



New class of Riemann-Liouville fractional boundary value equations on unbounded domain with mixed nonlocal conditions

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Abstract

In this paper, we analyze the solvability of a new class of fractional differential equations accompanied by novel nonlocal multi-point boundary conditions with classical Riemann-Liouville fractional derivative on an unbounded domain. The existence of solutions to the given equation is established by using the concept of complete continuity, iterative solutions and partially ordered set. Examples are presented to verify and highlight the utility of the obtained results.

Keywords. Riemann-Liouville fractional derivative, Positive solution, Infinite interval, Fixed point theorems, Partially ordered set.

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1. INTRODUCTION

Boundary value problems defined on unbounded intervals frequently emerge in various disciplines such as physics, chemistry, biology, engineering, and the study of dynamical systems. For instance, consider the following problem

$$\begin{cases} -\zeta'' + c\zeta' + \mu\zeta = g(t, \zeta(t)), & t \in (0, +\infty), \\ \zeta(0) = 0, \quad \zeta(+\infty) = 0. \end{cases} \quad (1.1)$$

This equation reduces to the classical Fisher-Kolmogorov model [15] when $\mu = 0$. In [13], the authors examined a boundary value problem on the half-line related to epidemic spread within a fixed population, using a generalized Fisher equation of the form (1.1) with $f(t, \xi(t)) = \xi, h(\xi)$. Furthermore, in [14, 34], the electrical potential of an isolated neutral atom was analyzed through the study of the following boundary value problem:

$$\begin{cases} -\zeta'' = \sqrt{\frac{\zeta^3}{z}}, & 0 < z < +\infty, \\ \zeta(0) = 1, \quad \zeta(+\infty) = 0. \end{cases}$$

For a detailed account of the work concerning the solvability of second-order ordinary differential equations defined on an unbounded domain; for further discussion, see the articles [2–4, 12, 17, 36, 42] and the references cited therein.

Recently, in [18], by using coincidence degree arguments, the authors conducted an analysis of the solvability of the problem given below:

$$\begin{cases} (r(z)\zeta^{(n-1)}(z))' = g(z, \zeta(z), \zeta'(z), \dots, \zeta^{(n-1)}(z)) = 0, & z \in (0, +\infty), \\ \zeta^i(0) = 0, \quad i = 1, 2, \dots, n-3, \\ \zeta^{(n-2)}(0) = \sum_{j=1}^k \alpha_j \int_0^{\mu_j} \zeta(s) ds, \quad \lim_{z \rightarrow +\infty} \{r(z)\zeta^{(n-1)}(z)\} = 0, \end{cases}$$

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where $\alpha_i \in \mathbb{R}$, $g : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Carathéodory's function, $0 < \mu_1 < \dots < \mu_k < 1$, $r \in C[0, +\infty) \cap C^{n-2}(0, +\infty)$, $r > 0$ and $\sum_{j=1}^k \alpha_j \mu_j = (n-1)!$.

In [36], the authors employ the method of lower and upper solutions applied to the following problem

$$\begin{cases} \zeta''(z) + m(z)g(z, \zeta(z), \zeta'(z)) = 0, & z \in (0, +\infty), \\ a\zeta(0) - b\zeta'(0) = \zeta_0 \geq 0, & \lim_{z \rightarrow +\infty} \zeta'(z) = m > 0. \end{cases}$$

Also, the authors in [8] considered the following problem:

$$\begin{cases} \zeta''(z) - \varphi^2 \zeta(z) + g(z, e^{-\varphi z} \zeta(z), e^{-\varphi z} \zeta'(z)) = 0, & z \in (0, +\infty), \\ \zeta(0) - \frac{1}{\varphi} \zeta'(0) = \int_0^{+\infty} e^{-2\varphi s} \zeta(s) ds, & \lim_{z \rightarrow +\infty} e^{-\varphi z} \zeta(s) = M, \end{cases}$$

where $\varphi > 0$, $\varphi \neq \frac{1}{6}$, $M \in \mathbb{R}$ and $g \in C([0, +\infty) \times \mathbb{R}^2; \mathbb{R})$.

On the other hand, fractional differential equations appear in the a variety of practical phenomena occurring in the areas of applied mathematics, physics, biology, engineering and other fields; see, for instance, [20, 28, 30]. For some more results concerning the fractional differential equations and related topic; for example, see [6, 11, 19, 29, 31, 32, 38]. Also, there have fewer studies on fractional differential equations on semi-infinite domains, the reader may consult to [9, 10, 21–24, 26, 33, 35, 37, 39] as well as the references listed within those works. In [33], using the Schauder's fixed point theorem, the authors studied the solvability of the following problem

$$\begin{cases} D_{0+} \zeta(z) = g(z, \zeta(z), D_{0+}^{-1} \zeta(z)), & z \in (0, +\infty), \\ \zeta(0) = 0, \quad D_{0+}^{-1} \zeta(\infty) = \zeta_\infty, \quad \zeta_\infty \in \mathbb{R}, \end{cases}$$

where D_{0+} denotes the Riemann-Liouville fractional derivative of order $1 < \leq 2$, and g is continuous mapping from $[0, +\infty) \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

In [40], the authors applied the generalized Avery-Henderson fixed point theorem to establish the solvability for the Hadamard-type fractional differential equation posed on an unbounded domain:

$$\begin{cases} {}^H D_{1+} \zeta(z) + \omega(z)\theta(z, \zeta(z)) = 0, & z \in (1, +\infty), \\ \zeta(1) = \zeta'(1) = 0, \\ {}^H D_{1+}^{-1} \zeta(\infty) = \delta \sum_{i=1}^n \sigma_i \zeta(\eta_i) + \sum_{i=1}^k \gamma_i {}^H I_{1+}^{\alpha_i} \zeta(\kappa_i), \end{cases}$$

where ${}^H D_{1+}$ is the the Hadamard fractional derivative of order $2 < \leq 3$. ${}^H I_{1+}^{\alpha_i}$ is the Hadamard integral of order $\alpha_i > 0$ ($i = 1, 2, \dots, k$), and θ is a continuous function.

