



Two compact techniques for numerical solution of the Klein-Gordon equation

Emre Kirli¹, Dursun Irk², and Melis Zorsahin^{2,*}

¹Department of Mathematics, Aksaray University, 68100, Aksaray, Türkiye.

²Department of Mathematics and Computer Sciences, Eskisehir Osmangazi University, 26040, Eskisehir, Türkiye.

Abstract

In this study, two compact numerical techniques are proposed for solving the nonlinear Klein-Gordon equation (KGE). The first method, denoted as M_1 , combines the cubic B-spline collocation method with the Crank-Nicolson (CN) time integration scheme to provide a fully discrete formulation. The second method, referred to as M_2 , employs a fourth-order accurate spatial discretization based on cubic B-spline basis functions, along with a one-step implicit time integration scheme of fourth-order accuracy. Consequently, the M_1 and M_2 methods achieve second- and fourth-order accuracy in both space and time, respectively. To assess the accuracy and efficiency of the proposed techniques, two benchmark test problems are considered. The L_∞ error norms, along with spatial and temporal convergence rates, are computed. The numerical results confirm that the observed convergence rates closely match the theoretical predictions. Overall, the findings demonstrate that both methods are effective, with the M_2 technique exhibiting superior accuracy compared to existing approaches.

Keywords. Cubic B-spline, Klein-Gordon equation, Collocation method, Finite element.

2010 Mathematics Subject Classification. 65D07, 65L60.

1. INTRODUCTION

The KGE which plays key role in quantum mechanics and mathematical physics is expressed in the form:

$$w_{tt} + \alpha w_{xx} + H(w) = f(x, t), \quad (1.1)$$

along with the initial conditions (ICs) and boundary conditions (BCs)

$$w(x, 0) = \Psi_0(x), \quad (1.2)$$

$$w_t(x, 0) = \Psi_1(x), \quad a \leq x \leq b, \quad (1.3)$$

$$w(a, t) = \Phi_0(t), \quad w(b, t) = \Phi_1(t), \quad (1.4)$$

where α is a nonzero real constant and $H(w)$ is the nonlinear function. The choice of $H(w)$ forms various equations, for example: $\sin(w)$, $\sinh(w)$, w^k where k is an integer or the summations of these functions. Nonlinear phenomena, which emerge across a wide range of scientific disciplines including fluid dynamics, plasma physics, mathematical biology and chemical kinetics are modeled using partial differential equations (PDEs). The KGE is one of the important models used in solid state physics, nonlinear optics and quantum field theory [24]. A variety of the techniques have been implemented to solve the KGE such as Legendre pseudospectral method [14], decomposition method [10], finite difference method [4], He's variational iteration method [21], radial basis function approximation method [3], lattice Boltzmann method [15], Taylor matrix method [2], Adomian's decomposition and variational iterative methods [18], Legendre wavelet method [6], finite difference time domain method and Gautschi-type exponential integrator Fourier pseudospectral methods [1], differential quadrature method [23], meshless method [7], element-free kp-Ritz method [8], multiquadric Quasi-interpolation method [19], Haar wavelet method [22], perturbation iteration transform method [11].

Received: 29 September 2025; Accepted: 19 October 2026.

* Corresponding author. Email: mzorsahin@ogu.edu.tr.

Moreover, several numerical techniques based on finite element method (FEM) have been applied to get approximate solution of the nonlinear KGE. Yang [25] used high order local discontinuous Galerkin method to solve numerically the KGE. Khalifa and Elgamal [12] developed based on FEM by using the quadratic interpolation functions. Rashidinia et al [17] proposed the cubic B-spline collocation method to obtain the approximate solution of nonlinear KGE. The collocation method based on exponential cubic B-spline functions was used by Ersoy Hepson et al. [5]. Selvitopi and Yazıcı [20] applied Galerkin FEM to KGE. A high order accurate numerical method was presented to solve KGE by Irk et al. [9] in which fourth-order one-step implicit scheme was proposed to discrete the temporal domain discretization and the spatial domain is discretized using Galerkin FEM.

The objective of this study is to examine the impact of increasing the accuracy in both spatial and temporal discretizations for the approximate solution of the KGE. To this end, we derive two compact numerical methods, denoted as M_1 and M_2 . In the M_1 method, a fully discrete scheme is formulated by applying the standard cubic B-spline collocation method (SCBCM) for spatial discretization and the CN scheme for temporal integration. In contrast, the M_2 method employs a fourth-order accurate scheme in which the spatial domain is discretized using cubic B-spline basis functions, while a fourth-order one-step integration method is formulated for the temporal discretization.

The remainder of the paper is structured as follows. In Section 2, the temporal discretization procedure is rigorously described, and the fourth-order cubic B-spline collocation method for spatial approximation is presented. Section 3 presents the implementation of the proposed methods for solving the KGE given in Eq. (2.1). Section 4 provides numerical experiments, with corresponding results summarized in tabular form. Finally, Section 5 concludes the paper with a brief summary of the findings and contributions of this work.

2. PROPOSED METHODS

In this study, we are interested in the nonlinear KGE given by

$$w_{tt} + \alpha w_{xx} + \beta w + \gamma w^3 = f(x, t), \quad (2.1)$$

where β, γ are constants and $f(x, t)$ is known function. The nonlinear function is chosen as $\beta w + \gamma w^3$. Setting $w_t(x, t) = v(x, t)$, the KGE (2.1) is transformed into the following system of first order (in time) equations

$$w_t = v, \quad (2.2)$$

$$v_t = -\alpha w_{xx} - \beta w - \gamma w^3 + f(x, t). \quad (2.3)$$

The BCs and ICs are rearranged to be written as

$$\begin{aligned} w(a, t) &= \Phi_0(t), & w(b, t) &= \Phi_1(t), \\ v(a, t) &= \frac{\partial \Phi_0}{\partial t}(t), & v(b, t) &= \frac{\partial \Phi_1}{\partial t}(t), \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} w(x, 0) &= \Psi_0(x), \\ v(x, 0) &= \Psi_1(x). \end{aligned} \quad (2.5)$$

The spatial-temporal domain $\Omega = [a, b] \times [0, T]$ is uniformly discretized, defining grid points of the form with the grid points (x_r, t_n) where $x_r = a + rh$, $r = 0, 1, 2, \dots, N$, $t_n = n\Delta t$, $n = 0, 1, 2, \dots$. Here, h denotes the spatial mesh size and Δt represents the temporal step size.

