



Lie symmetry analysis, optimal system, invariant solutions and conservation laws of the Manokov-Santini system

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Abstract

This article utilizes the symmetry method to perform an extensive analyses on the Manokov-Santini system. We show that it possesses infinite dimensional Lie algebra which are not closed in general except for a subalgebra. We use the generalized Lie algebra to exhaustively unearth the invariant solutions of the system under consideration using 2-level application of the Lie method. Furthermore, we show that the multipliers of the Manokov-Santini system are infinite and of first order. These facts are used to construct nontrivial conservation laws.

Keywords. Conserve vectors, Manokov-Santini system, Symmetry method, Exact solutions, Nonlinear system.

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1. INTRODUCTION

The work of Euler, d'Alembert, Lagrange, Laplace, Fourier, Riemann, Maxwell, Einstein and several others have ignited the study of partial differential equations (PDEs) from the 18th century up to the present time. Their work have not only shown, but have established PDEs as undisputable mechanism for understanding the intricate languages of the world and the universe. The importance of PDEs in almost every field of science, engineering and applied disciplines cannot be overemphasised. They have been used to extensively study a wide variety of physically significant problems arising from various fields of study each modeling different phenomena, and their solutions have given birth to fundamental laws in various fields of natural and applied sciences, see [7] for a summarized details.

There are a number of powerful and versatile methods that have been developed for the computational solution of PDEs, the theory of continuous groups of transformations is one them. Its application to the study of differential equations (DEs) was laid down by Sophus Lie in the later half of nineteenth century. Lie showed that the algebra of vector fields (infinitesimal generators) that leave a given DE (system of DEs) invariant could be found by solving auxiliary system of linear homogeneous equations. The infinitesimal generators are the cornerstones of Lie group theory because they contained almost all the information of the group they represent. For instance, they enable complicated nonlinear conditions of invariance under a group action to be replaced by relatively simple linear infinitesimal conditions which provide the key to the explicit determination of the symmetry groups (of systems) of DEs [17]. The infinitesimal generators are then further used to obtain either invariant solutions, lower the order of DEs or reduce the number of independent variables for the given DE (system of DEs). Lie's method provide a unification of many integral methods used for solving DEs in a systematic and algorithmic way. It has been used, by Lie himself, to show that most second order ODEs can be reduced to one of the following canonical forms:

$$y'' = f(y'), \quad y'' = \frac{1}{x}f(y'), \quad y'' = f(x), \quad y'' = f(x)y',$$

which greatly help in standardizing the method for solving such DEs. As for nonlinear PDEs, Lie's method is the only universal method that has had tremendous success in enabling researchers to find their analytical solutions as is

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evidenced from an increasing number of research papers currently published in the literature. Lucid explanation of Lie's theory and its application to DEs can be found in the following monographs and textbooks [1–3, 9, 11, 12, 17–19, 22].

The Manakov-Santini (MS) system

$$\begin{aligned} u_{xt} &= u_{yy} + u_x^2 + uu_{xx} + v_x u_{xy} - v_y u_{xx}, \\ v_{xt} &= v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} \end{aligned} \quad (1.1)$$

was proposed in [14–16] and it is a generalization of dispersionless Khokhlov-Zabolotskaya equation to the case of general (non-Hamiltonian) vector fields in the Lax pair [4]. System (1.1) has been shown by Plansangkate [20] to be a reduction of a general anti-self-dual conformal structure under two separate symmetry assumptions. Both the Pavlov equation ($u = 0$) and the dispersionless Kadomtsev-Petviashvili equation ($v = 0$) are special cases of system (1.1). Different research studies have been carried out on the MS system [4–6, 8, 10, 13, 20], but none has investigated it via the Lie group analysis except recently in [21]. The analysis carried out in [21] was limited to infinitesimals of first degree polynomials and to their optimal classifications up to 2-dimensional subalgebra. They also presented zero-order multiplier for the MS system which only lead to single conservation law.

In this article, we shall carry out a more comprehensive Lie group analysis on the MS system. We shall show that the MS system possesses infinite Lie algebra which shall be used to systematically exhibit infinitely many generalized nontrivial invariant solutions. In order to maximize the solution space, each of the reduced systems shall also be subjected to the Lie symmetry method. Although tedious, this 2-level application of the Lie method exhaustively unearth the invariant solutions of the MS system under consideration. We shall also show that the multipliers of the MS system are infinite and of first order with respect to the spatial independent variables. These facts shall be used to construct conservation laws for the MS system (1.1).

2. LIE ALGEBRA

Let

$$\mathcal{X} = \xi(x, y, t, u, v)\partial_x + \zeta(x, y, t, u, v)\partial_y + \tau(x, y, t, u, v)\partial_t + \eta(x, y, t, u, v)\partial_u + \phi(x, y, t, u, v)\partial_v,$$

be the generator of transformation group admitted by the system (1.1), and $\mathcal{X}^{(2)}$ be its second order prolongation. Then the invariance condition

$$\mathcal{X}^{(2)} \left[\begin{array}{l} u_{xt} = u_{yy} + u_x^2 + uu_{xx} + v_x u_{xy} - v_y u_{xx}, \\ v_{xt} = v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} \end{array} \right] \Big|_{(1.1)} = 0,$$

leads to the following determining equations:

$$\tau_x = 0, \quad (2.1)$$

$$\tau_y = 0, \quad (2.2)$$

$$\tau_u = 0, \quad (2.3)$$

$$\tau_v = 0, \quad (2.4)$$

$$\zeta_x = 0, \quad (2.5)$$

$$\zeta_u = 0, \quad (2.6)$$

$$\zeta_v = 0, \quad (2.7)$$

$$\xi_u = 0, \quad (2.8)$$

$$\xi_v = 0, \quad (2.9)$$

$$\eta_v = 0, \quad (2.10)$$

$$\phi_u = 0, \quad (2.11)$$

$$\zeta_{yy} = 0, \quad (2.12)$$

$$\xi_x - 2\zeta_y + \tau_t = 0, \quad (2.13)$$



$$\xi_{yy} - 2\zeta_{yt} + \tau_{tt} = 0, \quad (2.14)$$

$$\eta_u - 2\zeta_y + 2\tau_t = 0, \quad (2.15)$$

$$\eta_x - \zeta_{yt} + \tau_{tt} = 0, \quad (2.16)$$

$$\eta_y - \xi_{yt} + \zeta_{tt} = 0, \quad (2.17)$$

$$\phi_v - 3\zeta_y + 2\tau_t = 0, \quad (2.18)$$

$$\phi_x - 2\xi_y + \zeta_t = 0, \quad (2.19)$$

$$\phi_y + 2u(\zeta_y - \tau_t) - \xi_t - \eta = 0. \quad (2.20)$$

The solution to the linear system of PDEs (2.1-2.20) is given by:

$$\begin{aligned} \tau &= \alpha(t), \\ \zeta &= \beta(t)y + \gamma(t), \\ \xi &= \left(\beta'(t) - \frac{1}{2}\alpha''(t) \right) y^2 + (2\beta(t) - \alpha'(t))x + \lambda(t)y + \delta(t), \\ \eta &= \frac{1}{2}(\beta''(t) - \alpha'''(t))y^2 + (\lambda'(t) - \gamma''(t))y + 2(\beta(t) - \alpha'(t))u + (\beta'(t) - \alpha''(t))x + \rho(t) - \delta'(t), \\ \phi &= \left(\frac{1}{2}\beta''(t) - \frac{1}{3}\alpha'''(t) \right) y^3 + \left(\lambda'(t) - \frac{1}{2}\gamma''(t) \right) y^2 + (3\beta'(t) - 2\alpha''(t))xy \\ &\quad + (2\lambda(t) - \gamma'(t))x + (3\beta(t) - 2\alpha'(t))v + \rho(t)y + \sigma(t). \end{aligned}$$

The Lie algebra of the MS system are spanned by the following vector fields

$$\begin{aligned} \mathcal{X}_1 &= \alpha(t)\partial_t - \left(\frac{1}{2}\alpha''(t)y^2 + \alpha'(t)x \right) \partial_x - \left(\frac{1}{2}\alpha'''(t)y^2 + \alpha''(t)x + 2\alpha'(t)u \right) \partial_u \\ &\quad - \left(\frac{1}{3}\alpha'''(t)y^3 + 2\alpha''(t)xy + 2\alpha'(t)v \right) \partial_v, \\ \mathcal{X}_2 &= (\beta'(t)y^2 + 2\beta(t)x) \partial_x + \beta(t)y\partial_y + \left(\frac{1}{2}\beta''(t)y^2 + \beta'(t)x + 2\beta(t)u \right) \partial_u \\ &\quad + \left(\frac{1}{2}\beta''(t)y^3 + 3\beta'(t)xy + 3\beta(t)v \right) \partial_v, \\ \mathcal{X}_3 &= \gamma(t)\partial_y - \gamma''(t)y\partial_u - \left(\frac{1}{2}\gamma''(t)y^2 + \gamma'(t)x \right) \partial_v, \\ \mathcal{X}_4 &= \lambda(t)y\partial_x + \lambda'(t)y\partial_u + (\lambda'(t)y^2 + 2\lambda(t)x) \partial_v, \\ \mathcal{X}_5 &= \delta(t)\partial_x - \delta'(t)\partial_u, \\ \mathcal{X}_6 &= \rho(t)\partial_u + \rho(t)y\partial_v, \\ \mathcal{X}_7 &= \sigma(t)\partial_v. \end{aligned} \quad (2.21)$$

As can be seen from vector fields (2.21) that the Lie algebra of the MS system is infinite. Specific algebras can be generated by specializing the arbitrary functions. For instance, if we take them to be polynomials of degree two, then



the MS system will be spanned by the following 24 vector fields:

$$\begin{aligned}
\mathcal{A}_1 &= \partial_t, \mathcal{A}_2 = \partial_x, \mathcal{A}_3 = \partial_y, \mathcal{A}_4 = \partial_v, \mathcal{A}_5 = t\partial_v, \mathcal{A}_6 = t\partial_x + y\partial_v, \\
\mathcal{A}_7 &= \partial_u + y\partial_v, \mathcal{A}_8 = y\partial_x + 2x\partial_v, \mathcal{A}_9 = t\partial_y - x\partial_v, \\
\mathcal{A}_{10} &= 2x\partial_x + y\partial_y + 2u\partial_u + 3v\partial_v, \mathcal{A}_{11} = y\partial_y + 2t\partial_t - 2u\partial_u - v\partial_v, \\
\mathcal{A}_{12} &= ty\partial_x + y\partial_u + (y^2 + 2xt)\partial_v, \mathcal{A}_{13} = t\partial_u + ty\partial_v, \\
\mathcal{A}_{14} &= (y^2 + 2xt)\partial_x + ty\partial_y + (x + 2tu)\partial_u + 3(xy + tv)\partial_v, \\
\mathcal{A}_{15} &= -x\partial_x + t\partial_t - 2u\partial_u - 2v\partial_v, \mathcal{A}_{16} = \frac{1}{2}t^2\partial_y - y\partial_u - \frac{1}{2}y^2\partial_v, \\
\mathcal{A}_{17} &= \frac{1}{2}t^2\partial_x - t\partial_u, \mathcal{A}_{18} = t\partial_x - \partial_u, \mathcal{A}_{19} = \frac{1}{2}t^2y\partial_x + ty\partial_u + (xt^2 + ty^2)\partial_v, \\
\mathcal{A}_{20} &= \frac{1}{3}t^3\partial_y - 2ty\partial_u - (xt^2 + ty^2)\partial_v, \\
\mathcal{A}_{21} &= \frac{1}{2}t^2\partial_t - \left(xt + \frac{1}{2}y^2\right)\partial_x - (x + 2tu)\partial_u - 2(xy + tv)\partial_v, \\
\mathcal{A}_{22} &= \frac{1}{3}t^3\partial_t - (xt^2 + ty^2)\partial_x - (2xt + y^2 + 2t^2u)\partial_u - \left(4xyt + \frac{2}{3}y^3 + 2t^2v\right)\partial_v, \\
\mathcal{A}_{23} &= \left(\frac{2}{3}xt^3 + t^2y^2\right)\partial_x + \frac{1}{3}t^3y\partial_y + \left(xt^2 + ty^2 + \frac{2}{3}t^2u\right)\partial_u + (3xt^2y + ty^3 + t^3v)\partial_v, \\
\mathcal{A}_{24} &= (xt^2 + ty^2)\partial_x + \frac{1}{2}t^2y\partial_y + \left(xt + \frac{1}{2}y^2 + t^2u\right)\partial_u + \left(3xyt + \frac{1}{2}y^3 + \frac{3}{2}t^2v\right)\partial_v.
\end{aligned} \tag{2.22}$$

Their commutation relations are not closed even with respect to first order polynomials¹. We shall use some of the algebras $\mathcal{A}_j$ to construct invariant solutions that can not be deduced from those obtained via the algebras $\mathcal{X}_i$.

TABLE 1. The commutator table.

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9$	$\mathcal{A}_{10}$	$\mathcal{A}_{11}$
$\mathcal{A}_1$	0	0	0	0	$\mathcal{A}_4$	$\mathcal{A}_2$	0	0	$\mathcal{A}_3$	0	$\mathcal{A}_1$
$\mathcal{A}_2$	0	0	0	0	0	0	0	$2\mathcal{A}_4$	$-\mathcal{A}_4$	$\mathcal{A}_2$	0
$\mathcal{A}_3$	0	0	0	0	0	$\mathcal{A}_4$	$\mathcal{A}_4$	$\mathcal{A}_2$	0	$\frac{1}{2}\mathcal{A}_3$	$\frac{1}{2}\mathcal{A}_3$
$\mathcal{A}_4$	0	0	0	0	0	0	0	0	0	$\frac{3}{2}\mathcal{A}_4$	$-\frac{1}{2}\mathcal{A}_4$
$\mathcal{A}_5$	$-\mathcal{A}_4$	0	0	0	0	0	0	0	0	$\frac{3}{2}\mathcal{A}_5$	$-\frac{3}{2}\mathcal{A}_5$
$\mathcal{A}_6$	$-\mathcal{A}_2$	0	$-\mathcal{A}_4$	0	0	0	0	$2\mathcal{A}_5$	$-2\mathcal{A}_5$	$\mathcal{A}_6$	$-\mathcal{A}_6$
$\mathcal{A}_7$	0	0	$-\mathcal{A}_4$	0	0	0	0	0	$-\mathcal{A}_5$	$\mathcal{A}_7$	$-\mathcal{A}_7$
$\mathcal{A}_8$	0	$-2\mathcal{A}_4$	$-\mathcal{A}_2$	0	0	$-2\mathcal{A}_5$	0	0	$-\mathcal{A}_6$	$\frac{1}{2}\mathcal{A}_8$	$-\frac{1}{2}\mathcal{A}_8$
$\mathcal{A}_9$	$-\mathcal{A}_3$	$\mathcal{A}_4$	0	0	0	$2\mathcal{A}_5$	$\mathcal{A}_5$	$\mathcal{A}_6$	0	$\frac{1}{2}\mathcal{A}_9$	$-\frac{1}{2}\mathcal{A}_9$
$\mathcal{A}_{10}$	0	$-\mathcal{A}_2$	$-\frac{1}{2}\mathcal{A}_3$	$-\frac{3}{2}\mathcal{A}_4$	$-\frac{3}{2}\mathcal{A}_5$	$-\mathcal{A}_6$	$-\mathcal{A}_7$	$-\frac{1}{2}\mathcal{A}_8$	$-\frac{1}{2}\mathcal{A}_9$	0	0
$\mathcal{A}_{11}$	$-\mathcal{A}_1$	0	$-\frac{1}{2}\mathcal{A}_3$	$\frac{1}{2}\mathcal{A}_4$	$\frac{3}{2}\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\frac{1}{2}\mathcal{A}_8$	$\frac{1}{2}\mathcal{A}_9$	0	0

3. CLASSIFICATION OF SUBALGEBRAS

The first 11 Lie algebras $\mathcal{A}_1 - \mathcal{A}_{11}$ in (2.22) are closed as shown in Table 1, the construction of optimal system of various dimensional subalgebra of the Lie algebra $\mathcal{L}_{11}$ can be constructed using Ovsiannikov's two-step algorithm [11]

¹The vector fields generated with respect to first order polynomials are $\mathcal{A}_1 - \mathcal{A}_{13}$.



(see also [12]) or the method utilised in [9, 14] which is the one we shall employ to establish optimal system for the Lie algebra $\mathcal{L}_{11}$. We would like to emphasize that there is no any method that gives unique representations of subalgebras. Thus, the ones presented in this paper are assumed up to conjugacy and also, all c_k in the representations $\mathcal{A}_i + c_k \mathcal{A}_j$ will be equated to ± 1 .

