



Stability and existence results for a system of Hilfer-Hadamard fractional differential equations with the p-laplacian operator

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Abstract

In this paper, we establish the existence and uniqueness of solutions for a class of Hilfer-Hadamard fractional differential equations involving the p-Laplacian operator. The analysis is carried out using Banach's and Schauder's fixed point theorems. Moreover, we investigate the stability of the system in the framework of Ulam-Hyers and generalized Ulam-Hyers stability concepts. To demonstrate the applicability of the theoretical results, a representative example is provided.

Keywords. Hilfer-Hadamard fractional, Arzelà-Ascoli theorem, p-Laplacian, Fixed point Theorem.

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1. INTRODUCTION

The investigation of existence and uniqueness of solutions for fractional differential equations (FDEs) plays a crucial role in the theory of such systems. In addition, the stability analysis of FDEs is fundamental for understanding the behavior of solutions under small perturbations, which is essential for ensuring the robustness and reliability of models in practical applications. Due to their wide-ranging applications in real-world phenomena as well as various fields of science and engineering, FDEs have garnered significant attention from researchers. For further details, we refer the reader to [2, 3, 6, 7, 9] and the references therein.

Since 2017, considerable attention has been devoted to the study of the Hilfer-Hadamard fractional derivative, particularly its properties and applications in the context of differential equations. In this regard, researchers have extensively examined the existence and uniqueness of solutions to equations involving this operator. Moreover, such equations have been analyzed within various stability frameworks, including Ulam-Hyers stability and its generalized forms. For relevant contributions, see [1, 11].

In recent years, fractional differential equations involving p -Laplacian operators have gained significant importance due to their ability to model a variety of propagating phenomena. These equations have found widespread applications in fields such as fluid mechanics, chemistry, and other related areas. For further details, refer to [4, 5, 8, 16].

Throughout this paper, the term Hilfer-Hadamard is abbreviated as H-H.

In [17], Zada et al. investigated a nonlinear consecutively coupled system of Caputo fractional differential equations subject to boundary conditions involving Riemann-Liouville fractional integrals of the form

$$\begin{cases} ({}^c D^q + k {}^c D^{q-1})x(t) = f(t, x(t), y(t)), & t \in [0, T], & 2 < q \leq 3, \\ ({}^c D^p + k {}^c D^{p-1})y(t) = g(t, x(t), y(t)), & t \in [0, T], & 2 < p \leq 3, \\ x(0) = 0, & x(T) = \sum_{i=1}^n \alpha_i I^{\rho_i} y(\eta_i), \\ y(0) = 0, & y(T) = \sum_{j=1}^m \beta_j I^{\gamma_j} x(\theta_j), \end{cases}$$

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where ${}^c D^{(\cdot)}$ give the meaning the caputo fractional derivatives of order p and q , $I^{(\cdot)}$ give the meaning the Riemann-Liouville fractional integral of order ρ_i and $\gamma_j > 0$, $\eta_i, \theta_j \in (0, T)$, $k \in \mathbb{R}^+$. The functions $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are assumed to be continuous, and $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ are real constants such that

$$\sum_{i=1}^n \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \cdot \sum_{j=1}^m \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \neq T^2.$$

In [10], the authors investigated the existence and uniqueness of solutions for a system of H-H consecutively coupled fractional differential equations of the following form

$$\begin{cases} ({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1})u(t) = f(t, u(t), v(t)), & t \in [1, e], \\ ({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2})v(t) = g(t, u(t), v(t)), & t \in [1, e], \\ u(1) = 0, & u(e) = A_1, \\ v(1) = 0, & v(e) = A_2, \end{cases}$$

where ${}_H D^{\alpha_i, \beta_i}$ is the H-H fractional derivative of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}$, $k_1, k_2, A_1, A_2 \in \mathbb{R}^+$ and $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions.

Newly, in [13], the authors transformed a coupled system of consecutive Hilfer fractional differential equations from the referenced paper into a coupled system of consecutive H-H fractional differential equation

$$\begin{cases} ({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1})u(t) = f(t, u(t), v(t)), & t \in [1, e], \\ ({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2})v(t) = g(t, u(t), v(t)), & t \in [1, e], \\ u(1) = 0, & u(e) = \lambda v(\theta), & 1 < \theta < e, \\ v(1) = 0, & v(e) = \mu u(\eta), & 1 < \eta < e, \end{cases}$$

where ${}_H D^{\alpha_i, \beta_i}$ give the meaning the H-H fractional derivatives of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}$, $k_1, k_2 \in \mathbb{R}^+$ and $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions, and λ and μ are real constants.

In [14], the authors developed the results of [13] to a coupled system of consecutive H-H fractional differential equations

$$\begin{cases} ({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1})u(t) = f(t, u(t), v(t)), & t \in [1, e], \\ ({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2})v(t) = g(t, u(t), v(t)), & t \in [1, e], \\ u(1) = 0, & u(e) = \sum_{i=1}^n \lambda_i I_{1+}^{\delta_i} v(\theta_i) = \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i-1} v(s) ds, \\ v(1) = 0, & v(e) = \sum_{j=1}^n \mu_j I_{1+}^{\sigma_j} u(\eta_j) = \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j-1} u(s) ds, \end{cases}$$

where $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions and ${}_H D^{\alpha_i, \beta_i}$ give the meaning the H-H fractional derivatives of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}$, $k_1, k_2 \in \mathbb{R}^+$ and I^{δ_i} . I^{σ_j} are the Riemann-Liouville fractional integral of positive order θ_i and $\eta_j \in (1, e)$, and λ_i, μ_j , $1 \leq i, j \leq n$ are real constants.

Inspired by research conducted in this regard, in the present paper we develop the results of [13] and [14] to a coupled system of consecutive H-H fractional differential equations with the p -laplacian operator

$$\begin{cases} {}_H D_{1+}^{\gamma_1, \beta_1} \left(\varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1})u(t) \right) \right) = f(t, u(t), v(t)), & t \in [1, e], \\ {}_H D_{1+}^{\gamma_2, \beta_2} \left(\varphi_{p_2} \left(({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2})v(t) \right) \right) = g(t, u(t), v(t)), & t \in [1, e], \\ u(1) = 0, & u(e) = \lambda v(\theta), & ({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1})u(1) = 0, \\ v(1) = 0, & v(e) = \mu u(\eta), & ({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2})v(1) = 0, \end{cases} \quad (1.1)$$



where ${}_H D^{\alpha_i, \beta_i}$ and ${}_H D^{\gamma_i, \beta_i}$ give the meaning the H-H fractional derivatives of order $\alpha_i \in (1, 2]$, $\gamma_i \in (0, 1]$, type $\beta_i \in [0, 1]$ and $\varphi_{p_i}(s) := |s|^{p_i-2}s$, $p_i > 1$, $k_1, k_2 \in \mathbb{R}^+$, $\lambda, \mu \in \mathbb{R}$, $\theta, \eta \in (1, e)$ and $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions.

The continuation of this article is as follows: In Section 2, we revisit essential definitions and fundamental characteristics pertinent to the study. Section 3 establishes several sufficient criteria ensuring the existence of solutions for the boundary value problem (1.1) and we rigorously analyze the stability of solutions, focusing on both Ulam-Hyers and generalized Ulam-Hyers stability concepts. Finally, Section 4 demonstrates an illustrative example to substantiate and elucidate the theoretical findings presented.

2. PRELIMINARIES

We commence by recalling essential definitions, fundamental theorems, and pivotal Hyers-Ulam stability results as delineated in the seminal references [3, 6, 7], which form the theoretical backbone of the present investigation. Consider the space $L^1[a, b]$ as a Banach space for Lebesgue integrable functions, and also consider $AC[a, b]$ as the space of absolutely continuous functions, and $AC_\delta^n[a, b]$ as the space of n -times δ -absolutely continuous functions differentiable on the interval $[a, b]$. Everywhere in this paper, $[\alpha]$ denotes the integer part of the real number α and we define $n = [\alpha] + 1$.

Definition 2.1. (The Riemann-Liouville fractional integral) The Riemann-Liouville integral of order $\alpha > 0$ of a function $f : [a, \infty) \rightarrow \mathbb{R}$ is defined as

$$I_{a+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a.$$

Definition 2.2. (Hadamard fractional integral) The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ for a function $f : [a, \infty) \rightarrow \mathbb{R}$ is defined as

$${}_H I_{a+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log_e \frac{t}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t > a.$$

provided the integral exists.