2.1. Time Discretization. Utilizing a one-step finite difference scheme of fourth-order accuracy developed in [9], the temporal discretizations of Eqs. (2.2) and (2.3) are given by

$$w^{n+1} = w^n + \theta_1 w_t^{n+1} + \theta_2 w_t^n + \theta_3 w_{tt}^{n+1} + \theta_4 w_{tt}^n, \quad (2.6)$$

and

$$v^{n+1} = v^n + \theta_1 v_t^{n+1} + \theta_2 v_t^n + \theta_3 v_{tt}^{n+1} + \theta_4 v_{tt}^n, \quad (2.7)$$



where

$$w_{tt} = -\alpha w_{xx} - \beta w - \gamma w^3 + f(x, t), \quad (2.8)$$

$$v_{tt} = -\alpha v_{xx} - \beta v - \gamma 3w^2 v + f_t(x, t), \quad (2.9)$$

derived using Eqs. (2.2) and (2.3), where the quantities θ_i , $i = 1, 2, 3, 4$ represent unknown parameters to be determined later.

Substituting Eqs. (2.2) and (2.8) into Eq. (2.6) results in

$$\begin{aligned} w^{n+1} = & w^n + \theta_1 v^{n+1} + \theta_2 v^n + \theta_3 (-\alpha w_{xx}^{n+1} - \beta w^{n+1} - \gamma (w^{n+1})^3 + f(x, t_{n+1})) \\ & + \theta_4 (-\alpha w_{xx}^n - \beta w^n - \gamma (w^n)^3 + f(x, t_n)). \end{aligned} \quad (2.10)$$

Eq. (2.10) can be simplified to be written as

$$\begin{aligned} & w^{n+1} (1 + \theta_3 \beta + \theta_3 \gamma (w^{n+1})^2) + \alpha \theta_3 w_{xx}^{n+1} - \theta_1 v^{n+1} \\ & = w^n (1 - \theta_4 \beta - \theta_4 \gamma (w^n)^2) - \alpha \theta_4 w_{xx}^n + \theta_2 v^n + \theta_4 f(x, t_n) + \theta_3 f(x, t_{n+1}). \end{aligned} \quad (2.11)$$

In a similar way, using Eqs. (2.3) and (2.9) in Eq. (2.7) yields

$$\begin{aligned} v^{n+1} = & v^n + \theta_1 (-\alpha w_{xx}^{n+1} - \beta w^{n+1} - \gamma (w^{n+1})^3 + f(x, t_{n+1})) + \theta_2 (-\alpha w_{xx}^n - \beta w^n \\ & - \gamma (w^n)^3 + f(x, t_n)) + \theta_3 (-\alpha v_{xx}^{n+1} - \beta v^{n+1} - \gamma 3(w^{n+1})^2 v^{n+1} + f_t(x, t_{n+1})) \\ & + \theta_4 (-\alpha v_{xx}^n - \beta v^n - \gamma 3(w^n)^2 v^n + f_t(x, t_n)). \end{aligned}$$

After some arrangements, we have

$$\begin{aligned} & w^{n+1} (\theta_1 \beta + \theta_1 \gamma (w^{n+1})^2) + \alpha \theta_1 w_{xx}^{n+1} + \alpha \theta_3 v_{xx}^{n+1} + v^{n+1} (1 + \theta_3 \beta + 3\theta_3 \gamma (w^{n+1})^2) \\ & = w^n (-\theta_2 \beta - \theta_2 \gamma (w^n)^2) - \alpha \theta_2 w_{xx}^n + v^n (1 - \theta_4 \beta - 3\theta_4 \gamma (w^n)^2) \\ & - \alpha \theta_4 v_{xx}^n + \theta_1 f(x, t_{n+1}) + \theta_2 f(x, t_n) + \theta_3 f_t(x, t_{n+1}) + \theta_4 f_t(x, t_n). \end{aligned} \quad (2.12)$$

Lemma 2.1. Let $w, v, f \in C^6(\Omega)$ Then, for

$$\theta_1 = \theta_2 = \frac{\Delta t}{2}, \quad \theta_3 = -\theta_4 = -\frac{(\Delta t)^2}{12}.$$

the developed computational schemes (2.11) and (2.12) are consistent and has fourth-order accuracy in temporal direction for the norm $\|\cdot\|_\infty$.

Proof. Reported in [9]. □

It is evident that a suitable choice of the parameters in Eqs. (2.6) and (2.7) as

$$\theta_1 = \theta_2 = \frac{\Delta t}{2}, \quad \theta_3 = \theta_4 = 0,$$

yields CN scheme of second-order accuracy.

2.2. Fourth-Order Cubic B-spline Collocation Method (FOCBCM). In SCBCM, the cubic B-spline (CBS) approximations of unknown function and its first-order derivative are of the fourth order accuracy, whereas the approximation for second-order spatial derivative is of second-order accuracy. In contrast, in FOCBCM, the fourth-order approximation for second-order spatial derivative has been developed by enforcing the interpolatory and specific boundary constraints.



2.2.1. *Cubic B-spline Functions.* The typical third degree B-spline functions for $r = -1, 0, \dots, N+1$ are defined as

$$\phi_r^3(x) = \frac{1}{h^3} \begin{cases} (g_{r-2})^3, & x_{r-2} \leq x < x_{r-1}, \\ h^3 + 3h^2g_{r-1} + 3h(g_{r-1})^2 - 3(g_{r-1})^3, & x_{r-1} \leq x < x_r, \\ h^3 - 3h^2g_{r+1} - 3h(g_{r+1})^2 + 3(g_{r+1})^3, & x_r \leq x < x_{r+1}, \\ -(g_{r+2})^3, & x_{r+1} \leq x < x_{r+2}, \\ 0, & \text{Otherwise,} \end{cases} \quad (2.13)$$

where $g_r = x - x_r$. The collection $\{\phi_{-1}, \phi_0, \dots, \phi_{N+1}\}$ of cubic B-spline functions constitutes a basis for the approximation space defined on the interval $[a, b]$. Let $K(x, t)$ and $P(x, t)$ be CBS approximate solutions corresponding to analytical solutions $w(x, t)$ and $v(x, t)$, respectively. Then, they can be expressed as follows:

$$\begin{aligned} w(x, t) &\approx K(x, t) = \sum_{r=-1}^{N+1} a_r(t) \phi_r(x), \\ v(x, t) &\approx P(x, t) = \sum_{r=-1}^{N+1} b_r(t) \phi_r(x), \end{aligned} \quad (2.14)$$

where a_r and b_r , $r = -1, 0, \dots, N+1$, are unknown time-dependent parameters that are to be determined. The approximate solutions $K(x, t)$ and $P(x, t)$ fulfill the below mentioned interpolation conditions and BCS

$$K(x_r, t) = w(x_r, t), \quad (2.15)$$

$$P(x_r, t) = v(x_r, t), \quad 0 \leq r \leq N \quad (2.16)$$

$$K''(x_r, t) = w''(x_r, t) - \frac{h^2}{12} w^{(4)}(x_r, t), \quad r = 0, N, \quad (2.17)$$

$$P''(x_r, t) = v''(x_r, t) - \frac{h^2}{12} v^{(4)}(x_r, t), \quad r = 0, N. \quad (2.18)$$

