



An appropriate numerical method for fractional hematopoietic stem cells model

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Abstract

In this paper we introduce a numerical method for solving fractional hematopoietic stem cells model (FHSCM). The main idea is to convert FHSCM to a system of Volterra integral equation of weakly singular type, to which a well known block-by-block method is applied. A convergence analysis is provided and since for considered FHSCM, we do not have information about the exact solution, we provide the estimated error. Finally, numerical simulations are given to confirm application.

Keywords. Caputo fractional derivative, Hematopoietic stem cells, Block by block method, Error estimation.

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1. INTRODUCTION

Fractional calculus is a branch of mathematical analysis which deals with differential and integral operators of arbitrary (non-integer) real or complex order. One can find a detailed description of the subject in [15, 21]. The fractional differential equations (FDEs) are a form of differential equations in which the derivatives are in fractional rather than integer order. The study of FDEs has gained considerable attention in various fields of science and engineering due to their capability to describe complex phenomena with long-range interactions, memory effects and anomalous diffusion. These equations have applications in physics, biology, engineering, finance, and more, where standard integer order differential equations may not capture the underlying dynamics accurately. The modeling of fractional order systems enables a more accurate representation of real-world phenomena and aids in developing simulation and prediction algorithms.

In many cases, finding an analytical solution to the complicated FDEs arising in different models is almost impossible. Therefore, many researchers seek numerical methods for solving such equations. For example, Mukhtar et al. [19] presented intelligent supervised neural networks with back propagation using Bayesian regularization optimization algorithm, for studying the dynamics of fractional order Parkinson's disease model. Manohara and Kumbinarasaiah [16] considered the epidemic model of measles and the smoking model as well. They employed the Fibonacci wavelets operational matrix of integration and developed the Fibonacci wavelet collocation method to solve a nonlinear coupled system of fractional differential equations obtained from epidemic measles and smoking models. Farman et al. [9] developed a fractional-order dynamic model of Methicillin-Resistant Staphylococcus aureus infection and presented the existence, uniqueness, and generalized Ulama-Hyers-Rassias stability for this model. Ravi Kanth and Devi [24] described and analyzed numerical simulations based on the Adams-Bashforth approach for fractal-fractional order autoimmune disease framework, to explore the function of viruses in the progression of autoimmune disease. El-Mesady et al. [8] presented a Caputo fractional nonlinear model for simulating the spread of a novel Lumpy Skin Disease virus with the interaction between the cattle and vectors' populations. They investigated the stability and the equilibrium points for the models (both free state and endemic) to understand the influence of some of the key parameters on

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the spread of the virus. Avazzadeh et al. [3] presented a fractional-order mathematical model for fascioliasis disease. They used a set of generalized Fibonacci polynomials as basic functions and some innovative operational matrices of derivatives in the Caputo sense to solve the fascioliasis disease model. Hassani et al. [10] constructed an optimization technique supported by a class of basis functions, consisting of the generalized shifted Jacobi polynomials for solving the fractional HIV/AIDS epidemic model. Din et al. [7] formulated a co-infection model for the transmission dynamics of two virus hepatitis B and coronavirus to study the effect of white noise intensities. They presented the model equilibria and basic reproduction number and proved the local stability of the equilibria points. Cui et al. [5] proposed a fractional epidemic model of coronavirus with vaccination effects. They composed the fractional dynamics of the interaction among the human population and the external environmental factors of infected people. Naik et al. [20] investigated the dynamics of a developed co-infection model of HIV and Hepatitis C virus. They employed the operational matrices to calculate the basic reproduction number for the cases of Human immunodeficiency virus and Hepatitis C virus. Rajan and Kuppusamy [23] presented a fractional order Human Papillomavirus infection model with the impact of vaccination in the Caputo sense. They proved the existence and uniqueness of the solution using the fixed point theorem and applied the Adams-Bashforth-Moulton for solving this model. Kumar et al. [14] applied the Genocchi wavelets for numerical simulations of the fractional order SEIR epidemic of measles. Amit Prakash [2] used the fractional Adams-Bashforth method to solve the childhood disease model with non-local operator.

In this paper, we propose an appropriate numerical method for FHSCM by reducing it to a system of equivalent Volterra integral equations. The mathematical model of hematopoietic stem cells model (HSCM) with integer orders introduced by Korpusik and Kolev in [13] and Mohammad et al. [18] presented the HSCM fractional model based on the Atangana-Baleanu fractional derivative. We consider the Caputo fractional model of HSCM and convert it to a system of Volterra integral equation of weakly singular type, to which a well known block-by-block method is applied. This method was first introduced by Young [26] in application to the product integration technique and has the advantages of being self-starting, performance solving problems on large intervals, flexible in implementation, high convergence order and producing a block of values for the unknown function simultaneously [11, 12].

This paper is organized as follows: after introducing the problem in Section 2 we state some basic concepts about fractional calculus in Section 3. The description of the proposed numerical method is given in Section 4. Convergence of the method is then demonstrated in Section 5 and the order of convergence is obtained. The error estimator is discussed in Section 6, since for considered model, we do not have information about the exact solution of this problem. In Section 7, some numerical simulations are proposed to confirm the effectiveness of the method. Finally, Section 8 offers concluding remarks.

2. PROBLEM STATEMENT

The primary objective of this section is to introduce the mathematical model of FHSCM, which will be the focus of our study throughout the paper.

Problem: The mathematical model of HSCM with integer orders is subdivided into five categories: i) concentration of uninfected cytotoxic T lymphocytes (CTL) cells denoted as $X(t)$, ii) concentration of infected CTL cells denoted as $Y(t)$, iii) concentration of precursor CTL cells denoted as $W(t)$, iv) concentrations of effector T lymphocytes cells denoted as $Z(t)$, and v) concentration of CTL cells produced by HSC denoted as $H(t)$ [13]. The integer-order HSCM is represented by a nonlinear system of differential equations as follows [13]:

$$\begin{cases} \frac{dX(t)}{dt} = \eta - lX(t) - \delta X(t)Y(t), \\ \frac{dY(t)}{dt} = \delta X(t)Y(t) - aY(t) - bY(t)Z(t), \\ \frac{dW(t)}{dt} = \omega vH(t) + cW(t)X(t)Y(t) - qW(t)Y(t) - bW(t), \\ \frac{dZ(t)}{dt} = qW(t)Y(t) - d_1Z(t), \\ \frac{dH(t)}{dt} = -vH(t), \\ X(t_0) = X_0, \quad Y(t_0) = Y_0, \quad W(t_0) = W_0, \quad Z(t_0) = Z_0, \quad H(t_0) = H_0. \end{cases} \quad (2.1)$$

Here, it is important to note that a comprehensive nomenclature for the model (2.1) is detailed in Table 1. To achieve



TABLE 1. Explanation of the parameters mentioned in (2.1).