Definition 2.3. (Hadamard fractional derivative) The Hadamard fractional derivative of order $\alpha > 0$, applied to the function $f : [a, \infty) \rightarrow \mathbb{R}$, is defined as

$${}_H D_{a+}^\alpha f(t) := \delta^n ({}_H I_{a+}^{n-\alpha} f(t)), \quad n-1 < \alpha < n,$$

where $\delta^n = (t \frac{d}{dt})^n$.

Definition 2.4. (H-H fractional derivative) Let $n-1 < \alpha \leq n$, $0 \leq \beta \leq 1$, and $f \in L^1[a, b]$. The H-H fractional derivative of order α and type β of f is defined as

$$\begin{aligned} ({}_H D_{a+}^{\alpha, \beta} f)(t) &= ({}_H I_{a+}^{\beta(n-\alpha)} \delta^n {}_H I_{a+}^{(n-\alpha)(1-\beta)} f)(t) \\ &= ({}_H I_{a+}^{\beta(n-\alpha)} \delta^n {}_H I_{a+}^{n-\gamma} f)(t), \quad \gamma = \alpha + n\beta - \alpha\beta, \\ &= ({}_H I_{a+}^{\beta(n-\alpha)} {}_H D_{a+}^\gamma f)(t). \end{aligned}$$

Theorem 2.5. For $\alpha > 0$, and $0 < a < b < \infty$, if $f \in L^1[a, b]$ and $({}_H I_{a+}^{n-\alpha} f)(t) \in AC_\delta^n[a, b]$, then

$$({}_H I_{a+}^\alpha {}_H D_{a+}^\alpha f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)} ({}_H I_{a+}^{n-\alpha} f))(a)}{\Gamma(\alpha-j)} \left(\log \frac{t}{a} \right)^{\alpha-j-1}.$$



Theorem 2.6. For $\alpha > 0$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n - 1 < \gamma \leq n$, and $0 < a < b < \infty$, if $f \in L^1[a, b]$ and $({}_H I_{a+}^{n-\gamma} f)(t) \in AC_\delta^n[a, b]$, ergo

$${}_H I_{a+}^\alpha ({}_H D_{a+}^{\alpha, \beta} f)(t) = ({}_H I_{a+}^\gamma {}_H D_{a+}^\gamma f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)} ({}_H I_{a+}^{n-\gamma} f))(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a} \right)^{\gamma-j-1}.$$

Theorem 2.7. [12] Suppose that the nonempty closed set D is a subset of the Banach space X , and $F : D \rightarrow D$ a strict contraction, i.e., for all $x, y \in D$, there exists $k \in (0, 1)$ such that $\|F(x) - F(y)\| \leq k\|x - y\|$. Ergo, under these conditions, F admits a unique fixed point in D . This result is classically known as the Banach Fixed Point Theorem.

Theorem 2.8. [12] (Schauder's fixed point) Assume that D is a nonempty closed convex subset of a Banach space X . If $T : D \rightarrow D$ is a compact operator, then T has at least one fixed point in D .

Theorem 2.9. [12] If X is a compact metric space, then a closed subspace of $C(X; \mathbb{R})$ is compact $\Leftrightarrow$ it is bounded and equicontinuous. This result is classically known as the Arzelà-Ascoli Theorem.

Lemma 2.10. [4] Let $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$ is a p -Laplacian operator, where $\varphi_p(x) = |x|^{p-2}x$, and $(\varphi_p)^{-1}(s) = \varphi_q(s)$, $\frac{1}{p} + \frac{1}{q} = 1$. If $x > 0$, we have $\varphi_p(x) = x^{p-1}$. The p -Laplacian operator has the following characteristics:

(i) If $1 < p < 2$, $xy > 0$ and $0 < m \leq |x|, |y|$, ergo

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)m^{p-2}|x - y|.$$

(ii) If $p \geq 2$ and $|x|, |y| \leq M$, ergo

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x - y|.$$

We state the following for the stability of the solution of system (1.1).

Suppose $\alpha_1, \alpha_2 \in (1, 2]$, $\beta_1, \beta_2 \in [0, 1]$, $\gamma_1, \gamma_2 \in (0, 1]$, $\epsilon_1, \epsilon_2 > 0$, $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous functions. In addition to the system of fractional differential problems (1.1), we consider the following systems of inequalities

$$\begin{cases} \left| {}_H D_{1+}^{\gamma_1, \beta_1} \left(\varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1}) x(t) \right) \right) - f(t, x(t), y(t)) \right| \leq \epsilon_1, \\ \left| {}_H D_{1+}^{\gamma_2, \beta_2} \left(\varphi_{p_2} \left(({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2}) y(t) \right) \right) - g(t, x(t), y(t)) \right| \leq \epsilon_2, \end{cases} \quad (2.1)$$

with boundary conditions

$$\begin{cases} x(1) = 0, & x(e) = \lambda y(\theta), & ({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1}) x(1) = 0, \\ y(1) = 0, & y(e) = \mu x(\eta), & ({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2}) y(1) = 0, \end{cases} \quad (2.2)$$

where $\theta, \eta \in (1, e)$ and λ, μ are real constant. We characterize $Y = C^1([1, e], \mathbb{R})$, in the following Ulam stability definitions. In the vector $v = (v_1, v_2) > 0$, this means that each component is positive, in other words, $v_1, v_2 > 0$.

Remark 2.11. A function vector $(x, y) \in Y \times Y$ is a solution of inequalities (2.1) if and only if there exists a function $v_1, v_2 \in C([1, e], \mathbb{R})$ such that $|v_1(t)| \leq \epsilon_1$, $|v_2(t)| \leq \epsilon_2$, $t \in [1, e]$, and

$$\begin{cases} {}_H D_{1+}^{\gamma_1, \beta_1} \left(\varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_{1H} D_{1+}^{\alpha_1-1, \beta_1}) x(t) \right) \right) = f(t, x(t), y(t)) + v_1(t), \\ {}_H D_{1+}^{\gamma_2, \beta_2} \left(\varphi_{p_2} \left(({}_H D_{1+}^{\alpha_2, \beta_2} + k_{2H} D_{1+}^{\alpha_2-1, \beta_2}) y(t) \right) \right) = g(t, x(t), y(t)) + v_2(t). \end{cases} \quad (2.3)$$

Definition 2.12. [15] (Ulam-Hyers stable) The problem (1.1) is said to be Ulam-Hyers stable, if there exists a constant vector $c_{f,g} = (c_f, c_g) > 0$ such that, for each $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for each approximate solution $(x, y) \in Y \times Y$ of inequalities (2.1) with (2.2), there exists a solution $(u, v) \in Y \times Y$ of problem (1.1) satisfying

$$\|(x, y) - (u, v)\| \leq c_{f,g} \epsilon^T, \quad t \in [1, e].$$



Also, the problem (1.1) is said to be generalized Ulam-Hyers stable, if there exists a continuous vector function $\theta_{f,g} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\theta_{f,g}(0) = 0$ such that, for each approximate solution $(x, y) \in Y \times Y$ of inequalities (2.1) with (2.2), there exists a solution $(u, v) \in Y \times Y$ of problem (1.1) satisfying

$$\|(x, y) - (u, v)\| \leq \theta_{f,g}(\epsilon), \quad t \in [1, e].$$

3. MAIN RESULTS

Herein, we discuss the existence and uniqueness of solution for the boundary value problems (1.1) by some fixed point theorems such as Banach and Schauder. Finally, we check the stability of the system (1.1).