In [10], Using the Avery-Peterson fixed point theorem, the authors analyzed the existence of positive solutions for the following problem

$$\begin{cases} D_{0+} \zeta(s) + \omega(s)g(s, \zeta(s), \zeta'(s)) = 0, & s \in (0, +\infty), \\ \zeta(0) = \zeta'(0) = 0, \\ \lim_{s \rightarrow +\infty} D_{0+}^{-1} \zeta(s) = \int_0^{+\infty} q(s) \zeta'(s) ds + \delta \sum_{i=1}^m \sigma_i D_{0+}^\rho \zeta(\alpha_i), \end{cases}$$

where $2 < \leq 3$ and $0 \leq \rho \leq \nu - 1$, and θ is a continuous function.

Motivated by the studies mentioned above, this work develops and examines a new category of Riemann-Liouville fractional equations posed on the half-line, incorporating both the Riemann-Liouville fractional derivative and its



associated integral operator defined by

$$\begin{cases} -(e^{\lambda t} D_{0+}^{\sigma} \zeta(t))'' + \lambda^2 (e^{\lambda t} D_{0+}^{\sigma} \zeta(t)) + \mathcal{V}(t, \zeta(t)) = 0, & 0 < t < +\infty, \\ D_{0+}^{\sigma} \zeta(0) - \frac{1}{\lambda} (e^{\lambda t} D_{0+}^{\sigma} \zeta)'(0) = \int_0^{+\infty} e^{-\lambda s} D_{0+}^{\sigma} \zeta(s) ds - \sum_{i=1}^k \delta_i (e^{\lambda t} D_{0+}^{\sigma} \zeta)'(\mathfrak{R}_i), \\ \lim_{t \rightarrow +\infty} D_{0+}^{\sigma} \zeta(t) = 0, \\ \zeta(0) = 0, \quad D_{0+}^{\sigma-1} \zeta(+\infty) = \int_0^{+\infty} \mathfrak{g}(s) \zeta(s) ds + \delta \zeta(\mathfrak{R}) + \gamma I_{0+}^{\beta} \zeta(\kappa), \end{cases} \quad (1.2)$$

where $1 < \sigma \leq 2$, D_{0+}^{σ} is the Riemann-Liouville fractional derivative of order σ , I_{0+}^{β} denotes the Riemann-Liouville integral of order $\beta > 0$, $\lambda > 0$ and $\delta_i, \mathfrak{R}_i$ ($i = 1, 2, \dots, k$) are positive constants with

$$\frac{6\lambda - 1}{6\lambda} + \sum_{i=1}^k \frac{\delta_i}{2\lambda} e^{-\lambda \mathfrak{R}_i} > 0,$$

$1 < \kappa \leq \mathfrak{R} < +\infty$, $\gamma, \delta \geq 0$, $\mathfrak{g}$ is continuous mapping on $(0, +\infty)$, and $\mathcal{V}$ is continuous mapping from $(0, +\infty) \times [0, +\infty)$ into $[0, +\infty)$.

The present problem is posed on the half-line $(0, +\infty)$ and contains nonlocal boundary conditions expressed through fractional integrals and fractional derivatives. In such a setting, the Riemann-Liouville operator is particularly suitable because it preserves the integral-memory structure of the model and leads to a solution representation entirely formulated in terms of fractional integrals. This feature is essential in establishing the positivity properties of the Green kernel and in applying the fixed-point techniques developed in the manuscript.

The proposed model (1.2) substantially extends the existing works in [33, 35, 40] by introducing a more general and structurally richer class of Riemann-Liouville fractional boundary value problems on the half-line. In contrast to these studies, which consider standard fractional equations with either classical, multi-point, or integral-type boundary conditions, the present model incorporates a hybrid structure involving exponentially weighted fractional operators, distributed integral constraints over $(0, +\infty)$, and multi-point evaluations of transformed fractional derivatives. This combination of weighted nonlocal terms and fractional integral interactions is not present in the cited literature and leads to a more flexible analytical framework. Consequently, model (1.2) can not be reduced to any of the previously studied problems and therefore represents a genuinely new and more general class of fractional differential equations. Also, In particular, it combines a Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$ with an exponentially weighted differential operator $e^{\lambda t} D_{0+}^{\sigma} \zeta(t)$, leading to a hybrid fractional-exponential operator that does not appear in the cited works.

In this work, we introduce a new class of fractional differential equations accompanied by novel boundary conditions that, to the best of our knowledge, have not been previously studied in the existing literature. The proposed framework extends the current theory of fractional calculus and provides a broader analytical structure. The newly introduced boundary conditions offer fresh perspectives for the study of existence of solutions. We believe that this contribution opens a promising new direction for researchers in the field of fractional differential equations and may stimulate further theoretical and applied investigations in related areas of mathematical analysis and applied sciences. The used method is standard, but its configuration in the present problem is new. The obtained results are new and contribute to this new research topic concerning the study of positive solutions of boundary value problems.

Recent advances in network science have significantly expanded the study of complex dynamical systems through the incorporation of nonlocal interactions, memory-dependent processes, and fractional-order dynamics. In particular, fractional differential equations have emerged as effective mathematical tools for modeling diffusion [5], synchronization [41], anomalous transport [25], and long-memory behaviors in complex networks [1]. Although several studies have investigated fractional dynamical models under classical boundary conditions, comparatively little attention has been devoted to the analysis of Riemann-Liouville fractional differential equations with novel boundary structures. Moreover, existing literature remains largely focused on standard formulations and does not adequately address the broader analytical framework required for modern interconnected systems. Motivated by these limitations, the present work introduces a new class of fractional differential equations equipped with generalized boundary conditions, thereby



extending current theoretical developments and providing a more flexible framework for future research in network science and applied nonlinear analysis.