The linear transformations of the vector

$$\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4, \mathbf{v}^5, \mathbf{v}^6, \mathbf{v}^7, \mathbf{v}^8, \mathbf{v}^9, \mathbf{v}^{10}, \mathbf{v}^{11}),$$

are found from their generators

$$\mathcal{G}_i = c_{ij}^k \mathbf{v}^j \partial_{\mathbf{v}^k}, i = 1, 2, \dots, 11, \quad (3.1)$$

where the c_{ij}^k are the structure constants. From the commutation relations of Table 1, the non-zero structure constants $c_{ij}, j = 1, 2, \dots, 11$ are:

$$\begin{aligned} c_{15}^4 &= 1, c_{16}^2 = 1, c_{19}^3 = 1, c_{1(11)}^1 = 1, c_{28}^4 = 2, c_{29}^4 = -1, c_{2(10)}^2 = 1, c_{36}^4 = 1, \\ c_{37}^4 &= 1, c_{38}^2 = 1, c_{3(10)}^3 = \frac{1}{2}, c_{3(11)}^3 = \frac{1}{2}, c_{4(10)}^4 = \frac{3}{2}, c_{4(11)}^4 = -\frac{1}{2}, c_{5(10)}^5 = \frac{3}{2}, \\ c_{5(11)}^5 &= -\frac{3}{2}, c_{68}^5 = 2, c_{69}^5 = -2, c_{6(10)}^6 = 1, c_{6(11)}^6 = -1, c_{79}^5 = -1, c_{7(10)}^7 = 1, \\ c_{7(11)}^7 &= -1, c_{89}^6 = -1, c_{8(10)}^8 = \frac{1}{2}, c_{8(11)}^8 = -\frac{1}{2}, c_{9(10)}^9 = \frac{1}{2}, c_{9(11)}^9 = -\frac{1}{2}. \end{aligned} \quad (3.2)$$

Substituting each of the structure constants (3.2) in the generator (3.1), we have

$$\begin{aligned} \mathcal{G}_1 &= \mathbf{v}^5 \partial_{\mathbf{v}^4} + \mathbf{v}^6 \partial_{\mathbf{v}^2} + \mathbf{v}^9 \partial_{\mathbf{v}^3} + \mathbf{v}^{11} \partial_{\mathbf{v}^1}, \\ \mathcal{G}_2 &= 2\mathbf{v}^8 \partial_{\mathbf{v}^4} - \mathbf{v}^9 \partial_{\mathbf{v}^4} + \mathbf{v}^{10} \partial_{\mathbf{v}^2}, \\ \mathcal{G}_3 &= \mathbf{v}^6 \partial_{\mathbf{v}^4} + \mathbf{v}^7 \partial_{\mathbf{v}^4} + \mathbf{v}^8 \partial_{\mathbf{v}^2} + \frac{1}{2} \mathbf{v}^{10} \partial_{\mathbf{v}^3} + \frac{1}{2} \mathbf{v}^{11} \partial_{\mathbf{v}^3}, \\ \mathcal{G}_4 &= \frac{3}{2} \mathbf{v}^{10} \partial_{\mathbf{v}^4} - \frac{1}{2} \mathbf{v}^{11} \partial_{\mathbf{v}^4}, \\ \mathcal{G}_5 &= -\mathbf{v}^1 \partial_{\mathbf{v}^4} + \frac{3}{2} \mathbf{v}^{10} \partial_{\mathbf{v}^5} - \frac{3}{2} \mathbf{v}^{11} \partial_{\mathbf{v}^5}, \\ \mathcal{G}_6 &= -\mathbf{v}^1 \partial_{\mathbf{v}^2} - \mathbf{v}^3 \partial_{\mathbf{v}^4} + 2\mathbf{v}^8 \partial_{\mathbf{v}^5} - 2\mathbf{v}^9 \partial_{\mathbf{v}^5} + \mathbf{v}^{10} \partial_{\mathbf{v}^6} - \mathbf{v}^{11} \partial_{\mathbf{v}^6}, \\ \mathcal{G}_7 &= -\mathbf{v}^3 \partial_{\mathbf{v}^4} - \mathbf{v}^9 \partial_{\mathbf{v}^5} + \mathbf{v}^{10} \partial_{\mathbf{v}^7} - \mathbf{v}^{11} \partial_{\mathbf{v}^7}, \\ \mathcal{G}_8 &= -2\mathbf{v}^2 \partial_{\mathbf{v}^4} - \mathbf{v}^3 \partial_{\mathbf{v}^2} - 2\mathbf{v}^6 \partial_{\mathbf{v}^5} - \mathbf{v}^9 \partial_{\mathbf{v}^6} + \frac{1}{2} \mathbf{v}^{10} \partial_{\mathbf{v}^8} - \frac{1}{2} \mathbf{v}^{11} \partial_{\mathbf{v}^8}, \\ \mathcal{G}_9 &= -\mathbf{v}^1 \partial_{\mathbf{v}^3} + \mathbf{v}^2 \partial_{\mathbf{v}^4} + 2\mathbf{v}^6 \partial_{\mathbf{v}^5} + \mathbf{v}^7 \partial_{\mathbf{v}^5} + \mathbf{v}^8 \partial_{\mathbf{v}^6} + \frac{1}{2} \mathbf{v}^{10} \partial_{\mathbf{v}^9} - \frac{1}{2} \mathbf{v}^{11} \partial_{\mathbf{v}^9}, \\ \mathcal{G}_{10} &= -\mathbf{v}^2 \partial_{\mathbf{v}^2} - \frac{1}{2} \mathbf{v}^3 \partial_{\mathbf{v}^3} - \frac{3}{2} \mathbf{v}^4 \partial_{\mathbf{v}^4} - \frac{3}{2} \mathbf{v}^5 \partial_{\mathbf{v}^5} - \mathbf{v}^6 \partial_{\mathbf{v}^6} - \mathbf{v}^7 \partial_{\mathbf{v}^7} - \frac{1}{2} \mathbf{v}^8 \partial_{\mathbf{v}^8} - \frac{1}{2} \mathbf{v}^9 \partial_{\mathbf{v}^9}, \\ \mathcal{G}_{11} &= -\mathbf{v}^1 \partial_{\mathbf{v}^1} - \frac{1}{2} \mathbf{v}^3 \partial_{\mathbf{v}^3} + \frac{1}{2} \mathbf{v}^4 \partial_{\mathbf{v}^4} + \frac{3}{2} \mathbf{v}^5 \partial_{\mathbf{v}^5} + \mathbf{v}^6 \partial_{\mathbf{v}^6} + \mathbf{v}^7 \partial_{\mathbf{v}^7} + \frac{1}{2} \mathbf{v}^8 \partial_{\mathbf{v}^8} + \frac{1}{2} \mathbf{v}^9 \partial_{\mathbf{v}^9}. \end{aligned} \quad (3.3)$$

We next solve the systems of Equations (3.3) to construct inner automorphism via the solutions of Lie equations. For the automorphism $\mathcal{U}_k : k = 1, 2, \dots, 11$, the Lie equations with respect to the parameter $\mathbf{a}_k$ together with the initial condition $\bar{\mathbf{v}}|_{\mathbf{a}_k=0} = \mathbf{v}$ are:

$$\begin{aligned} \frac{\partial \bar{\mathbf{v}}^1}{\partial \mathbf{a}_1} &= \mathbf{v}^{11}, \frac{\partial \bar{\mathbf{v}}^2}{\partial \mathbf{a}_1} = \mathbf{v}^6, \frac{\partial \bar{\mathbf{v}}^3}{\partial \mathbf{a}_1} = \mathbf{v}^9, \frac{\partial \bar{\mathbf{v}}^4}{\partial \mathbf{a}_1} = \mathbf{v}^5, \frac{\partial \bar{\mathbf{v}}^5}{\partial \mathbf{a}_1} = 0, \frac{\partial \bar{\mathbf{v}}^6}{\partial \mathbf{a}_1} = 0, \frac{\partial \bar{\mathbf{v}}^7}{\partial \mathbf{a}_1} = 0, \frac{\partial \bar{\mathbf{v}}^8}{\partial \mathbf{a}_1} = 0, \\ \frac{\partial \bar{\mathbf{v}}^9}{\partial \mathbf{a}_1} &= 0, \frac{\partial \bar{\mathbf{v}}^{10}}{\partial \mathbf{a}_1} = 0, \frac{\partial \bar{\mathbf{v}}^{11}}{\partial \mathbf{a}_1} = 0, \end{aligned} \quad (3.4)$$



$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_2} &= 0, \frac{\partial \bar{v}^2}{\partial a_2} = v^{10}, \frac{\partial \bar{v}^3}{\partial a_2} = 0, \frac{\partial \bar{v}^4}{\partial a_2} = 2v^8 - v^9, \frac{\partial \bar{v}^5}{\partial a_2} = 0, \frac{\partial \bar{v}^6}{\partial a_2} = 0, \frac{\partial \bar{v}^7}{\partial a_2} = 0, \frac{\partial \bar{v}^8}{\partial a_2} = 0, \\ \frac{\partial \bar{v}^9}{\partial a_2} &= 0, \frac{\partial \bar{v}^{10}}{\partial a_2} = 0, \frac{\partial \bar{v}^{11}}{\partial a_2} = 0, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_3} &= 0, \frac{\partial \bar{v}^2}{\partial a_3} = v^8, \frac{\partial \bar{v}^3}{\partial a_3} = \frac{1}{2}(v^{10} + v^{11}), \frac{\partial \bar{v}^4}{\partial a_3} = 2v^6 + v^7, \frac{\partial \bar{v}^5}{\partial a_3} = 0, \frac{\partial \bar{v}^6}{\partial a_3} = 0, \frac{\partial \bar{v}^7}{\partial a_3} = 0, \\ \frac{\partial \bar{v}^8}{\partial a_3} &= 0, \frac{\partial \bar{v}^9}{\partial a_3} = 0, \frac{\partial \bar{v}^{10}}{\partial a_3} = 0, \frac{\partial \bar{v}^{11}}{\partial a_3} = 0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_4} &= 0, \frac{\partial \bar{v}^2}{\partial a_4} = v^8, \frac{\partial \bar{v}^3}{\partial a_4} = 0, \frac{\partial \bar{v}^4}{\partial a_4} = 0, \frac{\partial \bar{v}^5}{\partial a_4} = 0, \frac{\partial \bar{v}^6}{\partial a_4} = 0, \frac{\partial \bar{v}^7}{\partial a_4} = \frac{1}{2}(3v^{10} - v^{11}), \\ \frac{\partial \bar{v}^8}{\partial a_4} &= 0, \frac{\partial \bar{v}^9}{\partial a_4} = 0, \frac{\partial \bar{v}^{10}}{\partial a_4} = 0, \frac{\partial \bar{v}^{11}}{\partial a_4} = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_5} &= 0, \frac{\partial \bar{v}^2}{\partial a_5} = 0, \frac{\partial \bar{v}^3}{\partial a_5} = 0, \frac{\partial \bar{v}^4}{\partial a_5} = -v^1, \frac{\partial \bar{v}^5}{\partial a_5} = \frac{3}{2}(v^{10} - v^{11}), \frac{\partial \bar{v}^6}{\partial a_5} = 0, \frac{\partial \bar{v}^7}{\partial a_5} = 0, \\ \frac{\partial \bar{v}^8}{\partial a_5} &= 0, \frac{\partial \bar{v}^9}{\partial a_5} = 0, \frac{\partial \bar{v}^{10}}{\partial a_5} = 0, \frac{\partial \bar{v}^{11}}{\partial a_5} = 0, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_6} &= 0, \frac{\partial \bar{v}^2}{\partial a_6} = -v^1, \frac{\partial \bar{v}^3}{\partial a_6} = 0, \frac{\partial \bar{v}^4}{\partial a_6} = -v^3, \frac{\partial \bar{v}^5}{\partial a_6} = 2(v^8 - v^9), \frac{\partial \bar{v}^6}{\partial a_6} = v^{10} - v^{11}, \\ \frac{\partial \bar{v}^7}{\partial a_6} &= 0, \frac{\partial \bar{v}^8}{\partial a_6} = 0, \frac{\partial \bar{v}^9}{\partial a_6} = 0, \frac{\partial \bar{v}^{10}}{\partial a_6} = 0, \frac{\partial \bar{v}^{11}}{\partial a_6} = 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_7} &= 0, \frac{\partial \bar{v}^2}{\partial a_7} = 0, \frac{\partial \bar{v}^3}{\partial a_7} = 0, \frac{\partial \bar{v}^4}{\partial a_7} = -v^3, \frac{\partial \bar{v}^5}{\partial a_7} = -v^9, \frac{\partial \bar{v}^6}{\partial a_7} = 0, \frac{\partial \bar{v}^7}{\partial a_7} = v^{10} - v^{11}, \\ \frac{\partial \bar{v}^8}{\partial a_7} &= 0, \frac{\partial \bar{v}^9}{\partial a_7} = 0, \frac{\partial \bar{v}^{10}}{\partial a_7} = 0, \frac{\partial \bar{v}^{11}}{\partial a_7} = 0, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_8} &= 0, \frac{\partial \bar{v}^2}{\partial a_8} = -v^3, \frac{\partial \bar{v}^3}{\partial a_8} = 0, \frac{\partial \bar{v}^4}{\partial a_8} = -2v^2, \frac{\partial \bar{v}^5}{\partial a_8} = -2v^6, \frac{\partial \bar{v}^6}{\partial a_8} = -v^9, \frac{\partial \bar{v}^7}{\partial a_8} = 0, \\ \frac{\partial \bar{v}^8}{\partial a_8} &= \frac{1}{2}(v^{10} - v^{11}), \frac{\partial \bar{v}^9}{\partial a_8} = 0, \frac{\partial \bar{v}^{10}}{\partial a_8} = 0, \frac{\partial \bar{v}^{11}}{\partial a_8} = 0, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_9} &= 0, \frac{\partial \bar{v}^2}{\partial a_9} = 0, \frac{\partial \bar{v}^3}{\partial a_9} = -v^1, \frac{\partial \bar{v}^4}{\partial a_9} = v^2, \frac{\partial \bar{v}^5}{\partial a_9} = 2v^6 + v^7, \frac{\partial \bar{v}^6}{\partial a_9} = v^8, \\ \frac{\partial \bar{v}^7}{\partial a_9} &= \frac{1}{2}(v^{10} - v^{11}), \frac{\partial \bar{v}^8}{\partial a_9} = 0, \frac{\partial \bar{v}^9}{\partial a_9} = 0, \frac{\partial \bar{v}^{10}}{\partial a_9} = 0, \frac{\partial \bar{v}^{11}}{\partial a_9} = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_{10}} &= 0, \frac{\partial \bar{v}^2}{\partial a_{10}} = -v^2, \frac{\partial \bar{v}^3}{\partial a_{10}} = -\frac{1}{2}v^3, \frac{\partial \bar{v}^4}{\partial a_{10}} = -\frac{3}{2}v^4, \frac{\partial \bar{v}^5}{\partial a_{10}} = -\frac{3}{2}v^5, \frac{\partial \bar{v}^6}{\partial a_{10}} = -v^6, \\ \frac{\partial \bar{v}^7}{\partial a_{10}} &= -v^7, \frac{\partial \bar{v}^8}{\partial a_{10}} = -\frac{1}{2}v^8, \frac{\partial \bar{v}^9}{\partial a_{10}} = -\frac{1}{2}v^9, \frac{\partial \bar{v}^{10}}{\partial a_{10}} = 0, \frac{\partial \bar{v}^{11}}{\partial a_{10}} = 0, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \frac{\partial \bar{v}^1}{\partial a_{11}} &= -v^1, \frac{\partial \bar{v}^2}{\partial a_{11}} = 0, \frac{\partial \bar{v}^3}{\partial a_{11}} = -\frac{1}{2}v^3, \frac{\partial \bar{v}^4}{\partial a_{11}} = \frac{1}{2}v^4, \frac{\partial \bar{v}^5}{\partial a_{11}} = -\frac{3}{2}v^5, \frac{\partial \bar{v}^6}{\partial a_{11}} = v^6, \\ \frac{\partial \bar{v}^7}{\partial a_{11}} &= v^7, \frac{\partial \bar{v}^8}{\partial a_{11}} = \frac{1}{2}v^8, \frac{\partial \bar{v}^9}{\partial a_{11}} = \frac{1}{2}v^9, \frac{\partial \bar{v}^{10}}{\partial a_{11}} = 0, \frac{\partial \bar{v}^{11}}{\partial a_{11}} = 0. \end{aligned} \quad (3.14)$$

3.1. Inner Automorphism. The solutions to the Lie equations (3.4)–(3.14), with initial condition $\bar{\mathbf{v}}|_{\mathbf{a}_k=0} = \mathbf{v}$, give the following inner automorphisms.