Parameters	The physical explanation	Value
η	The rate at which uninfected cells are produced	0.20
l	The rate at which uninfected CTL cells die	0.50
δ	Infection rate	0.15
a	The rate at which infected cells die	0.80
b	Effector CTL rate of the infected cells	0.016
ω	Fractions of HSCs	0.0005
v	Rate of precursor CTL cells	0.005
c	Fraction of precursor CTL cells	0.10
q	The molarity rate of precursor CTL cells	0.10
d_1	The molarity rate of effector CTL cells	0.50

a more accurate approximation of HSCM (2.1), we substitute ordinary time derivatives $\frac{d}{dt}$ with Caputo ${}_0^C D_t^\zeta$ fractional derivatives. Following an assessment of the memory effect within our methodology, we introduce an alternative HSCM (2.1) by transitioning to Caputo fractional time derivatives. In our modeling, $\frac{d}{dt}$ and $\frac{d^{\zeta_i}}{dt^{\zeta_i}}$ have units of $(\text{time})^{-1}$ and $(\text{time})^{-\zeta_i}$ respectively, where $i = 1, 2, 3, 4, 5$, $0 < \zeta_i < 1$, and ν is considered a parameter with the dimension of time. Consequently, the unit of $\left(\frac{1}{\nu^{1-\zeta_i}} \frac{d^{\zeta_i}}{dt^{\zeta_i}}\right)$, where $i = 1, 2, 3, 4, 5$, will be $(\text{time})^{-1}$ [22]. The fractional form of Eq. (2.1) is subsequently defined as follows [1]:

$$\begin{cases} \frac{1}{\nu^{1-\zeta_1}} D_t^{\zeta_1} X(t) = \eta - lX(t) - \delta X(t)Y(t), \\ \frac{1}{\nu^{1-\zeta_2}} D_t^{\zeta_2} Y(t) = \delta X(t)Y(t) - aY(t) - bY(t)Z(t), \\ \frac{1}{\nu^{1-\zeta_3}} D_t^{\zeta_3} W(t) = \omega v H(t) + cW(t)X(t)Y(t) - qW(t)Y(t) - bW(t), \\ \frac{1}{\nu^{1-\zeta_4}} D_t^{\zeta_4} Z(t) = qW(t)Y(t) - d_1 Z(t), \\ \frac{1}{\nu^{1-\zeta_5}} D_t^{\zeta_5} H(t) = -vH(t), \\ X(t_0) = X_0, \quad Y(t_0) = Y_0, \quad W(t_0) = W_0, \quad Z(t_0) = Z_0, \quad H(t_0) = H_0. \end{cases} \quad (2.2)$$

3. DEFINITIONS AND PRELIMINARIES

In this section, we recall the main definitions and outcomes of fractional calculus that will be used throughout the following sections.

Definition 3.1. ([21]) The operator J_a^α of order $\alpha \geq 0$ of the function y is given by

$$J_a^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t \frac{y(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0, \\ y(t), & \alpha = 0, \end{cases}$$

and called the Riemann-Liouville fractional integral operator of order α .

Definition 3.2. ([21]) The Caputo fractional derivative of order $n - 1 < \alpha \leq n$ of the sufficiently differentiable function y is defined as

$${}_a^C D_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds, & n - 1 < \alpha < n, \\ y^{(n)}(t), & \alpha = n. \end{cases}$$



Proposition 3.3. ([21]) *Based on the previous definitions, we have:*

$${}_0^C D_t^\alpha (J_0^\alpha y(t)) = y(t), \quad (3.1)$$

$$J_0^\alpha ({}_0^C D_t^\alpha y(t)) = y(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} y^{(i)}(0). \quad (3.2)$$

4. DESCRIPTION OF THE NUMERICAL METHOD

In this section, we introduce a numerical method for FHSCM.

4.1. Conversion of fractional hematopoietic stem cells model to a system of weakly singular Volterra integral equation. The FHSCM (2.2) can be rewritten as

$$\begin{aligned} {}_0^C D_t^{\zeta_1} X(t) &= \nu^{1-\zeta_1} (\eta - lX(t) - \delta X(t)Y(t)), \\ {}_0^C D_t^{\zeta_2} Y(t) &= \nu^{1-\zeta_2} (\delta X(t)Y(t) - aY(t) - bY(t)Z(t)), \\ {}_0^C D_t^{\zeta_3} W(t) &= \nu^{1-\zeta_3} (\omega vH(t) + cW(t)X(t)Y(t) - qW(t)Y(t) - bW(t)), \\ {}_0^C D_t^{\zeta_4} Z(t) &= \nu^{1-\zeta_4} (qW(t)Y(t) - d_1 Z(t)), \\ {}_0^C D_t^{\zeta_5} H(t) &= -\nu^{1-\zeta_5} vH(t), \\ X(t_0) &= X_0, Y(t_0) = Y_0, W(t_0) = W_0, Z(t_0) = Z_0, H(t_0) = H_0. \end{aligned} \quad (4.1)$$

We apply the operator $J_0^{\zeta_i}$ ($i = 1, 2, \dots, 5$) to both sides of Equations (4.1), and using Proposition 3.3, we obtain the equivalent system

$$\begin{aligned} X(t) &= X_0 + \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX(s) - \delta X(s)Y(s)) ds, \\ Y(t) &= Y_0 + \frac{1}{\Gamma(\zeta_2)} \int_0^t (t-s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X(s)Y(s) - aY(s) - bY(s)Z(s)) ds, \\ W(t) &= W_0 + \frac{1}{\Gamma(\zeta_3)} \int_0^t (t-s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega vH(s) + cW(s)X(s)Y(s) - qW(s)Y(s) - bW(s)) ds, \\ Z(t) &= Z_0 + \frac{1}{\Gamma(\zeta_4)} \int_0^t (t-s)^{\zeta_4-1} \nu^{1-\zeta_4} (qW(s)Y(s) - d_1 Z(s)) ds, \\ H(t) &= H_0 - \frac{1}{\Gamma(\zeta_5)} \int_0^t (t-s)^{\zeta_5-1} \nu^{1-\zeta_5} vH(s) ds, \end{aligned} \quad (4.2)$$

where $t \in [0, T]$.