Lemma 3.1. *Let the parameters satisfy $1 < \alpha_1, \alpha_2 \leq 2$, $0 \leq \beta_1, \beta_2 \leq 1$, $0 < \gamma_1, \gamma_2 \leq 1$, $p_1, p_2 > 1$ with functions $F, G \in C([1, e], \mathbb{R})$. Then, for each $t \in [1, e]$, the pair (u, v) is a solution of the coupled system*

$$\begin{cases} {}_H D_{1+}^{\gamma_1, \beta_1} \left(\varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1}) u(t) \right) \right) = F(t), & t \in [1, e], \\ {}_H D_{1+}^{\gamma_2, \beta_2} \left(\varphi_{p_2} \left(({}_H D_{1+}^{\alpha_2, \beta_2} + k_2 {}_H D^{\alpha_2-1, \beta_2}) v(t) \right) \right) = G(t), & t \in [1, e], \\ u(1) = 0, \quad u(e) = \lambda v(\theta), \quad ({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1}) u(1) = 0, \\ v(1) = 0, \quad v(e) = \mu u(\eta), \quad ({}_H D_{1+}^{\alpha_2, \beta_2} + k_2 {}_H D^{\alpha_2-1, \beta_2}) v(1) = 0, \end{cases} \quad (3.1)$$

if and only if

$$\begin{aligned} u(t) = & \frac{(\log t)^{\sigma_2-1}}{1-AB} \left[k_1 \int_1^e \frac{u(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} G)|_{t=\theta} \right. \\ & - {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} F)|_{t=e} + A \left(-k_2 \int_1^e \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} G)|_{t=e} \right. \\ & \left. \left. + \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} F)|_{t=\eta} \right) \right] \\ & - k_1 \int_1^t \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} F), \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} v(t) = & \frac{(\log t)^{\sigma_4-1}}{1-AB} \left[k_2 \int_1^e \frac{v(s)}{s} ds - k_1 \mu \int_1^\eta \frac{u(s)}{s} ds + \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} F)|_{t=\eta} \right. \\ & - {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} G)|_{t=e} + B \left(-k_1 \int_1^e \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} F)|_{t=e} \right. \\ & \left. \left. + \lambda k_2 \int_1^\theta \frac{v(s)}{s} ds - \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} G)|_{t=\theta} \right) \right] \\ & - k_2 \int_1^t \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} G), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} A &= -\lambda(\log \theta)^{\sigma_4-1}, & \sigma_4 &= \alpha_2 + 2\beta_2 - \alpha_2\beta_2, \\ B &= -\mu(\log \eta)^{\sigma_2-1}, & \sigma_2 &= \alpha_1 + 2\beta_1 - \alpha_1\beta_1. \end{aligned}$$

Proof. By the effect of the Hadamard fractional integral of order γ_1 on the sides of the first equation (3.1) and according to Theorem 2.6 and considering $\sigma_1 = \gamma_1 + \beta_1 - \gamma_1\beta_1$,

$$\begin{aligned} \varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1}) u(t) \right) &= \frac{{}_H I_{1+}^{1-\gamma_1} \varphi_{p_1} \left(({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1}) u(t) \right) (1)}{\Gamma(\sigma_1)} (\log t)^{\sigma_1-1} + {}_H I_{1+}^{\gamma_1} F(t) \\ &= c_0 (\log t)^{\sigma_1-1} + {}_H I_{1+}^{\gamma_1} F(t), \end{aligned}$$



by the condition (3.1) in $t = 1$, we have $c_0 = 0$, ergo

$$\varphi_{p_1}(({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1})u(t)) = {}_H I_{1+}^{\gamma_1} F(t). \quad (3.4)$$

By the effect of φ_{q_1} both sides of the (3.4),

$$({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1})u(t) = \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)), \quad (3.5)$$

taking the ${}_H I_{1+}^{\alpha_1}$ both sides of the (3.5),

$$({}_H I_{1+}^{\alpha_1} {}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H I_{1+}^{\alpha_1} {}_H D^{\alpha_1-1, \beta_1})u(t) = {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)).$$

Ergo, with option $\sigma_2 = \alpha_1 + 2\beta_1 - \alpha_1\beta_1$, we arrive at the following relationship

$$u(t) - \frac{\delta I^{2-\sigma_2} u(1)}{\Gamma(\sigma_2)} (\log t)^{\sigma_2-1} - \frac{I^{2-\sigma_2} u(1)}{\Gamma(\sigma_2-1)} (\log t)^{\sigma_2-2} + k_1 {}_H I_{1+}^1 ({}_H I_{1+}^{\sigma_2-1} {}_H D_{1+}^{\sigma_2-1})u(t) = {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)),$$

by Theorem 2.5, we get

$$\begin{aligned} u(t) - \frac{\delta I^{2-\sigma_2} u(1)}{\Gamma(\sigma_2)} (\log t)^{\sigma_2-1} - \frac{I^{2-\sigma_2} u(1)}{\Gamma(\sigma_2-1)} (\log t)^{\sigma_2-2} \\ + k_1 {}_H I_{1+}^1 \left(u(t) - \frac{I^{2-\sigma_2} u(1)}{\Gamma(\sigma_2-1)} (\log t)^{\sigma_2-2} \right) u(t) = {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)). \end{aligned} \quad (3.6)$$

Ergo,

$$u(t) = c_1 (\log t)^{\sigma_2-1} + c_2 \left((\log t)^{\sigma_2-2} + k_1 \int_1^e \frac{(\log s)^{\sigma_2-2}}{s} ds \right) - k_1 \int_1^t \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)),$$

by the boundary condition $u(1) = 0$, we have $c_2 = 0$, now

$$u(t) = c_1 (\log t)^{\sigma_2-1} - k_1 \int_1^t \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t)). \quad (3.7)$$

In like manner,

$$v(t) = d_1 (\log t)^{\sigma_4-1} - k_2 \int_1^t \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} G(t)). \quad (3.8)$$

Next, the second boundary conditions of (3.1) together with (3.7) and (3.8) yields the system

$$c_1 - d_1 \lambda (\log \theta)^{\sigma_4-1} = k_1 \int_1^e \frac{u(s)}{s} ds - \lambda k_2 \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} F(t))|_{t=\theta} - {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t))|_{t=e},$$

and

$$c_1 (-\mu (\log \eta)^{\sigma_2-1}) + d_1 = k_2 \int_1^e \frac{v(s)}{s} ds - \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} F(t))|_{t=e} + \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t))|_{t=\eta},$$

respectively. By changing around the above equations, we obtain the following linear system of equations

$$\begin{cases} c_1 + d_1 A = J_1, \\ c_1 B + d_1 = J_2, \end{cases} \quad (3.9)$$

where

$$\begin{aligned} J_1 &= k_1 \int_1^e \frac{u(s)}{s} ds - \lambda k_2 \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} F(t))|_{t=\theta} - {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t))|_{t=e}, \\ J_2 &= k_2 \int_1^e \frac{v(s)}{s} ds - \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} F(t))|_{t=e} + \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} F(t))|_{t=\eta}, \end{aligned}$$

solving system (3.9),

$$c_1 = \frac{J_1 - A J_2}{1 - AB}, \quad d_1 = \frac{J_2 - B J_1}{1 - AB}.$$



Substituting c_1 and d_1 back into Equations (3.7) and (3.8), respectively, we get the integral equations (3.10) and (3.11).

The converse, by applying the Hadamard fractional differential operators directly to (3.10) and (3.11), and by using rules between Hadamard-type derivatives and integrals, together with substituting the boundary values into these expressions, it follows that the integral formulations not only reproduce the coupled fractional differential system but also satisfy all boundary conditions. Hence, the validity of the converse direction of the lemma is obtained through a direct calculation. $\square$

We name the Banach space $X = C([1, e], \mathbb{R})$ with the norm $\|u\| = \max_{t \in [1, e]} |u(t)|$, also the product space $X \times X$ with the norm $\|(u, v)\| = \|u\| + \|v\|$ represents a Banach space. In view of Lemma 3.1, we name an operator $T : X \times X \rightarrow X \times X$ by $T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t))$, where

$$\begin{aligned} T_1(u, v)(t) = & \frac{(\log t)^{\sigma_2-1}}{1-AB} \left[k_1 \int_1^e \frac{u(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g)|_{t=\theta} \right. \\ & - {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f)|_{t=e} + A \left(-k_2 \int_1^e \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g)|_{t=e} \right. \\ & \left. \left. + \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f)|_{t=\eta} \right) \right] \\ & - k_1 \int_1^t \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f), \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} T_2(u, v)(t) = & \frac{(\log t)^{\sigma_4-1}}{1-AB} \left[k_2 \int_1^e \frac{v(s)}{s} ds - k_1 \mu \int_1^\eta \frac{u(s)}{s} ds + \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f)|_{t=\eta} \right. \\ & - {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g)|_{t=e} + B \left(-k_1 \int_1^e \frac{u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f)|_{t=e} \right. \\ & \left. \left. + \lambda k_2 \int_1^\theta \frac{v(s)}{s} ds - \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g)|_{t=\theta} \right) \right] \\ & - k_2 \int_1^t \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g). \end{aligned} \quad (3.11)$$

Now let's consider the following conditions to examine next theorem.