Theorem 2.2. Let $K(x, t)$ and $P(x, t)$ denote the cubic B-spline interpolants (CSI) of the sufficiently smooth functions $w(x, t)$ and $v(x, t)$, respectively, which satisfy the interpolation conditions given in Eqs. (2.15)–(2.18). Under these assumptions, the following relations hold at the mesh points x_r , for $r = 0, 1, \dots, N$:

$$K''(x_r, t) = w''(x_r, t) - \frac{h^2}{12} w^{(4)}(x_r, t) + O(h^4), \quad (2.19)$$

$$P''(x_r, t) = v''(x_r, t) - \frac{h^2}{12} v^{(4)}(x_r, t) + O(h^4), \quad (2.20)$$

$$K'(x_r, t) = w'(x_r, t) + O(h^4), \quad (2.21)$$

$$P'(x_r, t) = v'(x_r, t) + O(h^4), \quad (2.22)$$

and

$$\begin{aligned} \|K^l - w^l\|_\infty &= O(h^{4-l}), \\ \|P^l - v^l\|_\infty &= O(h^{4-l}), \quad l = 0, 1, 2 \end{aligned}$$

where K^l , w^l , P^l and v^l denote the l th derivative w.r.t. 'x'.

Proof. For the proof see [16]. □

Based on the previous theorem, a fourth-order approximation for $K''(x, t)$ and $P''(x, t)$ is now pursued. As a first step, we define the difference operator ξ as follows:

$$\xi M(x_r, t) = M(x_{r-1}, t) - 2M(x_r, t) + M(x_{r+1}, t), \quad r = 1, 2, \dots, N-1 \quad (2.23)$$

for a smooth function M evaluated at the spatial mesh points.



Lemma 2.3. Let $K(x, t)$ and $P(x, t)$ denote the CSI of the functions $w(x, t)$ and $v(x, t)$, respectively, where both functions are assumed to be six times continuously differentiable over the domain Ω . Suppose further that they satisfy the interpolation conditions given in (2.15)–(2.18). Then, the following relations hold:

$$w^{(4)}(x_r, t) = \frac{1}{h^2} \xi K''(x_r, t) + O(h^2), \quad r = 1, 2, \dots, N-1, \quad (2.24)$$

$$v^{(4)}(x_r, t) = \frac{1}{h^2} \xi P''(x_r, t) + O(h^2), \quad r = 1, 2, \dots, N-1. \quad (2.25)$$

Proof. For the proof see [13]. □

Lemma 2.4. Let $K(x, t)$ and $P(x, t)$ denote the CSI of the functions $w(x, t)$ and $v(x, t)$, respectively, where both functions are assumed to be six times continuously differentiable over the domain Ω . Suppose further that they satisfy the interpolation conditions given in (2.15)–(2.18). Then, the following relations hold:

$$w^{(4)}(x_r, t) = \frac{2\xi K''(x_1, t) - \xi K''(x_2, t)}{h^2} + O(h^2), \quad r = 0, \quad (2.26)$$

$$v^{(4)}(x_r, t) = \frac{2\xi P''(x_1, t) - \xi P''(x_2, t)}{h^2} + O(h^2), \quad r = 0, \quad (2.27)$$

$$w^{(4)}(x_r, t) = \frac{2\xi K''(x_{N-1}, t) - \xi K''(x_{N-2}, t)}{h^2} + O(h^2), \quad r = N, \quad (2.28)$$

$$v^{(4)}(x_r, t) = \frac{2\xi P''(x_{N-1}, t) - \xi P''(x_{N-2}, t)}{h^2} + O(h^2), \quad r = N. \quad (2.29)$$

Proof. For the proof see [13]. □

Corollary 2.5. If $w(x, t)$ and $v(x, t) \in C^6(\Omega)$, then the below mentioned approximations are valid at the mesh points x_r

$$\begin{aligned} w''(x_0, t) &= \frac{1}{12} [14K''(x_0, t) - 5K''(x_1, t) + 4K''(x_2, t) - K''(x_3, t)] + O(h^4), \\ w''(x_r, t) &= \frac{1}{12} [K''(x_{r-1}, t) + 10K''(x_r, t) + K''(x_{r+1}, t)] + O(h^4), \quad r = 1, 2, \dots, N-1, \\ w''(x_N, t) &= \frac{1}{12} [14K''(x_N, t) - 5K''(x_{N-1}, t) + 4K''(x_{N-2}, t) - K''(x_{N-3}, t)] + O(h^4), \\ v''(x_0, t) &= \frac{1}{12} [14P''(x_0, t) - 5P''(x_1, t) + 4P''(x_2, t) - P''(x_3, t)] + O(h^4), \\ v''(x_j, t) &= \frac{1}{12} [P''(x_{r-1}, t) + 10P''(x_r, t) + P''(x_{r+1}, t)] + O(h^4), \quad r = 1, 2, \dots, N-1, \\ v''(x_N, t) &= \frac{1}{12} [14P''(x_N, t) - 5P''(x_{N-1}, t) + 4P''(x_{N-2}, t) - P''(x_{N-3}, t)] + O(h^4). \end{aligned}$$

Let $K(x, t)$ and $P(x, t)$ denote fourth-order cubic B-spline approximate solutions to functions $w(x, t)$ and $v(x, t)$, respectively. Then, the values of $K(x, t)$ and $P(x, t)$ together with their first and second spatial derivatives are as noted below:

$$\begin{aligned} K(x_r, t) &= a_{r-1} + 4a_r + a_{r+1} + O(h^4), \quad r = 0, 1, \dots, N, \\ K'(x_r, t) &= \frac{-3}{h} (a_{r-1} - a_{r+1}) + O(h^4), \quad r = 0, 1, \dots, N, \\ K''(x_0, t) &= \frac{1}{2h^2} (14a_{-1} - 33a_0 + 28a_1 - 14a_2 + 6a_3 - a_4) + O(h^4), \\ K''(x_r, t) &= \frac{1}{2h^2} (a_{r-2} + 8a_{r-1} - 18a_r + 8a_{r+1} + a_{r+2}) + O(h^4), \quad r = 1, 2, \dots, N-1, \end{aligned}$$



$$\begin{aligned}
K''(x_N, t) &= \frac{1}{2h^2} (-a_{N-4} + 6a_{N-3} - 14a_{N-2} + 28a_{N-1} - 33a_N + 14a_{N+1}) + O(h^4), \\
P(x_r, t) &= b_{r-1} + 4b_r + b_{r+1} + O(h^4), \quad r = 0, 1, \dots, N, \\
P'(x_r, t) &= \frac{-3}{h} (b_{r-1} - b_{r+1}) + O(h^4), \quad r = 0, 1, \dots, N, \\
P''(x_0, t) &= \frac{1}{2h^2} (14b_{-1} - 33b_0 + 28b_1 - 14b_2 + 6b_3 - b_4) + O(h^4), \\
P''(x_r, t) &= \frac{1}{2h^2} (b_{r-2} + 8b_{r-1} - 18b_r + 8b_{r+1} + b_{r+2}) + O(h^4), \quad r = 1, 2, \dots, N-1, \\
P''(x_N, t) &= \frac{1}{2h^2} (-b_{N-4} + 6b_{N-3} - 14b_{N-2} + 28b_{N-1} - 33b_N + 14b_{N+1}) + O(h^4).
\end{aligned} \tag{2.30}$$