$$\begin{aligned}
 \mathfrak{U}_1 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{a}_1 \mathbf{v}^{11} + \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{a}_1 \mathbf{v}^6 + \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{a}_1 \mathbf{v}^9 + \mathbf{v}^3, \bar{\mathbf{v}}^4 = \mathbf{a}_1 \mathbf{v}^5 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \mathbf{v}^5, \bar{\mathbf{v}}^6 = \mathbf{v}^6, \\ \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_2 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{a}_2 \mathbf{v}^{10} + \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = \mathbf{a}_2 (2\mathbf{v}^8 - \mathbf{v}^9) + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \mathbf{v}^5, \bar{\mathbf{v}}^6 = \mathbf{v}^6, \\ \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_3 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{a}_3 \mathbf{v}^8 + \mathbf{v}^2, \bar{\mathbf{v}}^3 = \frac{1}{2} \mathbf{a}_3 (\mathbf{v}^{10} + \mathbf{v}^{11}) + \mathbf{v}^3, \bar{\mathbf{v}}^4 = \frac{1}{2} \mathbf{a}_3 (\mathbf{v}^6 + \mathbf{v}^7) + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \mathbf{v}^5, \\ \bar{\mathbf{v}}^6 = \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_4 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = \frac{1}{2} \mathbf{a}_4 (3\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \mathbf{v}^5, \bar{\mathbf{v}}^6 = \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{v}^7, \\ \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_5 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = -\mathbf{a}_5 \mathbf{v}^1 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \frac{3}{2} \mathbf{a}_5 (\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^5, \bar{\mathbf{v}}^6 = \mathbf{v}^6, \\ \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_6 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = -\mathbf{a}_6 \mathbf{v}^1 + \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = -\mathbf{a}_6 \mathbf{v}^3 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = 2\mathbf{a}_6 (\mathbf{v}^8 - \mathbf{v}^9) + \mathbf{v}^5, \\ \bar{\mathbf{v}}^6 = \mathbf{a}_6 (\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_7 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = -\mathbf{a}_7 \mathbf{v}^3 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = -\mathbf{a}_7 \mathbf{v}^9 + \mathbf{v}^5, \\ \bar{\mathbf{v}}^6 = \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{a}_7 (\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_8 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = -\mathbf{a}_8 \mathbf{v}^3 + \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3, \bar{\mathbf{v}}^4 = -2\mathbf{a}_8 \mathbf{v}^2 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = -2\mathbf{a}_8 \mathbf{v}^6 + \mathbf{v}^5, \\ \bar{\mathbf{v}}^6 = -\mathbf{a}_8 \mathbf{v}^9 + \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \frac{1}{2} \mathbf{a}_8 (\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^8, \bar{\mathbf{v}}^9 = \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_9 : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{v}^2, \bar{\mathbf{v}}^3 = -\mathbf{a}_9 \mathbf{v}^1 + \mathbf{v}^3, \bar{\mathbf{v}}^4 = \mathbf{a}_9 \mathbf{v}^2 + \mathbf{v}^4, \bar{\mathbf{v}}^5 = \mathbf{a}_9^2 \mathbf{v}^8 + \mathbf{a}_9 (2\mathbf{v}^6 + \mathbf{v}^7) + \mathbf{v}^5, \\ \bar{\mathbf{v}}^6 = \mathbf{a}_9 \mathbf{v}^8 + \mathbf{v}^6, \bar{\mathbf{v}}^7 = \mathbf{v}^7, \bar{\mathbf{v}}^8 = \mathbf{v}^8, \bar{\mathbf{v}}^9 = \frac{1}{2} \mathbf{a}_9 (\mathbf{v}^{10} - \mathbf{v}^{11}) + \mathbf{v}^9, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_{10} : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1, \bar{\mathbf{v}}^2 = \mathbf{v}^2 e^{-\mathbf{a}_{10}}, \bar{\mathbf{v}}^3 = \mathbf{v}^3 e^{-\frac{1}{2} \mathbf{a}_{10}}, \bar{\mathbf{v}}^4 = \mathbf{v}^4 e^{-\frac{3}{2} \mathbf{a}_{10}}, \bar{\mathbf{v}}^5 = \mathbf{v}^5 e^{-\frac{3}{2} \mathbf{a}_{10}}, \bar{\mathbf{v}}^6 = \mathbf{v}^6 e^{-\mathbf{a}_{10}}, \\ \bar{\mathbf{v}}^7 = \mathbf{v}^7 e^{-\mathbf{a}_{10}}, \bar{\mathbf{v}}^8 = \mathbf{v}^8 e^{-\frac{1}{2} \mathbf{a}_{10}}, \bar{\mathbf{v}}^9 = \mathbf{v}^9 e^{-\frac{1}{2} \mathbf{a}_{10}}, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}, \end{cases} \\
 \mathfrak{U}_{11} : & \begin{cases} \bar{\mathbf{v}}^1 = \mathbf{v}^1 e^{-\mathbf{a}_{11}}, \bar{\mathbf{v}}^2 = \mathbf{v}^2, \bar{\mathbf{v}}^3 = \mathbf{v}^3 e^{-\frac{1}{2} \mathbf{a}_{11}}, \bar{\mathbf{v}}^4 = \mathbf{v}^4 e^{\frac{1}{2} \mathbf{a}_{11}}, \bar{\mathbf{v}}^5 = \mathbf{v}^5 e^{-\frac{3}{2} \mathbf{a}_{11}}, \bar{\mathbf{v}}^6 = \mathbf{v}^6 e^{\mathbf{a}_{11}}, \\ \bar{\mathbf{v}}^7 = \mathbf{v}^7 e^{\mathbf{a}_{11}}, \bar{\mathbf{v}}^8 = \mathbf{v}^8 e^{\frac{1}{2} \mathbf{a}_{11}}, \bar{\mathbf{v}}^9 = \mathbf{v}^9 e^{\frac{1}{2} \mathbf{a}_{11}}, \bar{\mathbf{v}}^{10} = \mathbf{v}^{10}, \bar{\mathbf{v}}^{11} = \mathbf{v}^{11}. \end{cases}
 \end{aligned}$$

3.2. Optimal Classification. Let $\mathcal{A} \in \mathcal{L}_{11}$ be any arbitrary operator written in the form

$$\mathcal{A} = \mathbf{v}^1 \mathcal{A}_1 + \mathbf{v}^2 \mathcal{A}_2 + \dots + \mathbf{v}^{11} \mathcal{A}_{11}, \quad (3.15)$$

where $\mathbf{v}^j : j = 1, 2, \dots, 11$ are arbitrary constants. The operator (3.15) is mapped by the automorphisms $\mathfrak{U}_j : j = 1, 2, \dots, 11$ to the operator $\bar{\mathcal{A}} \in \mathcal{L}_{11}$ defined as

$$\bar{\mathcal{A}} = \bar{\mathbf{v}}^1 \mathcal{A}_1 + \bar{\mathbf{v}}^2 \mathcal{A}_2 + \dots + \bar{\mathbf{v}}^{11} \mathcal{A}_{11}.$$

We shall use these facts to obtain the optimal system of one-dimensional subalgebras of system (1.1) by reducing the vector

$$\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^{11}), \quad (3.16)$$

to its lowest non-zero components, via the transformations $\mathfrak{U}_1$ to $\mathfrak{U}_{11}$.

Case 3.1. $\bar{\mathbf{v}}^{11} \neq 0$.

By letting $\mathbf{a}_1 = -\frac{\bar{\mathbf{v}}^1}{\bar{\mathbf{v}}^{11}}$ in $\mathfrak{U}_1$, we can render $\bar{\mathbf{v}}^1 = 0$. Thus, vector (3.16) is reduced to the form

$$\mathbf{v} = (0, \mathbf{v}^2, \mathbf{v}^3, \dots, \mathbf{v}^{11}). \quad (3.17)$$



If we now consider $\bar{v}^{10} \neq 0$, then the actions of $\mathfrak{U}_2$ through $\mathfrak{U}_9$ make

$$\bar{v}^2 = \bar{v}^3 = \dots = \bar{v}^9 = 0.$$

Thus we have the representations

$$\mathcal{A}_{10} \pm \mathcal{A}_{11}.$$

If $\bar{v}^{10} = 0$, then

$$\bar{v}^3, \bar{v}^4, \bar{v}^5, \bar{v}^6, \bar{v}^7, \bar{v}^8, \text{ and } \bar{v}^9$$

can all be made zero through the respective action of the transformations

$$\mathfrak{U}_3, \mathfrak{U}_4, \mathfrak{U}_5, \mathfrak{U}_6, \mathfrak{U}_7, \mathfrak{U}_8 \text{ and } \mathfrak{U}_9.$$

These give rise to the representations

$$\mathcal{A}_2 \pm \mathcal{A}_{11}, \mathcal{A}_{11}.$$

Case 3.2. $\bar{v}^{11} = 0$.

If we assume $\bar{v}^{10} \neq 0$, then we have the representations

$$\mathcal{A}_1 \pm \mathcal{A}_{10}, \mathcal{A}_{10}.$$

Case 3.2.1. If $\bar{v}^{10} = 0$, then we can let $\bar{v}^9 \neq 0$. So, $\bar{v}^3, \bar{v}^5$ and $\bar{v}^6$ will respectively become zero via the transformations $\mathfrak{U}_1, \mathfrak{U}_7$ and $\mathfrak{U}_8$. This reduces vector (3.16) to the form

$$\mathbf{v} = (v^1, v^2, 0, v^4, 0, 0, v^7, v^8, v^9, 0, 0), \bar{v}^9 \neq 0. \quad (3.18)$$

If we take $\bar{v}^8 \neq 0$ from the reduced vector (3.18) and subject it to the transformations $\mathfrak{U}_2$ and $\mathfrak{U}_3$ with

$$a_2 = -\frac{v^4}{2v^8 - v^9} \text{ and } a_3 = -\frac{v^2}{v^8}$$

respectively, we obtain

$$\mathbf{v} = (v^1, 0, 0, 0, 0, 0, v^7, v^8, v^9, 0, 0). \quad (3.19)$$

The reduced vector (3.19) cannot be further simplified through any of the transformations $\mathfrak{U}_k : k = 1, 2, \dots, 11$. Hence, the following representatives for the optimal system are obtained.

$$\mathcal{A}_8 \pm \mathcal{A}_9, \mathcal{A}_1 \pm \mathcal{A}_8 \pm \mathcal{A}_9, \mathcal{A}_7 \pm \mathcal{A}_8 \pm \mathcal{A}_9, \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_8 \pm \mathcal{A}_9.$$

If we had taken $\bar{v}^8 = 0$ from the reduced vector (3.18), then v^4 will become zero by the action of the transformation $\mathfrak{U}_2$, and this leads to the reduced vector

$$\mathbf{v} = (v^1, v^2, 0, 0, 0, 0, v^7, 0, v^9, 0, 0), \bar{v}^9 \neq 0. \quad (3.20)$$

If $\bar{v}^1 \neq 0$ in (3.20), then $\bar{v}^2 = 0$ via the action of $\mathfrak{U}_6$. This gives rise to the representations

$$\mathcal{A}_1 \pm \mathcal{A}_9, \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_9.$$

If we take $\bar{v}^1 = 0$ in (3.20), then the representations are

$$\mathcal{A}_9, \mathcal{A}_2 \pm \mathcal{A}_9, \mathcal{A}_7 \pm \mathcal{A}_9, \mathcal{A}_2 \pm \mathcal{A}_7 \pm \mathcal{A}_9.$$

Case 3.2.2. For $\bar{v}^9 = 0$, vector (3.16) will be reduced to the form

$$\mathbf{v} = (v^1, v^2, v^3, v^4, v^5, v^6, v^7, v^8, 0, 0, 0). \quad (3.21)$$

For $\bar{v}^8 \neq 0$, the transformations $\mathfrak{U}_2, \mathfrak{U}_3, \mathfrak{U}_6$ and $\mathfrak{U}_9$ will make $\bar{v}^2 = \bar{v}^4 = \bar{v}^5 = \bar{v}^6 = 0$. Thus, the above vector becomes

$$\mathbf{v} = (v^1, 0, v^3, 0, 0, 0, v^7, v^8, 0, 0, 0).$$

If $\bar{v}^1 \neq 0$, then $\bar{v}^1 = 0$ by $\mathfrak{U}_9$. Hence, we have

$$\mathcal{A}_1 \pm \mathcal{A}_8, \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_8.$$



If $\mathfrak{v}^1 = 0$, the representations are

$$\mathcal{A}_8, \mathcal{A}_3 \pm \mathcal{A}_8, \mathcal{A}_7 \pm \mathcal{A}_8, \mathcal{A}_3 \pm \mathcal{A}_7 \pm \mathcal{A}_8.$$

For $\mathfrak{v}^8 = 0$, the reduced vector (3.21) becomes

$$\mathfrak{v} = (\mathfrak{v}^1, \mathfrak{v}^2, \mathfrak{v}^3, \mathfrak{v}^4, \mathfrak{v}^5, \mathfrak{v}^6, \mathfrak{v}^7, 0, 0, 0, 0).$$

This vector will be simplified under the following cases.

3.2.2.1. $\mathfrak{v}^6 \neq 0, \mathfrak{v}^7 \neq 0$.

Transformations $\mathfrak{U}_1, \mathfrak{U}_3$ and $\mathfrak{U}_8$ will render $\bar{\mathfrak{v}}^2, \bar{\mathfrak{v}}^4$ and $\bar{\mathfrak{v}}^5$ zero respectively leading to the reduced vector

$$\mathfrak{v} = (\mathfrak{v}^1, 0, \mathfrak{v}^3, 0, 0, \mathfrak{v}^6, \mathfrak{v}^7, 0, 0, 0, 0).$$

If $\mathfrak{v}^1 \neq 0$, the action of $\mathfrak{U}_9$ will make $\mathfrak{v}^3 = 0$. This results into the representations

$$\mathcal{A}_1 \pm \mathcal{A}_6 \pm \mathcal{A}_7.$$

If $\mathfrak{v}^1 = 0$, then we have

$$\mathcal{A}_6 \pm \mathcal{A}_7, \mathcal{A}_3 \pm \mathcal{A}_6 \pm \mathcal{A}_7.$$

The details of the remaining cases are similar to the above, their results are as follows.

3.2.2.2. $\mathfrak{v}^6 \neq 0, \mathfrak{v}^7 = 0$.

The representations are

$$\mathcal{A}_6, \mathcal{A}_1 \pm \mathcal{A}_6, \mathcal{A}_3 \pm \mathcal{A}_6.$$

3.2.2.3. $\mathfrak{v}^6 = 0, \mathfrak{v}^7 \neq 0$.

The representations are

$$\mathcal{A}_7, \mathcal{A}_1 \pm \mathcal{A}_7, \mathcal{A}_2 \pm \mathcal{A}_7, \mathcal{A}_3 \pm \mathcal{A}_7.$$

3.2.2.4. $\mathfrak{v}^6 = 0, \mathfrak{v}^7 = 0$.

The representations are

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_1 \pm \mathcal{A}_5, \mathcal{A}_2 \pm \mathcal{A}_5, \mathcal{A}_3 \pm \mathcal{A}_5, \mathcal{A}_4 \pm \mathcal{A}_5.$$

Theorem 3.1. *The optimal system of one-dimensional subalgebras $\mathfrak{L}_j^1$ of Lie symmetry algebra $\mathcal{L}_{11}$ are given by the following presentations (up to conjugacy)*

$\mathfrak{L}_1 = \mathcal{A}_1,$	$\mathfrak{L}_{15} = \mathcal{A}_1 \pm \mathcal{A}_8,$	$\mathfrak{L}_{29} = \mathcal{A}_7 \pm \mathcal{A}_9,$
$\mathfrak{L}_2 = \mathcal{A}_2,$	$\mathfrak{L}_{16} = \mathcal{A}_1 \pm \mathcal{A}_9,$	$\mathfrak{L}_{30} = \mathcal{A}_8 \pm \mathcal{A}_9,$
$\mathfrak{L}_3 = \mathcal{A}_3,$	$\mathfrak{L}_{17} = \mathcal{A}_1 \pm \mathcal{A}_{10},$	$\mathfrak{L}_{31} = \mathcal{A}_{10} \pm \mathcal{A}_{11},$
$\mathfrak{L}_4 = \mathcal{A}_4,$	$\mathfrak{L}_{18} = \mathcal{A}_2 \pm \mathcal{A}_5,$	$\mathfrak{L}_{32} = \mathcal{A}_1 \pm \mathcal{A}_6 \pm \mathcal{A}_7,$
$\mathfrak{L}_5 = \mathcal{A}_5,$	$\mathfrak{L}_{19} = \mathcal{A}_2 \pm \mathcal{A}_7,$	$\mathfrak{L}_{33} = \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_8,$
$\mathfrak{L}_6 = \mathcal{A}_6,$	$\mathfrak{L}_{20} = \mathcal{A}_2 \pm \mathcal{A}_9,$	$\mathfrak{L}_{34} = \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_9,$
$\mathfrak{L}_7 = \mathcal{A}_7,$	$\mathfrak{L}_{21} = \mathcal{A}_2 \pm \mathcal{A}_{11},$	$\mathfrak{L}_{35} = \mathcal{A}_1 \pm \mathcal{A}_8 \pm \mathcal{A}_9,$
$\mathfrak{L}_8 = \mathcal{A}_8,$	$\mathfrak{L}_{22} = \mathcal{A}_3 \pm \mathcal{A}_5,$	$\mathfrak{L}_{36} = \mathcal{A}_2 \pm \mathcal{A}_7 \pm \mathcal{A}_9,$
$\mathfrak{L}_9 = \mathcal{A}_9,$	$\mathfrak{L}_{23} = \mathcal{A}_3 \pm \mathcal{A}_6,$	$\mathfrak{L}_{37} = \mathcal{A}_3 \pm \mathcal{A}_6 \pm \mathcal{A}_7,$
$\mathfrak{L}_{10} = \mathcal{A}_{10},$	$\mathfrak{L}_{24} = \mathcal{A}_3 \pm \mathcal{A}_7,$	$\mathfrak{L}_{38} = \mathcal{A}_3 \pm \mathcal{A}_7 \pm \mathcal{A}_8,$
$\mathfrak{L}_{11} = \mathcal{A}_{11},$	$\mathfrak{L}_{25} = \mathcal{A}_3 \pm \mathcal{A}_8,$	$\mathfrak{L}_{39} = \mathcal{A}_7 \pm \mathcal{A}_8 \pm \mathcal{A}_9,$
$\mathfrak{L}_{12} = \mathcal{A}_1 \pm \mathcal{A}_5,$	$\mathfrak{L}_{26} = \mathcal{A}_4 \pm \mathcal{A}_5,$	$\mathfrak{L}_{40} = \mathcal{A}_1 \pm \mathcal{A}_7 \pm \mathcal{A}_8 \pm \mathcal{A}_9,$
$\mathfrak{L}_{13} = \mathcal{A}_1 \pm \mathcal{A}_6,$	$\mathfrak{L}_{27} = \mathcal{A}_6 \pm \mathcal{A}_7,$	
$\mathfrak{L}_{14} = \mathcal{A}_1 \pm \mathcal{A}_7,$	$\mathfrak{L}_{28} = \mathcal{A}_7 \pm \mathcal{A}_8,$	

4. REDUCTIONS AND INVARIANT SOLUTIONS

Lie algebras are the main building blocks for the construction of exact solutions that leave the underlying (system of) differential equations invariant. We shall use the generalized Lie algebra (2.21) to find the solutions of the Manakov-Santini system (1.1). Where necessary, the symmetries for each of the reduced equations shall be found in order to maximize the solution space.