The limitations of the works on the fractional differential equations involving the Riemann-Liouville derivative are as follows: the use of Riemann-Liouville fractional derivatives in the formulation of fractional differential equations provides a mathematically rigorous framework for describing nonlocal and memory-dependent phenomena. However, several inherent limitations should be noted when employing this approach in both theoretical and applied contexts. First, the Riemann-Liouville derivative requires initial conditions expressed in terms of fractional integrals or fractional derivatives of the unknown function. Such conditions are generally non-local and lack a direct physical interpretation, which limits their applicability in modeling real-world processes where initial states are typically specified in classical terms. Second, solutions of Riemann-Liouville fractional differential equations often exhibit singular behavior at the initial point due to the intrinsic structure of the operator. This weak singularity may not be consistent with the expected smoothness of physical processes and can complicate both qualitative and quantitative analysis. Third, the formulation is not naturally compatible with standard boundary or initial value problem settings, as the incorporation of fractional integral conditions introduces additional analytical constraints. This often restricts the direct use of classical existence, uniqueness, and variational techniques. Fourth, the nonlocal nature of the operator leads to significant analytical and computational difficulties. Closed-form solutions are rare, particularly for nonlinear or nonlocal boundary value problems, and numerical implementations are computationally expensive due to the global memory dependence of the operator. Finally, the lack of a straightforward semigroup property further limits the applicability of standard operator-theoretic methods commonly used in integer-order differential equations. In summary, while the Riemann-Liouville derivative provides a powerful theoretical tool for capturing memory effects, its practical use is constrained by issues related to initial condition interpretation, boundary compatibility, singular behavior, and computational complexity. These limitations often motivate the adoption of alternative formulations, such as the Caputo derivative, in applied mathematical modeling.

The structure of the paper is as follows: Section 2 introduces essential preliminaries and supporting lemmas. Section 3 is devoted to establishing the principal results, and Section 4 provides examples that clarify their application.

2. PRELIMINARIES

This section introduces key concepts from fractional calculus pertinent to our study and proves an auxiliary lemma, which is essential for reformulating the problem as a fixed point problem.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\sigma > 0$, of function $g \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\sigma} f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} g(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\sigma > 0$ of a continuous function $g : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$D_{0+}^{\sigma} g(t) = \frac{1}{\Gamma(n-\sigma)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\sigma-1} g(s) ds,$$

where $n = [\sigma] + 1$.

Lemma 2.3. ([20]). The equality $D_{0+}^{\sigma} I_{0+}^{\sigma} g(t) = g(t)$, $\sigma > 0$ holds for $g \in L^1(0, 1)$.

Lemma 2.4. ([20]). Let $\sigma > 0$. Then the differential equation

$$D_{0+}^{\sigma} \zeta = 0,$$

has a unique solution $\zeta(t) = c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \dots + c_n t^{\sigma-n}$, $c_i \in \mathbb{R}$, $i = 1, \dots, n$, there $n-1 < \sigma \leq n$.

Lemma 2.5. ([20]). Let $\sigma > 0$. Then the following equality holds for $\zeta \in L^1(0, 1)$, $D_{0+}^{\alpha} u \in L^1(0, 1)$;

$$I_{0+}^{\sigma} D_{0+}^{\sigma} \zeta(t) = \zeta(t) + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \dots + c_n t^{\sigma-n},$$

$c_i \in \mathbb{R}$, $i = 1, \dots, n$, there $n-1 < \sigma \leq n$.



Define a real Banach space as follows:

$$X := \left\{ \zeta \in C(0, +\infty) : \frac{\zeta(z)}{1+z^{\sigma-1}} < +\infty \right\},$$

endowed with the norm

$$\|\zeta\|_X := \sup_{z \in (0, +\infty)} \left| \frac{\zeta(z)}{1+z^{\sigma-1}} \right|, \zeta \in X.$$

To find a solution of (1.2), let

$$\varpi(t) = -e^{\lambda t} D_{0+}^{\sigma} \zeta(t), \quad (2.1)$$

then, the problem

$$\begin{cases} -(e^{\lambda t} D_{0+}^{\sigma} \zeta(t))'' + \lambda^2 (e^{\lambda t} D_{0+}^{\sigma} \zeta(t)) + h(\zeta(t)) = 0, & t \in (0, +\infty), \\ D_{0+}^{\sigma} \zeta(0) - \frac{1}{\lambda} (e^{\lambda t} D_{0+}^{\sigma} \zeta)'(0) = \int_0^{+\infty} e^{-\lambda s} D_{0+}^{\sigma} \zeta(s) ds - \sum_{i=1}^k \delta_i (e^{\lambda t} D_{0+}^{\sigma} \zeta)'(\mathfrak{R}_i), \\ \lim_{t \rightarrow +\infty} D_{0+}^{\sigma} \zeta(t) = 0, \end{cases}$$

where $h : (0, +\infty) \rightarrow \mathbb{R}^+$ is a specified function satisfying $\int_0^{+\infty} h(s) ds < \infty$, is transformed into the following problem

$$\begin{cases} \varpi'' - \lambda^2 \varpi + h(t) = 0, & t \in (0, +\infty), \\ \varpi(0) - \frac{1}{\lambda} \varpi'(0) = \int_0^{+\infty} e^{-2\lambda s} \varpi(s) ds + \sum_{i=1}^k \delta_i \varpi'(\mathfrak{R}_i), & \lim_{t \rightarrow +\infty} e^{-\lambda t} \varpi(t) = 0. \end{cases} \quad (2.2)$$

Lemma 2.6. Assuming $h \in C(0, +\infty)$, Equation (2.2) admits a unique solution

$$Z(t) = \int_0^{+\infty} G(t, s) h(s) ds, \quad (2.3)$$

where

$$G(t, s) = \frac{e^{-\lambda t}}{12\lambda^2 \Theta} (3e^{-\lambda s} - 2e^{-2\lambda s}) + \frac{e^{-\lambda t}}{4\Theta} \sum_{i=1}^k \delta_i U(\mathfrak{R}_i, s) + \frac{1}{2\lambda} \begin{cases} e^{\lambda(s-t)}, & s \leq t, \\ e^{\lambda(t-s)}, & s > t, \end{cases} \quad (2.4)$$

with

$$\Theta = \left(\frac{6\lambda - 1}{6\lambda} + \sum_{i=1}^k \frac{\delta_i}{2\lambda} e^{-\lambda \mathfrak{R}_i} \right)^{-1},$$

and

$$U(\mathfrak{R}_i, s) = \begin{cases} -e^{\lambda(s-\mathfrak{R}_i)}, & s \leq \mathfrak{R}_i, \\ e^{\lambda(\mathfrak{R}_i-s)}, & s \geq \mathfrak{R}_i. \end{cases}$$

Proof. As argued in [8], we firstly solve the following problem:

$$\begin{cases} \varpi''(z) - \lambda^2 \varpi(z) + h(z) = 0, & 0 < z < +\infty, \\ \varpi(0) - \frac{1}{\lambda} \varpi'(0) = A, & \lim_{z \rightarrow +\infty} e^{-\lambda z} \varpi(z) = 0, \end{cases} \quad (2.5)$$

where $A \in \mathbb{R}$.

