t)$:

$$u(x, t) = \text{Principal} + \text{inflow} - \text{outflow}. \quad (3.1)$$

- *Principal* = $x_0(1 + i) = x_0$. where i = interest rate on capital held by the fund, which is being neglected at this stage.
- Inflow can be modelled as a function of the number of new investors, N_{new} , and their average investment, I_{avg} as:
 $\text{Inflow} = N_{new} \cdot I_{avg} = x_{new} = u_1(x, t)$
- Outflow is the sum of withdrawals and interest payments to existing investors, N_{old} :
 $\text{Outflow} = N_{old} \cdot I_{avg} \cdot (1 + r) = x_{old} = u_2(x, t)$
 where r is the promised interest rate, assuming both the withdrawals and interest payments treated together as implicit value.

So,

$$u(x, t) = x_0 + u_1(x, t) - u_2(x, t). \quad (3.2)$$

Now, let α be a constant such that $u_1(x, t) = \alpha u_2(x, t)$. But α cannot be negative, so we have:

$$u_1(x, t) = \alpha^2 u_2(x, t), \quad u_1, u_2 \in u. \quad (3.3)$$

So, the resulting PDE:

$$u + u_{xt} - \alpha^2 u_{xt} = 0, \quad (3.4)$$

i.e.,

$$\frac{\partial^2 u}{\partial x \partial t} - \alpha^2 \frac{\partial^2 u}{\partial x \partial t} + u = 0, \quad t > 0, \quad (3.5)$$

or,

$$(1 - \alpha^2) \frac{\partial^2 u}{\partial x \partial t} + u = 0, \quad (3.6)$$

which implies that:

$$\frac{\partial^2 u}{\partial x \partial t} + \frac{1}{1 - \alpha^2} u = 0 \quad \text{or} \quad u_{xt} + \frac{1}{1 - \alpha^2} u = 0, \quad (3.7)$$



which can be rewritten as:

$$u_{xt} + \beta u = 0, \quad \beta = \frac{1}{1 - \alpha^2}, \quad t > 0. \quad (3.8)$$

Thus, the governing equation is:

$$u_{xt} + \beta u = 0, \quad \text{where } \beta = \frac{1}{1 - \alpha^2}, \quad 0 < x < L, \quad t > 0. \quad (3.9)$$

Subject to the boundary conditions:

$$u(0, t) = u(L, t) = 0,$$

and initial condition:

$$u(x, 0) = f(x), \quad 0 < x < L,$$

where

- $u(x_0)$ = principal,
- α = constant,
- x = cashflow,
- t = time (month).

3.2. Analytical solution of model. In view of the peculiarity of the exact or analytical solution to the model, we shall use the Laplace transform in time which is best suitable under the circumstance. The Laplace transform is a mathematical operation named after Pierre-Simon Laplace that converts a function, often of time, into a function of a complex variable(s) in the Laplace domain. This integral transform is primarily used to simplify differential equations by transforming them into algebraic equations, which are generally easier to solve, and then the solution is converted back to the original domain using the inverse Laplace transform [13].

So, let $\mathcal{U}(x, s) = \mathcal{L}\{u(x, t)\}$. Then:

$$\mathcal{L}\{u_{xt}\} = \frac{\partial}{\partial x}(s\mathcal{U}(x, s) - u(x, 0)) = s\mathcal{U}_x(x, s) - f'(x). \quad (3.10)$$

The transformed PDE becomes:

$$s\mathcal{U}_x(x, s) - f'(x) + \beta\mathcal{U}(x, s) = 0. \quad (3.11)$$

or,

$$s\mathcal{U}_x + \beta\mathcal{U} = f'(x). \quad (3.12)$$

This is a first-order ODE in x . The integrating factor is:

$$\mu(x) = \exp\left(\int \frac{\beta}{s} dx\right) = e^{\frac{\beta}{s}x}. \quad (3.13)$$

Thus,

$$e^{\frac{\beta}{s}x} s\mathcal{U}_x + e^{\frac{\beta}{s}x} \beta\mathcal{U} = f'(x) e^{\frac{\beta}{s}x}, \quad (3.14)$$

which yields:

$$\frac{d}{dx} \left(\mathcal{U} e^{\frac{\beta}{s}x} \right) = \frac{1}{s} f'(x) e^{\frac{\beta}{s}x}, \quad (3.15)$$

and

$$\mathcal{U}(x, s) e^{\frac{\beta}{s}x} - \mathcal{U}(0, s) = \frac{1}{s} \int_0^x f'(\tau) e^{\frac{\beta}{s}\tau} d\tau. \quad (3.16)$$

Now, $\mathcal{U}(0, s) = \mathcal{L}\{u(0, t)\} = 0$. So:

$$\mathcal{U}(x, s) = \frac{1}{s} e^{-\frac{\beta}{s}x} \int_0^x f'(\tau) e^{\frac{\beta}{s}\tau} d\tau = \frac{1}{s} \int_0^x f'(\tau) e^{\frac{\beta(\tau-x)}{s}} d\tau. \quad (3.17)$$



The inverse transform becomes:

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} e^{\frac{k}{s}} \right\} = I_0(2\sqrt{kt}), \quad \text{for } k > 0, \quad (3.18)$$

where $k = \beta(\tau - x)$.