4.1. Solutions through subalgebra $\mathcal{X}_1$. The characteristic equations are

$$\begin{aligned}\frac{dt}{\alpha(t)} &= -\frac{dx}{\frac{1}{2}\alpha''(t)y^2 + \alpha'(t)x} = -\frac{du}{\frac{1}{2}\alpha'''(t)y^2 + \alpha''(t)x + 2\alpha'(t)u} \\ &= -\frac{dv}{\frac{1}{3}\alpha'''(t)y^3 + 2\alpha''(t)xy + 2\alpha'(t)v}.\end{aligned}$$

Solving these first order system of PDEs, we obtain the invariants

$$\begin{aligned}r &= \alpha(t)x + \frac{\alpha'(t)}{2}y^2, \quad s = y, \quad u = -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2 + \frac{1}{\alpha(t)^2}p(r, s), \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{1}{\alpha(t)^2}q(r, s).\end{aligned}\tag{4.1}$$

These invariants reduced the MS system to

$$\begin{aligned}(q_s - p)p_{rr} - q_r p_{rs} - p_{ss} - p_s^2 &= 0, \\ (q_s - p)q_{rr} - q_r q_{rs} - q_{ss} &= 0.\end{aligned}\tag{4.2}$$

The generators of the reduced equation (4.2) are:

$$\begin{aligned}\mathcal{X}_{1_1} &= \partial_r, \quad \mathcal{X}_{1_2} = \partial_s, \quad \mathcal{X}_{1_3} = \partial_q, \quad \mathcal{X}_{1_4} = \partial_p + s\partial_q, \quad \mathcal{X}_{1_5} = s\partial_r + 2r\partial_q, \\ \mathcal{X}_{1_6} &= r\partial_r + 2p\partial_p + 2q\partial_q, \quad \mathcal{X}_{1_7} = s\partial_s - 2p\partial_p - q\partial_q.\end{aligned}$$

4.1.1. Subalgebra $\mathcal{X}_{1_1} + \mathcal{X}_{1_2} + \mathcal{X}_{1_3} + \mathcal{X}_{1_4}$. The invariants are

$$\omega = r - cs, \quad p(r, s) = s + f(\omega), \quad q(r, s) = \frac{s^2}{2} + s + g(\omega),\tag{4.3}$$

where c is an arbitrary constant. They reduced system (4.2) to the system of ordinary differential equations

$$\begin{aligned}(1 - c^2 - f)f'' - f'^2 &= 0, \\ (1 - c^2 - f)g'' - 1 &= 0,\end{aligned}$$

whose solution is ($k_1 \neq 0$)

$$\begin{aligned}f(\omega) &= 1 - c^2 \pm \sqrt{1 + c^4 - 2c^2 + 2k_1\omega + 2k_2}, \\ g(\omega) &= k_3\omega + k_4 \mp \frac{\sqrt{(1 + c^4 - 2c^2 + 2k_1\omega + 2k_2)^3}}{3k_1^2}.\end{aligned}$$

Using the invariants (4.3), the solutions of the reduced system (4.2) is

$$\begin{aligned}p(r, s) &= s + 1 - c^2 \pm \sqrt{1 + c^4 - 2c^2 + 2k_1(r - cs) + 2k_2}, \\ q(r, s) &= \frac{s^2}{2} + s + k_3(r - cs) + k_4 \mp \frac{\sqrt{(1 + c^4 - 2c^2 + 2k_1(r - cs) + 2k_2)^3}}{3k_1^2}.\end{aligned}$$



Subsequently, the solutions of system (1.1) are given by

$$\begin{aligned} u &= -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2 + \frac{y + 1 - c^2 \pm \sqrt{1 + c^4 - 2c^2 + 2k_1 \left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2 - cy \right) + 2k_2}}{\alpha(t)^2}, \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{\frac{1}{2}y^2 + y + k_3 \left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2 - cy \right)}{\alpha(t)^2} + k_4 \\ &\quad \mp \frac{\sqrt{\left(1 + c^4 - 2c^2 + 2k_1 \left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2 - cy \right) + 2k_2 \right)^3}}{3k_1^2\alpha(t)^2}. \end{aligned} \quad (4.4)$$

4.1.2. *Subalgebra* $\mathcal{X}_{15}$. The invariants

$$\omega = s, \quad p(r, s) = f(\omega), \quad q(r, s) = \frac{r^2}{s} + s + g(\omega), \quad (4.5)$$

reduced system (4.2) to

$$f'' = 0, \quad g'' - g' + f = 0,$$

with solution

$$f(\omega) = k_1\omega + k_2, \quad g(\omega) = \frac{1}{3}k_3\omega^3 + k_1\omega^2 + k_2\omega + k_4.$$

Using both the invariants (4.1) and (4.5), we obtain

$$\begin{aligned} u &= -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2 + \frac{1}{\alpha(t)^2}(k_1y + k_2), \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{1}{\alpha(t)^2} \left(\frac{\left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2 \right)^2}{y} + \frac{1}{3}k_3y^3 + k_1y^2 + k_2y + k_4 \right). \end{aligned}$$

4.1.3. *Subalgebra* $\mathcal{X}_{16}$. The invariants are

$$\omega = s, \quad p(r, s) = r^2f(\omega), \quad q(r, s) = r^2g(\omega). \quad (4.6)$$

They lead to the reduced system

$$\begin{aligned} f'' - 2fg' + 4gf' + 6f^2 &= 0, \\ \frac{1}{2}g'' + gg' + fg &= 0. \end{aligned} \quad (4.7)$$

We could not solve this system except for the following special cases:

$$\text{If } f = 0, \text{ then } g = \tanh(\omega + k_1).$$

$$\text{If } g = 0, \text{ then } f = -\wp(\omega + k_1, 0, k_2).$$

where $\wp$ is the WeierstrassP function. Effecting these in the invariants (4.1) and (4.6) lead to the following solutions of system (1.1):

$$\begin{aligned} u &= -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2 - \frac{\left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2 \right)^2}{\alpha(t)^2} \wp(y + k_1, 0, k_2), \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3. \end{aligned}$$



$$u = -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2,$$

$$v = -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{\left(\alpha(t)x + \frac{\alpha'(t)}{2}y^2\right)^2}{\alpha(t)^2} \tanh(y + k_1).$$

4.1.4. *Subalgebra* $\mathcal{X}_{17}$. The characteristic equations are

$$\frac{ds}{s} = -\frac{dp}{2p} = -\frac{dq}{q} = \frac{dr}{0},$$

with solution

$$\omega = r, \quad p(r, s) = \frac{1}{s^2}f(\omega), \quad q(r, s) = \frac{1}{s}g(\omega).$$

The reduced system, via the above invariants, is

$$\begin{aligned} (f + g)f'' + f'^2 - 2f'g' + 6f &= 0, \\ (f + g)g'' - g'^2 + 2g &= 0. \end{aligned} \tag{4.8}$$

Here too, we shall consider two special cases. For $f = 0$, system (4.8) becomes

$$gg'' - g'^2 + 2g = 0,$$

with solutions¹

$$g(\omega) = \frac{k_1}{2k_2} \left(2k_1 k_2 e^{\frac{\omega}{2k_1}} - e^{-\frac{\omega}{2k_1}} \right)^2, \quad g(\omega) = \frac{k_1}{2k_2} \left(k_2 e^{\frac{\omega}{2k_1}} - 2k_1 e^{-\frac{\omega}{2k_1}} \right)^2,$$

where $k_1 > 0, k_2 > 0$ are arbitrary constants. Thus, the solutions of the Manokov-Santini system under the generator $\mathcal{X}_{17}$ are:

$$\begin{aligned} u &= -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2, \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{k_1}{2k_2\alpha(t)^2y} \left(2k_1 k_2 e^{\frac{\alpha(t)x + \frac{\alpha'(t)}{2}y^2}{2k_1}} - e^{-\frac{\alpha(t)x + \frac{\alpha'(t)}{2}y^2}{2k_1}} \right)^2, \\ u &= -\frac{\alpha'(t)}{\alpha(t)}x - \frac{\alpha''(t)}{2\alpha(t)}y^2, \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3 + \frac{k_1}{2k_2\alpha(t)^2y} \left(k_2 e^{\frac{\alpha(t)x + \frac{\alpha'(t)}{2}y^2}{2k_1}} - 2k_1 e^{-\frac{\alpha(t)x + \frac{\alpha'(t)}{2}y^2}{2k_1}} \right)^2. \end{aligned}$$

For $g = 0$, system (4.8) becomes

$$ff'' + f'^2 + 6f = 0$$

whose solutions are implicitly given by

$$\pm \int^{f(\omega)} \frac{z}{k_1 - 4z^3} dz - \omega - k_2 = 0.$$

Therefore, the solutions of the MS system are implicitly given by

$$\begin{aligned} \pm \int^h \frac{z}{k_1 - 4z^3} dz - \alpha(t)x - \frac{\alpha'(t)}{2}y^2 - k_2 &= 0, \\ v &= -\frac{\alpha'(t)^2}{3\alpha(t)^2}y^3 - \frac{2\alpha'(t)}{\alpha(t)}xy - \frac{\alpha''(t)}{3\alpha(t)}y^3. \end{aligned}$$

¹Both solutions are the same only when $k_1 = \frac{1}{2}$.



where $\mathfrak{h} = y^2 \alpha(t)^2 \left(u + \frac{\alpha'(t)}{\alpha(t)} x + \frac{\alpha''(t)}{2\alpha(t)} y^2 \right)$. For $k_1 = 0$, we have

$$u = -\frac{\alpha'(t)}{\alpha(t)} x - \frac{\alpha''(t)}{2\alpha(t)} y^2 - \frac{1}{y^2 \alpha(t)^2} \left(\alpha(t)x + \frac{\alpha'(t)}{2} y^2 + k_2 \right)^2,$$

$$v = -\frac{\alpha'(t)^2}{3\alpha(t)^2} y^3 - \frac{2\alpha'(t)}{\alpha(t)} xy - \frac{\alpha''(t)}{3\alpha(t)} y^3.$$

4.1.5. *Subalgebra* $\mathcal{X}_{16} + \mathcal{X}_{17}$. The characteristic equations are

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dq}{q} = \frac{dp}{0},$$

with solution

$$\omega = \frac{r}{s}, \quad p(r, s) = f(\omega), \quad q(r, s) = rg(\omega). \quad (4.9)$$

The reduced system, via the above invariants, is

$$(f - \omega g + \omega^2) f'' - (\omega g' + g - 2\omega) g' + f'^2 = 0, \quad (4.10)$$

$$(\omega g'' + 2g')(f - \omega g + \omega^2) = 0. \quad (4.11)$$

From (4.11), we have

$$g = \frac{k_1}{\omega} + k_2, \quad k_1 \neq 0, \quad (4.12)$$

or

$$f = \omega g - \omega^2. \quad (4.13)$$

The substituting of (4.12) into Eq. (4.10) results to

$$(f + \omega^2 - k_2 \omega - k_1) f'' + (2\omega - k_2) f' + f'^2 = 0,$$

whose solution is implicitly given by

$$\frac{\frac{k_2 - 2\omega}{2} \text{Bi} \left(\frac{k_2^2}{4} + k_1 - f(\omega) \right) + \text{Bi} \left(1, \frac{k_2^2}{4} + k_1 - f(\omega) \right)}{\frac{k_2 - 2\omega}{2} \text{Ai} \left(\frac{k_2^2}{4} + k_1 - f(\omega) \right) + \text{Ai} \left(1, \frac{k_2^2}{4} + k_1 - f(\omega) \right)} = c_1,$$

where $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy function of the first and second kind respectively. The solution of the system (1.1) is obtained by using the invariants (4.9) and (4.1).

Eq. (4.10) is satisfied identically by the substitution (4.13). Hence the solution to the system (4.2) is

$$p(r, s) = \frac{r}{s} g \left(\frac{r}{s} \right) - \frac{r^2}{s^2},$$

$$q(r, s) = rg \left(\frac{r}{s} \right).$$

Using the invariants (4.1), the solution of system (1.1) is given by

$$u = -\frac{\alpha'(t)}{\alpha(t)} x - \frac{\alpha''(t)}{2\alpha(t)} y^2 - \left(\frac{\alpha(t)x + \frac{\alpha'(t)}{2} y^2}{\alpha(t)y} \right)^2 + \frac{\alpha(t)x + \frac{\alpha'(t)}{2} y^2}{\alpha(t)^2 y} g \left(\frac{\alpha(t)x + \frac{\alpha'(t)}{2} y^2}{y} \right),$$

$$v = -\left(\frac{\alpha'(t)^2}{3\alpha(t)^2} + \frac{\alpha''(t)}{3\alpha(t)} \right) y^3 - \frac{2\alpha'(t)}{\alpha(t)} xy + \frac{\alpha(t)x + \frac{\alpha'(t)}{2} y^2}{\alpha(t)^2} g \left(\frac{\alpha(t)x + \frac{\alpha'(t)}{2} y^2}{y} \right).$$



4.2. **Solutions through subalgebra $\mathcal{X}_2$.** The characteristic equations are

$$\frac{dx}{\beta'(t)y^2 + 2\beta(t)x} = \frac{dy}{\beta(t)y} = \frac{du}{\frac{1}{2}\beta''(t)y^2 + \beta'(t)x + 2\beta(t)u} = \frac{dv}{\frac{1}{2}\beta''(t)y^3 + 3\beta'(t)xy + 3\beta(t)v}.$$

The solution to the above system of ODEs give rise to the invariants

$$r = \frac{x}{y^2} - \frac{\beta'(t) \ln y}{\beta(t)}, \quad s = t, \quad u = -\frac{1}{2} \left(\frac{\beta'(t)}{\beta(t)} y \ln y \right)^2 + \frac{\beta'(t)}{\beta(t)} x \ln y + \frac{\beta''(t)}{2\beta(t)} y^2 \ln y + y^2 p(r, s),$$

$$v = -\frac{3}{2} y \left(\frac{\beta'(t)}{\beta(t)} y \ln y \right)^2 + \frac{3\beta'(t)}{\beta(t)} xy \ln y + \frac{\beta''(t)}{2\beta(t)} y^3 \ln y + y^3 q(r, s).$$

These invariants reduced system (1.1) to¹

$$\begin{aligned} & \left(3q - p - 4r^2 + \frac{\beta''}{2\beta} - \frac{\beta'}{\beta} r - \frac{\beta'^2}{\beta^2} \right) p_{rr} + p_{rs} - p_r^2 + \left(2r + \frac{3\beta''}{\beta} \right) p_r \\ & - \frac{\beta'}{\beta} q_r - 2p + \frac{\beta'}{\beta} r - \frac{3\beta''}{2\beta} + \frac{\beta'^2}{\beta^2} = 0, \\ & \left(3q - p - 4r^2 + \frac{\beta''}{2\beta} - \frac{\beta'}{\beta} r - \frac{\beta'^2}{\beta^2} \right) q_{rr} + q_{rs} - q_r^2 + 2 \left(3r + \frac{\beta''}{\beta} \right) q_r \\ & - 6q - \frac{3\beta'}{\beta} r - \frac{5\beta''}{2\beta} + \frac{3\beta'^2}{\beta^2} = 0. \end{aligned} \tag{4.14}$$

The infinitesimals of system (4.14) are two but both lead to the same reduced equation. So we shall present only one of them which is

$$\begin{aligned} \mathcal{X}_{2_1} = & \left(\left(\frac{\beta\beta''}{2\beta'^2} r + \frac{\beta\beta'''}{4\beta'^2} - \frac{\beta\beta''^2}{2\beta'^3} \right)' + \frac{\beta''}{\beta'^2} \left(\frac{\beta''}{4} + 2\beta p \right) - \frac{\beta''}{4\beta} + \frac{\beta'}{2\beta} + r - 2p \right) \partial_p \\ & \left(\left(\frac{\beta\beta''}{2\beta'^2} r + \frac{\beta\beta'''}{12\beta'^2} - \frac{\beta\beta''^2}{6\beta'^3} - \frac{\beta''}{6\beta'} \right)' + \frac{\beta''}{\beta'^2} \left(\frac{\beta''}{4} + 2\beta q \right) - \frac{\beta''}{4\beta} + \frac{\beta'}{2\beta} + 3r - 2p \right) \partial_q \\ & + \left(\frac{\beta\beta''}{\beta'^2} r - r + 1 \right) \partial_r + \frac{\beta}{\beta'} \partial_s. \end{aligned}$$

The invariants are obtained as

$$\begin{aligned} p(r, s) &= \frac{\beta'}{\beta} rs - \frac{\beta'^2}{2\beta^2} s^2 + \frac{\beta''}{2\beta'} r + \frac{\beta'}{2\beta} - \frac{3\beta''^2}{8\beta'^2} + \frac{\beta'''}{4\beta'} - \frac{\beta''}{4\beta} + \frac{\beta'^2}{\beta^2} f(\omega), \\ q(r, s) &= \frac{3\beta'}{\beta} rs - \frac{3\beta'^2}{2\beta^2} s^2 + \frac{\beta''}{2\beta'} r - \frac{\beta''^2}{8\beta'^2} + \frac{\beta'''}{12\beta'} - \frac{\beta''}{4\beta} + \frac{\beta'}{2\beta} + \frac{\beta'^2}{\beta^2} g(\omega), \\ \omega &= \frac{\beta}{\beta'} r - s. \end{aligned} \tag{4.15}$$

They reduced system (4.14) to system of ODEs

$$\begin{aligned} (3g - f - 4\omega^2 - 1)f'' + 2(\omega + 1)f' - f'^2 - g' - 2f + \omega + 1 &= 0, \\ (3g - f - 4\omega^2 - 1)g'' + (6\omega + 1)g' - g'^2 - 6g - 3\omega + 3 &= 0. \end{aligned}$$

This system could not be solved for arbitrary $f(\omega)$ and $g(\omega)$, but a particular solution is found to be

$$f(\omega) = \frac{1}{2}(\omega^2 + 1), \quad g(\omega) = \frac{1}{2}(3\omega^2 + 1).$$

¹Here, ' denotes derivative with respect to s and $\beta = \beta(s)$.