4.2. The proposed numerical method. Let the system (4.2) has a smooth solution (for non-smooth solution we can use the smoothing transformation, for more details see [11]) and for a given number $N \in \mathbb{N}$, we consider a uniform partition of the interval $[0, T]$ with the step size $h = \frac{T}{N}$ in which $t_i = ih$, $i = 0, 1, \dots, N$. For given real numbers c_j with $0 = c_0 < c_1 < \dots < c_p = 1$, define the set $\Pi_n := \{t_{n,j}\}$ of mesh points by $t_{n,j} := t_n + c_j h$, $j = 0, 1, \dots, p$, $n = 0, 1, \dots, N-1$ where $c_j = \frac{j}{p}$. For the grid point $t = t_{n,j}$, from (4.2) we have

$$\begin{aligned} X(t_{n,j}) &= X_0 + \frac{1}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX(s) - \delta X(s)Y(s)) ds \\ &\quad + \frac{1}{\Gamma(\zeta_1)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX(s) - \delta X(s)Y(s)) ds, \\ Y(t_{n,j}) &= Y_0 + \frac{1}{\Gamma(\zeta_2)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X(s)Y(s) - aY(s) - bY(s)Z(s)) ds, \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{\Gamma(\zeta_2)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X(s)Y(s) - aY(s) - bY(s)Z(s)) ds, \\
W(t_{n,j}) &= W_0 + \frac{1}{\Gamma(\zeta_3)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v H(s) + cW(s)X(s)Y(s) - qW(s)Y(s) - bW(s)) ds, \\
& + \frac{1}{\Gamma(\zeta_3)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v H(s) + cW(s)X(s)Y(s) - qW(s)Y(s) - bW(s)) ds, \\
Z(t_{n,j}) &= Z_0 + \frac{1}{\Gamma(\zeta_4)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_4-1} \nu^{1-\zeta_4} (qW(s)Y(s) - d_1 Z(s)) ds, \\
& + \frac{1}{\Gamma(\zeta_4)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_4-1} \nu^{1-\zeta_4} (qW(s)Y(s) - d_1 Z(s)) ds, \\
H(t_{n,j}) &= H_0 - \frac{1}{\Gamma(\zeta_5)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_5-1} \nu^{1-\zeta_5} v H(s) ds - \frac{1}{\Gamma(\zeta_5)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_5-1} \nu^{1-\zeta_5} v H(s) ds, \\
& j = 0, 1, \dots, p, \quad n = 0, 1, \dots, N-1.
\end{aligned} \tag{4.3}$$

In the following, we describe a method with at least five order of convergence, then let $p = 4$ and consider the notations $X_{n,j} \simeq X(t_{n,j})$, $Y_{n,j} \simeq Y(t_{n,j})$, ..., $H_{n,j} \simeq H(t_{n,j})$ for the approximation values of the unknown functions. We approximate the integrals in the intervals $[t_i, t_{i+1}]$ by using the two-step Romberg quadrature rule (which is equivalent to Boole's rule)

$$\int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta-1} \phi(s) ds \simeq h \sum_{k=0}^4 w_k (t_{n,j} - t_{i,k})^{\zeta-1} \phi_{i,k}, \tag{4.4}$$

where $w_0 = w_4 = 7/90$, $w_2 = 2/15$, $w_1 = w_3 = 16/45$. This quadrature rule cannot be used for the integrals in the interval $[t_n, t_{n,j}]$, then we use product integration method

$$\begin{aligned}
\int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta-1} \phi(s) ds &\simeq \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta-1} \sum_{k=0}^4 L_k(s) \phi_{n,k} ds \\
&= \sum_{k=0}^4 \phi_{n,k} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta-1} L_k(s) ds,
\end{aligned} \tag{4.5}$$

where $L_k(s) = \prod_{q=0, q \neq k}^4 \frac{s - t_{n,q}}{t_{n,k} - t_{n,q}}$ and the last integral can be calculated exactly.

Substituting the approximations (4.4) and (4.5) into equation (4.3) yields

$$\begin{aligned}
X_{n,j} &= X_0 + \frac{1}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k (t_{n,j} - t_{i,k})^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX_{i,k} - \delta X_{i,k} Y_{i,k}) \\
& + \frac{1}{\Gamma(\zeta_1)} \sum_{k=0}^4 \nu^{1-\zeta_1} (\eta - lX_{n,k} - \delta X_{n,k} Y_{n,k}) \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds, \\
Y_{n,j} &= Y_0 + \frac{1}{\Gamma(\zeta_2)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k (t_{n,j} - t_{i,k})^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X_{i,k} Y_{i,k} - aY_{i,k} - bY_{i,k} Z_{i,k}) \\
& + \frac{1}{\Gamma(\zeta_2)} \sum_{k=0}^4 \nu^{1-\zeta_2} (\delta X_{n,k} Y_{n,k} - aY_{n,k} - bY_{n,k} Z_{n,k}) \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_2-1} L_k(s) ds, \\
W_{n,j} &= W_0 + \frac{1}{\Gamma(\zeta_3)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k (t_{n,j} - t_{i,k})^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v H_{i,k} + cW_{i,k} X_{i,k} Y_{i,k} - qW_{i,k} Y_{i,k} - bW_{i,k}) \\
& + \frac{1}{\Gamma(\zeta_3)} \sum_{k=0}^4 \nu^{1-\zeta_3} (\omega v H_{n,k} + cW_{n,k} X_{n,k} Y_{n,k} - qW_{n,k} Y_{n,k} - bW_{n,k}) \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_3-1} L_k(s) ds,
\end{aligned}$$



$$\begin{aligned}
Z_{n,j} &= Z_0 + \frac{1}{\Gamma(\zeta_4)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_4-1} \nu^{1-\zeta_4} (qW_{i,k}Y_{i,k} - d_1Z_{i,k}) \\
&\quad + \frac{1}{\Gamma(\zeta_4)} \sum_{k=0}^4 \nu^{1-\zeta_4} (qW_{n,k}Y_{n,k} - d_1Z_{n,k}) \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_4-1} L_k(s) ds, \\
H_{n,j} &= H_0 - \frac{1}{\Gamma(\zeta_5)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_5-1} \nu^{1-\zeta_5} vH_{i,k} - \frac{1}{\Gamma(\zeta_5)} \sum_{k=0}^4 \nu^{1-\zeta_5} vH_{n,k} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_5-1} L_k(s) ds, \\
j &= 0, 1, \dots, 4, \quad n = 0, 1, \dots, N-1.
\end{aligned} \tag{4.6}$$

Finally, in each step for $n = 0, 1, \dots, N-1$, we obtain the unknown values of functions $X(t), Y(t), \dots, H(t)$ at the mesh points by solving the algebraic system (4.6).

5. CONVERGENCE ANALYSIS

We first recall a generalized discrete Gronwall lemma form [17].