(H_1) $f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, and there exists positive constant $L_f, L_g > 0$ such that, for $u_i, v_i \in \mathbb{R}$ ($i = 1, 2$),

$$\begin{aligned} |f(t, u_1, u_2) - f(t, v_1, v_2)| & \leq L_f (|u_1 - v_1| + |u_2 - v_2|), \\ |g(t, u_1, u_2) - g(t, v_1, v_2)| & \leq L_g (|u_1 - v_1| + |u_2 - v_2|). \end{aligned}$$



(H₂) There exists a finite solution $R > 0$ of the following inequality

$$\begin{aligned} R \geq & \frac{1}{|1-AB|} \left[((1+|A\mu|+|\mu|+|B|)k_1 + (1+|B\lambda|+|\lambda|+|A|)k_2)R \right. \\ & \left. + \frac{1+|A\mu|+|\mu|+|B|}{\Gamma(\alpha_1+1)} M_f^{q_1-1} + \frac{1+|B\lambda|+|\lambda|+|A|}{\Gamma(\alpha_2+1)} M_g^{q_2-1} \right] \\ & + (k_1+k_2)R + \frac{M_g^{q_2-1}}{\Gamma(\alpha_2+1)} + \frac{M_f^{q_1-1}}{\Gamma(\alpha_1+1)}, \end{aligned}$$

where $M_f = \frac{L_f R + f_0}{\Gamma(\gamma_1+1)}$, $f_0 = \sup_{t \in [1,e]} |f(t, 0, 0)|$, $M_g = \frac{L_g R + g_0}{\Gamma(\gamma_2+1)}$, and $g_0 = \sup_{t \in [1,e]} |g(t, 0, 0)|$.

(H₃) For $t \in [1, e]$, there exist nonnegative constants $a_i, b_i \geq 0$, ($i = 1, 2$) such that,

$$\begin{aligned} |f(t, u, v)| & \leq a_1 |u|^{p_1-1} + b_1 |v|^{p_1-1}, \\ |g(t, u, v)| & \leq a_2 |u|^{p_2-1} + b_2 |v|^{p_2-1}. \end{aligned}$$

For brevity, we set

$$\begin{aligned} n_1 &= \frac{1}{|1-AB|} \left[(1+|\mu A|)k_1 + (|\lambda|+|A|)k_2 + \frac{(|\lambda|+|A|)(q_2-1)M_g^{q_2-2}L_g}{\Gamma(\gamma_2+\alpha_2+1)} \right. \\ & \quad \left. + \frac{(1+|\mu A|)(q_1-1)M_f^{q_1-2}L_f}{\Gamma(\gamma_1+\alpha_1+1)} \right] + k_1 + \frac{(q_1-1)M_f^{q_1-2}L_f}{\Gamma(\gamma_1+\alpha_1+1)}, \\ n_2 &= \frac{1}{|1-AB|} \left[(1+|\lambda B|)k_2 + (|\mu|+|B|)k_1 + \frac{(1+|B\lambda|)(q_2-1)M_g^{q_2-2}L_g}{\Gamma(\gamma_2+\alpha_2+1)} \right. \\ & \quad \left. + \frac{(|\mu|+|B|)(q_1-1)M_f^{q_1-2}L_f}{\Gamma(\gamma_1+\alpha_1+1)} \right] + k_2 + \frac{(q_2-1)M_g^{q_2-2}L_g}{\Gamma(\gamma_2+\alpha_2+1)}, \\ n_3 &= \frac{1}{|1-AB|} \left[k_1 + |\lambda|k_2 + |A|k_2 + |A\mu|k_1 + \frac{|\lambda|+|A|}{\Gamma(\alpha_2+1)} \left(\frac{a_2+b_2}{\Gamma(\gamma_2+1)} \right)^{q_2-1} \right. \\ & \quad \left. + \frac{1+|A\mu|}{\Gamma(\alpha_1+1)} \left(\frac{a_1+b_1}{\Gamma(\gamma_1+1)} \right)^{q_1-1} \right] + k_1 + \frac{1}{\Gamma(\alpha_1+1)} \left(\frac{a_1+b_1}{\Gamma(\gamma_1+1)} \right)^{q_1-1}, \\ n_4 &= \frac{1}{|1-AB|} \left[k_2 + |\mu|k_1 + |B|k_1 + |B\lambda|k_2 + \frac{|\mu|+|B|}{\Gamma(\alpha_1+1)} \left(\frac{a_1+b_1}{\Gamma(\gamma_1+1)} \right)^{q_1-1} \right. \\ & \quad \left. + \frac{1+|B\lambda|}{\Gamma(\alpha_2+1)} \left(\frac{a_2+b_2}{\Gamma(\gamma_2+1)} \right)^{q_2-1} \right] + k_2 + \frac{1}{\Gamma(\alpha_2+1)} \left(\frac{a_2+b_2}{\Gamma(\gamma_2+1)} \right)^{q_2-1}. \end{aligned}$$

Theorem 3.2. Suppose that $1 < p_i \leq 2$, $i = \{1, 2\}$ and H_1, H_2 are true. If $n_1 + n_2 < 1$ then the system (1.1) has a unique solution on the interval $[1, e]$.

Proof. We consider the set $B_R = \{(u, v) \in X \times X \mid \|u, v\| \leq R\}$. For each $t \in [1, e]$, $(u, v) \in B_R$, we have the following estimation

$$\begin{aligned} |{}_H I_{1+}^{\gamma_1} f(t, u, v)| & \leq {}_H I_{1+}^{\gamma_1} |f(t, u, v) - f(t, 0, 0) + f(t, 0, 0)| \\ & \leq {}_H I_{1+}^{\gamma_1} (|f(t, u, v) - f(t, 0, 0)| + |f(t, 0, 0)|) \\ & \leq {}_H I_{1+}^{\gamma_1} (L_f(|u|+|v|) + f_0) \\ & \leq \frac{L_f R + f_0}{\Gamma(\gamma_1+1)} = M_f, \end{aligned}$$



in like manner,

$$|{}_H I_{1+}^{\gamma_2} g(t, u, v)| \leq \frac{L_g R + g_0}{\Gamma(\gamma_2 + 1)} = M_g.$$

For $q_i \geq 2$ ($i = 1, 2$),

$$\begin{aligned} |T_1(u, v)(t)| &\leq \frac{1}{|1 - AB|} \left[k_1 \int_1^e \frac{|u(s)|}{s} ds + k_2 |\lambda| \int_1^\theta \frac{|v(s)|}{s} ds + |\lambda| {}_H I_{1+}^{\alpha_2} \varphi_{q_2}(|{}_H I_{1+}^{\gamma_2} g|)|_{t=\theta} \right. \\ &\quad + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}(|{}_H I_{1+}^{\gamma_1} f|)|_{t=e} + |A| \left(k_2 \int_1^e \frac{|v(s)|}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}(|{}_H I_{1+}^{\gamma_2} g|)|_{t=e} \right. \\ &\quad \left. \left. + |\mu| k_1 \int_1^\eta \frac{|u(s)|}{s} ds + |\mu| {}_H I_{1+}^{\alpha_1} \varphi_{q_1}(|{}_H I_{1+}^{\gamma_1} f|)|_{t=\eta} \right) \right] \\ &\quad + k_1 \int_1^t \frac{|u(s)|}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1}(|{}_H I_{1+}^{\gamma_1} f|) \\ &\leq \frac{1}{|1 - AB|} \left[k_1 R + k_2 |\lambda| R + \frac{|\lambda| M_g^{q_2-1}}{\Gamma(\alpha_2 + 1)} + \frac{M_f^{q_1-1}}{\Gamma(\alpha_1 + 1)} \right. \\ &\quad \left. + |A| \left(k_2 R + \frac{M_g^{q_2-1}}{\Gamma(\alpha_2 + 1)} + |\mu| k_1 R + \frac{|\mu| M_f^{q_1-1}}{\Gamma(\alpha_1 + 1)} \right) \right] + k_1 R + \frac{M_f^{q_1-1}}{\Gamma(\alpha_1 + 1)} \\ &\leq \frac{1}{|1 - AB|} \left[(k_1 + |\lambda| k_2 + |A| k_2 + |A \mu| k_1) R \right. \\ &\quad \left. + \frac{|\lambda| + |A|}{\Gamma(\alpha_2 + 1)} M_g^{q_2-1} + \frac{1 + |A \mu|}{\Gamma(\alpha_1 + 1)} M_f^{q_1-1} \right] + k_1 R + \frac{M_f^{q_1-1}}{\Gamma(\alpha_1 + 1)}, \end{aligned}$$