The general form of the solution for the homogeneous equation

$$\varpi''(z) - \lambda^2 \varpi(z) = 0, \quad 0 < z < +\infty,$$

is

$$\varpi(z) = d_1 e^{-\lambda z} + d_2 e^{\lambda z}, \quad d_1, d_2 \in \mathbb{R}.$$



Obviously, the solution of problem (2.5) has as follows:

$$\varpi(z) = \int_0^{+\infty} \varrho(z, s)h(s)ds + \frac{A}{2}e^{-\lambda z}, \quad (2.6)$$

where

$$\varrho(z, s) = \begin{cases} C_1(s)e^{-\lambda z} + C_2(s)e^{\lambda z}, & z < s, \\ C_3(s)e^{-\lambda z} + C_4(s)e^{\lambda z}, & z \geq s. \end{cases}$$

By noting that ϱ is continuous and that its partial derivative with respect to z experiences a unit jump at $z = s$ (see [7]), we obtain

$$\varrho(z, s) = \frac{1}{2\lambda} \begin{cases} e^{\lambda(z-s)}, & z < s, \\ e^{\lambda(s-z)}, & z \geq s. \end{cases} \quad (2.7)$$

Since

$$\int_0^{+\infty} \varpi(s)e^{-2\lambda s}ds = \int_0^{+\infty} \left(e^{-2\lambda s} \int_0^{+\infty} \varrho(s, r)h(r)dr \right) ds + \frac{A}{2} \int_0^{+\infty} e^{-3\lambda s}ds, \quad (2.8)$$

and

$$\sum_{i=1}^k \delta_i \varpi'(\mathfrak{R}_i) = \sum_{i=1}^k \delta_i \int_0^{+\infty} \frac{\partial \varrho}{\partial t}(\mathfrak{R}_i, s)h(s)ds - \sum_{i=1}^k \frac{A\delta_i}{2\lambda} e^{-\lambda \mathfrak{R}_i}. \quad (2.9)$$

In (2.6), putting

$$A = \int_0^{+\infty} e^{-2\lambda s} \varpi(s)ds + \sum_{i=1}^k \delta_i \varpi'(\mathfrak{R}_i),$$

so by (2.8) and (2.9), we get

$$\begin{aligned} A &= \frac{1}{\Theta} \int_0^{+\infty} \left(\int_0^{+\infty} e^{-2\lambda s} \varrho(s, r)ds \right) h(r)dr + \frac{1}{\Theta} \sum_{i=1}^k \delta_i \int_0^{+\infty} \frac{\partial \varrho}{\partial z}(\mathfrak{R}_i, s)h(s)ds \\ &= \frac{1}{2\lambda^2 \Theta} \int_0^{+\infty} \left(e^{-\lambda r} - \frac{2}{3}e^{-2\lambda r} \right) h(r)dr \frac{1}{2\Theta} \sum_{i=1}^k \delta_i \left[e^{\lambda \mathfrak{R}_i} \int_{\mathfrak{R}_i}^{+\infty} e^{-\lambda s} h(s)ds - e^{-\lambda \mathfrak{R}_i} \int_0^{\mathfrak{R}_i} e^{\lambda s} h(s)ds \right]. \end{aligned} \quad (2.10)$$

Now, by replacing (2.10) in (2.6), we obtain

$$\begin{aligned} \varpi(z) &= \int_0^{+\infty} \varrho(z, s)h(s)ds + \frac{e^{-\lambda z}}{4\lambda^2 \Theta} \int_0^{+\infty} \left(e^{-\lambda r} - \frac{2}{3}e^{-2\lambda r} \right) h(r)dr \\ &\quad + \frac{e^{-\lambda z}}{4\Theta} \sum_{i=1}^k \delta_i \left[e^{\lambda \mathfrak{R}_i} \int_{\mathfrak{R}_i}^{+\infty} e^{-\lambda s} h(s)ds - e^{-\lambda \mathfrak{R}_i} \int_0^{\mathfrak{R}_i} e^{\lambda s} h(s)ds \right]. \end{aligned}$$

Therefor, we have the conclusion. $\square$

Lemma 2.7. *Let*

$$\mathfrak{A} := \Gamma(\sigma) - \frac{\gamma \Gamma(\sigma) \kappa^{\sigma+\beta-1}}{\Gamma(\sigma+\beta)} - \delta \mathfrak{R}^{\sigma-1} > 0,$$

$$\mathfrak{A}_1 := \mathfrak{A} - \int_0^\infty t^{\sigma-1} \mathfrak{g}(t)dt > 0.$$

Then the problem stated below

$$\begin{cases} D_{0+}^\sigma \zeta(t) + e^{-\lambda t} \varpi(t) = 0, & 1 < \sigma < 2, \quad t \in (0, +\infty), \\ \zeta(0) = 0, \quad D_{0+}^{\sigma-1} \zeta(+\infty) = \int_0^{+\infty} \mathfrak{g}(s) \zeta(s)ds + \delta \zeta(\mathfrak{R}) + \gamma I_{0+}^\beta \zeta(\kappa), \end{cases} \quad (2.11)$$



admits the following solution:

$$\zeta(t) = \int_0^{+\infty} \Xi(t, s) e^{-\lambda s} \varpi(s) ds, \quad (2.12)$$

where

$$\Xi(t, s) = \Xi_1(t, s) + \Xi_2(t, s), \quad (2.13)$$

with

$$\begin{aligned} \Xi_1(t, s) &= \Upsilon(t, s, \sigma) + \frac{\gamma t^{\sigma-1}}{\mathfrak{A}} \Upsilon(\kappa, s, \sigma + \beta) + \frac{\delta t^{\sigma-1}}{\mathfrak{A}} \Upsilon(\mathfrak{R}, s, \sigma), \\ \Xi_2(t, s) &= \frac{t^{\sigma-1}}{\mathfrak{A}_1} \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t) dt, \end{aligned}$$

and

$$\Upsilon(t, s, \sigma) = \frac{1}{\Gamma(\sigma)} \begin{cases} t^{\sigma-1} - (t-s)^{\sigma-1}, & 0 \leq s \leq t \leq +\infty, \\ t^{\sigma-1}, & 0 \leq t \leq s \leq +\infty. \end{cases}$$