3. IMPLEMENTATION OF THE SUGGESTED METHODS

Inserting Eq. (2.14) into Eq. (2.11), using (2.30) and choosing the following parameters

$$\begin{aligned}
c_1 &= 1 + \beta\theta_3 + \gamma\theta_3(K^2)_r^{n+1}, & c_2 &= 1 - \beta\theta_4 - \gamma\theta_4(K^2)_r^n, \\
c_3 &= \beta\theta_1 + \gamma\theta_1(K^2)_r^{n+1}, & c_4 &= 1 + \beta\theta_3 + 3\gamma\theta_3(K^2)_r^{n+1}, \\
c_5 &= -\beta\theta_2 - \gamma\theta_2(K^2)_r^n, & c_6 &= 1 - \beta\theta_4 - 3\gamma\theta_4(K^2)_r^n,
\end{aligned} \tag{3.1}$$

we get

$$\begin{aligned}
&(c_1 + \frac{7\alpha\theta_3}{h^2})a_{-1}^{n+1} + (4c_1 - \frac{33\alpha\theta_3}{2h^2})a_0^{n+1} \\
&+ (c_1 + \frac{14\alpha\theta_3}{h^2})a_1^{n+1} - \frac{7\alpha\theta_3}{h^2}a_2^{n+1} + \frac{3\alpha\theta_3}{h^2}a_3^{n+1} - \frac{\alpha\theta_3}{2h^2}a_4^{n+1} \\
&- \theta_1 b_{-1}^{n+1} - 4\theta_1 b_0^{n+1} - \theta_1 b_1^{n+1} = t(x_0, t_n), \\
&\frac{\alpha\theta_3}{2h^2}a_{r-2}^{n+1} + (c_1 + \frac{4\alpha\theta_3}{h^2})a_{r-1}^{n+1} + (4c_1 - \frac{9\alpha\theta_3}{h^2})a_r^{n+1} \\
&+ a_{r+1}^{n+1}(c_1 + \frac{4\alpha\theta_3}{h^2}) + \frac{\alpha\theta_3}{2h^2}a_{r+2}^{n+1} - \theta_1 b_{r-1}^{n+1} - 4\theta_1 b_r^{n+1} \\
&- \theta_1 b_{r+1}^{n+1} = t(x_r, t_n), \quad 1 \leq r \leq N-1, \\
&-\frac{\alpha\theta_3}{2h^2}a_{N-4}^{n+1} + \frac{3\alpha\theta_3}{h^2}a_{N-3}^{n+1} - \frac{7\alpha\theta_3}{h^2}a_{N-2}^{n+1} \\
&+ (c_1 + \frac{14\alpha\theta_3}{h^2})a_{N-1}^{n+1} + (4c_1 - \frac{33\alpha\theta_3}{2h^2})a_N^{n+1} + (c_1 + \frac{7\alpha\theta_3}{h^2})a_{N+1}^{n+1} \\
&- \theta_1 b_{N-1}^{n+1} - 4\theta_1 b_N^{n+1} - \theta_1 b_{N+1}^{n+1} = t(x_N, t_n)
\end{aligned} \tag{3.2}$$

where

$$t(x_r, t_n) = c_2 K_r^n - \alpha\theta_4(K_r'')^n + \theta_2 P_r^n + \theta_4 f(x_r, t_n) + \theta_3 f(x_r, t_{n+1}), \quad 0 \leq r \leq N.$$

In a similar way, plugging Eq. (2.14) into Eq. (2.12), employing (2.30) and taking the parameters given in Eq. (3.1), we obtain

$$\begin{aligned}
&(c_3 + \frac{7\alpha\theta_1}{h^2})a_{-1}^{n+1} + (4c_3 - \frac{33\alpha\theta_1}{2h^2})a_0^{n+1} + (c_3 + \frac{14\alpha\theta_1}{h^2})a_1^{n+1} - \frac{7\alpha\theta_1}{h^2}a_2^{n+1} \\
&+ \frac{3\alpha\theta_1}{h^2}a_3^{n+1} - \frac{\alpha\theta_1}{2h^2}a_4^{n+1} + (c_4 + \frac{7\alpha\theta_3}{h^2})b_{-1}^{n+1} + (4c_4 - \frac{33\alpha\theta_3}{2h^2})b_0^{n+1} \\
&+ (c_4 + \frac{14\alpha\theta_3}{h^2})b_1^{n+1} - \frac{7\alpha\theta_3}{h^2}b_2^{n+1} + \frac{3\alpha\theta_3}{h^2}b_3^{n+1} - \frac{\alpha\theta_3}{2h^2}b_4^{n+1} = r(x_0, t_n), \\
&\frac{\alpha\theta_1}{2h^2}a_{r-2}^{n+1} + (c_3 + \frac{4\alpha\theta_1}{h^2})a_{r-1}^{n+1} + (4c_3 - \frac{9\alpha\theta_1}{h^2})a_r^{n+1} + (c_3 + \frac{4\alpha\theta_1}{h^2})a_{r+1}^{n+1} \\
&+ \frac{\alpha\theta_1}{2h^2}a_{r+2}^{n+1} + \frac{\alpha\theta_3}{2h^2}b_{r-2}^{n+1} + (c_4 + \frac{4\alpha\theta_3}{h^2})b_{r-1}^{n+1} + (4c_4 - \frac{9\alpha\theta_3}{h^2})b_r^{n+1} \\
&+ (c_4 + \frac{4\alpha\theta_3}{h^2})b_{r+1}^{n+1} + \frac{\alpha\theta_3}{2h^2}b_{r+2}^{n+1} = r(x_r, t_n), \quad 1 \leq r \leq N-1, \\
&\frac{\alpha\theta_1}{2h^2}a_{N-4}^{n+1} + \frac{3\alpha\theta_1}{h^2}a_{N-3}^{n+1} - \frac{7\alpha\theta_1}{h^2}a_{N-2}^{n+1} + (c_3 + \frac{14\alpha\theta_1}{h^2})a_{N-1}^{n+1} \\
&+ (4c_3 - \frac{33\alpha\theta_1}{2h^2})a_N^{n+1} + (c_3 + \frac{7\alpha\theta_1}{h^2})a_{N+1}^{n+1} + \frac{-\alpha\theta_3}{2h^2}b_{N-4}^{n+1} + \frac{3\alpha\theta_3}{h^2}b_{N-3}^{n+1} \\
&- \frac{7\alpha\theta_3}{h^2}b_{N-2}^{n+1} + (c_4 + \frac{14\alpha\theta_3}{h^2})b_{N-1}^{n+1} + (4c_4 - \frac{33\alpha\theta_3}{2h^2})b_N^{n+1} \\
&+ (c_4 + \frac{14\alpha\theta_3}{2h^2})b_{N+1}^{n+1} = r(x_N, t_n)
\end{aligned} \tag{3.3}$$



where for $0 \leq r \leq N$

$$r(x_r, t_n) = c_5 K_r^n - \alpha \theta_2 \left(K'' \right)_r^n + c_6 P_r^n - \alpha \theta_4 \left(P'' \right)_r^n + \theta_1 f(x_r, t_{n+1}) + \theta_2 f(x_r, t_n) + \theta_3 f_t(x_r, t_{n+1}) + \theta_4 f_t(x_r, t_n).$$