also,

$$\begin{aligned} |T_2(u, v)(t)| &\leq \frac{1}{|1 - AB|} \left[(k_2 + |\mu| k_1 + |B| k_1 + |B \lambda| k_2) R \right. \\ &\quad \left. + \frac{|\mu| + |B|}{\Gamma(\alpha_1 + 1)} M_f^{q_1-1} + \frac{1 + |B \lambda|}{\Gamma(\alpha_2 + 1)} M_g^{q_2-1} \right] + k_2 R + \frac{M_g^{q_2-1}}{\Gamma(\alpha_2 + 1)}, \end{aligned}$$

ergo,

$$\begin{aligned} |T(u, v)| &= |T_1(u, v)| + |T_2(u, v)| \\ &\leq \frac{1}{|1 - AB|} \left[((1 + |A \mu| + |\mu| + |B|) k_1 + (1 + |B \lambda| + |\lambda| + |A|) k_2) R \right. \\ &\quad \left. + \frac{1 + |A \mu| + |\mu| + |B|}{\Gamma(\alpha_1 + 1)} M_f^{q_1-1} + \frac{1 + |B \lambda| + |\lambda| + |A|}{\Gamma(\alpha_2 + 1)} M_g^{q_2-1} \right] \\ &\quad + (k_1 + k_2) R + \frac{M_g^{q_2-1}}{\Gamma(\alpha_2 + 1)} + \frac{M_f^{q_1-1}}{\Gamma(\alpha_1 + 1)} \\ &\leq R. \end{aligned}$$

Which indicates that $T(B_R) \subset B_R$. For each $t \in [1, e]$, $(u_1, v_1), (u_2, v_2) \in B_R$,

$$|T_1(u_1, v_1)(t) - T_1(u_2, v_2)(t)| = \left| \frac{(\log t)^{\sigma_2-1}}{1 - AB} \left[k_1 \int_1^e \frac{u_1(s) - u_2(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{v_1(s) - v_2(s)}{s} ds \right] \right|$$



$$\begin{aligned}
& + \lambda {}_H I_{1+}^{\alpha_2} (\varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g(t, u_1, v_1))|_{t=\theta} - \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g(t, u_2, v_2))|_{t=\theta}) \\
& - {}_H I_{1+}^{\alpha_1} (\varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_1, v_1))|_{t=e} - \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_2, v_2))|_{t=e}) \\
& + A \left(-k_2 \int_1^e \frac{v_1(s) - v_2(s)}{s} ds + {}_H I_{1+}^{\alpha_2} (\varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g(t, u_1, v_1))|_{t=e} \right. \\
& - \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g(t, u_2, v_2))|_{t=e}) + \mu k_1 \int_1^\eta \frac{u_1(s) - u_2(s)}{s} ds \\
& \left. - \mu {}_H I_{1+}^{\alpha_1} (\varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_1, v_1))|_{t=\eta} - \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_2, v_2))|_{t=\eta}) \right) \Bigg] \\
& - k_1 \int_1^t \frac{u_1(s) - u_2(s)}{s} ds + {}_H I_{1+}^{\alpha_1} (\varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_1, v_1)) - \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f(t, u_2, v_2))) \Bigg| \\
& \leq \frac{1}{|1 - AB|} \left[k_1 |u_1(s) - u_2(s)| + k_2 |\lambda| |v_1(s) - v_2(s)| \right. \\
& + |\lambda| (q_2 - 1) M_g^{q_2-2} {}_H I_{1+}^{\alpha_2} \left| {}_H I_{1+}^{\gamma_2} g(t, u_1, v_1)|_{t=\theta} - {}_H I_{1+}^{\gamma_2} g(t, u_2, v_2)|_{t=\theta} \right| \\
& + (q_1 - 1) M_f^{q_1-2} {}_H I_{1+}^{\alpha_1} \left| {}_H I_{1+}^{\gamma_1} f(t, u_1, v_1)|_{t=e} - {}_H I_{1+}^{\gamma_1} f(t, u_2, v_2)|_{t=e} \right| \\
& + |A| \left(k_2 |v_1(s) - v_2(s)| + (q_2 - 1) M_g^{q_2-2} {}_H I_{1+}^{\alpha_2} \left| {}_H I_{1+}^{\gamma_2} g(t, u_1, v_1)|_{t=e} \right. \right. \\
& \left. \left. - {}_H I_{1+}^{\gamma_2} g(t, u_2, v_2)|_{t=e} \right| + |\mu| k_1 |u_1(s) - u_2(s)| \right. \\
& \left. + |\mu| (q_1 - 1) M_f^{q_1-2} {}_H I_{1+}^{\alpha_1} \left| {}_H I_{1+}^{\gamma_1} f(t, u_1, v_1)|_{t=\eta} - {}_H I_{1+}^{\gamma_1} f(t, u_2, v_2)|_{t=\eta} \right| \right) \Bigg] \\
& + k_1 |u_1(s) - u_2(s)| + (q_1 - 1) M_f^{q_1-2} {}_H I_{1+}^{\alpha_1} \left| {}_H I_{1+}^{\gamma_1} f(t, u_1, v_1) - {}_H I_{1+}^{\gamma_1} f(t, u_2, v_2) \right| \\
& \leq \frac{1}{|1 - AB|} \left[(1 + |\mu A|) k_1 |u_1(s) - u_2(s)| + k_2 (|\lambda| + |A|) |v_1(s) - v_2(s)| \right. \\
& + (|\lambda| + |A|) (q_2 - 1) M_g^{q_2-2} L_g (|u_1 - u_2| + |v_1 - v_2|) \frac{B(\alpha_2, \gamma_2 + 1)}{\Gamma(\gamma_2 + 1) \Gamma(\alpha_2)} \\
& + (1 + |\mu A|) (q_1 - 1) M_f^{q_1-2} L_f (|u_1 - u_2| + |v_1 - v_2|) \frac{B(\alpha_1, \gamma_1 + 1)}{\Gamma(\gamma_1 + 1) \Gamma(\alpha_1)} \Bigg] \\
& + k_1 |u_1(s) - u_2(s)| + (q_1 - 1) M_f^{q_1-2} L_f (|u_1 - u_2| + |v_1 - v_2|) \frac{B(\alpha_1, \gamma_1 + 1)}{\Gamma(\gamma_1 + 1) \Gamma(\alpha_1)} \\
& \leq \left[\frac{1}{|1 - AB|} \left[(1 + |\mu A|) k_1 + (|\lambda| + |A|) k_2 + \frac{(|\lambda| + |A|) (q_2 - 1) M_g^{q_2-2} L_g}{\Gamma(\gamma_2 + \alpha_2 + 1)} \right. \right. \\
& \left. \left. + \frac{(1 + |\mu A|) (q_1 - 1) M_f^{q_1-2} L_f}{\Gamma(\gamma_1 + \alpha_1 + 1)} \right] + k_1 + \frac{(q_1 - 1) M_f^{q_1-2} L_f}{\Gamma(\gamma_1 + \alpha_1 + 1)} \right] (|u_1 - u_2| + |v_1 - v_2|) \\
& = n_1 (|u_1 - u_2| + |v_1 - v_2|), \tag{3.12}
\end{aligned}$$



also

$$\begin{aligned}
& |T_2(u_1, v_1)(t) - T_2(u_2, v_2)(t)| \\
& \leq \left[\frac{1}{|1 - AB|} \left[(1 + |\lambda B|)k_2 + (|\mu| + |B|)k_1 \right. \right. \\
& \quad \left. \left. + \frac{(1 + |B\lambda|)(q_2 - 1)M_g^{q_2-2}L_g}{\Gamma(\gamma_2 + \alpha_2 + 1)} + \frac{(|\mu| + |B|)(q_1 - 1)M_f^{q_1-2}L_f}{\Gamma(\gamma_1 + \alpha_1 + 1)} \right] \right. \\
& \quad \left. + k_2 + \frac{(q_2 - 1)M_g^{q_2-2}L_g}{\Gamma(\gamma_2 + \alpha_2 + 1)} \right] (|u_1 - u_2| + |v_1 - v_2|) \\
& = n_2(|u_1 - u_2| + |v_1 - v_2|).
\end{aligned} \tag{3.13}$$

We can deduce that $\|T(u_1, v_1) - T(u_2, v_2)\| \leq (n_1 + n_2)(\|u_1 - u_2\| + \|v_1 - v_2\|)$ from (3.12) and (3.13). Now we deduce that the operator T is contractible from the condition of the theorem $n_1 + n_2 < 1$. Ergo, operator T has a unique fixed point by Banach contraction mapping principle. Therefore, the system (1.1) has a unique solution on $[1, e]$. $\square$

Theorem 3.3. Assume that H_3 is satisfied, then the system (1.1) has at least one solution if $n_3 + n_4 < 1$.