For $|\alpha| < 1, \beta > 0$, and since $\tau \leq x, k \leq 0$. So, let $k = -\beta(x - \tau)$. Then

$$e^{\frac{\beta(\tau-x)}{s}} = e^{\frac{-\beta(x-\tau)}{s}}. \quad (3.19)$$

So,

$$\mathcal{U}(x, s) = \frac{1}{s} \int_0^x f'(\tau) e^{\frac{-\beta(x-\tau)}{s}} d\tau. \quad (3.20)$$

Then,

$$u(x, t) = \int_0^x f'(\tau) \mathcal{L}^{-1} \left\{ \frac{1}{s} e^{\frac{-\beta(x-\tau)}{s}} \right\} d\tau, \quad (3.21)$$

and,

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} e^{\frac{-k}{s}} \right\} = \mathcal{J}_0(2\sqrt{kt}), \quad \text{for } k > 0, \quad (3.22)$$

where $k = \beta(x - \tau) > 0$. Therefore,

$$u(x, t) = \int_0^x f'(\tau) \mathcal{J}_0(2\sqrt{\beta(x-\tau)t}) d\tau, \quad (3.23)$$

with $\beta = \frac{1}{1-\alpha^2}$:

$$u(x, t) = \int_0^x f'(\tau) \mathcal{J}_0 \left(2\sqrt{\frac{(x-\tau)t}{1-\alpha^2}} \right) d\tau. \quad (3.24)$$

This is the solution for $|\alpha| < 1$ with homogeneous Dirichlet conditions.

If $|\alpha| > 1$, we replace $\mathcal{J}_0$ with $\mathcal{I}_0$ and $\frac{1}{1-\alpha^2}$ with $\frac{1}{\alpha^2-1}$ to obtain:

$$u(x, t) = \int_0^x f'(\tau) \mathcal{I}_0 \left(2\sqrt{\frac{(x-\tau)t}{\alpha^2-1}} \right) d\tau. \quad (3.25)$$

3.3. Well-Posedness of Model's Governing PDE. To show that the governing equation of the model:

$$u_{xt} + \frac{1}{1-\alpha^2} u = 0, \quad 0 < x < L, \quad t > 0, \quad (3.26)$$

with the boundary conditions:

$$u(0, t) = 0, \quad u(L, 0) = 0, \quad t > 0, \quad (3.27)$$

and initial condition:

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (3.28)$$

is well-posed, we must demonstrate:

- (1) Existence of solution.
- (2) Uniqueness of solution.
- (3) Continuity.

We assume $|\alpha| < 1$, so that $\beta = \frac{1}{1-\alpha^2} > 0$. The case $|\alpha| > 1$ is similar but with the modified Bessel function $\mathcal{I}_0$



3.3.1. *Existence of Solution.* Given the solution by Laplace transform:

$$u(x, t) = \int_0^x f'(\tau) \mathcal{J}_0 \left(2\sqrt{\frac{(x-\tau)t}{1-\alpha^2}} \right) d\tau. \quad (3.29)$$

This is an explicit formula for the solution. To verify that it satisfies the PDE and conditions:

- **PDE:** Differentiate under the integral sign (assuming f is sufficiently smooth) and use properties of $\mathcal{J}_0$.
- **Initial Condition:** $t \rightarrow 0^+$, $\mathcal{J}_0(2\sqrt{\beta(x-\tau)t}) \rightarrow \mathcal{J}_0(0) = 1$, so,

$$u(x, 0) = \int_0^x f'(\tau) \cdot 1 d\tau = f(x) - f(0). \quad (3.30)$$

But we require $u(x, 0) = f(x)$. To resolve this, note that the boundary condition $u(0, t) = 0$, implies $f(0) = 0$. Indeed, from the solution at $x = 0$:

$$u(x, 0) = \int_0^0 f'(\tau) \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) d\tau = 0, \quad (3.31)$$

and at $t = 0$:

$$u(x, 0) = f(x) - f(0). \quad (3.32)$$

So, we must have $f(0) = 0$ for consistency. Thus, if $f(0) = 0$, then $u(x, 0) = f(x)$.

- **Boundary conditions:**
 - $u(0, t) = 0$ (by the limits of integrations).
 - $u(L, t) = \int_0^L f'(\tau) \mathcal{J}_0(2\sqrt{\beta(x-\tau)t}) d\tau$. This is not necessarily zero, but we require $u(L, t) = 0$. This imposes a condition on f : so, we must have $f(L) = 0$ as well. Indeed, if $f(0) = 0$ and $f(L) = 0$, then the solution $u(L, t) = 0$.

Therefore, for existence, we require: $f(0) = 0$, $f(L) = 0$.

With these conditions, the explicit formula gives a solution that satisfies the PDE, boundary conditions, and initial condition.