By merging Eqs. (3.2) and (3.3), a system comprising $2N + 2$ equations in $2N + 6$ unknowns is obtained, where the unknown parameters are as follows:

$$\mathbf{a} = (a_{-1}^{n+1}, a_0^{n+1}, \dots, a_{N+1}^{n+1})^T, \quad (3.4)$$

and

$$\mathbf{b} = (b_{-1}^{n+1}, b_0^{n+1}, \dots, b_{N+1}^{n+1})^T. \quad (3.5)$$

Using boundary conditions enable us to equalize the number of equations and unknown parameters and the unknown parameters a_{-1}^{n+1} , a_{N+1}^{n+1} , b_{-1}^{n+1} , b_{N+1}^{n+1} are eliminated with the help of the following BCs

$$\begin{aligned} w(a, t) &= \Phi_0(t), & u(b, t) &= \Phi_1(t), \\ v(a, t) &= \frac{\partial \Phi_0}{\partial t}(t), & v(b, t) &= \frac{\partial \Phi_1}{\partial t}(t). \end{aligned} \quad (3.6)$$

Therefore, we obtain solvable matrix system consisting of $2N + 2$ equation in $2N + 2$ unknown parameters. Then, the initial vector

$$(a_{-1}^0, b_{-1}^0, a_0^0, b_0^0, \dots, a_{N+1}^0, b_{N+1}^0)^T, \quad (3.7)$$

is determined with the use of BCs and ICs of the considered problem. After computing the initial parameters $\mathbf{a}^0$ and $\mathbf{b}^0$, the unknown vector $(a_{-1}^{n+1}, b_{-1}^{n+1}, \dots, a_{N+1}^{n+1}, b_{N+1}^{n+1})^T$ for $n = 0, 1, 2, \dots$ at any specified time level is obtained through an inner iteration procedure.

4. NUMERICAL EXPERIMENTS AND DISCUSSIONS

To evaluate the applicability and efficiency of the suggested computational techniques, two benchmark problems are considered. The L_∞ error norm is computed using the following expression:

$$L_\infty = \max_m |u - U|, \quad (4.1)$$

where u and U represent analytic solution and approximate solution, respectively. The pointwise rates of convergence in both time and space are evaluated using the following expressions:

$$order = \frac{\log \left| \frac{(L_\infty)_{h_i}}{(L_\infty)_{h_{i+1}}} \right|}{\log \left| \frac{h_i}{h_{i+1}} \right|}, \quad (4.2)$$

$$order = \frac{\log \left| \frac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}} \right|}{\log \left| \frac{\Delta t_i}{\Delta t_{i+1}} \right|}. \quad (4.3)$$

Here, $(L_\infty)_{h_i}$ and $(L_\infty)_{\Delta t_i}$ stand for the L_∞ error norm corresponding to spatial step size h_i and temporal step size Δt_i , respectively.

Note that when we use CN method and cubic B-spline collocation method for time and space discretizations, respectively, the obtained method is called as M_1 method which is $O(h^2 + \Delta t^2)$ accurate and when fourth-order one-step method and FOCBCM are employed for time and space discretizations, respectively, the obtained method is referred to as M_2 method which is $O(h^4 + \Delta t^4)$ accurate



4.1. **Test problem 1.** Consider the following nonlinear KGE

$$u_{tt} - \frac{5}{2}u_{xx} + u + \frac{3}{2}u^3 = 0,$$

over interval $[0, 1]$ for the space variable. The ICs and BCs are chosen as follows:

$$\begin{aligned} u(x, 0) &= B \tan(Kx), & u_t(x, 0) &= BcK \sec^2(Kx), \\ u(0, t) &= B \tan(Kct), & u(1, t) &= B \tan(K + Kct). \end{aligned}$$

The analytic solution of this problem is given by

$$u(x, t) = B \tan(K(x + ct)),$$

where $B = \sqrt{\frac{2}{3}}$ and $K = \sqrt{\frac{-1}{2(-2.5+c^2)}}$.

This problem is solved employing fixed $\Delta t = 0.001$ time step, while varying the spatial step size h with values $h = 0.01$, $h = 0.02$, $h = 0.05$. L_∞ errors for $c = 0.5$ and $c = 0.05$ are reported in Tables 1 and 2, respectively. In view of the results in Tables 1 and 2, the suggested methods, particularly M_2 method produce much better or almost similar results than methods given in [3, 9, 17, 19]. The evolution of the analytical and approxiamte solutions for $c = 0.5$ and $c = 0.05$ is depicted in Figures 1 and 3, respectively, at multiple time levels. The simulations of the absolute errors of the M_2 method for values of $c = 0.5$ and $c = 0.05$ are given in Figures 2 and 4 at $t = 4$.

TABLE 1. The error norms for test problem 1 with $c = 0.5, \Delta t = 0.001$.

Method	h	$t = 1$	$t = 2$	$t = 3$	$t = 4$
M_1	0.02	2.46×10^{-6}	9.23×10^{-6}	4.63×10^{-5}	8.21×10^{-4}
M_1	0.01	6.17×10^{-7}	2.31×10^{-6}	1.16×10^{-5}	2.04×10^{-4}
M_2	0.02	2.05×10^{-10}	1.22×10^{-9}	1.40×10^{-8}	1.21×10^{-6}
M_2	0.01	1.30×10^{-11}	7.75×10^{-11}	9.02×10^{-10}	8.76×10^{-8}
[3]	0.01	5.99×10^{-6}	2.19×10^{-5}	9.08×10^{-5}	8.29×10^{-4}
[19]	0.02	5.21×10^{-6}	2.18×10^{-5}	9.01×10^{-5}	8.23×10^{-4}
[17]	0.01	2.69×10^{-8}	8.74×10^{-8}	3.09×10^{-7}	1.93×10^{-6}
[9]	0.01	1.69×10^{-11}	8.18×10^{-11}	8.42×10^{-10}	7.63×10^{-8}

TABLE 2. The error norms for test problem 1 with $c = 0.05, \Delta t = 0.001$.