Proof. suppose there is $R' > 0$, such that $B_{R'} = \{(u, v) \in X \times X \mid \|u, v\| \leq R'\}$. It is clear that $B_{R'}$ is a bounded closed convex set in $X \times X$. The proof will be given step-by-step.

Step1. By the continuity of f, g , it is easy to see that T is continuous.

Step2. T maps bounded sets into bounded sets in $X \times X$. For each $t \in [1, e]$, $(u, v) \in B_{R'}$,

$$\begin{aligned}
|{}_H I_{1+}^{\gamma_1} f(t, u, v)| & \leq {}_H I_{1+}^{\gamma_1} |f(t, u, v)| \leq \frac{a_1 + b_1}{\Gamma(\gamma_1 + 1)} R'^{p_1-1}, \\
|{}_H I_{1+}^{\gamma_2} g(t, u, v)| & \leq {}_H I_{1+}^{\gamma_2} |g(t, u, v)| \leq \frac{a_2 + b_2}{\Gamma(\gamma_2 + 1)} R'^{p_2-1}.
\end{aligned}$$

Ergo,

$$\begin{aligned}
|T_1(u, v)(t)| & \leq \left[\frac{1}{|1 - AB|} \left[k_1 + |\lambda|k_2 + |A|k_2 + |A\mu|k_1 + \frac{|\lambda| + |A|}{\Gamma(\alpha_2 + 1)} \left(\frac{a_2 + b_2}{\Gamma(\gamma_2 + 1)} \right)^{q_2-1} \right. \right. \\
& \quad \left. \left. + \frac{1 + |A\mu|}{\Gamma(\alpha_1 + 1)} \left(\frac{a_1 + b_1}{\Gamma(\gamma_1 + 1)} \right)^{q_1-1} \right] + k_1 + \frac{1}{\Gamma(\alpha_1 + 1)} \left(\frac{a_1 + b_1}{\Gamma(\gamma_1 + 1)} \right)^{q_1-1} \right] R' \\
& = n_3 R',
\end{aligned}$$

also,

$$\begin{aligned}
|T_2(u, v)(t)| & \leq \left[\frac{1}{|1 - AB|} \left[k_2 + |\mu|k_1 + |B|k_1 + |B\lambda|k_2 + \frac{|\mu| + |B|}{\Gamma(\alpha_1 + 1)} \left(\frac{a_1 + b_1}{\Gamma(\gamma_1 + 1)} \right)^{q_1-1} \right. \right. \\
& \quad \left. \left. + \frac{1 + |B\lambda|}{\Gamma(\alpha_2 + 1)} \left(\frac{a_2 + b_2}{\Gamma(\gamma_2 + 1)} \right)^{q_2-1} \right] + k_2 + \frac{1}{\Gamma(\alpha_2 + 1)} \left(\frac{a_2 + b_2}{\Gamma(\gamma_2 + 1)} \right)^{q_2-1} \right] R' \\
& = n_4 R',
\end{aligned}$$

ergo, $\|T\| \leq \|T_1\| + \|T_2\| \leq (n_3 + n_4)R' \leq R'$, we deduce that $T(B_{R'}) \subset B_{R'}$.



Step3. We show that T is equicontinuous. For $1 \leq t_1 \leq t_2 \leq e$,

$$\begin{aligned}
& |T_1(u, v)(t_2) - T_1(u, v)(t_1)| \\
&= \left| \frac{(\log t_2)^{\sigma_2-1} - (\log t_1)^{\sigma_2-1}}{1 - AB} \left[k_1 \int_1^e \frac{u(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g) \right]_{t=\theta} \right. \\
&\quad - {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \Big|_{t=e} + A \left(-k_2 \int_1^e \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g) \Big|_{t=e} \right. \\
&\quad \left. + \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \Big|_{t=\eta} \right) \Big] \\
&\quad - k_1 \int_1^{t_2} \frac{u(s)}{s} ds + k_1 \int_1^{t_1} \frac{u(s)}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^{t_2} \left(\log \frac{t_2}{\tau} \right)^{\alpha_1-1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \frac{d\tau}{\tau} \\
&\quad - \frac{1}{\Gamma(\alpha_1)} \int_1^{t_1} \left(\log \frac{t_1}{\tau} \right)^{\alpha_1-1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \frac{d\tau}{\tau} \Big| \\
&= \left| \frac{(\log t_2)^{\sigma_2-1} - (\log t_1)^{\sigma_2-1}}{1 - AB} \left[k_1 \int_1^e \frac{u(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{v(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g) \right]_{t=\theta} \right. \\
&\quad - {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \Big|_{t=e} + A \left(-k_2 \int_1^e \frac{v(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2}({}_H I_{1+}^{\gamma_2} g) \Big|_{t=e} \right. \\
&\quad \left. + \mu k_1 \int_1^\eta \frac{u(s)}{s} ds - \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \Big|_{t=\eta} \right) \Big] - k_1 \int_{t_1}^{t_2} \frac{u(s)}{s} ds \\
&\quad + \frac{1}{\Gamma(\alpha_1)} \int_1^{t_1} \left(\left(\log \frac{t_2}{\tau} \right)^{\alpha_1-1} - \left(\log \frac{t_1}{\tau} \right)^{\alpha_1-1} \right) \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \frac{d\tau}{\tau} \\
&\quad + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{\tau} \right)^{\alpha_1-1} \varphi_{q_1}({}_H I_{1+}^{\gamma_1} f) \frac{d\tau}{\tau} \Big|.
\end{aligned}$$

In the above relationship, if t_2 approaches t_1 , we observe that $|T_1(u, v)(t_2) - T_1(u, v)(t_1)| \rightarrow 0$. In the same way, we have $|T_2(u, v)(t_2) - T_2(u, v)(t_1)| \rightarrow 0$. Ergo, $T = (T_1, T_2)$ is equicontinuous. By the Arzelà-Ascoli Theorem, it follows that T is compact on $B_{R'}$. As a result, by the Schauder fixed point theorem, T admits a fixed point (u, v) , which corresponds to a solution of the system (1.1). $\square$

Theorem 3.4. Assume that H_1 and $n_1 + n_2 < 1$ are satisfied, for $1 < p_i \leq 2$, $i = \{1, 2\}$, the boundary value problem defined by system (1.1) is Ulam-Hyers stable, and hence generalized Ulam-Hyers stable.