Lemma 5.1. [17] *If x_j , $j = 0, 1, \dots, N$, is a sequence of real numbers with*

$$|x_0| \leq \delta, \quad |x_i| \leq h^{1-\epsilon} M \sum_{j=0}^{i-1} \frac{1}{(i-j)^\epsilon} |x_j| + \delta, \quad i = 1, 2, \dots, N,$$

where $M > 0$ and independent of h , $\delta > 0$ and $0 \leq \epsilon < 1$ (independent of h) then

$$\|x\|_\infty \leq (\delta' + hM'T^{n-1-n\epsilon}\delta) \exp(M'T^{n-n\epsilon}),$$

where

$$M' = M, \quad n = 1,$$

$$M' = M^n \prod_{k=1}^{n-1} B(k(1-\epsilon), 1-\epsilon), \quad n \geq 2,$$

$$\delta' = \delta \left\{ (1 + h^{1-\epsilon} M) \sum_{j=0}^{n-2} \gamma^j + \gamma^{n-1} \right\}, \quad n \geq 1,$$

with $\gamma = \frac{MT^{1-\epsilon}}{1-\epsilon}$, $T = Nh$, Beta function B and n is the smallest positive integer such that $\epsilon \leq (n-1)/n$.

Theorem 5.2. *The approximation method provided by the system (4.6) is convergent and the order of convergence is at least $O(h^5)$.*

Proof. Let $EX_{n,j} = X(t_{n,j}) - X_{n,j}$ be the error of concentration of uninfected CTL cells, so by subtracting (4.6) from (4.3) we have

$$\begin{aligned}
EX_{n,j} &= \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} \eta ds - \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} (l + \delta Y(s)) X(s) ds \\
&\quad + \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} \eta ds - \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} (l + \delta Y(s)) X(s) ds \\
&\quad - \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} \eta + \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} (l + \delta Y_{i,k}) X_{i,k} \\
&\quad - \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{k=0}^4 \eta \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds + \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{k=0}^4 (l + \delta Y_{n,k}) X_{n,k} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds,
\end{aligned}$$



Next, by subtracting and adding

$$\begin{aligned} & \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} (l + \delta Y(t_{i,k})) X(t_{i,k}), \\ & \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \sum_{k=0}^4 (l + \delta Y(t_{n,k})) X(t_{n,k}) \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds, \end{aligned}$$

we get

$$\begin{aligned} EX_{n,j} = & \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(\sum_{i=0}^{n-1} \eta \left(\int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} ds - h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} \right) \right. \\ & - \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} (l + \delta Y(s)) X(s) ds - h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} (l + \delta Y(t_{i,k})) X(t_{i,k}) \right) \\ & - \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} [(l + \delta Y(t_{i,k})) X(t_{i,k}) - (l + \delta Y_{i,k}) X_{i,k}] \\ & + \eta \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} \left(1 - \sum_{k=0}^4 L_k(s) \right) ds \\ & - \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} [(l + \delta Y(s)) X(s) - \sum_{k=0}^4 (l + \delta Y(t_{n,k})) X(t_{n,k}) L_k(s)] ds \\ & \left. - \sum_{k=0}^4 [(l + \delta Y(t_{n,k})) X(t_{n,k}) - (l + \delta Y_{n,k}) X_{n,k}] \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds \right). \end{aligned}$$

Let numerical integration errors as

$$\begin{aligned} R_i &:= \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} ds - h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1}, \\ r_i &:= \int_{t_i}^{t_{i+1}} (t_{n,j} - s)^{\zeta_1-1} (l + \delta Y(s)) X(s) ds - h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} (l + \delta Y(t_{i,k})) X(t_{i,k}), \end{aligned}$$

therefore we will have

$$\begin{aligned} EX_{n,j} = & \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(\sum_{i=0}^{n-1} \eta R_i - \sum_{i=0}^{n-1} r_i \right. \\ & - \sum_{i=0}^{n-1} h \sum_{k=0}^4 w_k(t_{n,j} - t_{i,k})^{\zeta_1-1} (l(X(t_{i,k}) - X_{i,k}) + \delta Y(t_{i,k})X(t_{i,k}) \pm \delta Y(t_{i,k})X_{i,k} - \delta Y_{i,k}X_{i,k}) \\ & - \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} I_n(s) ds \\ & \left. - \sum_{k=0}^4 [l(X(t_{n,k}) - X_{n,k}) + \delta Y(t_{n,k})X(t_{n,k}) \pm \delta Y(t_{n,k})X_{n,k} - \delta Y_{n,k}X_{n,k}] \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds \right), \end{aligned}$$

where $I_n := (l + \delta Y(s)) X(s) - \sum_{k=0}^4 (l + \delta Y(t_{n,k})) X(t_{n,k}) L_k(s)$ represent the interpolation error.

By using the definition of grid points and error of unknown functions ($EY_{n,j} = Y(t_{n,j}) - Y_{n,j}$, $EX_{n,j} = X(t_{n,j}) - X_{n,j}$)



the above relation can be simplified as

$$\begin{aligned}
EX_{n,j} &= \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(\sum_{i=0}^{n-1} (\eta r_i - R_i) \right. \\
&\quad - h^{\zeta_1} \sum_{i=0}^{n-1} \sum_{k=0}^4 w_k (n + c_j - i - c_k)^{\zeta_1-1} (lEX_{i,k} + \delta Y(t_{i,k})EX_{i,k} + X_{i,k}EY_{i,k}) \\
&\quad - \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} I_n(s) ds \\
&\quad \left. - \sum_{k=0}^4 [lEX_{n,k} + \delta Y(t_{n,k})EX_{n,k} + X_{n,k}EY_{n,k}] \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} L_k(s) ds \right).
\end{aligned}$$

Without loss of generality, suppose that the approximation $X(t)$ has the maximum error, and $L := \max_{s,k} |L_k(s)|$, then

$$\begin{aligned}
|EX_{n,j}| &\leq \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(\sum_{i=0}^{n-1} |\eta r_i - R_i| + h^{\zeta_1} \sum_{i=0}^{n-1} \sum_{k=0}^4 w_k \frac{|l + \delta Y(t_{i,k}) + X_{i,k}|}{|n + c_j - i - c_k|^{1-\zeta_1}} |EX_{i,k}| \right. \\
&\quad + \int_{t_n}^{t_{n,j}} |(t_{n,j} - s)^{\zeta_1-1}| |I_n(s)| ds \\
&\quad \left. + \sum_{k=0}^4 |l + \delta Y(t_{n,k}) + X_{n,k}| |EX_{n,k}| L \int_{t_n}^{t_{n,j}} (t_{n,j} - s)^{\zeta_1-1} ds \right).
\end{aligned}$$