3.3.2. *Uniqueness of Solution.* Assume two solutions u_1 and u_2 satisfy the PDE and conditions. Let $v = u_1 + u_2$. Then v satisfies:

$$v_{xt} + \beta v = 0, \quad (3.33)$$

with

$$v(0, t) = 0, \quad v(L, t) = 0, \quad v(x, 0) = 0. \quad (3.34)$$

We show that $v \equiv 0$. Let us consider the energy functional:

$$E(t) = \frac{1}{2} \int_0^L v_x(x, t)^2 dx. \quad (3.35)$$

Differentiating with respect to t :

$$\frac{dE}{dt} = \int_0^L v_x v_{xt} dx. \quad (3.36)$$

From the PDE, $v_{xt} = -\beta v$. So,

$$\frac{dE}{dt} = -\beta \int_0^L v_x v dx. \quad (3.37)$$

Integrating, we have:

$$\int_0^L v_x v dx = \left[\frac{1}{2} v^2 \right]_0^L - \int_0^L v v_x dx = 0 - \int_0^L v v_x dx. \quad (3.38)$$



So,

$$2 \int_0^L v_x v dx = 0 \Rightarrow \int_0^L v_x v dx = 0. \quad (3.39)$$

Thus,

$$\frac{dE}{dt} = 0. \quad (3.40)$$

So $E(t)$ is a constant. But at $t = 0$, $v(x, 0) = 0$, so $v_x(x, 0) = 0$, and hence $E(0) = 0$. Therefore, $E(t) = 0$ for all t , which implies $v_x(x, t) = 0$ for all x, t . Thus, v is a constant in x . But $v(0, t) = 0$, so $v(x, t) = 0$ for all x, t . Hence $u_1 = u_2$. This proves uniqueness.

3.3.3. Continuity of Solution. Suppose we have two initial conditions $f(x)$ and $\tilde{f}(x)$, with corresponding solutions u and $\tilde{u}$. Let $w = u - \tilde{u}$. Then w satisfies:

$$v_{xt} + \beta w = 0, \quad (3.41)$$

with,

$$w(0, t) = 0, \quad w(L, t) = 0, \quad w(x, 0) = f(x) - \tilde{f}(x) = \delta(x). \quad (3.42)$$

Applying the part-integration, we have $w(x, t)$ as:

$$w(x, t) = \int_0^x \delta'(\tau) \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) d\tau. \quad (3.43)$$

We want to show that if δ is small, then w is small. Now, assume f and $\tilde{f}$ are differentiable and $f(0) = \tilde{f}(0)$, $f(L) = \tilde{f}(L) = 0$. Then Equation (3.43) follows. Now, let:

$$U = \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right), \quad dV = \delta'(\tau) d\tau. \quad (3.44)$$

Then,

$$dU = \frac{\partial}{\partial \tau} \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) d\tau, \quad V = \delta(\tau). \quad (3.45)$$

So,

$$w(x, t) = \left[\delta(\tau) \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) \right]_0^x - \int_0^x \delta(\tau) \frac{\partial}{\partial \tau} \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) d\tau. \quad (3.46)$$

At $\tau = x$, $\mathcal{J}_0(0) = 1$, we have, $\delta(x) \cdot 1$; and at $\tau = 0$, $\mathcal{J}_0(2\sqrt{\beta xt})$, and $\delta(0) = 0$.

Thus,

$$w(x, t) = \delta(x) - \int_0^x \delta(\tau) \frac{\partial}{\partial \tau} \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) d\tau. \quad (3.47)$$

Now, $\mathcal{J}_0$ is bounded ($|\mathcal{J}_0(z)| \leq 1$) and its derivative is bounded ($|\mathcal{J}_0'(z)| \leq 1$) for real z .

So,

$$\left| \frac{\partial}{\partial \tau} \mathcal{J}_0 \left(2\sqrt{\beta(x-\tau)t} \right) \right| = \left| \mathcal{J}_0' \left(2\sqrt{\beta(x-\tau)t} \right) \cdot \frac{-\beta t}{\sqrt{\beta(x-\tau)t}} \right| \leq \frac{\beta \sqrt{t}}{\sqrt{x-\tau}}. \quad (3.48)$$

Therefore,

$$|w(x, t)| \leq |\delta(x)| + \int_0^x |\delta(\tau)| \frac{\beta \sqrt{t}}{\sqrt{x-\tau}} d\tau. \quad (3.49)$$

Let $\|\delta\|_\infty = \max_{0 \leq x \leq L} |\delta(x)|$. Then,

$$|w(x, t)| \leq \|\delta\|_\infty + \|\delta\|_\infty \beta \sqrt{t} \int_0^x \frac{1}{\sqrt{x-\tau}} d\tau. \quad (3.50)$$



But,

$$\int_0^x \frac{1}{\sqrt{x-\tau}} d\tau = 2\sqrt{x} \leq 2\sqrt{L}. \quad (3.51)$$

So,

$$|w(c, t)| \leq \|\delta\|_\infty (1 + 2\beta\sqrt{Lt}). \quad (3.52)$$

Thus, for any finite time $T > 0$, we have:

$$\max_{0 \leq x \leq L, 0 \leq t \leq T} |w(x, t)| \leq \|\delta\|_\infty (1 + 2\beta\sqrt{LT}). \quad (3.53)$$

This shows that if the initial data are close, then the solutions are close uniformly in x and t for $t \in [0, T]$. Hence, the solution depends continuously on the initial data. Which concludes the assertion that the governing equation of the model is well-posed.