Proof. Suppose $\epsilon = (\epsilon_1, \epsilon_2) > 0$ is given, also (x, y) is a solution of the inequality (2.1) subject to the conditions (2.2). Ergo, by virtue of Remark 2.11 there exist functions $v_1, v_2 \in C^1([1, e], \mathbb{R})$ such that $|v_1(t)| \leq \epsilon_1, |v_2(t)| \leq \epsilon_2, t \in [1, e]$, and

$$\begin{cases} {}_H D_{1+}^{\gamma_1, \beta_1} \left(\varphi_{p_1}(({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1})x(t)) \right) = f(t, x(t), y(t)) + v_1(t), \\ {}_H D_{1+}^{\gamma_2, \beta_2} \left(\varphi_{p_2}(({}_H D_{1+}^{\alpha_2, \beta_2} + k_2 {}_H D^{\alpha_2-1, \beta_2})y(t)) \right) = g(t, x(t), y(t)) + v_2(t), \\ x(1) = 0, \quad x(e) = \lambda y(\theta), \quad ({}_H D_{1+}^{\alpha_1, \beta_1} + k_1 {}_H D^{\alpha_1-1, \beta_1})x(1) = 0, \\ y(1) = 0, \quad y(e) = \mu x(\eta), \quad ({}_H D_{1+}^{\alpha_2, \beta_2} + k_2 {}_H D^{\alpha_2-1, \beta_2})y(1) = 0. \end{cases} \quad (3.14)$$



By Lemma 3.1, the solution of (3.14) can be written as

$$\begin{aligned}
 x(t) = & \frac{(\log t)^{\sigma_2-1}}{1-AB} \left[k_1 \int_1^e \frac{x(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{y(s)}{s} ds + \lambda {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} (g + v_2))|_{t=\theta} \right. \\
 & - {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1))|_{t=e} + A \left(-k_2 \int_1^e \frac{y(s)}{s} ds + {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g + v_2)|_{t=e} \right. \\
 & \left. \left. + \mu k_1 \int_1^\eta \frac{x(s)}{s} ds - \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1))|_{t=\eta} \right) \right] \\
 & - k_1 \int_1^t \frac{x(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1)),
 \end{aligned} \tag{3.15}$$

other hand,

$$\begin{aligned}
 |{}_H I_{1+}^{\gamma_1} (f(t, x, y) + v_1)| & \leq |{}_H I_{1+}^{\gamma_1} f(t, x, y)| + |{}_H I_{1+}^{\gamma_1} v_1| \\
 & \leq M_f + \frac{\epsilon_1}{\Gamma(\gamma_1 + 1)} \leq M_f + \frac{1}{\Gamma(\gamma_1 + 1)} = m_f, \\
 |{}_H I_{1+}^{\gamma_2} (g(t, x, y) + v_2)| & \leq |{}_H I_{1+}^{\gamma_2} g(t, x, y)| + |{}_H I_{1+}^{\gamma_2} v_2| \\
 & \leq M_g + \frac{\epsilon_2}{\Gamma(\gamma_2 + 1)} \leq M_g + \frac{1}{\Gamma(\gamma_2 + 1)} = m_g.
 \end{aligned}$$

By virtue of Theorem 3.2, we denote by (u, v) the unique solution of system (1.1), also by Lemma 2.10,

$$\begin{aligned}
 |x(t) - u(t)| & = \left| \frac{(\log t)^{\sigma_2-1}}{1-AB} \left[k_1 \int_1^e \frac{x(s) - u(s)}{s} ds - k_2 \lambda \int_1^\theta \frac{y(s) - v(s)}{s} ds \right. \right. \\
 & \quad + \lambda ({}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} (g + v_2))|_{t=\theta} - {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g)|_{t=\theta}) \\
 & \quad - {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1))|_{t=e} + {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f)|_{t=e} \\
 & \quad - A k_2 \int_1^e \frac{y(s) - v(s)}{s} ds + A {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g + v_2)|_{t=e} \\
 & \quad - A {}_H I_{1+}^{\alpha_2} \varphi_{q_2} ({}_H I_{1+}^{\gamma_2} g)|_{t=e} + A \mu k_1 \int_1^\eta \frac{x(s) - u(s)}{s} ds \\
 & \quad \left. \left. - A \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1))|_{t=\eta} + A \mu {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f)|_{t=\eta} \right] \right. \\
 & \quad \left. - k_1 \int_1^t \frac{x(s) - u(s)}{s} ds + {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} (f + v_1)) - {}_H I_{1+}^{\alpha_1} \varphi_{q_1} ({}_H I_{1+}^{\gamma_1} f) \right| \\
 & \leq \frac{1}{|1-AB|} \left[k_1 |x - u| + k_2 |\lambda| |y - v| + |A| k_2 |y - v| + |A \mu| k_1 |x - u| \right. \\
 & \quad \left. + \frac{(|\lambda| + |A|)(q_2 - 1)m_g^{q_2-2}}{\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \epsilon_2 + \frac{(1 + |A \mu|)(q_1 - 1)m_f^{q_1-2}}{\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \epsilon_1 \right] \\
 & \quad + k_1 |x - u| + \frac{(q_1 - 1)m_f^{q_1-2}}{\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \epsilon_1.
 \end{aligned}$$



Ergo,

$$\begin{aligned} \|x - u\| &\leq \left(\frac{k_1(1 + |A\mu|)}{|1 - AB|} + k_1 \right) \|x - u\| + \frac{(|\lambda| + |A|)k_2}{|1 - AB|} \|y - v\| \\ &\quad + \frac{(|\lambda| + |A|)(q_2 - 1)m_g^{q_2-2}}{|1 - AB|\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \epsilon_2 + \left(\frac{(1 + |A\mu|)}{|1 - AB|} + 1 \right) \frac{(q_1 - 1)m_f^{q_1-2}}{\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \epsilon_1, \end{aligned} \quad (3.16)$$

in like manner,

$$\begin{aligned} \|y - v\| &\leq \left(\frac{k_2(1 + |B\lambda|)}{|1 - AB|} + k_2 \right) \|y - v\| + \frac{(|\mu| + |B|)k_1}{|1 - AB|} \|x - u\| \\ &\quad + \frac{(|\mu| + |B|)(q_1 - 1)m_f^{q_1-2}}{|1 - AB|\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \epsilon_1 + \left(\frac{(1 + |B\lambda|)}{|1 - AB|} + 1 \right) \frac{(q_2 - 1)m_g^{q_2-2}}{\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \epsilon_2. \end{aligned} \quad (3.17)$$

Solving inequalities (3.16) and (3.17) simultaneously for $\|x - u\|$ and $\|y - v\|$,

$$\|x - u\| \leq \frac{H_1\epsilon_1 + H_2\epsilon_2}{D}, \quad \|y - v\| \leq \frac{E_1\epsilon_1 + E_2\epsilon_2}{D},$$

where

$$\begin{aligned} D &= \left(1 - \frac{k_1(1 + |A\mu|)}{|1 - AB|} - k_1 \right) \left(1 - \frac{k_2(1 + |B\lambda|)}{|1 - AB|} - k_2 \right) \\ &\quad - \left(\frac{(|\mu| + |B|)k_1}{|1 - AB|} \right) \left(\frac{(|\lambda| + |A|)k_2}{|1 - AB|} \right), \\ H_1 &= \frac{(|\lambda| + |A|)k_2}{|1 - AB|} \frac{(|\mu| + |B|)(q_1 - 1)m_f^{q_1-2}}{|1 - AB|\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \\ &\quad + \left(1 - \frac{k_2(1 + |B\lambda|)}{|1 - AB|} - k_2 \right) \left(\frac{(1 + |A\mu|)}{|1 - AB|} + 1 \right) \frac{(q_1 - 1)m_f^{q_1-2}}{\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)}, \\ H_2 &= \left(\frac{(1 + |B\lambda|)}{|1 - AB|} + 1 \right) \frac{(q_2 - 1)m_g^{q_2-2}}{\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \left(\frac{(|\lambda| + |A|)k_2}{|1 - AB|} \right) \\ &\quad + \left(1 - \frac{k_2(1 + |B\lambda|)}{|1 - AB|} - k_2 \right) \left(\frac{(|\lambda| + |A|)(q_2 - 1)m_g^{q_2-2}}{|1 - AB|\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \right), \\ E_1 &= \left(\frac{(|\mu| + |B|)(q_1 - 1)m_f^{q_1-2}}{|1 - AB|\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \right) \left(1 - \frac{k_1(1 + |A\mu|)}{|1 - AB|} - k_1 \right) \\ &\quad + \left(\frac{(|\mu| + |B|)k_1}{|1 - AB|} \right) \left(\left(\frac{(1 + |A\mu|)}{|1 - AB|} + 1 \right) \frac{(q_1 - 1)m_f^{q_1-2}}{\Gamma(\gamma_1 + 1)\Gamma(\alpha_1 + 1)} \right), \\ E_2 &= \left(\frac{(1 + |B\lambda|)}{|1 - AB|} + 1 \right) \frac{(q_2 - 1)m_g^{q_2-2}}{\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \left(1 - \frac{k_1(1 + |A\mu|)}{|1 - AB|} - k_1 \right) \\ &\quad + \left(\frac{(|\mu| + |B|)k_1}{|1 - AB|} \right) \left(\frac{(|\lambda| + |A|)(q_2 - 1)m_g^{q_2-2}}{|1 - AB|\Gamma(\gamma_2 + 1)\Gamma(\alpha_2 + 1)} \right). \end{aligned}$$