In the following, we can write

$$\begin{aligned}
|EX_{n,j}| &\leq \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(nR|\eta - 1| \right. \\
&\quad + h^{\zeta_1} \sum_{i=0}^{n-1} \sum_{k=0}^4 w_k \frac{|l + \delta Y(t_{i,k}) + X_{i,k}|}{|n + c_j - i - c_k|^{1-\zeta_1}} |EX_{i,k}| \\
&\quad \left. + \frac{c_j^{\zeta_1}}{\zeta_1} Ih^{\zeta_1} + \frac{c_j^{\zeta_1}}{\zeta_1} h^{\zeta_1} L \sum_{k=0}^4 |l + \delta Y(t_{n,k}) + X_{n,k}| |EX_{n,k}| \right),
\end{aligned}$$

where $R := \max_{i=0,1,\dots,n-1} \{R_i, r_i\}$ and $I := \max_s |I_n(s)|$ are used to represent the maximum values of integration and interpolation errors, respectively. Equivalently, we have

$$|EX_{n,j}| \leq \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(|\eta - 1|nR + \frac{c_j^{\zeta_1}}{\zeta_1} Ih^{\zeta_1} \right) + m_3 h^{\zeta_1} \sum_{i=0}^{n-1} \sum_{k=0}^4 \frac{|EX_{i,k}|}{|n + c_j - i - c_k|^{1-\zeta_1}} + m_4 h^{\zeta_1} \sum_{k=0}^4 |EX_{n,k}|,$$

where $m_3 := \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \max_{i,k} w_k |l + \delta Y(t_{i,k}) + X_{i,k}|$ and $m_4 := \frac{\nu^{1-\zeta_1} c_j^{\zeta_1}}{\zeta_1 \Gamma(\zeta_1)} L \max_k |l + \delta Y(t_{n,k}) + X_{n,k}|$. Let $\max_j |EX_{n,j}| = |EX_{n,4}|$, and let h be small enough such that $4m_4 h^{\zeta_1} \leq \frac{1}{2}$. Then, we can conclude that

$$|EX_{n,4}| \leq 2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(|\eta - 1|nR + \frac{c_j^{\zeta_1}}{\zeta_1} Ih^{\zeta_1} \right) + 2m_3 h^{\zeta_1} \sum_{i=0}^n \sum_{k=0}^3 \frac{|EX_{i,k}|}{|n + c_j - i - c_k|^{1-\zeta_1}}, \quad n = 0, 1, \dots, N.$$

Letting $\delta := 2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(|\eta - 1|nR + \frac{c_j^{\zeta_1}}{\zeta_1} Ih^{\zeta_1} \right)$, $M := 2m_3$ and $\epsilon := 1 - \zeta_1$, the conditions of Lemma 5.1 will be hold, thus

$$\|EX\|_{\infty} \leq (\delta' + hM'T^{P-1-P\epsilon}\delta) \exp(M'T^{P-P\epsilon}),$$



where

$$M' = \begin{cases} M, & P = 1, \\ M^P \prod_{k=1}^{P-1} B(k(1-\epsilon), 1-\epsilon), & P \geq 2, \end{cases}$$

$$\delta' = \delta \left\{ (1 + h^{1-\epsilon} M) \sum_{j=0}^{P-2} \gamma^j + \gamma^{P-1} \right\}, \quad P \geq 1,$$

with $\gamma = \frac{MT^{1-\epsilon}}{1-\epsilon}$ and P is the smallest positive integer such that $\epsilon \leq (P-1)/P$. In the following, we can write

$$\begin{aligned} \|EX\|_{\infty} &\leq \delta \left((1 + h^{1-\epsilon} M) \sum_{j=0}^{P-2} \gamma^j + \gamma^{P-1} \right) \exp(M'T^{P-P\epsilon}) \\ &= 2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} \left(|\eta - 1| nR + \frac{c_j^{\zeta_1}}{\zeta_1} I h^{\zeta_1} \right) \left((1 + h^{1-\epsilon} M) \sum_{j=0}^{P-2} \gamma^j + \gamma^{P-1} \right) \exp(M'T^{P-P\epsilon}). \end{aligned}$$

Hence $\|EX\|_{\infty} \rightarrow 0$ as $h \rightarrow 0$, since R and I tend to zero as $h \rightarrow 0$. By some calculation we get

$$\begin{aligned} \|EX\|_{\infty} &\leq \left[2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} |\eta - 1| \left(\sum_{j=0}^{P-2} \gamma^j + \gamma^{P-1} \right) \frac{T}{h} R \right. \\ &\quad + 2 \frac{\nu^{1-\zeta_1} c_j^{\zeta_1}}{\zeta_1 \Gamma(\zeta_1)} \left(\sum_{j=0}^{P-2} \gamma^j + \gamma^{P-1} \right) I h^{\zeta_1} + 2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} M |\eta - 1| \left(\sum_{j=0}^{P-2} \gamma^j \right) T h^{-\epsilon} R \\ &\quad + 2 \frac{\nu^{1-\zeta_1} c_j^{\zeta_1}}{\zeta_1 \Gamma(\zeta_1)} M \left(\sum_{j=0}^{P-2} \gamma^j \right) I h^{1+\zeta_1-\epsilon} \\ &\quad \left. + 2 \frac{\nu^{1-\zeta_1}}{\Gamma(\zeta_1)} |\eta - 1| M' T^{P-1-P\epsilon} R + 2 \frac{\nu^{1-\zeta_1} c_j^{\zeta_1}}{\zeta_1 \Gamma(\zeta_1)} M' T^{P-1-P\epsilon} I h^{1+\zeta_1} \right] \exp(M'T^{P-P\epsilon}), \end{aligned}$$

for sufficiently smooth $X(t)$ (i.e., $X \in C^6[0, T]$, if the function lacks the required derivatives, theoretical analysis using the Peano kernel provides a rigorous way to express and bound the error), we have $R = O(h^6)$ and $I = O(h^5)$ and therefore the order of convergence of the introduced method is at least $O(h^5)$. $\square$

6. ERROR ESTIMATION

In this section, since the analytical solution of the fractional hematopoietic stem cells model (4.1) is not available, we derive an error estimator for the approximate solution of this model.