4. NUMERICAL SOLUTION OF THE MODEL USING THE 4TH-ORDER RUNGE-KUTTA (RK4) METHOD

The Runge-Kutta methods are a family of numerical methods used to approximate solutions of ordinary differential equations (ODEs). They are particularly effective for solving initial-value problems where the value of the dependent variable is known at a specific point, and the goal is to find its value at other points [10]. These methods are often used when analytical solutions to ODEs are difficult or impossible to obtain.

So, for a given time step $\Delta x = h$ and current state y_j at time t_j , the RK4 method calculates the next state y_{j+1} using a weighted average of four slope estimates:

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \quad (4.1)$$

where the slope vectors, k_1, k_2, k_3, k_4 are obtained as follows:

$$\mathbf{k}_1 = hf(t_j, y_j). \quad (4.2)$$

$$\mathbf{k}_2 = hf\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}\right). \quad (4.3)$$

$$\mathbf{k}_3 = hf\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}\right). \quad (4.4)$$

$$\mathbf{k}_4 = hf(t_j + h, y_j + k_3). \quad (4.5)$$

Now, for the governing PDE of the model:

$$u_{xt} + \frac{1}{1-\alpha^2}u = 0, \quad 0 < x < L, \quad t > 0, \quad (4.6)$$

with boundary conditions: $u(0, t) = 0$, $u(L, t) = 0$, and initial condition: $u(x, 0) = f(x)$.

4.0.1. *Discretisation.* We discretise the spatial domain $[0, L]$ into N points with step size:

$$\Delta x = \frac{L}{N-1}, \quad (4.7)$$

where, $x_j = j\Delta x$, $j = 0, 1, 2, \dots, N-1$.

Similarly, discretising time with step size Δt :

$$t_n = n\Delta t, \quad n = 0, 1, 2, \dots$$

Let $u_j^n \approx u(x_j, t_n)$. Thus, substituting into the PDE:

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = -\beta u, \quad \text{where } \beta = \frac{1}{1-\alpha^2}, \quad (4.8)$$



which gives:

$$\frac{\partial u}{\partial t} = -\beta \int_0^x u(\eta, t) d\eta + g(t), \quad (4.9)$$

where $g(t)$ is an arbitrary function. From the boundary condition $u(0, t) = 0$, we have:

$$\left. \frac{\partial u}{\partial t} \right|_{x=0} = g(t). \quad (4.10)$$

But $u(0, t) = 0$ implies $\frac{\partial u}{\partial t}(0, t) = 0$, so $g(t) = 0$. Therefore,

$$\frac{\partial u}{\partial t} = -\beta \int_0^x u(\eta, t) d\eta. \quad (4.11)$$

It follows that to apply RK4, we need to discretise the integral. Thus, using the trapezoidal rule, we have:

$$\int_0^{x_i} u(\eta, t) d\eta \approx \Delta x \left(\frac{u_0(t)}{2} + \sum_{j=1}^{i-1} u_j(t) + \frac{u_i(t)}{2} \right). \quad (4.12)$$

Since $u_0(t) = 0$, this simplifies to:

$$I_i(t) = \Delta x \left(\sum_{j=1}^{i-1} u_j(t) + \frac{u_i(t)}{2} \right). \quad (4.13)$$

So that,

$$\frac{\partial u_i}{\partial t} = -\beta I_i(t). \quad (4.14)$$

Thus, for each spatial point, x_i (except $i = 0$ and $i = N - 1$ which are fixed by the boundary conditions), we have:

$$\frac{du_i}{dt} = F_i(u) = -\beta \Delta x \left(\sum_{j=1}^{i-1} u_j + \frac{u_i}{2} \right). \quad (4.15)$$

Now, if we let $U = [u_1, u_2, \dots, u_{N-2}]^T$ (since $u_0 = 0$ and $u_{N-1} = 0$ are fixed).

Then,

$$\frac{dU}{dt} = F(U), \quad (4.16)$$

where F is a vector function with components:

$$F_i(U) = -\beta \Delta x \left(\sum_{j=1}^{i-1} u_j + \frac{u_i}{2} \right), \quad i = 1, 2, 3, \dots, N - 2. \quad (4.17)$$

Given U^n at time t_n , we apply RK4:

$$k_1 = F(U^n), \quad k_2 = F\left(U^n + \frac{\Delta t}{2} k_1\right), \quad k_3 = F\left(U^n + \frac{\Delta t}{2} k_2\right), \quad k_4 = F(U^n + \Delta t k_3), \quad (4.18)$$

which yields,

$$U^{n+1} = U^n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4). \quad (4.19)$$



4.1. Implementation using Python. The numerical solution of the Ponzi scheme model using the Runge-Kutta 4th order method (RK4), is implemented with Python programming language. The initial condition is $u(x, 0) = f(x)$, with boundary condition: $u(0, t) = 0$, and $u(L, t) = 0$.