Using the invariants (4.15) and (4.1), another solution of system (1.1) is obtained as:

$$\begin{aligned} u &= y^2 \left(\frac{1}{2} \left(\frac{x}{y^2} - \frac{\alpha'(t) \ln y}{\alpha(t)} \right)^2 + \frac{\beta''(t)}{2\beta'(t)} \left(\frac{x}{y^2} - \frac{\alpha'(t) \ln y}{\alpha(t)} \right) + \frac{\alpha'(t)}{2\alpha(t)} - \frac{3\alpha''(t)^2}{8\alpha'(t)^2} + \frac{\alpha'''(t)}{4\alpha'(t)} - \frac{\alpha''(t)}{4\alpha(t)} + \frac{\alpha'(t)^2}{2\alpha(t)^2} \right) \\ &\quad - \frac{\alpha'(t)^2}{2\alpha(t)^2} (y \ln y)^2 + \frac{\alpha'(t)}{\alpha(t)} x \ln y + \frac{\beta''(t)}{2\beta(t)} y^2 \ln y, \\ v &= y^3 \left(\frac{3}{2} \left(\frac{x}{y^2} - \frac{\alpha'(t) \ln y}{\alpha(t)} \right)^2 + \frac{\beta''(t)}{2\beta'(t)} \left(\frac{x}{y^2} - \frac{\alpha'(t) \ln y}{\alpha(t)} \right) + \frac{\alpha'(t)}{2\alpha(t)} - \frac{\alpha''(t)^2}{8\alpha'(t)^2} + \frac{\alpha'''(t)}{12\alpha'(t)} - \frac{\alpha''(t)}{4\alpha(t)} + \frac{\alpha'(t)^2}{2\alpha(t)^2} \right) \\ &\quad - \frac{3\alpha'(t)^2}{2\alpha(t)^2} y^3 (\ln y)^2 + \frac{3\alpha'(t)}{\alpha(t)} xy \ln y + \frac{\beta''(t)}{2\beta(t)} y^3 \ln y. \end{aligned}$$

4.3. Solutions through subalgebra $\mathcal{X}_3$. The characteristic equations are

$$\frac{dy}{\gamma(t)} = -\frac{du}{\gamma''(t)y} = -\frac{dv}{\frac{1}{2}\gamma''(t)y^2 + \gamma'(t)x}$$

with solution

$$r = x, \quad s = t, \quad u = -\frac{\gamma''(t)}{2\gamma(t)}y^2 + p(r, s), \quad v = -\frac{\gamma''(t)}{6\gamma(t)}y^3 - \frac{\gamma'(t)}{\gamma(t)}xy + q(r, s). \quad (4.16)$$

They reduced system (1.1) to

$$\begin{aligned} \left(p + \frac{\gamma'(s)}{\gamma(s)}r \right) p_{rr} - p_{rs} + p_r^2 - \frac{\gamma''(s)}{\gamma(s)} &= 0, \\ \left(p + \frac{\gamma'(s)}{\gamma(s)}r \right) q_{rr} - q_{rs} - \frac{\gamma'(s)}{\gamma(s)}q_r &= 0. \end{aligned} \quad (4.17)$$

We shall find the generators of system (4.17) by considering $\gamma(s)$ as a parameter. That is, the unknown become p, q and γ . After solving the determining equations that result from the invariance condition, the 28 infinitesimals are found as follows:

$$\begin{aligned} \mathcal{X}_{3_1} &= \partial_s + q\partial_q, \quad \mathcal{X}_{3_2} = \partial_s + \gamma\partial_\gamma, \quad \mathcal{X}_{3_3} = \gamma\partial_\gamma + \partial_q, \quad \mathcal{X}_{3_4} = \gamma\partial_\gamma + s\partial_q, \quad \mathcal{X}_{3_5} = \partial_s + r\partial_r + p\partial_p, \\ \mathcal{X}_{3_6} &= r\partial_r + p\partial_p + \partial_q, \quad \mathcal{X}_{3_7} = r\partial_r + p\partial_p + s\partial_q, \quad \mathcal{X}_{3_8} = c\partial_c + c\partial_q, \quad \mathcal{X}_{3_9} = r\partial_r + p\partial_p + \gamma\partial_\gamma, \\ \mathcal{X}_{3_{10}} &= (q+1)\partial_q, \quad \mathcal{X}_{3_{11}} = (q+s)\partial_q, \quad \mathcal{X}_{3_{12}} = \partial_s + \frac{r}{\gamma}\partial_q, \quad \mathcal{X}_{3_{13}} = \frac{1}{\gamma}\partial_r + q\partial_q, \quad \mathcal{X}_{3_{14}} = \frac{1}{\gamma}\partial_r + \gamma\partial_\gamma, \\ \mathcal{X}_{3_{15}} &= (q+c)\partial_q, \quad \mathcal{X}_{3_{16}} = \left(\frac{r}{\gamma} + s \right) \partial_q, \quad \mathcal{X}_{3_{17}} = \frac{1}{\gamma}\partial_r + \frac{r}{\gamma}\partial_q, \quad \mathcal{X}_{3_{18}} = \left(\frac{1}{\gamma} + r \right) \partial_r + p\partial_p, \\ \mathcal{X}_{3_{19}} &= \left(\frac{r}{\gamma} + 1 \right) \partial_q, \quad \mathcal{X}_{3_{20}} = \left(\frac{r}{\gamma} + \gamma \right) \partial_q, \quad \mathcal{X}_{3_{21}} = s\partial_s + r\partial_r + \frac{\gamma}{2}\partial_\gamma - \frac{q}{2}\partial_q, \\ \mathcal{X}_{3_{22}} &= \frac{s}{\gamma}\partial_r - \frac{1}{\gamma}\partial_p + q\partial_q, \quad \mathcal{X}_{3_{23}} = \frac{s}{\gamma}\partial_r + \gamma\partial_\gamma - \frac{1}{\gamma}\partial_p, \quad \mathcal{X}_{3_{24}} = s\partial_s + \frac{\gamma}{2}\partial_\gamma - p\partial_p + \frac{q}{2}\partial_q, \\ \mathcal{X}_{3_{25}} &= s\partial_s + \frac{3\gamma}{2}\partial_\gamma - p\partial_p - \frac{q}{2}\partial_q, \quad \mathcal{X}_{3_{26}} = \frac{s}{\gamma}\partial_r - \frac{1}{\gamma}\partial_p + \frac{r}{\gamma}\partial_q, \\ \mathcal{X}_{3_{27}} &= s\partial_s + \frac{\gamma}{2}\partial_\gamma - p\partial_p + \left(\frac{r}{\gamma} - \frac{q}{2} \right) \partial_q, \quad \mathcal{X}_{3_{28}} = \left(r + \frac{s}{\gamma} \right) \partial_r + \left(p - \frac{1}{\gamma} \right) \partial_p. \end{aligned}$$

4.3.1. Subalgebra $\mathcal{X}_{3_2} + \mathcal{X}_{3_3} + \mathcal{X}_{3_4} + \mathcal{X}_{3_8} + \mathcal{X}_{3_{14}}$. The characteristic equations are

$$\frac{ds}{1} = \frac{\gamma(t)dr}{1} = \frac{d(\gamma(t))}{5\gamma(t)} = \frac{dq}{1+s+\gamma(t)},$$

with solution

$$\omega = r + \frac{e^{-5s}}{5k}, \quad p(r, s) = f(\omega), \quad q(r, s) = s + \frac{s^2}{2} + \frac{ke^{5s}}{5} + g(\omega). \quad (4.18)$$



They reduced system (4.17) to

$$\begin{aligned}(5\omega + f)f'' + f'^2 - 25 &= 0, \\ (5\omega + f)g'' - 5g' &= 0,\end{aligned}$$

whose solution is

$$\begin{aligned}f(\omega) &= \frac{1}{10}W(e^{100\omega-1}) - 5\omega + \frac{1}{10}, \\ g(\omega) &= \frac{c_1}{150} \left(\sqrt{W(e^{100\omega-1})^3} - 3\sqrt{W(e^{100\omega-1})} \right) + c_2\end{aligned}$$

where $W(z)$ is the Lambert W function (which is also called omega function or product logarithm). Using the invariants (4.18) and (4.16), we obtain

$$\begin{aligned}u &= -\frac{25}{2}y^2 + \frac{W(e^{100x+\frac{20e^{-5t}}{k}-1})}{10} - 5x - \frac{e^{-5t}}{k} + \frac{1}{10}, \\ v &= -\frac{25}{6}y^3 - 5xy + t + \frac{t^2}{2} + \frac{ke^{5t}}{5} + \frac{c_1\sqrt{W(e^{100x+\frac{20e^{-5t}}{k}-1})^3}}{150} - \frac{c_1\sqrt{W(e^{100x+\frac{20e^{-5t}}{k}-1})}}{50} + c_2.\end{aligned}$$

4.3.2. Subalgebra $\mathcal{X}_{31} + \mathcal{X}_{35} + \mathcal{X}_{36} + \mathcal{X}_{37} + \mathcal{X}_{39} + \mathcal{X}_{310} + \mathcal{X}_{311}$. The characteristic equations are

$$\frac{ds}{2} = \frac{dr}{r} = \frac{dp}{4p} = \frac{dq}{2+2s+\gamma+3q},$$

with solution

$$\omega = re^{-2s}, \quad p(r, s) = e^{2s}f(\omega), \quad q(r, s) = -\frac{2(3s+5)}{9} + \frac{1}{2}e^{\frac{3}{2}s} \int \gamma(s)e^{-\frac{3}{2}s}ds + e^{\frac{3}{2}s}g(\omega). \quad (4.19)$$

Applying the invariants (4.19) on system (4.17), we obtain

$$\begin{aligned}\left(f + 2\gamma + \frac{\gamma'(s)}{\gamma(s)}\omega\right)f'' + f'^2 - \frac{\gamma''(s)}{\gamma(s)} &= 0, \\ \left(f + 2\gamma + \frac{\gamma'(s)}{\gamma(s)}\omega\right)g'' + \left(\frac{1}{2} - \frac{\gamma'(s)}{\gamma(s)}\right)g' &= 0.\end{aligned} \quad (4.20)$$

Theorem 4.1. The second order system of ordinary differential equation (4.20) is the reduced system of the second order partial differential equations (4.17) if and only if $\gamma(s) = e^{ks}$, where $k \neq 0$ is an arbitrary constant.

We shall consider three cases only: $k = -2, k = -1$ and $k = 1$.

Case 4.1. $\gamma(s) = e^{-2s}$

For this choice of γ , the solutions to the system (4.20) are:

$$\begin{aligned}f(\omega) &= \sqrt{4\omega^2 + 2m}, \\ g(\omega) &= \frac{(18\omega^2 + 5m)\sqrt{4\omega^2 + 2m} + 36\omega^3 + 18m\omega}{\sqrt{2\omega^2 + m}(\sqrt{4\omega^2 + 2m} + 2\omega)^{\frac{9}{4}}} + c_1, \\ f(\omega) &= -\sqrt{4\omega^2 + 2m}, \\ g(\omega) &= \frac{(4\omega^3 + 2m\omega + (2\omega^2 + 5m)\sqrt{4\omega^2 + 2m})(\sqrt{4\omega^2 + 2m} + 2\omega)^{\frac{1}{4}}}{\sqrt{4\omega^2 + 2m}} + c_1.\end{aligned}$$



Using the invariants (4.19) and (4.16), we obtain the solutions to system (1.1) under the aforementioned sectional subalgebra as follows:

$$\begin{aligned}
 u &= -2y^2 + e^{2t} \sqrt{4x^2 e^{-4t} + 2m}, \\
 v &= -\frac{2y^3}{3} + 2xy - \frac{2(3t+5)}{9} - \frac{e^{-2t}}{7} \\
 &\quad + \frac{\left((18x^2 e^{-4t} + 5m) \sqrt{4x^2 e^{-4t} + 2m} + 36x^3 e^{-6t} + 18mxe^{-2t}\right) e^{\frac{3}{2}t}}{\sqrt{2x^2 e^{-4t} + m} \left(\sqrt{4x^2 e^{-4t} + 2m} + 2xe^{-2t}\right)^{\frac{9}{4}}} + c_1, \\
 u &= -2y^2 - e^{2t} \sqrt{4x^2 e^{-4t} + 2m}, \\
 v &= -\frac{2y^3}{3} + 2xy - \frac{2(3t+5)}{9} - \frac{e^{-2t}}{7} \\
 &\quad + \frac{e^{\frac{3}{2}t} \left(4x^3 e^{-6t} + 2mxe^{-2t} + (2x^2 e^{-4t} + 5m) \sqrt{4x^2 e^{-4t} + 2m}\right) \sqrt[4]{2xe^{-2t} + \sqrt{4x^2 e^{-4t} + 2m}}}{\sqrt{4x^2 e^{-4t} + 2m}} + c_1.
 \end{aligned}$$

Case 4.2. $\gamma(s) = e^{-s}$

For this choice of γ , the solution to the system (4.20) is

$$\begin{aligned}
 f(\omega) &= \frac{1}{2} W(e^{4\omega-1}) - \omega + \frac{1}{2}, \\
 g(\omega) &= \frac{c_2 (3W(e^{4\omega-1}) - 1)}{\sqrt[4]{W(e^{4\omega-1})^3}} + c_1.
 \end{aligned}$$

The solution to system (1.1) is thus given by

$$\begin{aligned}
 u &= -\frac{y^2}{2} + \frac{e^{2t}}{2} W\left(e^{4xe^{-2t}-1}\right) - x + \frac{e^{2t}}{2}, \\
 v &= -\frac{y^3}{6} + xy - \frac{2(3t+5)}{9} - \frac{e^{-t}}{5} + e^{\frac{3}{2}t} \left(\frac{c_2 (3W(e^{4xe^{-2t}-1}) - 1)}{\sqrt[4]{W(e^{4xe^{-2t}-1})^3}} + c_1 \right).
 \end{aligned}$$

Case 4.3. $\gamma(s) = e^s$

The solution to the system (4.20), with respect to this choice of γ , is

$$\begin{aligned}
 f(\omega) &= \omega + \frac{\sqrt{1-64\omega}-1}{16}, \\
 g(\omega) &= \left(\frac{\sqrt{1-64\omega}-1}{9} + \frac{1}{5} \right) \sqrt[4]{\frac{(1-\sqrt{1-64\omega})^5}{32}}.
 \end{aligned}$$

The solution to system (1.1) is thus given by

$$\begin{aligned}
 u &= -\frac{y^2}{2} + x + \frac{e^{2t} (\sqrt{1-64xe^{-2t}} - 1)}{16}, \\
 v &= -\frac{y^3}{6} - xy - \frac{2(3t+5)}{9} - e^t + e^{\frac{3}{2}t} \left(\frac{\sqrt{1-64xe^{-2t}}}{9} + \frac{4}{45} \right) \sqrt[4]{\frac{(1-\sqrt{1-64xe^{-2t}})^5}{32}}.
 \end{aligned}$$

4.3.3. Subalgebra $\mathcal{X}_{3_{12}} + \mathcal{X}_{3_{13}} + \mathcal{X}_{3_{15}} + \mathcal{X}_{3_{16}}$. The characteristic equations are

$$\frac{ds}{1} = \frac{\gamma dr}{1} = \frac{dq}{\frac{2r}{\gamma} + s + \gamma + 2q}.$$



For $\gamma = e^{ks}$, the invariants are

$$\omega = r + \frac{e^{-ks}}{k}, \quad p(r, s) = f(\omega), \quad q(r, s) = -\frac{2re^{-ks}}{k+2} - \frac{e^{-2ks}}{(k+1)(k+2)} + \frac{e^{ks}}{k-2} - \frac{2s+1}{4} + e^{2s}g(\omega).$$

These invariants reduce system (4.17) to

$$(k\omega + f)f'' + f'^2 - k^2 = 0, \\ (k\omega + f)g'' - (k+2)g' = 0,$$

whose solution eventually gives rise to

$$u = -\frac{k^2}{2}y^2 + \frac{1}{2k} \left(-2k(kx + e^{-kt}) + W \left(e^{4k(kx + e^{-kt})-1} \right) + 1 \right), \\ v = ke^{2t}W \left(e^{4k(kx + e^{-kt})-1} \right)^{\frac{k+2}{2k}} \left(W \left(e^{4k(kx + e^{-kt})-1} \right) + \frac{2W \left(e^{4k(kx + e^{-kt})-1} \right) + 2}{k} + 3 \right) \\ - \frac{k^2}{6}y^3 - kxy - \frac{xe^{-kt}}{k+m} + h(t).$$

4.3.4. *Subalgebra* $\mathcal{X}_{317} + \mathcal{X}_{318} + \mathcal{X}_{319} + \mathcal{X}_{320} + \mathcal{X}_{321}$. The characteristic equations are

$$\frac{ds}{s} = \frac{dr}{2r + \frac{2}{\gamma}} = \frac{2d\gamma}{\gamma} = \frac{dp}{p} = \frac{dq}{1 + \gamma + \frac{3r}{\gamma} - \frac{q}{2}}.$$

The invariants are

$$\omega = \frac{r}{s^2} + \frac{4}{5\sqrt{s^5}}, \quad p(r, s) = sf(\omega), \quad q(r, s) = \frac{3r}{2\sqrt{s}} + \frac{6}{s} + \sqrt{s} + 2 + \frac{1}{\sqrt{s}}g(\omega), \quad \gamma = \sqrt{s}. \quad (4.21)$$

The reduced equations are

$$(5\omega + 2f)f'' + 2 \left(f' + \frac{1}{2} \right)^2 = 0, \\ (5\omega + 2f)g'' + 4g' = 0.$$

Its solution, coupled with the invariants (4.21) and (4.16), leads to the solution of system (1.1) as

$$u = \frac{y^2}{8t^2} + \frac{\left(\frac{x}{y} + \frac{4}{5\sqrt{t^3}} \right) \left(4 - W \left(-\frac{64x}{t^2} - \frac{1024}{5\sqrt{t^5}} \right) \right)}{2W \left(-\frac{64x}{t^2} - \frac{1024}{5\sqrt{t^5}} \right)}, \\ v = \frac{y^3}{24t^2} - \frac{xy}{2t} + \frac{x}{\sqrt{t}} + \frac{c_1 + c_2 W \left(-\frac{64x}{t^2} - \frac{1024}{5\sqrt{t^5}} \right) \left(2 + W \left(-\frac{64x}{t^2} - \frac{1024}{5\sqrt{t^5}} \right) \right)}{\sqrt{t}}.$$

4.3.5. *Subalgebra* $\mathcal{X}_{322} + \mathcal{X}_{323} + \mathcal{X}_{324} + \mathcal{X}_{325}$. The solutions

$$\omega = r + \frac{2}{\sqrt{s}}, \quad p(\omega) = \frac{2}{\sqrt{s^3}} + \frac{f(\omega)}{s}, \quad q(\omega) = \sqrt{s}g(\omega), \quad \gamma = \sqrt{s^3},$$

of the characteristic equations

$$\frac{ds}{2s} = \frac{\gamma dr}{2s} = \frac{d\gamma}{3\gamma} = -\frac{dp}{\frac{2}{\gamma} + 2p} = \frac{dq}{q},$$

reduced the system (4.17) to

$$\left(\frac{3}{2}\omega + f \right) f'' + f' (f' + 1) - \frac{3}{4} = 0,$$



$$\left(\frac{3}{2}\omega + f\right)g'' - 2g' = 0.$$

Its solution is

$$f(\omega) = \frac{W(e^{8\omega-1}) + 1}{4} - \frac{3}{2}\omega,$$

$$g(\omega) = c_1 + c_2 \left(\frac{1}{16} W(e^{8\omega-1})^2 + \frac{1}{8} W(e^{8\omega-1}) \right).$$