Let $E_n(X) = X(t) - X_n(t)$ denote the error function associated with the approximate solution $X_n(t)$ of (4.6) and the exact solution of (4.2). Therefore, $X_n(t), \dots, H_n(t)$ satisfy the following system

$$\begin{aligned} X_n(t) &= X_0 + \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX_n(s) - \delta X_n(s)Y_n(s)) ds + \gamma_{n1}(t), \\ Y_n(t) &= Y_0 + \frac{1}{\Gamma(\zeta_2)} \int_0^t (t-s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X_n(s)Y_n(s) - aY_n(s) - bY_n(s)Z_n(s)) ds + \gamma_{n2}(t), \\ W_n(t) &= W_0 + \gamma_{n3}(t) + \frac{1}{\Gamma(\zeta_3)} \int_0^t (t-s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v H_n(s) + cW_n(s)X_n(s)Y_n(s) - qW_n(s)Y_n(s) - bW_n(s)) ds, \\ Z_n(t) &= Z_0 + \frac{1}{\Gamma(\zeta_4)} \int_0^t (t-s)^{\zeta_4-1} \nu^{1-\zeta_4} (qW_n(s)Y_n(s) - d_1 Z_n(s)) ds + \gamma_{n4}(t), \end{aligned}$$



$$H_n(t) = H_0 - \frac{1}{\Gamma(\zeta_5)} \int_0^t (t-s)^{\zeta_5-1} \nu^{1-\zeta_5} v H_n(s) ds + \gamma_{n5}(t), \quad (6.1)$$

The perturbation terms $\gamma_{n1}(t), \dots, \gamma_{n5}(t)$ can be obtained from

$$\begin{aligned} \gamma_{n1}(t) &= X_n(t) - X_0 - \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lX_n(s) - \delta X_n(s)Y_n(s)) ds, \\ \gamma_{n2}(t) &= Y_n(t) - Y_0 - \frac{1}{\Gamma(\zeta_2)} \int_0^t (t-s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta X_n(s)Y_n(s) - aY_n(s) - bY_n(s)Z_n(s)) ds, \\ \gamma_{n3}(t) &= W_n(t) - W_0 - \frac{1}{\Gamma(\zeta_3)} \int_0^t (t-s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v H_n(s) + cW_n(s)X_n(s)Y_n(s) - qW_n(s)Y_n(s) - bW_n(s)) ds, \\ \gamma_{n4}(t) &= Z_n(t) - Z_0 - \frac{1}{\Gamma(\zeta_4)} \int_0^t (t-s)^{\zeta_4-1} \nu^{1-\zeta_4} (qW_n(s)Y_n(s) - d_1Z_n(s)) ds, \\ \gamma_{n5}(t) &= H_n(t) - H_0 + \frac{1}{\Gamma(\zeta_5)} \int_0^t (t-s)^{\zeta_5-1} \nu^{1-\zeta_5} v H_n(s) ds, \end{aligned} \quad (6.2)$$

Subtracting (6.1) from (4.2) leads to

$$\begin{aligned} E_n(X) &= \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1-1} \nu^{1-\zeta_1} (\eta - lE_n(X) - \delta E_n(X)E_n(Y)) ds - \gamma_{n1}(t), \\ E_n(Y) &= \frac{1}{\Gamma(\zeta_2)} \int_0^t (t-s)^{\zeta_2-1} \nu^{1-\zeta_2} (\delta E_n(X)E_n(Y) - aE_n(Y) - bE_n(Y)E_n(Z)) ds - \gamma_{n2}(t), \\ E_n(W) &= \frac{1}{\Gamma(\zeta_3)} \int_0^t (t-s)^{\zeta_3-1} \nu^{1-\zeta_3} (\omega v E_n(H) + cE_n(W)E_n(X)E_n(Y) - qE_n(W)E_n(Y) - bE_n(W)) ds - \gamma_{n3}(t), \\ E_n(Z) &= \frac{1}{\Gamma(\zeta_4)} \int_0^t (t-s)^{\zeta_4-1} \nu^{1-\zeta_4} (qE_n(W)E_n(Y) - d_1E_n(Z)) ds - \gamma_{n4}(t), \\ E_n(H) &= -\frac{1}{\Gamma(\zeta_5)} \int_0^t (t-s)^{\zeta_5-1} \nu^{1-\zeta_5} v E_n(H) ds - \gamma_{n5}(t). \end{aligned} \quad (6.3)$$

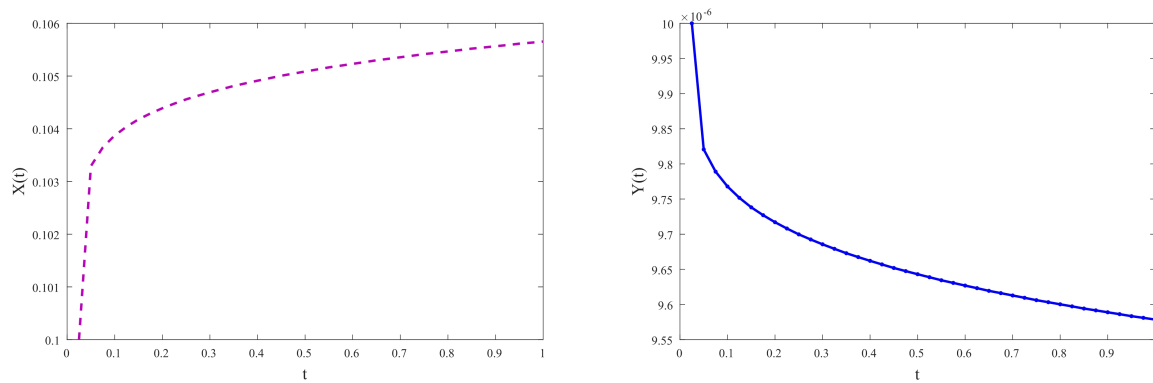
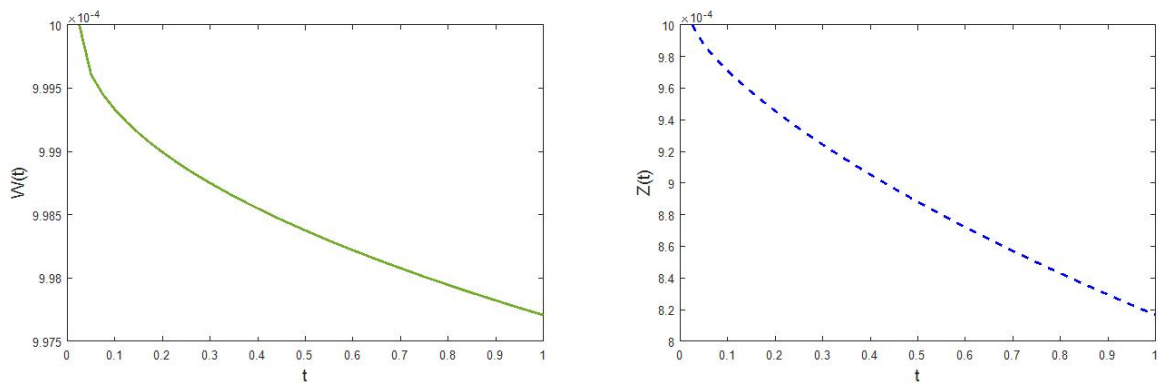
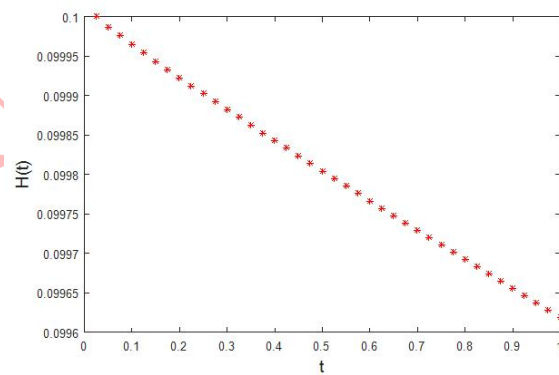
By solving the above system, we can obtain approximations $E_{n,N}(X), \dots, E_{n,N}(H)$ for the error functions $E_n(X), \dots, E_n(H)$; For this purpose, we use the same method as described in Section 4.