Method	h	$t = 1$	$t = 2$	$t = 3$	$t = 4$
M_1	0.05	3.61×10^{-6}	7.46×10^{-6}	8.67×10^{-6}	6.18×10^{-6}
M_1	0.01	1.45×10^{-7}	2.99×10^{-7}	3.49×10^{-7}	2.40×10^{-7}
M_2	0.05	1.15×10^{-9}	2.40×10^{-9}	2.83×10^{-9}	2.08×10^{-9}
M_2	0.01	2.10×10^{-12}	4.10×10^{-12}	4.87×10^{-12}	3.61×10^{-12}
[3]	0.01	3.64×10^{-7}	3.89×10^{-7}	4.21×10^{-7}	4.59×10^{-7}
[19]	0.05	2.17×10^{-7}	3.06×10^{-7}	3.70×10^{-7}	3.42×10^{-7}
[17]	0.01	1.19×10^{-8}	2.47×10^{-8}	2.89×10^{-8}	1.99×10^{-8}
[9]	0.01	4.05×10^{-12}	4.32×10^{-12}	5.07×10^{-12}	

Also, the spatial and temporal orders of convergence for both suggested techniques, together with the error norm L_∞ are provided in Tables 3 and 4, respectively. It is evident from Tables 3 and 4 that the accuracies of the M_1 and M_2 in both time and space are two and four, respectively. which is consistent with theoretical expectations. Also, Table 5 presents the convergence order of M_2 method using simultaneous refinement of h and Δt , which aligns with the expected theoretical convergence order of M_2 method. Therefore the proposed methods, especially M_2 are quite satisfactory.



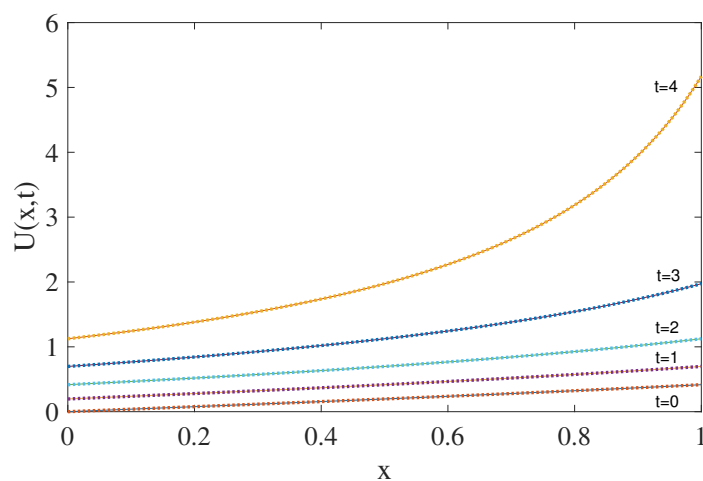


FIGURE 1. Exact (—) and numerical (···) solutions for the problem 1 at $t = 0, 1, 2, 3, 4$ with $c = 0.5$.

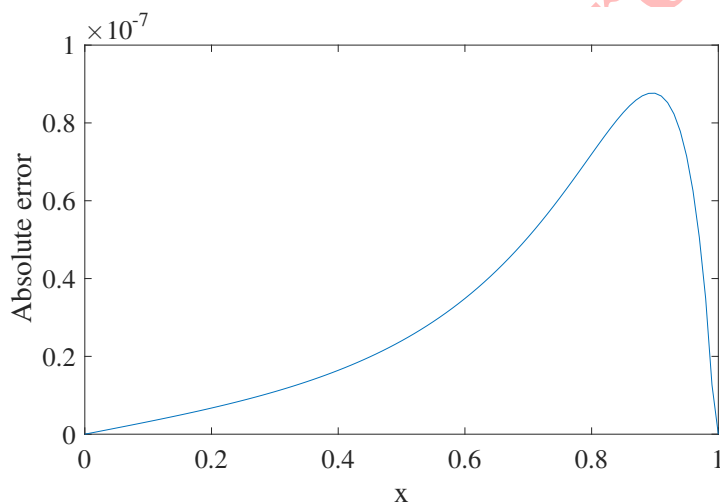


FIGURE 2. Absolute error for M_2 with $c = 0.5$, $h = \Delta t = 0.01$.

TABLE 3. The errors L_∞ and rate of spatial convergence with $\Delta t = 0.001$ and $c = 0.5$ at $t = 1$.

h	M_1	Order	M_2	Order
0.1	5.99×10^{-5}	—	8.99×10^{-8}	—
0.05	1.53×10^{-5}	1.97	7.31×10^{-9}	3.62
0.025	3.85×10^{-6}	1.99	4.96×10^{-10}	3.88
0.0125	9.63×10^{-7}	2.00	3.17×10^{-11}	3.97

4.2. **Test problem 2.** Consider the nonlinear KGE given by

$$u_{tt} - u_{xx} + u + u^3 = f(x, t),$$

with $f(x, t) = (x^2 - 2) \cosh(x + t) - 4x \sinh(x + t) + x^6 \cosh^3(x + t)$ over interval $[-1, 1]$ for space variable.



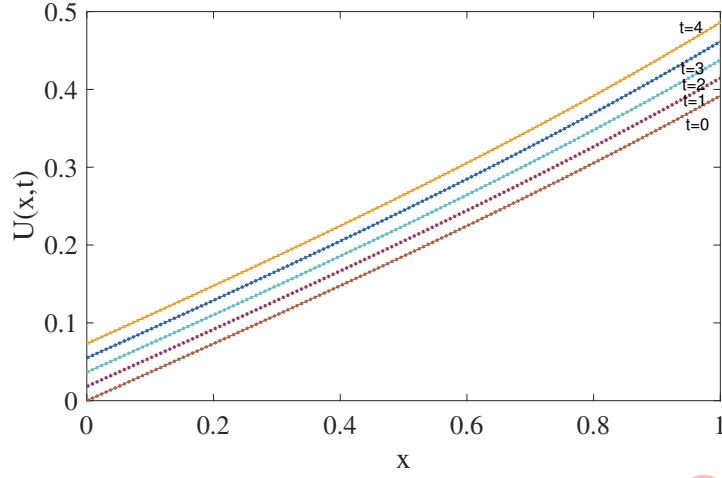


FIGURE 3. Exact (—) and numerical ($\cdots$) solution for the problem 1 at $t = 0, 1, 2, 3, 4$ with $c = 0.05$.