Proof. Following the discussion in [21], the solution of (2.11) takes the form

$$\zeta(t) = -\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds + c_1 t^{\sigma-1} + c_2 t^{\sigma-2}, \quad (2.14)$$

for some arbitrary real constants c_1 and c_2 . So, by conditions $\zeta(0) = 0$, we get $c_2 = 0$. Hence,

$$\zeta(t) = -\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds + c_1 t^{\sigma-1}, \quad (2.15)$$

$$D_{0+}^{\sigma-1} \zeta(t) = -\int_0^t e^{-\lambda s} \varpi(s) ds + c_1 \Gamma(\sigma). \quad (2.16)$$

Then, by (2.15), (2.16) and the second boundary condition in (2.11), we get

$$\begin{aligned} c_1 &= \frac{1}{\mathfrak{A}} \left\{ \int_0^{+\infty} \mathfrak{g}(t) \zeta(t) dt + \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds - \frac{\gamma}{\Gamma(\sigma + \beta)} \int_0^{\kappa} (\kappa - s)^{\sigma+\beta-1} e^{-\lambda s} \varpi(s) ds \right. \\ &\quad \left. - \frac{\delta}{\Gamma(\sigma)} \int_0^{\mathfrak{R}} (\mathfrak{R} - s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds \right\}. \end{aligned} \quad (2.17)$$

By substituting (2.17) into (2.15), we have

$$\begin{aligned} \zeta(t) &= \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t) \zeta(t) dt - \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds \\ &\quad - \frac{\gamma t^{\sigma-1}}{\mathfrak{A} \Gamma(\sigma + \beta)} \int_0^{\kappa} (\kappa - s)^{\sigma+\beta-1} e^{-\lambda s} \varpi(s) ds - \frac{\delta t^{\sigma-1}}{\mathfrak{A} \Gamma(\sigma)} \int_0^{\mathfrak{R}} (\mathfrak{R} - s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds \\ &= \frac{t^{\sigma-1}}{\Gamma(\sigma)} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds + \frac{t^{\sigma-1}(\Gamma(\sigma) - \mathfrak{A})}{\mathfrak{A} \Gamma(\sigma)} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds \\ &\quad + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t) \zeta(t) dt - \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds \\ &\quad - \frac{\gamma t^{\sigma-1}}{\mathfrak{A} \Gamma(\sigma + \beta)} \int_0^{\kappa} (\kappa - s)^{\sigma+\beta-1} e^{-\lambda s} \varpi(s) ds - \frac{\delta t^{\sigma-1}}{\mathfrak{A} \Gamma(\sigma)} \int_0^{\mathfrak{R}} (\mathfrak{R} - s)^{\sigma-1} e^{-\lambda s} \varpi(s) ds \\ &= \frac{t^{\sigma-1}}{\Gamma(\sigma)} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds + \frac{\gamma t^{\sigma-1} \kappa^{\sigma+\beta-1}}{\mathfrak{A} \Gamma(\sigma + \beta)} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds + \frac{\delta t^{\sigma-1} \mathfrak{R}^{\sigma-1}}{\mathfrak{A} \Gamma(\sigma)} \int_0^{+\infty} e^{-\lambda s} \varpi(s) ds \end{aligned}$$



$$\begin{aligned}
& + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt - \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\lambda s} \varpi(s)ds \\
& - \frac{\gamma t^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma+\beta)} \int_0^\kappa (\kappa-s)^{\sigma+\beta-1} e^{-\lambda s} \varpi(s)ds - \frac{\delta t^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma)} \int_0^{\mathfrak{R}} (\mathfrak{R}-s)^{\sigma-1} e^{-\lambda s} \varpi(s)ds \\
& = \int_0^t \Upsilon(t, s, \sigma) e^{-\lambda s} \varpi(s)ds + \frac{\gamma t^{\sigma-1} \kappa^{\sigma+\beta-1}}{\mathfrak{A}} \int_0^t \Upsilon(\kappa, s, \sigma+\beta) e^{-\lambda s} \varpi(s)ds \\
& + \frac{\delta t^{\sigma-1} \mathfrak{R}^{\sigma-1}}{\mathfrak{A}} \int_0^t \Upsilon(\mathfrak{R}, s, \sigma) e^{-\lambda s} \varpi(s)ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt \\
& = \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt &= \int_0^{+\infty} \mathfrak{g}(t) \left(\int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt \right) dt \\
&= \int_0^{+\infty} \mathfrak{g}(t) \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds dt + \frac{1}{\mathfrak{A}} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t)dt \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt.
\end{aligned}$$

So,

$$\int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt \left(1 - \frac{1}{\mathfrak{A}} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t)dt \right) = \int_0^{+\infty} \mathfrak{g}(t) \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds dt.$$

From $\mathfrak{A}_1 = \mathfrak{A} - \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t)dt$, it follows that

$$\begin{aligned}
\int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt &= \frac{\mathfrak{A}}{\mathfrak{A}_1} \int_0^{+\infty} \mathfrak{g}(t) \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds dt \\
&= \frac{\mathfrak{A}}{\mathfrak{A}_1} \int_0^{+\infty} e^{-\lambda s} \varpi(s) \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t)dt ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta(t) &= \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \int_0^{+\infty} \mathfrak{g}(t)\zeta(t)dt \\
&= \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \frac{t^{\sigma-1}}{\mathfrak{A}} \left[\frac{\mathfrak{A}}{\mathfrak{A}_1} \int_0^{+\infty} e^{-\lambda s} \varpi(s) \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t)dt ds \right] \\
&= \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \int_0^{+\infty} \frac{t^{\sigma-1}}{\mathfrak{A}_1} \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t)dt e^{-\lambda s} \varpi(s)ds \\
&= \int_0^{+\infty} \Xi_1(t, s) e^{-\lambda s} \varpi(s)ds + \int_0^{+\infty} \Xi_2(t, s) e^{-\lambda s} \varpi(s)ds \\
&= \int_0^{+\infty} \Xi(t, s) e^{-\lambda s} \varpi(s)ds
\end{aligned}$$