7. NUMERICAL SIMULATIONS

In this section, we develop numerical simulations for the fractional hematopoietic stem cell model (4.1). For this purpose, we consider the numerical values of the parameters presented in Table 1, $\nu = 0.02$ and the initial conditions $X(0) = 0.10$, $Y(0) = 0.00001$, $W(0) = 0.001$, $Z(0) = 0.001$ and $H(0) = 0.10$. We obtained the results using Maple 16 and the results were plotted with MATLAB. The obtained solutions of $X(t)$, $Y(t)$, $W(t)$, $Z(t)$, and $H(t)$ are respectively indicated in Figures 1-3, with step size $h = 0.1$ and values $\zeta_1 = 0.15$, $\zeta_2 = 0.24$, $\zeta_3 = 0.48$, $\zeta_4 = 0.76$ and $\zeta_5 = 0.93$. Also in Figures 4-6, the obtained solutions of $X(t)$, $Y(t)$, $W(t)$, $Z(t)$ and $H(t)$ are respectively indicated for step size $h = 0.2$ and values $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0.90$. The fractional orders ζ_i are chosen to illustrate the numerical method's performance across a range of memory parameters. Biologically, these orders reflect the degree of non-locality or memory in cell population dynamics, with values closer to 1 indicating weaker memory effects. The distinct values in Figures 1-3 demonstrate the method's capability when different cell types exhibit heterogeneous memory characteristics, while the uniform value $\zeta = 0.90$ in Figures 4-6 tests the method under homogeneous conditions. We emphasize that rigorous estimation of these parameters from experimental data is an important direction for future research. The figures show that infected and precursor CTL cells concentration ($Y(t), W(t)$), effector T lymphocytes CTL cells concentration ($Z(t)$) and CTL cells concentration produced by HSC ($H(t)$) decrease gradually while uninfected CTL cells concentration ($X(t)$) increase gradually.

To validate the theoretical error estimation framework proposed in Section 6, we present the estimated errors for this



FIGURE 1. Numerical solution for $X(t)$ and $Y(t)$ with $h = 0.1$.FIGURE 2. Numerical solution for $W(t)$ and $Z(t)$ with $h = 0.1$.FIGURE 3. Numerical solution for $H(t)$ with $h = 0.1$.

example in Table 2 for step size $h = 0.1$ at specific mesh points. Notably, the estimated errors serve as an a priori tool to predict the method's accuracy across different examples. Since the exact solution of the fractional hematopoietic

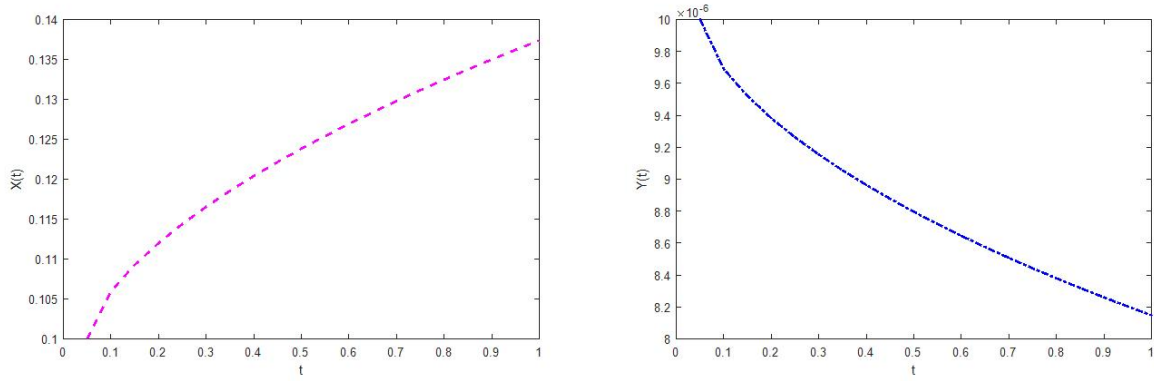


FIGURE 4. Numerical solution for $X(t)$ and $Y(t)$ for $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0.90$ and $h = 0.2$.

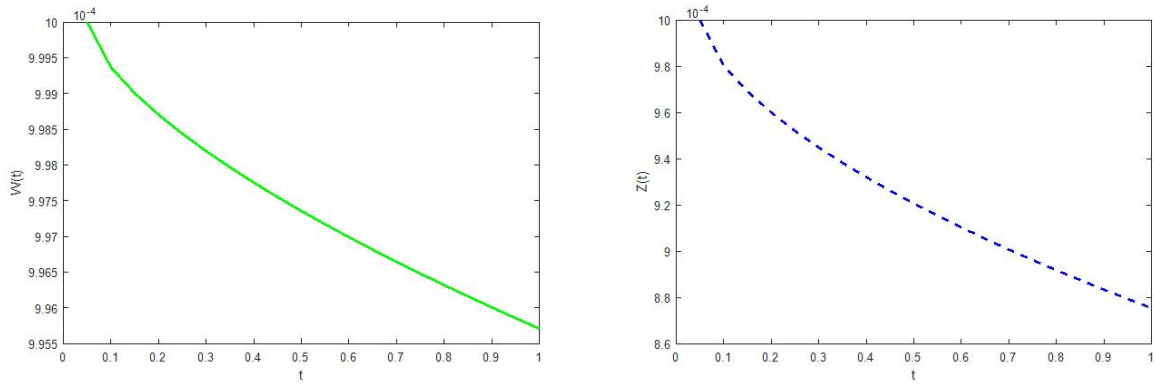


FIGURE 5. Numerical solution for $W(t)$ and $Z(t)$ for $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0.90$ and $h = 0.2$.