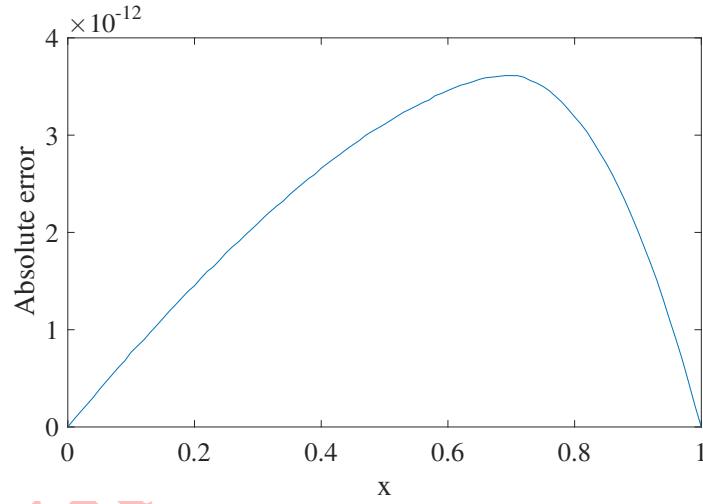


FIGURE 4. Absolute error for M_2 with $c = 0.05$, $h = \Delta t = 0.01$.

TABLE 4. The errors L_∞ and rate of temporal convergence with $h = 0.001$ and $c = 0.5$ at $t = 1$.

Δt	M_1	Order	M_2	Order
0.1	3.24×10^{-6}	—	4.12×10^{-10}	—
0.05	8.57×10^{-7}	1.92	2.67×10^{-11}	3.95
0.025	2.30×10^{-7}	1.90	1.68×10^{-12}	3.99
0.0125	5.96×10^{-8}	1.95	1.84×10^{-13}	3.19

The ICs and BCs are taken as follows:

$$\begin{aligned}
 u(x, 0) &= x^2 \cosh(x), & u_t(x, 0) &= x^2 \sinh(x), \\
 u(-1, t) &= \cosh(-1 + t), & u(1, t) &= \sinh(1 + t).
 \end{aligned}$$



TABLE 5. Convergence order for M_2 with $c = 0.1$ at $t = 1$.

h	Δt	M_2	Order
0.1	0.1	8.95×10^{-8}	—
0.05	0.05	7.32×10^{-9}	3.61
0.025	0.025	4.95×10^{-10}	3.89
0.0125	0.0125	3.17×10^{-11}	3.97

The analytical solution of this problem is

$$u(x, t) = x^2 \cosh(x + t).$$

The numerical computations are carried out by taking $h = 0.04$, $h = 0.01$ and $\Delta t = 0.0001$. Error norms L_∞ are presented in Table 6 which confirms that the suggested methods, particularly the M_2 method demonstrate superior accuracy compared to the results reported by other approaches given in [3, 17, 19]. Moreover, we compute the spatial and temporal convergence orders of both proposed methods. Table 7 gives the errors and the rate of spatial convergence for M_1 and M_2 methods. Table 8 displays the errors and the rate of temporal convergence for both proposed methods. The numerical finding verify that the M_1 method achieves second-order accuracy in both temporal and spatial directions, while the M_2 method exhibits fourth-order accuracy in both directions. The simulation of the analytical and approximate solutions at various time levels up to $t = 5$ is graphically illustrated in Figure 5. The absolute error corresponding to the M_2 method at $t = 5$ is depicted in Figure 6.

TABLE 6. The error norms for test problem 2 with $\Delta t = 0.0001$.

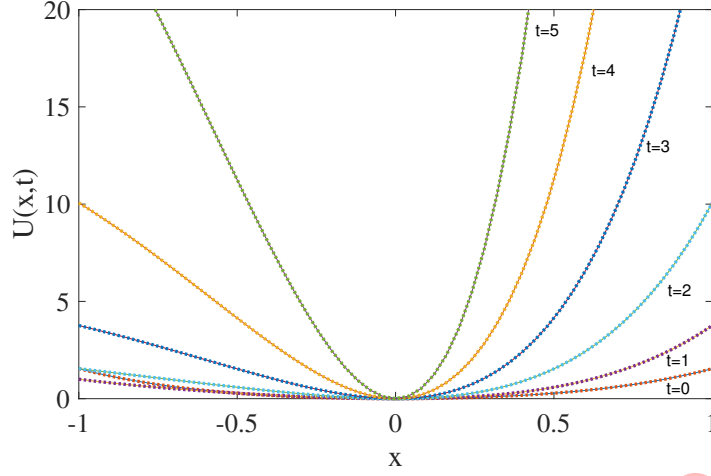
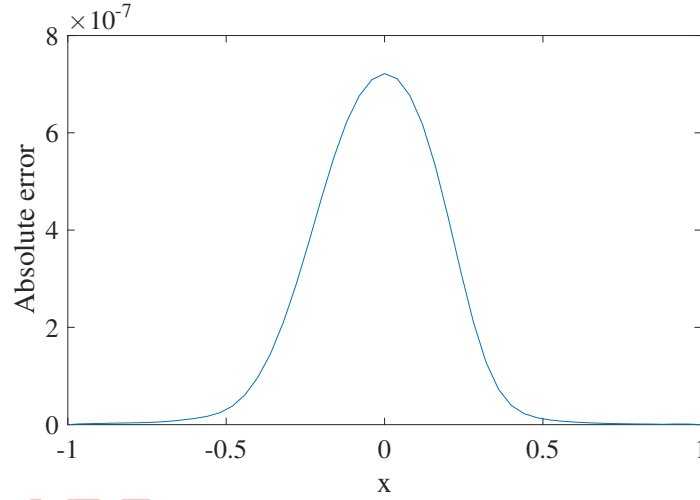
Method	h	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
M_1	0.04	1.01×10^{-3}	1.93×10^{-3}	2.22×10^{-3}	4.33×10^{-3}	5.36×10^{-3}
M_1	0.01	6.34×10^{-5}	1.21×10^{-4}	1.39×10^{-4}	2.71×10^{-4}	3.36×10^{-4}
M_2	0.04	1.26×10^{-7}	2.47×10^{-7}	2.94×10^{-7}	5.70×10^{-7}	7.22×10^{-7}
M_2	0.01	4.98×10^{-10}	9.68×10^{-10}	1.15×10^{-9}	2.23×10^{-9}	2.81×10^{-9}
[3]	0.01	5.07×10^{-5}	5.02×10^{-4}	2.06×10^{-3}	6.57×10^{-3}	1.90×10^{-2}
[19]	0.04	4.26×10^{-5}	4.93×10^{-4}	2.03×10^{-3}	6.41×10^{-3}	1.79×10^{-2}
[17]	0.01	3.56×10^{-6}	3.19×10^{-6}	3.96×10^{-6}	5.68×10^{-6}	6.33×10^{-6}

TABLE 7. The errors and rate of spatial convergence with $\Delta t = 0.001$ at $t = 1$.

h	M_1	Order	M_2	Order
0.1	6.34×10^{-3}	—	4.69×10^{-6}	—
0.05	1.58×10^{-3}	2.00	3.06×10^{-7}	3.94
0.025	3.97×10^{-4}	1.99	1.94×10^{-8}	3.98
0.0125	9.91×10^{-5}	2.00	1.22×10^{-9}	4.00

TABLE 8. The errors and rate of temporal convergence with $h = 0.001$ at $t = 1$.