Therefore,

$$u = -\frac{3}{8}t^{-2}y^2 + 2t^{-\frac{3}{2}} + \frac{W(e^{8x+\frac{16}{\sqrt{t}}-1}) + 1}{4t} - \frac{3}{2t} \left(x + \frac{2}{\sqrt{t}} \right),$$

$$v = -\frac{1}{8}t^{-2}y^3 - \frac{3}{2t}xy + c_1\sqrt{t} + \frac{c_2\sqrt{t}}{16} W(e^{8x+\frac{16}{\sqrt{t}}-1})^2 + \frac{c_2\sqrt{t}}{8} W(e^{8x+\frac{16}{\sqrt{t}}-1}).$$

4.3.6. *Subalgebra* $\mathcal{X}_{3_{26}} + \mathcal{X}_{3_{27}} + \mathcal{X}_{3_{28}}$. The characteristic equations are

$$\frac{ds}{s} = \frac{dr}{r + \frac{2s}{\gamma}} = \frac{2d\gamma}{\gamma} = -\frac{\gamma dp}{2} = \frac{dq}{\frac{2r}{\gamma} - \frac{q}{2}}.$$

Their solution give rise to the invariants

$$\omega = \frac{r}{s} + \frac{4}{\sqrt{s}}, \quad p(r, s) = \frac{4}{\sqrt{s}} + f(\omega), \quad q(r, s) = \frac{2r}{\sqrt{s}} - 8 + \frac{1}{\sqrt{s}}g(\omega), \quad \gamma = \sqrt{s}.$$

which reduced the system (4.17) to

$$\left(\frac{3}{2}\omega + f\right)f'' + \left(f' + \frac{1}{2}\right)^2 = 0,$$

$$\left(\frac{3}{2}\omega + f\right)g'' + 2g' = 0.$$

The solution of the above system of equations is

$$f(\omega) = \frac{\omega(2 - W(-8\omega))}{2W(-8\omega)}, \quad g(\omega) = c_1 + W(-8\omega)(2 + W(-8\omega)).$$

Thus, the solution of the MS system (1.1) is obtained as

$$u = \frac{y^2}{8t^2} + \frac{4}{\sqrt{t}} + \frac{(x + 4\sqrt{t}) \left(2 - W\left(-\frac{8x}{t} - \frac{32}{\sqrt{t}}\right) \right)}{2tW\left(-\frac{8x}{t} - \frac{32}{\sqrt{t}}\right)},$$

$$v = \frac{y^3}{24t^2} - \frac{xy}{2t} + \frac{2x}{\sqrt{t}} + \frac{c_1 + W\left(-\frac{8x}{t} - \frac{32}{\sqrt{t}}\right) \left(2 + W\left(-\frac{8x}{t} - \frac{32}{\sqrt{t}}\right) \right)}{\sqrt{t}}.$$

4.4. **Solutions through subalgebras** $\mathcal{X}_4$, $\mathcal{X}_5 + \mathcal{X}_6 + \mathcal{X}_7$. The calculations are straight forward, so we shall present solutions obtained without going through the details.

The solution of the MS system (1.1), via the generator $\mathcal{X}_4$, is obtained as

$$u = \frac{\lambda'(t)}{\lambda(t)}x + \frac{1}{2} \left(\frac{\lambda''(t)}{\lambda(t)} - \frac{2\lambda'(t)^2}{\lambda(t)^2} \right) y^2 + a(t)y + b(t),$$

$$v = \frac{x^2}{y} + \frac{\lambda'(t)}{\lambda(t)}xy + \frac{1}{3}c(t)y^3 + a(t)y^2 + b(t)y + d(t).$$



The solution of the MS system (1.1), via the generator $\mathcal{X}_5 + \mathcal{X}_6 + \mathcal{X}_7$, is given by

$$u = \frac{\rho(t) - \delta'(t)}{\delta(t)}x + \frac{(\delta(t)\rho(t))' - \delta(t)\delta''(t) - \rho(t)^2}{2\delta(t)^2}y^2 + a(t)y + b(t),$$

$$v = \frac{\sigma(t) + \rho(t)y}{\delta(t)}x + \left(\frac{\rho(t)}{6\delta(t)}\right)'y^3 + \left(\frac{\sigma(t)}{2\delta(t)}\right)'y^2 - \frac{1}{6}\left(\frac{\rho(t)}{\delta(t)}\right)^2y^3 - \frac{\rho(t)\sigma(t)}{2\delta(t)^2}y^2 + c(t)y + d(t).$$

Further background analysis shows that other solution variants are possible if the linear combinations of the generators (2.22) are used for reduction. For instance, the combination

$$\mathcal{A}_6 + \mathcal{A}_7 + \mathcal{A}_8 + \mathcal{A}_9 = (t + y)\partial_x + t\partial_y + \partial_u + (2y + x)\partial_v,$$

give rise to the reduced system of equations

$$\left(\frac{r}{s} - p - 1\right)p_{rr} + p_{rs} - p_r^2 + \frac{1}{s}p_r = 0,$$

$$\left(\frac{r}{s} - p - 1\right)q_{rr} + q_{rs} - \frac{1}{s} = 0, \quad (4.22)$$

which is a variant of system (4.17). The generators of the reduced system (4.22) are infinite which can be made finite by specializing the arbitrary functions involved. Without going through the details, some of the solutions of system (1.1) obtained via them, which can not be deduced directly from any of the previous ones, are as follows:

$$u = \frac{3\left(x - y - \frac{y^2}{2t}\right) + t}{t(c_1 - 3\ln t)} + \frac{y(t+1) - t}{t}, \quad (4.23)$$

$$v = \frac{y(yt^2 + t(2x + y) - \frac{2}{3}y^2)}{2t^2} + \left(3x - 3y - \frac{3y^2}{2t} + t\right)\left(\frac{\ln t + t}{3} + c_2\right) - \frac{t+1}{3},$$

$$u = \frac{y(t+1) - t}{t} + \frac{c_1 W\left(2t - x + y + \frac{y^2}{2t} + c_1\right) + 2t - x + y + \frac{y^2}{2t} + c_1}{tW\left(2t - x + y + \frac{y^2}{2t} + c_1\right)},$$

$$v = \frac{y(yt^2 + t(2x + y) - \frac{2}{3}y^2)}{2t^2} + \frac{9\left(2xt - (y+t)^2 + 1 - x + y + \frac{y^2}{2t}\right) + 16t}{18} \quad (4.24)$$

$$- \frac{\left(3W\left(2t - x + y + \frac{y^2}{2t} + c_1\right) + 2\right)\left(2t - x + y + \frac{y^2}{2t} + c_1\right)^3}{t^2\left(W\left(2t - x + y + \frac{y^2}{2t} + c_1\right)\right)^3}$$

$$+ \left(x - y - \frac{y^2}{2t} - \frac{2t}{3}\right)\ln t,$$

$$u = \frac{y(t+1) - t}{t} + \frac{c_1 W\left(-x + y + \frac{y^2}{2t} + c_1\right) - x + y + \frac{y^2}{2t} + c_1}{tW\left(-x + y + \frac{y^2}{2t} + c_1\right)},$$

$$v = \frac{y(yt^2 + t(2x + y) - \frac{2}{3}y^2)}{2t^2} + \left(x - y - \frac{y^2}{2t}\right)(\ln t + t + 1) + \frac{1}{2}t\ln t \quad (4.25)$$

$$- \frac{c_2 t}{2}W\left(-x + y + \frac{y^2}{2t} + c_1\right)\left(W\left(-x + y + \frac{y^2}{2t} + c_1\right) + 2\right) + c_3 t - \frac{1}{2},$$



$$\begin{aligned}
u &= \frac{y(t+1)-t}{t} + \frac{3c_1 W\left(9t^2\left(x-y-\frac{y^2}{2t}\right)-9c_1\right) + 3c_1 - 3t^2\left(x-y-\frac{y^2}{2t}\right)}{t^3 W\left(9t^2\left(x-y-\frac{y^2}{2t}\right)-9c_1\right)}, \\
v &= \frac{y\left(yt^2+t(2x+y)-\frac{2}{3}y^2\right)}{2t^2} + \left(x-y-\frac{y^2}{2t}\right)(c_3+t) + \frac{c_4}{t} \\
&+ \frac{\left(t^2\left(x-y-\frac{y^2}{2t}\right)-c_1\right) W\left(9t^2\left(x-y-\frac{y^2}{2t}\right)-9c_1\right)}{3t^2} + \frac{c_1-t^2\left(x-y-\frac{y^2}{2t}\right)}{3t^2} \\
&+ \left(\frac{t^2\left(x-y-\frac{y^2}{2t}\right)-c_1}{3t^2 W\left(9t^2\left(x-y-\frac{y^2}{2t}\right)-9c_1\right)}\right),
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
u &= \frac{y(t+1)-t}{t} + \frac{1}{3\ln t} W\left(e^{-\frac{3}{t}\left(x-y-\frac{y^2}{2t}\right)} \ln t\right), \\
v &= \frac{y\left(yt^2+t(2x+y)-\frac{2}{3}y^2\right)}{2t^2} + \frac{t^3-t-1}{3} + t\left(x-y-\frac{y^2}{2t}\right) \\
&+ \frac{t\ln t}{3} (\text{Ei}(1, \ln t) - \text{Ei}(1, -\ln t)) - \frac{c_1 t \ln t}{3} \ln\left(W\left(e^{-\frac{3}{t}\left(x-y-\frac{y^2}{2t}\right)} \ln t\right)\right) \\
&+ \frac{c_1 t \ln t}{3W\left(e^{-\frac{3}{t}\left(x-y-\frac{y^2}{2t}\right)} \ln t\right)} + t \ln t \left(\frac{x-y}{t} - \frac{y^2}{2t^2} - \frac{\ln(\ln t)}{3} + c_2\right),
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
u &= \frac{y(t+1)-t}{t} - \frac{1}{t \ln t} + \frac{1}{k \ln t} \left(W\left(ke^{-\frac{k}{t}\left(x-y-\frac{y^2}{2t}\right)-k\text{Ei}(1, \ln t)}\right) + 1\right), \\
v &= \frac{y\left(yt^2+t(2x+y)-\frac{2}{3}y^2\right)}{2t^2} + (t+1+\ln t) \left(x-y-\frac{y^2}{2t}\right) + t \ln t \text{Ei}(1, \ln t) \\
&- \frac{c_1 t \ln t}{k} \ln\left(W\left(ke^{-\frac{k}{t}\left(x-y-\frac{y^2}{2t}\right)-k\text{Ei}(1, \ln t)}\right)\right) + \frac{c_1 t \ln t}{kW\left(ke^{-\frac{k}{t}\left(x-y-\frac{y^2}{2t}\right)-k\text{Ei}(1, \ln t)}\right)},
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
u &= \frac{y(t+1)-t}{t} \pm \frac{1}{\ln t} \sqrt{c_1 \left(\frac{x-y-\frac{y^2}{2t}}{t \ln t} + \frac{\text{Ei}(1, \ln t)}{4} + \frac{1+t-\ln t}{4t(\ln t)^2}\right) + c_2} \\
&+ \frac{y+\frac{y^2}{2t}-x}{t \ln t} - \frac{t+1}{2t(\ln t)^2}, \\
v &= \frac{y\left(yt^2+t(2x+y)-\frac{2}{3}y^2\right)}{2t^2} + (c_3+t+\ln t) \left(x-y-\frac{y^2}{2t}\right) + h(t),
\end{aligned} \tag{4.29}$$

where Ei is the exponential integral which, for real non-zero of x , is defined as

$$\text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt$$

5. CONSERVATION LAWS

Conservation laws of various (physical) systems have been studied extensively in the literature due to their unmatched importance in the formulation of various fundamental laws of physics. There are various methods that are used to find the conservation laws of (system of) PDEs, the one we shall use to construct the conservation laws admitted by the MS system is the invariance and multiplier approach [17].



Let $\mathcal{P}$ and $\mathcal{R}$ be any two arbitrary multipliers of the system (1.1). This implies that

$$E_{u^j} (\mathcal{P} (-u_{xt} + u_{yy} + u_x^2 + uu_{xx} + v_x u_{xy} - v_y u_{xx}) + \mathcal{R} (-v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx})) = 0, \quad (5.1)$$

where E_{u^j} is the Euler operator with respect to u^j given by

$$E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u_i^j} + \dots + (-1)^k D_{i_1} \dots D_{i_k} \frac{\partial}{\partial u_{i_1 \dots i_k}^j} + \dots, j = 1, \dots, m. \quad (5.2)$$

If we let

$$\mathcal{P} = \mathcal{P}(x, y, t, u, v, u_x, v_x, u_y, v_y, u_{xx}, u_{yy}), \text{ and } \mathcal{R} = \mathcal{R}(x, y, t, u, v, u_x, v_x, u_y, v_y, v_{xx}, v_{yy}),$$

then solution to the determining equations (5.1) is given by

$$\begin{aligned} \mathcal{P} &= \frac{1}{6} a_1''(t) y^3 + \frac{1}{3} a_1'(t) (y^2 v_x + 3xy) + \frac{1}{2} a_3'(t) y v_x + \frac{1}{2} a_2'(t) y^2 \\ &\quad + \frac{1}{3} a_1(t) (y(v_x^2 + v_y) + 2xv_x + 3v) + a_3(t) (v_x^2 + v_y) + a_4(t) v_x \\ &\quad + a_5(t) y + \frac{1}{2} a_2(t) (yv_x + 2x) + a_6(t) + \frac{1}{2} c_1 (yv_x^2 + yv_y + 2xv_x + 3v), \\ \mathcal{R} &= \frac{1}{2} a_3''(t) y - \frac{1}{6} a_1''(t) y^2 + \frac{1}{3} a_1'(t) (y^2 u_x - 2yv_x - x) + \frac{1}{2} a_3'(t) (yu_x + 2v_x) \\ &\quad + a_4'(t) - \frac{1}{2} a_2'(t) y + \frac{1}{3} a_1(t) (y(2u_x v_x + u_y) + 2xu_x - 3v_x^2 - 2v_y - 2u) \\ &\quad + a_3(t) (2u_x v_x + u_y) + \frac{1}{2} a_2(t) (yu_x - 2v_x) + a_4(t) u_x - a_5(t) \\ &\quad + \frac{1}{2} c_1 (2yu_x v_x + yu_y + 2xu_x - 3v_x^2 - 2v_y - 2u). \end{aligned} \quad (5.3)$$

This give rise to the following seven multipliers:

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{2} (yv_x^2 + yv_y + 2xv_x + 3v), \\ \mathcal{R}_1 &= \frac{1}{2} (2yu_x v_x + yu_y + 2xu_x - 3v_x^2 - 2v_y - 2u), \\ \mathcal{P}_2 &= \frac{1}{6} a_1''(t) y^3 + \frac{1}{3} a_1'(t) (y^2 v_x + 3xy) + \frac{1}{3} a_1(t) (y(v_x^2 + v_y) + 2xv_x + 3v), \\ \mathcal{R}_2 &= -\frac{1}{6} a_1''(t) y^2 + \frac{1}{3} a_1'(t) (y^2 u_x - 2yv_x - x) + \frac{1}{3} a_1(t) y (2u_x v_x + u_y) + \frac{1}{3} a_1(t) (2xu_x - 3v_x^2 - 2v_y - 2u), \\ \mathcal{P}_3 &= \frac{1}{2} a_2'(t) y^2 + \frac{1}{2} a_2(t) (yv_x + 2x), \\ \mathcal{R}_3 &= -\frac{1}{2} a_2'(t) y + \frac{1}{2} a_2(t) (yu_x - 2v_x), \\ \mathcal{P}_4 &= \frac{1}{2} a_3'(t) y v_x + a_3(t) (v_x^2 + v_y), \\ \mathcal{R}_4 &= \frac{1}{2} a_3''(t) y + \frac{1}{2} a_3'(t) (yu_x + 2v_x) + a_3(t) (2u_x v_x + u_y), \\ \mathcal{P}_5 &= a_4(t) v_x, \\ \mathcal{R}_5 &= a_4'(t) + a_4(t) u_x, \\ \mathcal{P}_6 &= a_5(t) y, \\ \mathcal{R}_6 &= -a_5(t), \\ \mathcal{P}_7 &= a_6(t), \\ \mathcal{R}_7 &= 0. \end{aligned}$$