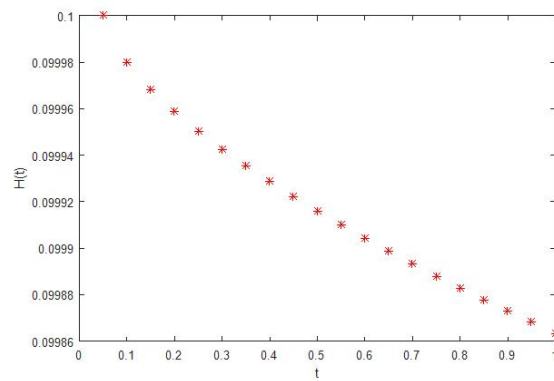


FIGURE 6. Numerical solution for $H(t)$ for $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0.90$ and $h = 0.2$.

stem cell model is unavailable, the proposed error estimator was validated using a reference solution computed on a



TABLE 2. The estimated errors for $h = 0.1$.

t	$E_n(X)$	$E_n(Y)$	$E_n(W)$	$E_n(Z)$	$E_n(H)$
0.1	1.23777e-12	2.74627e-11	1.57984e-08	4.17269e-12	4.79789e-11
0.2	1.91849e-12	2.33561e-10	2.47772e-08	3.87411e-12	4.48801e-11
0.3	2.47174e-11	2.33692e-10	3.22343e-08	4.16459e-12	4.41522e-11
0.4	2.95335e-11	9.35039e-10	3.88472e-08	5.15436e-12	4.78632e-11
0.5	3.38660e-11	9.44237e-10	4.48949e-08	3.34415e-12	4.07553e-11
0.6	4.78393e-12	3.89836e-10	5.05266e-08	5.23299e-12	5.29364e-11
0.7	6.15310e-12	1.91023e-10	5.58342e-08	3.92863e-12	4.14169e-11
0.8	4.49923e-11	1.90777e-10	6.08787e-08	4.92759e-12	4.73192e-11
0.9	4.82621e-11	1.91899e-10	6.57040e-08	2.93419e-11	4.32205e-11
1	5.13664e-12	2.88287e-10	7.03421e-08	3.93293e-12	4.01224e-11

much finer temporal mesh ($h_{\text{ref}} = 0.01$) and the results was reported in Table 3. In this Table, the reference error was evaluated as

$$E_{\text{ref}}(t_i) = |u_h(t_i) - u_{h_{\text{ref}}}(t_i)|,$$

where u_h denotes the numerical solution obtained with the mesh size used in the simulations ($h = 0.1$) and $u_{h_{\text{ref}}}$ is the numerical solution obtained with the refined mesh. The maximum norm of the reference error,

$$\|E_{\text{ref}}\|_{\infty} = \max_i |u_h(t_i) - u_{h_{\text{ref}}}(t_i)|,$$

was compared with the residual-based error estimator to assess its accuracy. For this purpose, efficiency index $R_u = \frac{\|E_N(u)\|_{\infty}}{\|E_{\text{ref}}\|_{\infty}}$ was calculated. This validation procedure is standard practice when an analytical solution is unavailable [4, 25].

Values of the efficiency index close to unity indicate good agreement between the estimated and reference errors. For the variables (X), (Y), and (W), the efficiency indices are 0.8331, 1.1223, and 1.1617, respectively, indicating that the estimator predicts the discretization error with high accuracy. In particular, the estimator slightly underestimates the error for (X) and slightly overestimates it for (Y) and (W). For the variables (Z) and (H), the efficiency indices are 0.7290 and 0.7118, respectively, showing a moderate underestimation of the reference error. Overall, the validation results demonstrate that the proposed error estimator provides a reliable approximation of the actual discretization error.

TABLE 3. Validation Table.

Variable	$\ E_N\ _{\infty}$	$\ E_{\text{ref}}\ _{\infty}$	R_u
X	4.82621e-11	5.79335e-11	0.8331
Y	9.44237e-10	8.41371e-10	1.1223
W	7.03421e-08	6.05492e-08	1.1617
Z	2.93419e-11	4.02481e-11	0.7290
H	5.52182e-11	7.75727e-11	0.7118

In order to validate the order of convergence, we consider a special version of the Bagley-Torvik equation [6]

$$AD_0^2 y(t) + BD_0^{\frac{3}{2}} y(t) + Cy(t) = f(t),$$

where A, B and C are certain constants and f is a given function. This equation arises, for example, in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [6]. Consider

$$D^2 y(t) + \frac{8}{17} D_0^{\frac{3}{2}} y(t) = f(t), \quad t \in (0, 1), \quad (7.1)$$



with the initial conditions $y(0) = 0$ and $y'(0) = 1$, where $f(x)$ is determined by using the exact solution $y(t) = t^5 - \frac{29}{10}t^4 + \frac{76}{25}t^3 - \frac{339}{250}t^2 + \frac{27}{125}t$. Assuming $X(t) = D^2y(t)$, leads to

$$\begin{cases} y(t) = t + J_0^2 X(t) = t + \int_0^t (t-s)X(s)ds, \\ D_0^{\frac{3}{2}} y(t) = \frac{1}{\Gamma(\frac{1}{2})} t^{-\frac{1}{2}} + J_0^{\frac{1}{2}} X(t) = \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{X(s)}{\sqrt{t-s}} ds. \end{cases} \quad (7.2)$$

Therefore, the integral equation equivalent to the Equation (7.1) will be

$$X(t) + \frac{8}{17\sqrt{\pi}} \int_0^t (t-s)^{-1/2} X(s)ds = f(t) - \frac{8}{17\sqrt{\pi}} t^{-1/2},$$

with exact solution $X(t) = 20t^3 - \frac{174}{5}t^2 + \frac{456}{25}t - \frac{339}{125}$. Table 4 presents the absolute errors and the ratios of errors for $h = 0.2$ and $h = 0.1$ at specific mesh points. The ratios of the errors are close to $2^5 = 32$, indicating the theoretical convergence order of at least $O(h^5)$.

TABLE 4. The numerical results for Equation (7.1).

t	$h = 0.2$	$h = 0.1$	Ratio
0.2	2.5718e-6	5.3837e-8	47.77
0.4	4.9151e-6	1.0921e-7	45.01
0.6	2.0578e-5	5.6213e-7	36.61
0.8	4.3152e-6	1.0946e-6	39.42
1	5.6691e-7	9.4123e-9	60.23

8. CONCLUSIONS

In this paper, we considered a fractional hematopoietic stem cell model under the Caputo fractional derivative concept and converted it into a system of Volterra integral equation of weakly singular type. In the following, a well known block-by-block method is applied for solving obtained system. We also provide the error analysis and convergence of the method, also, since the considered model lacks information about the exact solution, the error estimation was discussed. Finally, the numerical simulations are proposed to confirm the effectiveness of the method.

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