Δt	M_1	Order	M_2	Order
0.1	1.82×10^{-4}	—	2.77×10^{-8}	—
0.05	4.27×10^{-5}	2.09	1.73×10^{-9}	4.00
0.025	9.86×10^{-6}	2.12	1.07×10^{-10}	4.01
0.0125	2.02×10^{-6}	2.28	7.18×10^{-12}	3.89

FIGURE 5. Exact solution for the problem 2 at $t = 0, 1, 2, 3, 4, 5$.FIGURE 6. Absolute error for M_2 with $h = 0.01, \Delta t = 0.0001$.

5. CONCLUSION

In this work, we have developed two compact techniques for computing the numerical solution of KGE. The local truncation error analysis is performed for the temporal discretization scheme. It is numerically proved that the M_1 and M_2 methods exhibit second and fourth order accuracy in both spatial and temporal directions, respectively. To validate the effectiveness and enhanced numerical performance of the proposed methods, they are applied to two standard benchmark problems commonly used in the literature. The obtained results are compared with other techniques taken from the literature. Also, the proposed methods are compared with each other in terms of the advantages of the M_2 method. The results emphasize that the present techniques, particularly the M_2 method are capable of producing highly accurate solutions and M_2 technique demonstrates superior performance in addressing the approximate solution of KGE. The advantages of the suggested techniques over existing methods are that they are compact, easy to implement and, especially M_2 method provide far better results than the methods given by [3, 17, 19]. In conclusion,

it is seen that the M_2 method can be a rather good choice to achieve a high order of accuracy while dealing with similar types of nonlinear PDE.

REFERENCES

- [1] W. Bao and X. Dong, *Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime*, Numerische Mathematik, 120(2) (2012), 189–229.
- [2] B. Bülbül and M. Sezer, *A new approach to numerical solution of nonlinear Klein-Gordon equation*, Mathematical Problems in Engineering, 2013 (2013).
- [3] M. Dehghan and A. Shokri, *Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions*, Journal of Computational and Applied Mathematics, 230(2) (2009), 400–410.
- [4] D. B. Duncan, *Symplectic finite difference approximations of the nonlinear Klein-Gordon equation*, SIAM Journal on numerical analysis, 34(5) (1997), 1742–1760.
- [5] O. E. Hepson, A. Korkmaz, and I. Dag, *On the numerical solution of the Klein-Gordon equation by exponential cubic B-spline collocation method*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1) (2018), 412–421.
- [6] E. Hesameddini and S. Shekarpaz, *Wavelet solutions of the Klein-Gordon equation*, Journal of Mahani Mathematical Research, 1(1) (2012), 29–45.
- [7] A. Hussain, S. Haq, and M. Uddin, *Numerical solution of Klein-Gordon and sine-Gordon equations by meshless method of lines*, Engineering Analysis with Boundary Elements, 37(11) (2013), 1351–1366.
- [8] P. F. Guo, K. M. Liew, and P. Zhu, *Numerical solution of nonlinear Klein-Gordon equation using the element-free kp-Ritz method*, Applied Mathematical Modelling, 39(10-11) (2015), 2917–2928.
- [9] D. Irk, E. Kirli, and M. Z. Gorgulu, *A High Order Accurate Numerical Solution of the Klein-Gordon Equation*, Applied Mathematics & Information Sciences An International Journal, 16(2) (2022), 331–339.
- [10] D. Kaya and S. M. El-Sayed, *A numerical solution of the Klein-Gordon equation and convergence of the decomposition method*, Applied mathematics and computation, 156(2) (2004), 341–353.
- [11] M. Khalid, M. Sultana, F. Zaidi, and U. Arshad, *Solving linear and nonlinear Klein-Gordon equations by new perturbation iteration transform method*, TWMS Journal of Applied and Engineering Mathematics, 6(1) (2016), 115–125.
- [12] M. E. Khalifa and M. Elgamal, *A numerical solution to Klein-Gordon equation with Dirichlet boundary condition*, Applied mathematics and computation, 160(2) (2005), 451–475.
- [13] E. Kırıl, D. Irk, and M. Zorsahin, *An improved B-spline computational scheme for time fractional Burger equation*, ZAMM - Journal of Applied Mathematics and Mechanics, 105(e70237) (2025).
- [14] X. Li and B.Y. Guo, *A Legendre pseudospectral method for solving nonlinear Klein-Gordon equation*, Journal of Computational Mathematics, 15 (1997), 105–126.
- [15] Q. Li, Z. Ji, Z. Zheng, and H. Liu, *Numerical solution of nonlinear Klein-Gordon equation using Lattice Boltzmann method*, Applied Mathematics, 2(12) (2011), 1479–1485.
- [16] T. R. Lucas, *Error bounds for interpolating cubic splines under various end conditions*, SIAM Journal on Numerical Analysis, 11(3) (1974), 569–584.
- [17] J. Rashidinia, M. Ghasemi, and R. Jalilian, *Numerical solution of the nonlinear Klein-Gordon equation*, Journal of Computational and Applied Mathematics, 233(8) (2010), 1866–1878.
- [18] S. B. Sadigh, *Numerical Solution Of Klein-Gordon Equation By Using The Adomian's Decomposition And Variational Iterative Methods*, International Journal of Industrial Mathematics, 3 (2011), 79–89.
- [19] M. Sarboland and A. Aminataei, *Numerical solution of the nonlinear Klein-Gordon equation using multiquadric quasi-interpolation scheme*, Universal Journal of Applied Mathematics, 3(3) (2015), 40–49.
- [20] H. Selvitopi and M. Yazıcı, *Numerical results for the Klein-Gordon equation in de Sitter spacetime*, Mathematical Methods in the Applied Sciences, 42(16) (2019), 5446–5454.
- [21] F. Shakeri and M. Dehghan, *Numerical solution of the Klein-Gordon equation via He's variational iteration method*, Nonlinear Dynamics, 51 (2008), 89–97.



- [22] S. C. Shiralashetti, L. M. Angadi, A. B. Deshi, and M. H. Kantli, *Haar wavelet method for the numerical solution of Klein–Gordon equations*, Asian-European Journal of Mathematics, 9 (2016).
- [23] A. Verma, R. Jiwari, and S. Kumar, *A numerical scheme based on differential quadrature method for numerical simulation of nonlinear Klein–Gordon equation*, International Journal of Numerical Methods for Heat & Fluid Flow, 24(7) (2014), 1390–1404.
- [24] A. M. Wazwaz, *New travelling wave solutions to the Boussinesq and the Klein–Gordon equations*, Communications in Nonlinear Science and Numerical Simulation, 13(5) (2008), 889–901.
- [25] H. Yang, *High-order energy and linear momentum conserving methods for the Klein–Gordon equation*, Mathematics, 6(10) (2018).

Uncorrected Proof