Hence, we have the conclusion. □

Lemma 2.8. *The functions Ξ , Ξ_1 and Ξ_2 possess the following characteristics:*

- (i): Ξ is continuous on $(0, \infty) \times (0, \infty)$ and satisfies $\Xi(t, s) \geq 0$;
- (ii): $\Upsilon(t, s, \sigma)$ is increasing with respect to t ;
- (iii): $\frac{\Xi(t, s)}{1 + t^{\sigma-1}} \leq \frac{1}{\mathfrak{A}_1}$ for $t, s \in (0, +\infty)$;



(iv):

$$\min_{\kappa \leq t \leq k\kappa} \frac{\Xi(t, s)}{1 + t^{\sigma-1}} \geq \frac{\gamma \kappa^{\sigma-1} \Upsilon(\kappa, s, \sigma + \beta)}{(1 + (k\kappa)^{\sigma-1})} \left(\frac{1}{\mathfrak{A}_1} - \frac{1}{\mathfrak{A}} \right),$$

for $s \in (0, +\infty)$ and $k > 1$.

Proof. The proof of the parts (i) and (ii) are straightforward, so we omit them. To prove (iii), for $t, s \in (0, +\infty)$, we get

$$\begin{aligned} \frac{\Xi(t, s)}{1 + t^{\sigma-1}} &= \frac{\Xi_1(t, s)}{1 + t^{\sigma-1}} + \frac{\Xi_2(t, s)}{1 + t^{\sigma-1}} \\ &= \frac{\Upsilon(t, s, \sigma)}{1 + t^{\sigma-1}} + \frac{\gamma t^{\sigma-1}}{\mathfrak{A}(1 + t^{\sigma-1})} \Upsilon(\kappa, s, \sigma + \beta) \\ &\quad + \frac{\delta t^{\sigma-1}}{\mathfrak{A}(1 + t^{\sigma-1})} \Upsilon(\kappa, s, \sigma) + \frac{t^{\sigma-1}}{\mathfrak{A}_1(1 + t^{\sigma-1})} \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t) dt \\ &\leq \frac{1}{\Gamma(\sigma)} + \frac{\gamma \kappa^{\sigma+\beta-1}}{\mathfrak{A}\Gamma(\sigma + \beta)} + \frac{\delta \kappa^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma)} + \frac{1}{\mathfrak{A}_1\Gamma(\sigma)} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt \\ &\quad + \frac{1}{\mathfrak{A}_1} \frac{\gamma(\log \kappa)^{\sigma+\beta-1}}{\mathfrak{A}\Gamma(\sigma + \beta)} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt + \frac{1}{\mathfrak{A}_1} \frac{\delta(\log \kappa)^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma)} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt \\ &= \frac{1}{\Gamma(\sigma)} + \frac{\gamma \kappa^{\sigma+\beta-1}}{\mathfrak{A}\Gamma(\sigma + \beta)} + \frac{\delta \kappa^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma)} \\ &\quad + \frac{1}{\mathfrak{A}\mathfrak{A}_1\Gamma(\sigma)} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt \left(\mathfrak{A} + \frac{\gamma \Gamma(\sigma) \kappa^{\sigma+\beta-1}}{\Gamma(\sigma + \beta)} + \sigma \delta \kappa^{\sigma-1} \right) \\ &= \frac{1}{\Gamma(\sigma)} + \frac{\gamma \kappa^{\sigma+\beta-1}}{\mathfrak{A}\Gamma(\sigma + \beta)} + \frac{\sigma \delta \kappa^{\sigma-1}}{\mathfrak{A}\Gamma(\sigma)} + \frac{1}{\mathfrak{A}\mathfrak{A}_1} \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt \\ &= \frac{1}{\Gamma(\sigma)} + \frac{1}{\mathfrak{A}\Gamma(\sigma)} (\Gamma(\sigma) - \mathfrak{A}) + \frac{\mathfrak{A} - \mathfrak{A}_1}{\mathfrak{A}\mathfrak{A}_1} = \frac{1}{\mathfrak{A}_1}. \end{aligned}$$

To prove (iv), By lemma 2.8(ii), for $k > 1$ and $0 < t, s < +\infty$, we obtain

$$\begin{aligned} \min_{\kappa \leq t \leq k\kappa} \frac{\Xi(t, s)}{1 + t^{\sigma-1}} &\geq \min_{\kappa \leq t \leq k\kappa} \frac{t^{\sigma-1}}{\mathfrak{A}_1(1 + t^{\sigma-1})} \int_0^{+\infty} \Xi_1(t, s) \mathfrak{g}(t) dt \\ &\geq \frac{\kappa^{\sigma-1}}{\mathfrak{A}_1(1 + (k\kappa)^{\sigma-1})} \int_0^{+\infty} \left(\frac{\gamma t^{\sigma-1}}{\mathfrak{A}} \Upsilon(\kappa, s, \sigma + \beta) + \frac{\delta t^{\sigma-1}}{\mathfrak{A}} \Upsilon(\mathfrak{K}, s, \sigma) \right) \mathfrak{g}(t) dt \\ &\geq \frac{\kappa^{\sigma-1}}{\mathfrak{A}_1(1 + (k\kappa)^{\sigma-1})} \left(\frac{\gamma}{\mathfrak{A}} \Upsilon(\kappa, s, \sigma + \beta) + \frac{\delta t^{\sigma-1}}{\mathfrak{A}} \Upsilon(\mathfrak{K}, s, \sigma) \right) \int_0^{+\infty} t^{\sigma-1} \mathfrak{g}(t) dt \\ &\geq \frac{\kappa_1^{\sigma-1}}{(1 + (k\kappa_1)^{\sigma-1})} \left(\gamma \Upsilon(\kappa, s, \sigma + \beta) + \delta t^{\sigma-1} \Upsilon(\mathfrak{K}, s, \sigma) \right) \left(\frac{1}{\mathfrak{A}_1} - \frac{1}{\mathfrak{A}} \right) \\ &\geq \frac{\gamma \kappa^{\sigma-1} \Upsilon(\kappa, s, \sigma + \beta)}{(1 + (k\kappa)^{\sigma-1})} \left(\frac{1}{\mathfrak{A}_1} - \frac{1}{\mathfrak{A}} \right). \end{aligned}$$