Parameters

- x_0 - principal,
- x - cashflow,
- α - constant (rate of withdrawal of fund),
- - time (months).

TABLE 1. $u(x, t)$ at $\alpha = 0.50, \beta = 1.333333$.

x	t=0.00	t=6.00	t=12.00	t=18.00	t=24.00
0	0	0	0	0	0
300	0.309017	0	0	0	0
600	0.587785	0	0	0	0
900	0.809017	0	0	0	0
1200	0.951057	0	0	0	0
1500	1	-0.000002	0	0	0
1800	0.951057	-0.00113	0	0	0
2100	0.809017	-0.000321	0	0	0
2400	0.587785	-0.000513	0	0	0
2700	0.309017	-0.001177	0	0	0
3000	0	0	0	0	0

TABLE 2. $u(x, t)$ at $\alpha = 0.98, \beta = 25.252525$.

x	t=0.00	t=6.00	t=12.00	t=18.00	t=24.00
0	0	0	0	0	0
300	0.309017	nan	nan	nan	nan
600	0.587785	nan	nan	nan	nan
900	0.809017	nan	nan	nan	nan
1200	0.951057	nan	nan	nan	nan
1500	1	nan	nan	nan	nan
1800	0.951057	nan	nan	nan	nan
2100	0.809017	nan	nan	nan	nan
2400	0.587785	nan	nan	nan	nan
2700	0.309017	nan	nan	nan	nan
3000	0	0	0	0	0

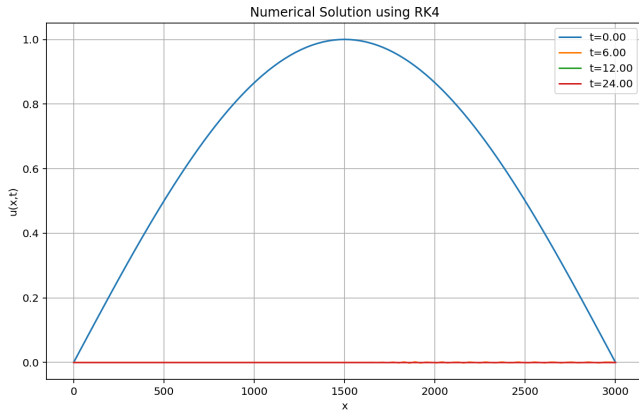


FIGURE 1. $u(x, t)$ of Table 1 when $0 \leq \alpha \leq 0.89$.

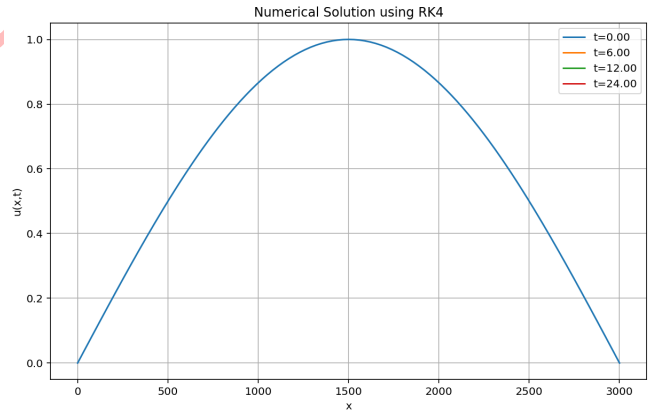


FIGURE 2. $u(x, t)$ of Table 2 when $0.97 \leq \alpha \leq 0.99$.

5. RESULTS AND DISCUSSION

The parameter α is identified as the rate of withdrawal of fund, which acts as a critical control parameter in the system. The equation itself,

$$u_{xt} + \beta u = 0, \quad \beta = \frac{1}{1 - \alpha^2}. \quad (5.1)$$



TABLE 3. $u(x,t)$ at $\alpha = 0.90, \beta = 5.263158$.

x	t=0.00	t=6.00	t=12.00	t=18.00	t=24.00
0	0	0	0	0	0
300	0.309017	0	0	0	0
600	0.587785	0	0	0	0
900	0.809017	0	0	0	0
1200	0.951057	0	0	0	0
1500	1	0	0	0	0
1800	0.951057	0	0	0	0
2100	0.809017	0	0	0	0
2400	0.587785	0	0	0	0
2700	0.309017	1162.694	0	0	0
3000	0	0	0	0	0

TABLE 4. $u(x,t)$ at $\alpha = 0.95, \beta = 10.256410$.

x	t=0.00	t=6.00	t=12.00	t=18.00	t=24.00
0	0	0	0	0	0
300	0.309017	3.13E+158	36552044697	nan	nan
600	0.587785	3.34E+180	nan	nan	nan
900	0.809017	1.65E+200	nan	nan	nan
1200	0.951057	2.58E+218	nan	nan	nan
1500	1	3.13E+235	nan	nan	nan
1800	0.951057	49738968361	nan	nan	nan
2100	0.809017	14670685600	nan	nan	nan
2400	0.587785	10245366831	nan	nan	nan
2700	0.309017	20335412338	nan	nan	nan
3000	0	0	0	0	0