Each of the multipliers $(\mathcal{P}_j, \mathcal{R}_j) : j = 1, 2, \dots, 7$ will generate a conservation law

$$D_t \mathcal{T} + D_k \mathcal{C}^k \Big|_{(1.1)} = 0, k = x, y,$$

where the densities $(\mathcal{T})$ and the fluxes $(\mathcal{C})$ are as follows:

$$\begin{aligned} \mathcal{T}_1 &= -\frac{1}{2} y u_x v_x^2 + \frac{1}{2} v_x^3 - x u_x v_x + \frac{1}{2} v_y (v_x - y u_x) - \frac{3}{2} v u_x, \\ \mathcal{C}_1^x &= v_x v_y^2 - u^2 v_x + \frac{1}{2} v_x^2 (y u_x - v_x) (u - v_y) + \frac{1}{2} (3 v u_x + 2 x u_x v_x) (u - v_y) \\ &\quad + \frac{1}{2} v_t (2u + v_y - y u_y) + \frac{1}{2} y (u_x v_y + u_y v_x) (u - v_y) - x u_y v_y, \\ \mathcal{C}_1^y &= -\frac{1}{2} v_x^4 - \frac{1}{2} v_y^2 - \frac{1}{2} u v_x^2 + x u_x v_x^2 - u v_y + \frac{1}{2} y u_x v_x^3 + \frac{1}{2} y u_y (v_x^2 + v_y) + \frac{3}{2} v u_y \\ &\quad + \frac{1}{2} u_x v_x (3v - y u) + \frac{1}{2} (y u_x - v_x) v_t + y u_x v_x v_y + x u_y v_x + x u_x v_y - \frac{3}{2} v_y v_x^2, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \mathcal{T}_2 &= -\frac{1}{3} a_1'(t) y^2 u_x v_x + a_1'(t) y u - \frac{1}{3} a_1(t) y u_x v_x^2 - \frac{1}{3} a_1(t) y u_x v_y - \frac{2}{3} a_1(t) x u_x v_x \\ &\quad - a_1(t) v u_x + \frac{1}{3} y a_1'(t) v_x^2 - \frac{1}{3} a_1'(t) v + \frac{1}{3} a_1(t) v_x^3 + \frac{2}{3} a_1(t) u v_x + \frac{1}{3} a_1(t) v_x v_y - \frac{2}{3} a_1(t) u v_x, \\ \mathcal{C}_2^x &= -\frac{1}{6} a_1''(t) y^3 u_t - a_1'(t) x y u_t - \frac{1}{3} a_1(t) y u_y v_t + \frac{1}{6} a_1''(t) y^2 v_t + \frac{1}{3} x a_1'(t) v_t + \frac{1}{3} a_1(t) v_t v_y \\ &\quad + \frac{2}{3} a_1(t) u v_t + \frac{1}{6} a_1''(t) y^3 u u_x + \frac{1}{6} a_1'(t) y^2 u u_x v_x + a_1'(t) x y u u_x + \frac{1}{3} a_1(t) y u u_x v_y \\ &\quad + \frac{2}{3} a_1(t) x u u_x v_x + a_1(t) u v u_x - a_1'(t) x y u_x v_y - \frac{1}{6} a_1''(t) y^3 u_x v_y - \frac{1}{3} a_1(t) y u_x v_y v_x^2 \\ &\quad - \frac{1}{3} a_1(t) u v_x^3 + \frac{1}{6} a_1''(t) y^2 v_x v_y - \frac{1}{3} a_1'(t) y^2 u_x v_x v_y + \frac{1}{3} a_1'(t) y v_y v_x^2 + \frac{1}{3} a_1'(t) x v_x v_y \\ &\quad - \frac{1}{3} a_1(t) y u_y v_x v_y - \frac{2}{3} a_1(t) x u_x v_x v_y + \frac{1}{3} a_1(t) v_y v_x^3 + \frac{2}{3} a_1(t) u v_x v_y + \frac{2}{3} a_1(t) v_x v_y^2 \\ &\quad - \frac{1}{3} a_1'(t) y u v_x^2 - \frac{1}{3} a_1'(t) x u v_x + \frac{1}{3} a_1(t) y u u_x v_x^2 + \frac{1}{3} a_1(t) y u u_y v_x - \frac{2}{3} a_1(t) u^2 v_x \\ &\quad - \frac{2}{3} a_1(t) u v_x v_y - a_1(t) v u_x v_y - \frac{1}{6} a_1''(t) y^2 u v_x - \frac{2}{3} a_1(t) x u_y v_y + \frac{1}{3} a_1'(t) y v_y^2 \\ &\quad - \frac{1}{3} a_1(t) y u_x v_y^2 - \frac{1}{3} a_1'(t) y^2 u_y v_y - \frac{2}{3} a_1'(t) y v u_y + \frac{1}{3} a_1'(t) y u v - \frac{1}{2} a_1'(t) y u^2, \\ \mathcal{C}_2^y &= \frac{1}{3} a_1(t) y u_x v_t - \frac{1}{3} a_1(t) v_x v_t + \frac{1}{6} a_1''(t) y^3 u_y + \frac{1}{3} a_1'(t) y^2 u_y v_x + a_1'(t) x y u_y \\ &\quad + \frac{1}{3} a_1(t) y u_y v_y + \frac{2}{3} a_1(t) x u_y v_x + a_1(t) v u_y + a_1'(t) x y u_x v_x + \frac{1}{6} a_1''(t) y^3 u_x v_x \\ &\quad - \frac{1}{6} a_1''(t) y^2 v_x^2 + \frac{1}{3} a_1'(t) y^2 u_x v_x^2 - \frac{1}{3} a_1'(t) y v_x^3 - \frac{1}{3} a_1'(t) x v_x^2 + \frac{1}{3} a_1(t) y u_x v_x^3 \\ &\quad + \frac{1}{3} a_1(t) y u_y v_x^2 + \frac{2}{3} a_1(t) x u_x v_x^2 - \frac{1}{3} a_1(t) v_x^4 - \frac{2}{3} a_1(t) u v_x^2 - a_1(t) v_y v_x^2 \\ &\quad - \frac{1}{6} a_1''(t) y^2 v_y + \frac{1}{3} a_1'(t) y^2 u_x v_y - \frac{2}{3} a_1'(t) y v_x v_y - \frac{1}{3} a_1'(t) x v_y + \frac{2}{3} a_1(t) y u_x v_x v_y \\ &\quad + \frac{2}{3} a_1(t) x u_x v_y - \frac{2}{3} a_1(t) u v_y - \frac{1}{3} a_1(t) v_y^2 - \frac{1}{3} a_1(t) y u u_x v_x + \frac{1}{3} a_1(t) u v_x^2 \\ &\quad + a_1(t) v u_x v_x + \frac{1}{3} a_1''(t) y v - \frac{1}{2} a_1''(t) y^2 u - a_1'(t) x u + \frac{2}{3} a_1'(t) y v u_x, \end{aligned} \quad (5.5)$$

$$\begin{aligned}
\mathcal{T}_3 &= -\frac{1}{2}a'_2(t)y^2u_x - \frac{1}{2}a_2(t)yu_xv_x - a_2(t)xu_x + \frac{1}{2}a'_2(t)yv_x + \frac{1}{2}a_2(t)v_x^2, \\
\mathcal{C}_3^x &= \frac{1}{2}a''_2(t)y^2u + a'_2(t)xu + \frac{1}{2}a'_2(t)y^2uu_x + \frac{1}{2}a_2(t)yuuv_x + a_2(t)xuu_x - \frac{1}{2}a_2(t)u^2 \\
&\quad - \frac{1}{2}a'_2(t)y^2u_xv_y - a_2(t)xu_xv_y - \frac{1}{2}a''_2(t)yv + \frac{1}{2}a_2(t)v_y^2 - \frac{1}{2}a'_2(t)yuuv_x - \frac{1}{2}a_2(t)uv_x^2 \\
&\quad + \frac{1}{2}a'_2(t)yv_xv_y - \frac{1}{2}a_2(t)yu_xv_xv_y + \frac{1}{2}a_2(t)v_yv_x^2 - \frac{1}{2}a_2(t)yu_yv_y + \frac{1}{2}a_2(t)uv_y, \\
\mathcal{C}_3^y &= \frac{1}{2}a'_2(t)y^2u_y - a'_2(t)yu + \frac{1}{2}a_2(t)yu_yv_x + a_2(t)xu_y + \frac{1}{2}a'_2(t)y^2u_xv_x + a_2(t)xu_xv_x \\
&\quad - \frac{1}{2}a'_2(t)yv_y + \frac{1}{2}a'_2(t)v + \frac{1}{2}a_2(t)yu_xv_y - a_2(t)v_xv_y - \frac{1}{4}a'_2(t)yv_x^2 + \frac{1}{4}a_2(t)yu_xv_x^2 \\
&\quad - \frac{1}{2}a_2(t)v_x^3 - \frac{1}{4}a'_2(t)yv_x^2 + \frac{1}{4}a_2(t)yu_xv_x^2 - \frac{1}{2}a_2(t)uv_x,
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\mathcal{T}_4 &= -\frac{1}{2}a'_3(t)yu_xv_x - a_3(t)u_xv_x^2 - a_3(t)u_xv_y - \frac{1}{2}a''_3(t)yv_x - \frac{1}{2}a'_3(t)v_x^2, \\
\mathcal{C}_4^x &= -a_3(t)u_yv_t + \frac{1}{2}a'''_3(t)yv - \frac{1}{2}a'_3(t)v_y^2 - a_3(t)u_xv_y^2 + \frac{1}{2}a''_3(t)yuv_x + \frac{1}{2}a'_3(t)yuuv_xv_x \\
&\quad + \frac{1}{2}a'_3(t)uv_x^2 + a_3(t)uu_xv_x^2 - \frac{1}{2}a''_3(t)yv_xv_y - \frac{1}{2}a'_3(t)yu_xv_xv_y - \frac{1}{2}a'_3(t)v_yv_x^2 \\
&\quad - a_3(t)u_yv_xv_y - a_3(t)u_xv_yv_x^2 + a_3(t)uu_xv_y - \frac{1}{2}a'_3(t)yu_yv_y + \frac{1}{2}a'_3(t)uv_y + a_3(t)uu_yv_x, \\
\mathcal{C}_4^y &= a_3(t)u_xv_t + \frac{1}{2}a''_3(t)yv_y - \frac{1}{2}a''_3(t)v + \frac{1}{2}a'_3(t)yu_xv_y + a'_3(t)v_xv_y + a_3(t)u_yv_y \\
&\quad + 2a_3(t)u_xv_xv_y + \frac{1}{2}a'_3(t)yu_xv_x^2 + \frac{1}{2}a'_3(t)v_x^3 + \frac{1}{2}a_3(t)u_yv_x^2 + a_3(t)u_xv_x^3 \\
&\quad + \frac{1}{2}a''_3(t)yv_x^2 + \frac{1}{2}a_3(t)yu_yv_x^2 + \frac{1}{2}a'_3(t)yu_yv_x - \frac{1}{2}a'_3(t)uv_x - a_3(t)uu_xv_x,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\mathcal{C}_4^y &= a_3(t)u_xv_t + \frac{1}{2}a''_3(t)yv_y - \frac{1}{2}a''_3(t)v + \frac{1}{2}a'_3(t)yu_xv_y + a'_3(t)v_xv_y + a_3(t)u_yv_y \\
&\quad + 2a_3(t)u_xv_xv_y + \frac{1}{2}a'_3(t)yu_xv_x^2 + \frac{1}{2}a'_3(t)v_x^3 + \frac{1}{2}a_3(t)u_yv_x^2 + a_3(t)u_xv_x^3 \\
&\quad + \frac{1}{2}a''_3(t)yv_x^2 + \frac{1}{2}a_3(t)yu_yv_x^2 + \frac{1}{2}a'_3(t)yu_yv_x - \frac{1}{2}a'_3(t)uv_x - a_3(t)uu_xv_x,
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
\mathcal{T}_5 &= -a_4(t)u_xv_x - a'_4(t)v_x, \\
\mathcal{C}_5^x &= a''_4(t)v - a_4(t)u_yv_y + a_4(t)u_xv_x(u - v_y) + a'_4(t)uv_x - a_4(t)v_xv_y, \\
\mathcal{C}_5^y &= a_4(t)u_yv_x + \frac{1}{2}a_4(t)u_xv_x^2 + a'_4(t)v_y + a_4(t)u_xv_y + \frac{1}{2}(a'_4(t) + a_4(t)u_x)v_x^2 + \frac{1}{2}a_4(t)v_x^2,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\mathcal{T}_6 &= -a_5(t)yu_x + a_5(t)v_x, \\
\mathcal{C}_6^x &= a'_5(t)yu + a_5(t)yuuv_x - a'_5(t)v - a_5(t)uv_x + a_5(t)v_xv_y - a_5(t)yu_xv_y, \\
\mathcal{C}_6^y &= a_5(t)yu_y - a_5(t)u - a_5(t)v_y - a_5(t)v_x^2 + a_5(t)yu_xv_x,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
\mathcal{T}_7 &= -a_6(t)u_x, \\
\mathcal{C}_7^x &= a'_6(t)u + a_6(t)uu_x - a_6(t)u_xv_y, \\
\mathcal{C}_7^y &= a_6(t)u_y + a_6(t)u_xv_x.
\end{aligned} \tag{5.11}$$

6. ANALYSES OF RESULTS

Each of the solutions obtained in this research work exhibits various phenomena depending on the parametric choice of the independent variables and their assigned values. Apart from generating different wave profiles, the explicit choice of $\alpha(t)$ will also enable the locations of (local) extremum and saddle points. For instance, let us consider solution (4.4)



with $\alpha(t) = t^m (m \neq 0)$, $k_1 \neq 0$, $k_2 = k_4 = 0$ and $k_3 = 1$:

$$\begin{aligned} u &= -\frac{m}{t}x - \frac{m(m-1)}{2t^2}y^2 + \frac{y+1-c^2 + \sqrt{(c^2-1)^2 + 2k_1 \left(xt^m + \frac{mt^{m-1}}{2}y^2 - cy\right)}}{t^{2m}}, \\ v &= -\frac{m^2}{3t^2}y^3 - \frac{2m}{t}xy - \frac{m(m-1)}{3t^2}y^3 + \frac{\frac{y^2}{2} + y + k_3 \left(xt^m + \frac{mt^{m-1}}{2}y^2 - cy\right)}{t^{2m}} \\ &\quad - \frac{\sqrt{\left((c^2-1)^2 + 2k_1 \left(xt^m + \frac{mt^{m-1}}{2}y^2 - cy\right)\right)^3}}{3k_1^2 t^{2m}}. \end{aligned} \quad (6.1)$$

Taking t as parameter, we have:

$$\begin{aligned} u_x &= -\frac{m}{t} + \frac{k_1}{t^m \sqrt{(c^2-1)^2 + 2k_1 \left(xt^m + \frac{mt^{m-1}}{2}y^2 - cy\right)}}, \\ u_y &= \frac{m(1-m)}{t^2}y + \frac{1}{t^{2m}} + \frac{k_1(mt^{m-1}y - c)}{t^{2m} \sqrt{(c^2-1)^2 + 2k_1 \left(xt^m + \frac{mt^{m-1}}{2}y^2 - cy\right)}}. \end{aligned}$$

Solving $\nabla u = 0$ give rise to the critical points

$$(x_j, y_j) = \left(\frac{k_1 + 2m(m-1)c}{2m^2 t^{3m-2}} + \frac{(2-m)c^2}{2t^{2m-1}} - \frac{1}{2mt^{4m-3}} - \frac{(c^2-1)^2}{2k_1 t^m}, \frac{mct^{m+1} - t^2}{mt^{2m}} \right),$$

subject to the condition that $\frac{k_1 t^{1-m}}{m} > 0$. Now, $u_{xx}|_{(x_j, y_j)} = -\frac{(mt^{m-1})^2}{\frac{k_1 t^{1-m}}{m}} < 0$ for all m, k and t . Using the discriminant test, we obtain

$$D|_{(x_j, y_j)} = (u_{xx}u_{yy} - u_{xy}^2)|_{(x_j, y_j)} = -\frac{m^3}{k_1} \left(m^2 t^{3m-5} - \frac{k_1(m-1)t^{2m-4}}{\left| \frac{k_1 t^{1-m}}{m} \right|} \right) = -\frac{m^4}{k_1} t^{3m-5}.$$

From the condition $\frac{k_1 t^{1-m}}{m} > 0$, we establish the following:

1. $D|_{(x_j, y_j)} > 0$ if and only if

- (i) $k_1 < 0, m < 0$ and $t > 0$.
- (ii) $k_1 < 0, m < 0$ and $t < 0$ provided m is odd.
- (iii) $k_1 > 0, m < 0$ and $t < 0$ provided m is even.

2. $D|_{(x_j, y_j)} < 0$ if and only if

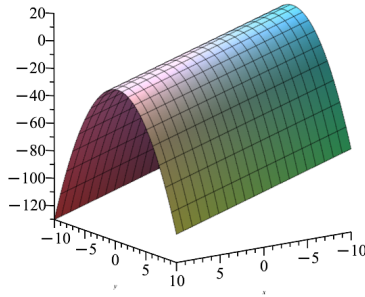
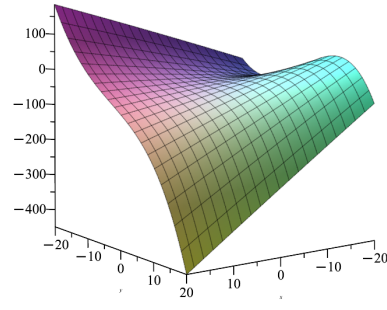
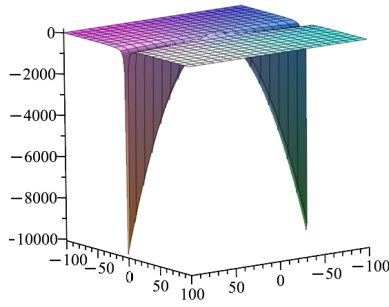
- (i) $k_1 > 0, m > 0$ and $t > 0$.
- (ii) $k_1 > 0, m > 0$ and $t < 0$ provided m is odd.
- (iii) $k_1 < 0, m > 0$ and $t < 0$ provided m is even.

Hence (x_j, y_j) are local maximum points of u with the conditions in Figure 4A and saddle points with the conditions in Figure 4B.

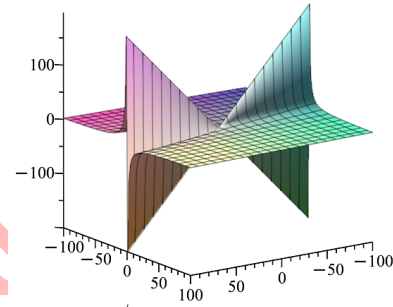
Similarly for v , the critical points are obtained from $\nabla v = 0$ as

$$(x_k, y_k) = \left(\frac{t^{1-2m} [4k_1 m(1-2m)t^{2m} + 2(2k_1 m + m - 1)t^{m+1} - t^2] \mathfrak{Y} + (4m-1)k_1 t^m - t}{8k_1 m^2 t^m - 2mt - 2t}, \mathfrak{Y}t \right),$$



(A) Wave profile of u when $t = 1, c = -20$.(B) Wave profile of v when $t = 10, c = 7.9$.FIGURE 1. 3D profiles of solution (4.4) showing saddle and relative maximum points with $\alpha(t) = t^2$.