In consequence, we get that

$$\|x - u\| + \|y - v\| \leq \frac{H_1 + E_1}{D} \epsilon_1 + \frac{H_2 + E_2}{D} \epsilon_2,$$

since $c_{f,g} = (c_f, c_g) = (\frac{H_1+E_1}{D}, \frac{H_2+E_2}{D}) > 0$, for each $\epsilon = (\epsilon_1, \epsilon_2) > 0$, we get that

$$\|(x, y) - (u, v)\| = \|x - u\| + \|y - v\| \leq c_f \epsilon_1 + c_g \epsilon_2 = c_{f,g} \epsilon^T.$$



Therefore, the system (1.1) exhibits Ulam–Hyers stability. Furthermore, it satisfies generalized Ulam–Hyers stability characterized by the estimate $\|v - u\| \leq \theta_{f,g}(\epsilon)$, where $\theta_{f,g}(\epsilon) = c_{f,g}\epsilon^T$, $\theta_{f,g}(0) = 0$. $\square$

4. EXAMPLE

To demonstrate the applicability and validate the theoretical findings presented in the preceding section, we develop a comprehensive example that thoroughly applies the derived results.

Example 4.1. Suppose

$$\begin{cases} {}_H D_{1+}^{\frac{1}{3}, \frac{1}{2}} \left(\varphi_2 \left(({}_H D_{1+}^{\frac{3}{2}, \frac{1}{2}} + \frac{1}{20} {}_H D_{1+}^{\frac{1}{2}, \frac{1}{2}}) u(t) \right) \right) = f(t, u, v), & t \in [1, e], \\ {}_H D_{1+}^{\frac{1}{4}, \frac{1}{2}} \left(\varphi_2 \left(({}_H D_{1+}^{\frac{5}{4}, \frac{1}{2}} + \frac{1}{10} {}_H D_{1+}^{\frac{1}{4}, \frac{1}{2}}) v(t) \right) \right) = g(t, u, v), \\ u(1) = 0, \quad u(e) = v(1.2), \quad ({}_H D_{1+}^{\frac{3}{2}, \frac{1}{2}} + \frac{1}{20} {}_H D_{1+}^{\frac{1}{2}, \frac{1}{2}}) u(1) = 0, \\ v(1) = 0, \quad v(e) = 2u(1.1), \quad ({}_H D_{1+}^{\frac{5}{4}, \frac{1}{2}} + \frac{1}{10} {}_H D_{1+}^{\frac{1}{4}, \frac{1}{2}}) v(1) = 0, \end{cases} \quad (4.1)$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{3}, \quad \gamma_2 = \frac{1}{4}, \quad \beta_1 = \beta_2 = \frac{1}{2}, \quad \alpha_1 = \frac{3}{2}, \quad \alpha_2 = \frac{5}{4}, \quad p_1 = p_2 = q_1 = q_2 = 2, \quad k_1 = \frac{1}{20}, \\ k_2 &= \frac{1}{10}, \quad \lambda = 1, \quad \mu = 2, \quad \theta = 1.2, \quad \eta = 1.1. \end{aligned}$$

case1. For

$$f(t, u, v) = \frac{1}{e^t (1 + \frac{1}{10}|u| + \frac{1}{10}|v|)}, \quad g(t, u, v) = \frac{(1 + \log t)|u|}{100 + |u|} + \frac{|v|}{(8 + t)^3(1 + |v|)},$$

by computation,

$$\begin{aligned} f_0 &= 0.3679, \quad g_0 = 0, \quad A = -0.3452, \quad B = -0.3431, \\ |f(t, u_1, u_2) - f(t, v_1, v_2)| &\leq 0.0368(|u_1 - v_1| + |u_2 - v_2|), \\ |g(t, u_1, u_2) - g(t, v_1, v_2)| &\leq 0.0200(|u_1 - v_1| + |u_2 - v_2|), \\ M_f &= 0.041202R + 0.411774, \quad M_g = 0.022064R, \end{aligned}$$

according to H_2 ,

$$R > 26.478,$$

also

$$n_1 + n_2 = 0.886765 < 1.$$

As a result, all of the assumptions we made in Theorem 3.2 are satisfied. Hence, the system (4.1) in $R > 26.478$ has a unique solution on $[1, e]$.

Moreover, for Ulam–Hyers stability, considering $R = 27$, according to Theorem 3.4,

$$\begin{aligned} M_f &= 1.535028, \quad m_f = 2.654878, \quad M_g = 0.595728, \quad m_g = 1.698998, \\ D &= 0.61845, \quad H_1 = 2.1766, \quad H_2 = 1.4852, \quad E_1 = 2.2400, \quad E_2 = 2.2934, \end{aligned}$$

ergo,

$$\begin{cases} c_f = \frac{H_1 + E_1}{D} \approx 7.140 > 0, \\ c_g = \frac{H_2 + E_2}{D} \approx 6.109 > 0, \end{cases} \implies c_{f,g} = (7.140, 6.109) > 0.$$

Therefore, the system (4.1) is both Ulam–Hyers stable and generalized Ulam–Hyers stable. The simulation diagrams are shown in Figures 1–2.



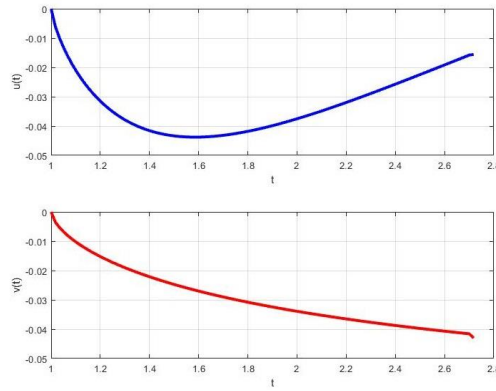
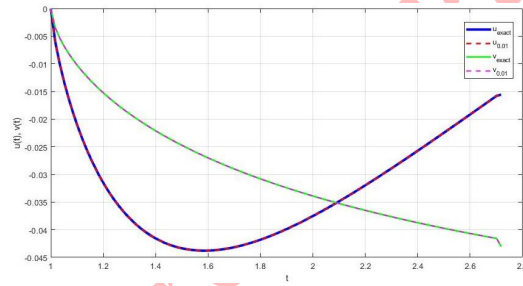


FIGURE 1. Numerical simulation of solutions in case 1.

FIGURE 2. Numerical simulation of stability with $\epsilon = 0.01$ in case 1.

Case2. For

$$f(t, u, v) = \frac{1}{200}e^{-t}|u| + \frac{1}{200}\cos t|v| \leq 0.00184|u| + 0.005|v|,$$

$$g(t, u, v) = \frac{1}{25}\sqrt[3]{t}|u| + \frac{1}{200}\sin t|v| \leq 0.05582|u| + 0.005|v|,$$

where

$$a_1 = 0.00184, \quad a_2 = 0.05582, \quad b_1 = b_2 = 0.005,$$

and

$$A = -0.3452, \quad B = -0.3431, \quad |1 - AB| = 0.8816,$$

as a result

$$n_3 + n_4 = 0.95559 < 1.$$

Therefore, everything assumed in Theorem 3.3 is satisfied, and then there exists a solution of system (4.1).



5. CONCLUSION

In this paper, we have studied a class of Hilfer–Hadamard fractional differential equations involving the p -Laplacian operator. By applying Banach’s and Schauder’s fixed point theorems, we established sufficient conditions for the existence and uniqueness of solutions and we found that if the problem has a unique solution according to the assumptions of Theorem 3.2, it is also Ulam–Hyers stable and generalized Ulam–Hyers stable on $[1, e]$. The incorporation of the p -Laplacian operator provides a significant extension of previous studies and enriches the theoretical framework for nonlinear fractional systems. An illustrative example was presented to confirm the validity and applicability of the obtained results.

For future work, it would be of interest to extend the current results to boundary value problems or coupled systems that involve more general fractional operators or variable-order derivatives. Furthermore, constructing efficient numerical algorithms to approximate the analytical solutions could offer deeper insights into the qualitative behavior of such nonlinear fractional models and their potential applications in physics and engineering.

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