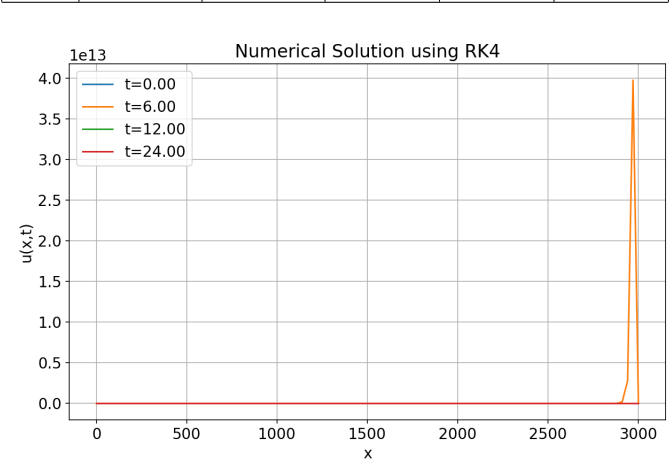


FIGURE 3. $u(x,t)$ of Table 3 when $\alpha = 0.90$.

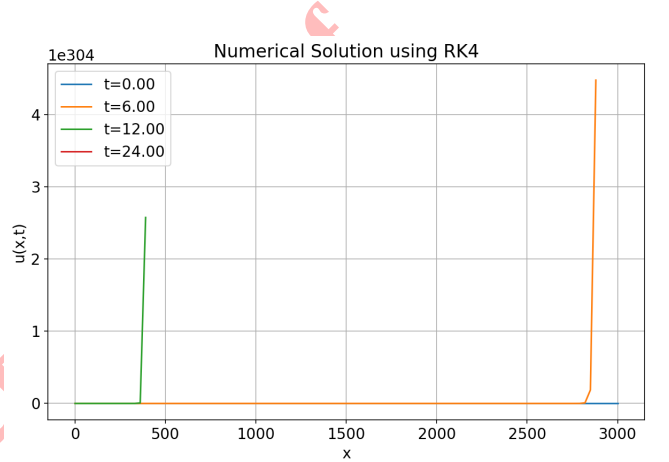


FIGURE 4. $u(x,t)$ of Table 4 when $\alpha = 0.95$.

TABLE 5. $u(x,t)$ at $\alpha = 0.96, \beta = 12.755102$.

x	t=0.00	t=6.00	t=12.00	t=18.00	t=24.00
0	0	0	0	0	0
300	0.309017	489878654712	nan	nan	nan
600	0.587785	506572312503	nan	nan	nan
900	0.809017	nan	nan	nan	nan
1200	0.951057	nan	nan	nan	nan
1500	1	nan	nan	nan	nan
1800	0.951057	nan	nan	nan	nan
2100	0.809017	nan	nan	nan	nan
2400	0.587785	nan	nan	nan	nan
2700	0.309017	nan	nan	nan	nan
3000	0	0	0	0	0

is a hyperbolic PDE. The sign and magnitude of the coefficient β are determined by α , which dramatically influences the solution's behaviour. The parameter, α can equally be referred to as the rate of growth of the Ponzi scheme fund

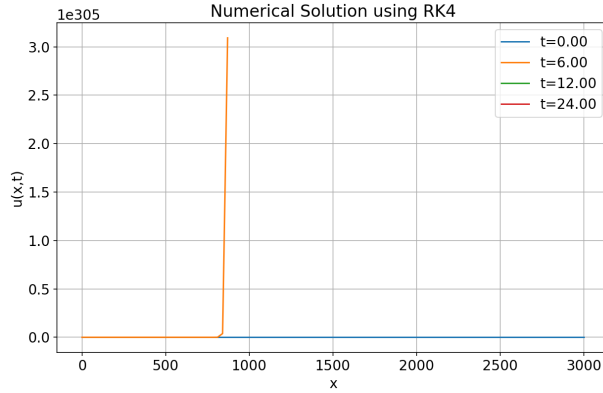


FIGURE 5. $u(x,t)$ of Table 5 when $\alpha = 0.96$

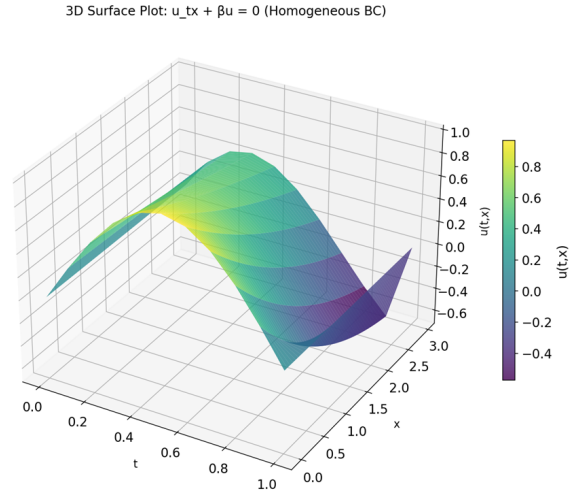


FIGURE 6. 3D surface plot of Table 6 at $\alpha = 0.5$.