(A) Parabolic-double solitons.



(B) Double-edged solitons.

FIGURE 2. 3D profiles for u of solution (4.4) showing different forms of solitons with $\alpha(t) = t^2$.

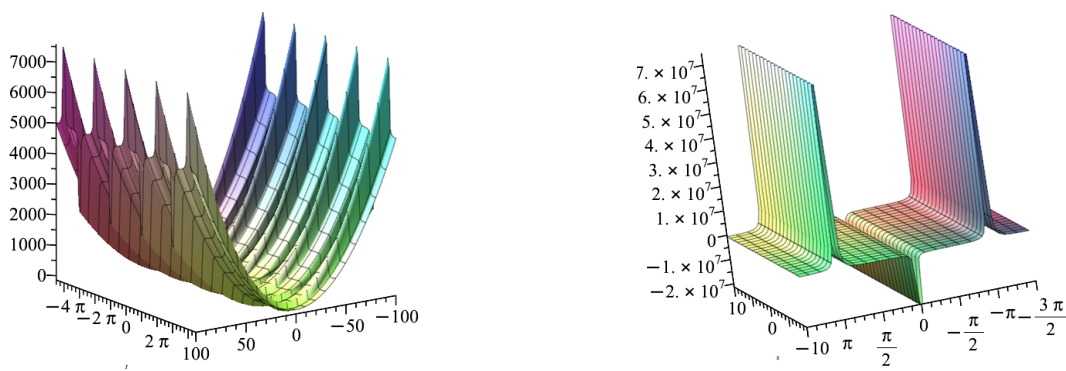
where $\mathfrak{Y} = \text{RootOf}[(4k_1m^3t^{3m} - m^2t^{1+2m} - mt^{1+2m})Z^2 + (-4k_1m^2t^{2m} + 4mt^{1+m} - t^2)Z + k_1mt^m - t]$. The discriminant is too lengthy to be written down but the test shows that v also has only saddle or relative maximum points. To show this explicitly, let us choose $m = 2$ and $t = \frac{1}{2}$. Thus the critical points become¹

$$(x_{k_i}, y_{k_i}) = \left(\frac{\mp 8k_1\sqrt{3-2k_1} + 80k_1^2 - 50k_1 + 12}{(8k_1-3)^2}, \frac{4k_1-3 \pm \sqrt{3-2k_1}}{8k_1-3} \right), \frac{3}{8} \neq k_1 \leq \frac{3}{2}.$$

For $k = \{-\frac{1}{2}, -3\}$, v has a relative maximum at $\{(\frac{65}{49}, \frac{3}{7}), (\frac{106}{81}, \frac{4}{9})\}$ and saddle points at $\{(1, 1), (\frac{10}{9}, \frac{2}{3})\}$ respectively. There are infinitely many of such points for both u and v as shown in Figure 1 with $k_1 = \frac{1}{2}, k_2 = 0, k_3 = 1$ and $k_4 = 0$.

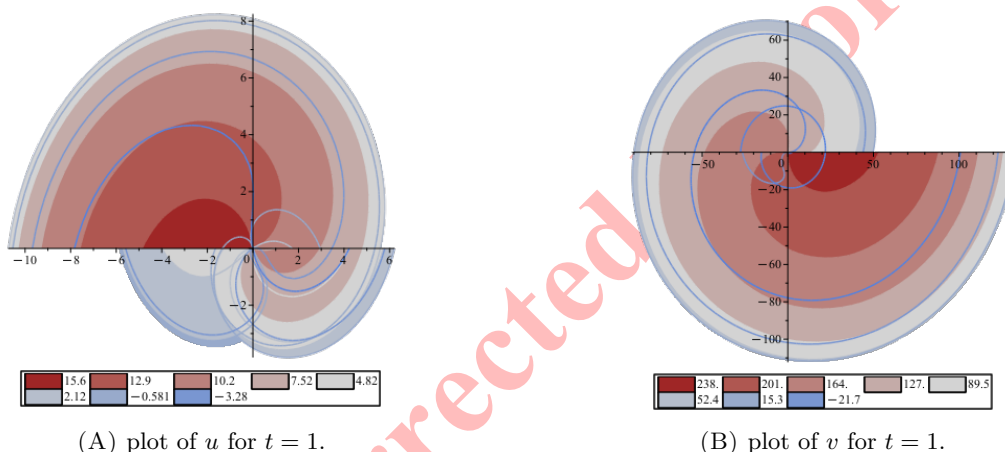
There are various wave profiles that solution (4.4) exhibits depending on the choice of its arbitrary function and constants coupled with the range of the independent variables. Taking $\alpha(t) = t^2, x = -100, k_1 = -10$ and $c = -100$, Figure 2A shows a parabolic-double solitons for u plotted in the range $-100 \leq t, y \leq 100$ and it happens when $t \rightarrow 0$. The scenario is different when y is taken as parameter with $y = 2, c = 40$ and $k_1 = 1$ as can be seen in Figure 2B. When $\alpha(t) = \sin(t)$, solution (4.4) possesses infinitely many periodic solitons each occurring at $t = k\pi, k \in \mathbb{Z}$ as shown in Figure 3. The contour plots, in spherical coordinates, and the density plots are shown in Figure 4 to Figure 6. The contours in Figure 4 are plotted in the interval $-\pi \leq x, y \leq \pi$, those in Figure 5 are plotted in the intervals $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\pi \leq y \leq \pi$ while those in Figure 6 are plotted on $-\pi \leq x \leq \pi, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. The intervals for all the plots in Figure 7 to Figure 9 is $[-\pi, \pi]$ for both x and y . Maple 2023 was used to generate all figures in this manuscript.

¹Complex values are excluded.



(A) 3D profile of u when $x = 20, c = 100, k_1 = 100$. (B) 3D profile of v when $y = -100, c = -41, k_1 = 66, k_3 = 1$.

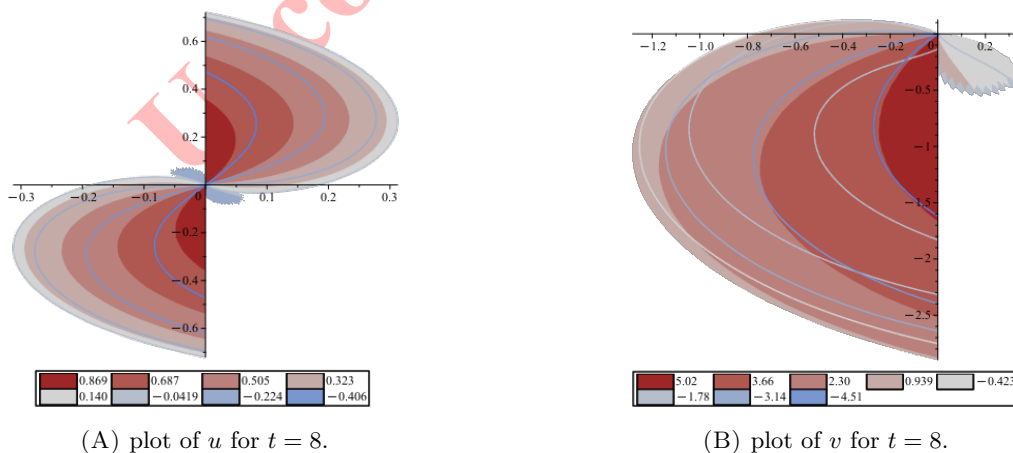
FIGURE 3. Periodic solitons of solution (4.4) when $\alpha(t) = \sin t$.



(A) plot of u for $t = 1$.

(B) plot of v for $t = 1$.

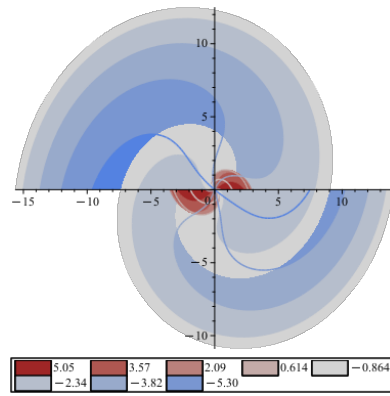
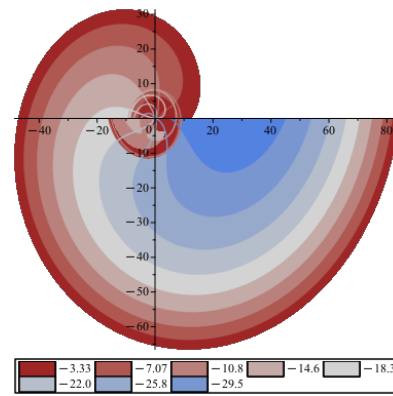
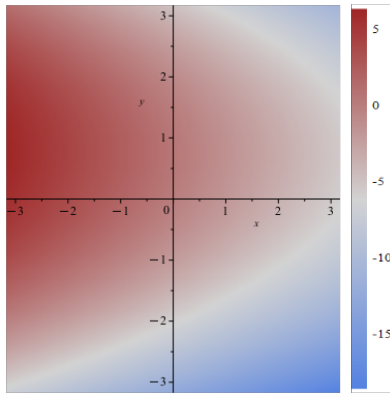
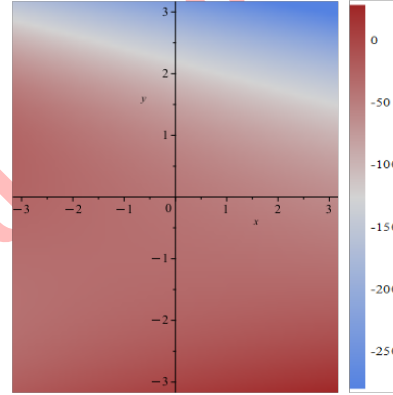
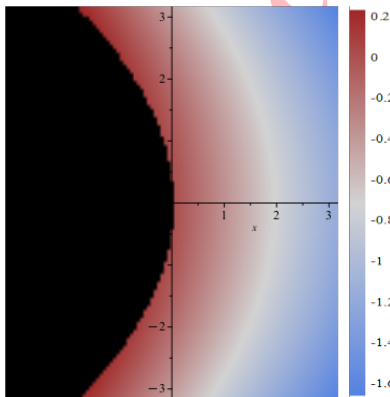
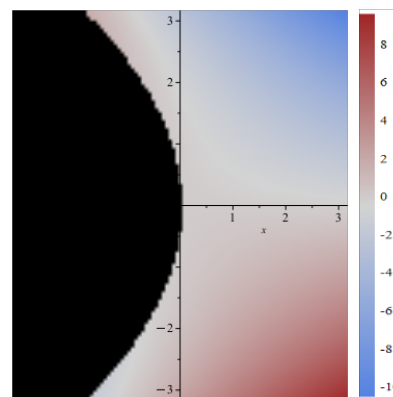
FIGURE 4. Contour plots of solution (4.4) for $\alpha(t) = t^2, c = -2, k_1 = \frac{1}{2}, k_3 = 1$.



(A) plot of u for $t = 8$.

(B) plot of v for $t = 8$.

FIGURE 5. Contour plots of solution (4.4) for $\alpha(t) = t^3, c = -2, k_1 = 1, k_3 = 1$.

(A) plot of u for $t = 1$.(B) plot of v for $t = 1$.FIGURE 6. Contour plots of solution (4.4) for $\alpha(t) = \tan t, c = -2, k_1 = \frac{1}{2}, k_3 = 1$.(A) plot of u for $t = 1$.(B) plot of v for $t = 1$.FIGURE 7. Density plots of solution (4.4) for $\alpha(t) = t^2, c = -2, k_1 = \frac{1}{2}, k_3 = 1$.(A) plot of u for $t = 8$.(B) plot of v for $t = 8$.FIGURE 8. Density plots of solution (4.4) for $\alpha(t) = t^3, c = -2, k_1 = 1, k_3 = 1$.

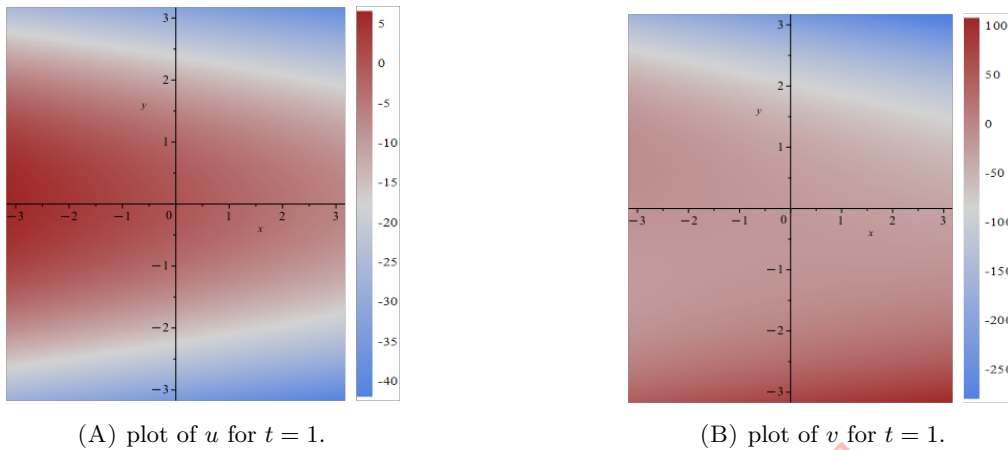


FIGURE 9. Density plots of solution (4.4) for $\alpha(t) = \tan t, c = -2, k_1 = \frac{1}{2}, k_3 = 1$.

7. CONCLUSIONS

The Lie symmetry group, a powerful and versatile method, has been used to carry out an extensive analyses on the MS system. Similar work have been carried out by authors of [21], but results obtained in this article is completely different from theirs. For clarity, the entire analysis carried out in [21] was based on 11-dimensional Lie algebra which were used mainly for obtaining (up to) 2-dimensional group classifications and reduced equations. The authors also stated clearly in the last paragraph of [21] that: “Although, exact solutions corresponding to every reduction in Table 2 and reduction for every element in Θ_2 could have been written in this work, but we have only preferred to concentrate on Lie symmetry analysis and multi-dimensional group classifications of...”

In this work, the goals of the research are to find the complete Lie algebra of the MS system, identify a closed subalgebra and classify it using alternative method, construct explicit nontrivial invariant solutions and conservation laws for the MS system. All the aforementioned objectives were successfully achieved. We have shown that the MS system possesses infinite vector fields ($\mathcal{X}_i : i = 1, \dots, 7$) which are not closed in general even when the arbitrary functions are taken to be first order polynomials. When the arbitrary function are quadratic polynomials, the MS system are spanned by Lie algebra $\mathcal{A}_j : j = 1, 2, \dots, 24$. The entire infinite Lie algebra ($\mathcal{X}_k : k = 1, \dots, 7$) were used to construct infinitely many, nontrivial analytic and special function solutions to the MS system. We further established how some of the $\mathcal{A}_j$ could lead to invariant solutions (4.23)-(4.29) of the MS system that can not be deduced directly from those obtained via the $\mathcal{X}_i$. Two-level application of the Lie group method was used to obtained the invariant solutions to the MS system presented in this research work. That is, the obtained reduced equations are also subjected to the Lie group method. This way, although tedious, made it possible to solve all of them successfully. Finally, we employed the invariance and multiplier approach to established first order multipliers for the MS system and used them to construct nontrivial conservation laws which are infinite with respect to the time-independent variable.

DISCLOSURE STATEMENT

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REFERENCES

- [1] G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*, Springer-Verlag, New York, 1974.
- [2] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.



- [3] G. W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer, New York, 2002.
- [4] L. Bogdanov, *On a class of reductions of the Manakov–Santini hierarchy connected with the interpolating system*, J. Phys. A: Math. Theor., *43* (2010), 115206.
- [5] L. V. Bogdanov, *Dispersionless BKP Equation, the Manakov-Santini System and Einstein–Weyl Structures*, Symmetry, *13* (2021), 1699.
- [6] L. V. Bogdanov, *Matrix extension of the Manakov-Santini system and an integrable Chiral model on an Einstein–Weyl Background*, Theor. Math. Phys., *201* (2019), 1701-1709.
- [7] H. Brezis and Felix Browder, *Partial Differential Equations in the 20th Century*, Adv. Math., *135* (1998), 76-144.
- [8] M. S. Bruzón et al., *1 + 1 spectral problems arising from the Manakov-Santini system*, J. Phys. A: Math. Theor., *43* (2010), 495204.
- [9] B. J. Cantwell, *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge, 2002.
- [10] J. H. Changa and Y. T. Chen, *Hodograph solutions for the Manakov–Santini equation*, J. Math. Phys., *51* (2010), 042701.
- [11] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel Publishing Company, Dordrecht, 1985.
- [12] N. H. Ibragimov, *Handbook of Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, 1996.
- [13] A. H. Kara, *On the Reduction of Some Dispersionless Integrable Systems*, Acta Appl. Math., *132* (2014), 371-376.
- [14] S. V. Manakov and P. M. Santini, *Cauchy problem on the plane for the dispersionless Kadomtsev-Petviashvili equation*, JETP letters, *83* (2006), 462-466.
- [15] S. V. Manakov and P. M. Santini, *A hierarchy of integrable partial differential equations in 2+1 dimensions associated with one-parameter families of one-dimensional vector fields*, Theor. Math. Phys., *152* (2007), 1004-1011.
- [16] S. V. Manakov and P. M. Santini, *On the solutions of the dKP equation: the nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking*, J. Phys. A: Math. Theor., *41* (2008), 055204.
- [17] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.
- [18] P. J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.
- [19] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
- [20] P. Plansangkate, *From the conformal self-duality equations to the Manakov–Santini system*, J. Geom. Phys., *145* (2019), 103468.
- [21] M. Singh and S. F. Tian, *Lie symmetries, group classification and conserved quantities of dispersionless Manakov–Santini system in (2+1)-dimension*, Indian J. Pure Appl. Math., *54* (2023), 312-329.
- [22] H. Stephani, *Differential Equations. Their Solutions Using Symmetries*, Cambridge University Press, Cambridge, 1989.