So, we have the conclusion. $\square$

3. MAIN RESULTS

Consider the operator $\mathcal{G} : \mathcal{Q} \rightarrow X$ defined as

$$\mathcal{G}\zeta(t) = \int_0^{+\infty} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta(r)) dr ds, \quad (3.1)$$



where $\mathcal{Q} \subset X$ is a cone as follows:

$$\mathcal{Q} := \{\zeta \in X : \zeta(r) \geq 0, r \in (0, +\infty)\}.$$

It is evident that finding solutions to problem (1.2) is equivalent to identifying fixed points of $\mathcal{G}$ within $\mathcal{Q}$.

Lemma 3.1. *Let $\mathfrak{W} \subset X$ be a bounded set. Then $\mathfrak{W}$ is relatively compact in X if the following conditions hold:*

- (i): *The family of functions $\left\{ \frac{\zeta(t)}{1+t^{\sigma-1}} : \zeta \in \mathfrak{W} \right\}$ is equicontinuous on all compact subsets of $(0, \infty)$;*
- (ii): *For each $\varepsilon > 0$, there exists a constant $\Phi_{\mathfrak{W}} > 0$ satisfying*

$$\left| \frac{\zeta(z_1)}{1+z_1^{\sigma-1}} - \frac{\zeta(z_2)}{1+z_2^{\sigma-1}} \right| < \varepsilon, \quad \forall z_1, z_2 \geq \Phi_{\mathfrak{W}}, \zeta \in \mathfrak{W}.$$

Proof. The demonstration follows the approach of [27, Lemma 3.1] and is therefore omitted. $\square$

Lemma 3.2. *Suppose that the following assumption is satisfied:*

- (H1): *The function $\mathcal{V}$ is continuous mapping from $(0, +\infty) \times \mathbb{R}$ into $\mathbb{R}$ and there exists a positive constant Ω_1 such that $|\mathcal{V}(z, (1+z^{\sigma-1})\vartheta)| < \Omega_1$ for all $0 < z < +\infty$ and positive bounded ϑ .*

Then, $\mathcal{G} : \mathcal{Q} \rightarrow \mathcal{Q}$ is completely continuous.

Proof. Obviously, $\mathcal{G}\zeta(t) \geq 0$ on $(0, +\infty)$ for every $\zeta \in \mathcal{Q}$. We begin by showing that the operator $\mathcal{G}$ sends bounded sets to bounded sets. Using straightforward calculations, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} G(s, r) dr ds = \frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3}. \quad (3.2)$$

Also, by (H1), for any $\zeta \in \mathcal{Q}$, we have

$$|\mathcal{V}(z, \zeta(z))| = \left| \mathcal{V}\left(z, (1+z^{\sigma-1}) \frac{\zeta(z)}{1+z^{\sigma-1}}\right) \right| < \Omega_1. \quad (3.3)$$

So, in view of Lemma 2.8(iii), (3.2) and (3.3), we get

$$\begin{aligned} \frac{|(\mathcal{G}\zeta)(t)|}{1+t^{\sigma-1}} &= \int_0^{+\infty} \frac{\Xi(t, s)}{1+t^{\sigma-1}} e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta(r)) dr ds \\ &\leq \frac{\Omega_1}{\mathfrak{A}_1} \left(\int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} G(s, r) dr ds \right) \\ &\leq \frac{\Omega_1}{\mathfrak{A}_1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) < +\infty. \end{aligned}$$

Thus, for all $0 < t < +\infty$, we have $(\mathcal{G}\zeta)(t) \in X$. This confirms that $\mathcal{G}$ is well-defined. Moreover, by the argument presented above, $\mathcal{G}$ carries bounded sets into bounded sets.

Next, we assert that $\mathcal{G}$ is continuous. To show this, let $\{\zeta_n\} \subset \mathcal{Q}$ be a sequence converging to $\hat{\zeta}$ in $\mathcal{Q}$ as $n \rightarrow \infty$. Then, applying (H1), Lemma 2.8(iii), and (g1), we deduce that

$$\begin{aligned} \int_0^{+\infty} \frac{\Xi(t, s)}{1+t^{\sigma-1}} e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta_n(r)) dr ds &\leq \frac{\Omega_1}{\mathfrak{A}_1} \left(\int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} G(s, r) dr ds \right) \\ &\leq \frac{\Omega_1}{\mathfrak{A}_1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) < +\infty. \end{aligned}$$

Therefore, utilizing the continuity of $\mathcal{V}$ together with the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \rightarrow +\infty} \int_0^{+\infty} \frac{\Xi(t, s)}{1+t^{\sigma-1}} e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta_n(r)) dr ds$$



$$= \int_0^{+\infty} \frac{\Xi(t, s)}{1 + t^{\sigma-1}} e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \widehat{\zeta}(r)) dr ds.$$

Therefore, $\mathcal{G}$ is continuous. So, $\mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.