TABLE 6. $u(x,t)$ at $\alpha = 0.90, \beta = 5.263158$.

	x=0.00	x=0.6	x=1.20	x=1.80	x=2.40	x=3.00
t=0.000	0	0	0	0	0	0
t=0.240	0.68454711	0.62908849	0.576587368	0.526925165	0.479987158	0.436093016
t=0.500	1	0.8020415	0.627237668	0.473621626	0.339358735	0.223823387
t=0.760	0.68454711	0.35702972	0.095001221	-0.11042645	-0.26721047	-0.381488877
t=1.000	0	0	0	0	0	0

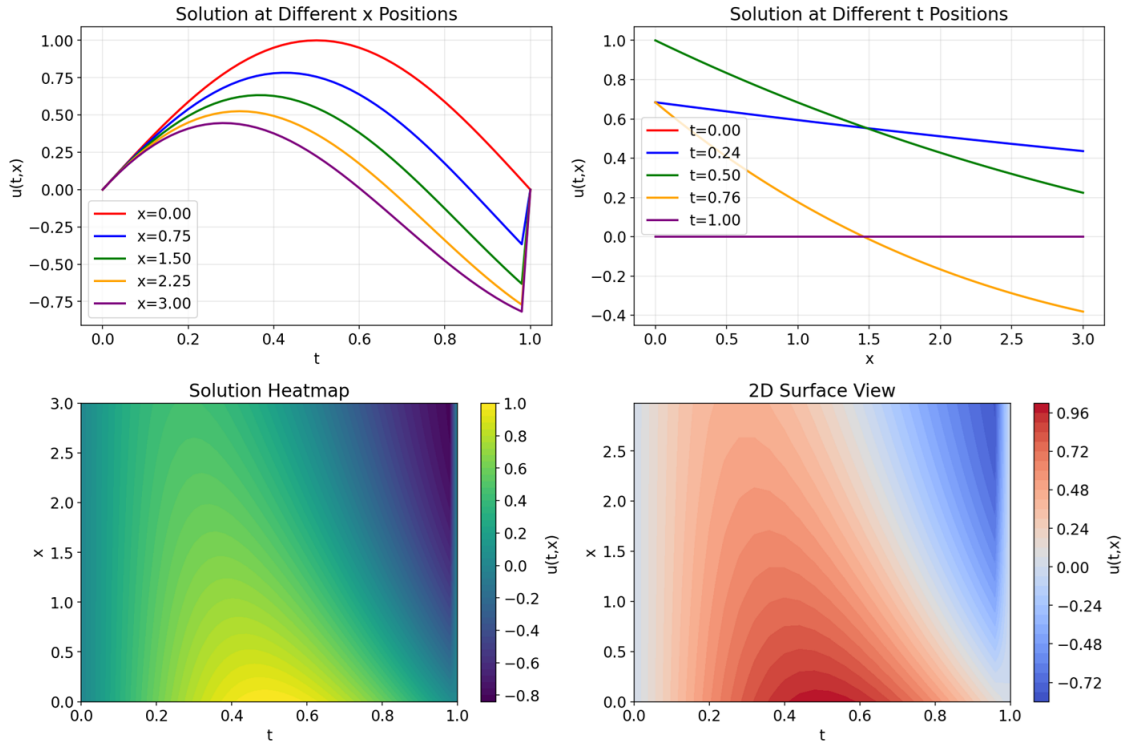
TABLE 7. $u(t,x)$ at sampled values, $\alpha = 0.75, \beta = 2.369281$.

	x=0.00	x=0.6	x=1.20	x=1.80	x=2.40	x=3.00
t=0.000	0	0	0	0	0	0
t=0.240	0.684547106	0.629088494	0.576587368	0.526925165	0.479987158	0.436093016
t=0.500	1	0.802041502	0.627237668	0.473621626	0.339358735	0.223823387

at any time, it describes the rate of withdrawals taking place for matured investments visa vis the rate of recruitment of new investors and re-investment by existing investors. The graphs are interpreted below:

(1) For Figure 1, $0 \leq \alpha \leq 0.89$ and Figures 6 ($\alpha = 0.5$) and 7 ($\alpha = 0.75$):

- Stable oscillatory phase ($\alpha < 0.9$): The coefficient β is positive. This positivity leads to oscillatory or wave-like behaviour of the rate of withdrawal of fund in the solution.
- On Figure 1, the 2D plots show $u(x)$, cashflow at a specific time and $u(t)$ at a specific point x which captures damped oscillations. The solution starts from the initial condition $f(x)$ and oscillates while decaying in amplitude over time due to the dissipative nature of the term u_{xt} combined with the restoring force of βu . The boundary conditions force the solution to be zero at both ends as α is restricted to $\alpha < 1$.
- 3D Surface plot (Figures 6 and 7): It is observed that the surface resembles a damped wave. The initial bump from $f(x)$ oscillates with the waves reflecting at the boundaries $x = 0$ and $x = L$ and interfering with each other. The amplitude of the oscillations decreases as time t increases, which indicates that after the critical point of x , the rate of withdrawals is gradually overshadowing the rate of recruitment

FIGURE 7. 3D surface plot of Table 7 at $\alpha = 0.75$

of investments leading eventually to the collapse of the scheme. This represents a stable, well-behaved system where perturbations settle down.

(2) Figures 3, 4, and 5: The transition zone ($\alpha = 0.90, 0.95, 0.96$):

- Transition to stiffness ($0.9 \leq \alpha \leq 0.96$): The system becomes stiff, exhibiting high-frequency oscillations and posing challenges for numerical simulation.
- As $\alpha \rightarrow 1$ from below, $\alpha^2 \rightarrow 1$, making the denominator $(1 - \alpha^2) \rightarrow 0$ from the positive side. This means that the coefficient β becomes very large and positive.
- This is a critical transition region. The system's behaviour becomes more extreme:
 - Amplified oscillations: The restoring force βu becomes immense. This leads to oscillations with significantly higher frequencies and much slower decay. The solution appears as very rapid, indicating sharp oscillations in space (cashflow) and/or time.
 - Numerical stiffness: Observe that the regime is stiff and the RK4 method might require an extremely small-time step or refined meshes to remain stable and accurate, otherwise, it might result to spurious numerical oscillations or instabilities in the plots which suggests that the solution's character changes noticeably in the narrow parameter range.

(3) Figure 2: Near-singular phase ($0.97 \leq \alpha \leq 0.99$):

- This is the ultra-critical region, closest to $\alpha = 1$. Here, β is an extremely large positive number.
- The behaviour observed in Figures 3–5 is now extreme and may lead to a breakdown of the model $\alpha = 1$:
 - Here, we observed solutions dominated by very high-frequency, low-amplitude oscillations almost immediately. The initial condition $f(x)$ seem to shatter into rapid oscillations.
 - In addition, we observed that the system is highly sensitive. The immense coefficient β means any small value of u produces a very strong response in the u_{xt} term. This indicates that the model is approaching a resonance or a singularity at $\alpha = 1$, where the solution ceases to be well-defined

(blow-up), which implies the collapse of the scheme as more and more old investors cannot withdraw their investments plus returns or interest at maturity.

- It is equally observed that the RK4 method struggles here, and the graph in this range shows numerical artifacts or computational errors which indicates extreme behaviour of the model.

6. CONCLUSIONS AND RECOMMENDATIONS

It has been observed that the parameter α acts as a classic bifurcation parameter. The scheme undergoes a qualitative change in behaviour as the rate of withdrawal, α increases. The entire scheme is discovered to have three distinct phases. The first phase, stable oscillatory ($0 \leq \alpha \leq 0.89$), has a solution that behaves like a damped wave, stable and well-controlled. The second phase, transition to stiffness ($0.9 \leq \alpha \leq 0.96$), portrays a system that becomes stiff, thus exhibiting high-frequency oscillations and posing challenges for numerical simulation. The last phase, near-singular ($\alpha \leq 0.97$) system, is driven by an enormous coefficient, leading to extreme behaviour that leads to a breakdown of the model at $\alpha = 1$.

The financial interpretation of the model encapsulates $u(x, t)$ as the cashflow of the Ponzi scheme, α as the rate at which investors withdraw their capital plus interest at a given time, t (usually months) of maturity of their investment. The influence of the α is very key in the system as the solution starts from the initial condition $f(x)$ and oscillates while decaying in amplitude over time due to the dissipative nature of the term u_{xt} combined with the restoring force of βu . A low withdrawal rate ($\alpha \leq 0.89$) leads to predictable, stable behaviour, but the scheme collapses over a medium to long term as investors re-invest their capital and new investors are being added to the scheme i.e., net cashflow is positive and growing until it reaches the critical point where decline sets-in, resulting to negative cashflow. Alternatively, high withdrawal rate ($\alpha \leq 0.9$), leads to highly volatile and unpredictable behaviour. The system becomes unstable and difficult to manage, potentially crashing over a short time as the scheme reaches critical point over a short period, and net cashflow decelerates sharply leading to the eventual crash of the scheme. Another obvious characteristic is the critical point - this point represents the extrema or turning point of the curve $u(x, t)$. At the initial stage, the net present value (NPV) of the cashflow will continue to increase as more investors are recruited and in addition to re-investment of matured investment. However, at critical point, the NPV turns negative as more investors are withdrawing their capital than the addition of capital from new recruitment. So, for investors who have risk appetite and wish to invest in a Ponzi scheme, the ability to decipher the critical point will give the investor the best time to pull out of the scheme before the NPV starts to decelerate and eventually crashes.

Future research should consider the performance stability analysis on the numerical scheme to understand the time step requirements as $\alpha \rightarrow 1$. In addition, to demonstrate the real-life application of the model, the return of the principal, i and the interest, r payable to investors should be captured and incorporated into the scheme.

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Uncorrected Proof