Assume that $\mathcal{J} \subset (0, +\infty)$ be a compact interval and $\mathcal{K}$ a bounded subset of $\mathcal{Q}$. In view of (H1), for all $\zeta \in \mathcal{K}$ and $\tau_1 < \tau_2$ belongs in $\mathcal{J}$, we obtain

$$\begin{aligned} & \left| \frac{1}{1 + \tau_2^{\sigma-1}} (\mathcal{G}\zeta)(\tau_2) - \frac{1}{1 + \tau_1^{\sigma-1}} (\mathcal{G}\zeta)(\tau_1) \right| \\ & \leq \int_0^{+\infty} \left| \frac{\Xi(\tau_2, s)}{1 + \tau_2^{\sigma-1}} - \frac{\Xi(\tau_1, s)}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) |\mathcal{V}(r, \zeta(r))| dr ds \\ & \leq \Omega_1 \int_0^{+\infty} \left| \frac{\Xi(\tau_2, s)}{1 + \tau_2^{\sigma-1}} - \frac{\Xi(\tau_1, s)}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) dr ds \\ & \leq \Omega_1 \int_0^{\tau_2} \left| \frac{\Xi(\tau_2, s)}{1 + \tau_2^{\sigma-1}} - \frac{\Xi(\tau_1, s)}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) dr ds \\ & + \Omega_1 \int_{\tau_2}^{+\infty} \left| \frac{\Xi(\tau_2, s)}{1 + \tau_2^{\sigma-1}} - \frac{\Xi(\tau_1, s)}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) dr ds \\ & \leq \Omega_1 \int_0^{\tau_2} \left| \frac{\Xi(\tau_2, s)}{1 + \tau_2^{\sigma-1}} - \frac{\Xi(\tau_1, s)}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) dr ds \\ & + \frac{\Omega_1}{\mathfrak{A}_1} \int_{\tau_2}^{+\infty} \left| \frac{\tau_2^{\sigma-1}}{1 + \tau_2^{\sigma-1}} - \frac{\tau_1^{\sigma-1}}{1 + \tau_1^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) dr ds. \end{aligned}$$

Since $\frac{\zeta(\tau, s)}{1 + \tau^{\sigma-1}}$ and $\frac{\tau^{\sigma-1}}{1 + \tau^{\sigma-1}}$ are uniformly continuous on $\mathcal{J}$, so, we get

$$\left| \frac{(\mathcal{G}\zeta)(\tau_2)}{1 + \tau_2^{\sigma-1}} - \frac{(\mathcal{G}\zeta)(\tau_1)}{1 + \tau_1^{\sigma-1}} \right| \rightarrow 0, \quad \text{as } \tau_1 \rightarrow \tau_2.$$

Thus, $\frac{(\mathcal{G}\zeta)(\tau)}{1 + \tau^{\sigma-1}}$ is equicontinuous on $\mathcal{J}$.

We now show that $\mathcal{G}$ is equiconvergent at ∞ . So, for every $\zeta \in \mathcal{Q}$, By (H1), Lemma 2.8(iii) and (3.2), we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left| \frac{(\mathcal{G}\zeta)(t)}{1 + t^{\sigma-1}} \right| & \leq \lim_{t \rightarrow +\infty} \int_0^{+\infty} \left| \frac{\Xi(t, s)}{1 + t^{\sigma-1}} \right| e^{-\lambda s} \int_0^{+\infty} G(s, r) |\mathcal{V}(r, \zeta(r))| dr ds \\ & \leq \frac{\Omega_1}{\mathfrak{A}_1} \lim_{t \rightarrow +\infty} \left(\int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} G(s, r) dr ds \right) \\ & \leq \frac{\Omega_1}{\mathfrak{A}_1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) < +\infty. \end{aligned}$$

This demonstrates that $\mathcal{G}$ is equiconvergent as $t \rightarrow +\infty$. Consequently, applying Lemma 3.1, we conclude that $\mathcal{G}$ is completely continuous. Thus, the result follows. $\square$

3.1. Existence of two solutions.

Theorem 3.3. Assume that (H1) holds, along with the following conditions:

(H2): For each $0 < t < +\infty$, the function $\mathcal{V}(t, \zeta)$ is nondecreasing with respect to ζ ;

(H3): There exists a constant $\chi_1 > 0$ for which

$$\sup_{t \in (0, +\infty)} \mathcal{V} \left(t, (1 + t^{\sigma-1}) \mathfrak{A}_1^{-1} \left[\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right] \chi_1 \right) \leq \frac{\chi_1}{2}.$$



Under these conditions, problem (1.2) admits at least two solutions, denoted by $\hat{\zeta}^*$ and $\tilde{\zeta}^*$, with norms bounded by Υ , that is, $\|\hat{\zeta}^*\|_X, \|\tilde{\zeta}^*\|_X \leq \Upsilon$, where

$$\Upsilon := \mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \chi_1. \quad (3.4)$$

Proof. Lemma 3.2 implies that $\mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$. Set $S_\Upsilon := \{\zeta \in \mathcal{Q} : \|\zeta\|_X \leq \Upsilon\}$ and take $\zeta \in S_\Upsilon$. So, we get

$$\begin{aligned} |\mathcal{V}(t, \zeta)| &= \left| \mathcal{V} \left(t, (1+t^{\sigma-1}) \mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \mathfrak{A}_1 \right. \right. \\ &\quad \times \left. \left. \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right)^{-1} \frac{1}{1+t^{\sigma-1}} \zeta \right) \right| \\ &\leq \frac{\mathfrak{A}_1}{2} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right)^{-1} \|\zeta\|_X \\ &\leq \frac{\mathfrak{A}_1}{2} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right)^{-1} \Upsilon. \end{aligned} \quad (3.5)$$

Also, Lemma 2.8(iii) and (3.2) imply that

$$\begin{aligned} \|\mathcal{G}\zeta\|_X &= \sup_{t \in (0, +\infty)} \left| \frac{(\mathcal{G}\zeta)(t)}{1+t^{\sigma-1}} \right| \leq \sup_{t \in (0, +\infty)} \int_0^{+\infty} \frac{1}{1+t^{\sigma-1}} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta(r)) dr ds \\ &\leq \frac{1}{\mathfrak{A}_1} \int_0^{+\infty} e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, \zeta(r)) dr ds \\ &\leq \frac{1}{2} \frac{1}{\mathfrak{A}_1} \mathfrak{A}_1 \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right)^{-1} \\ &\quad \times \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \Upsilon \\ &= \frac{\Upsilon}{2} < \Upsilon. \end{aligned} \quad (3.6)$$

Hence, $\mathcal{G} : S_\Upsilon \rightarrow S_\Upsilon$.

We now define

$$\tilde{\zeta}_0(t) = 0, \quad \hat{\zeta}_0(t) := (1+t^{\sigma-1}) \mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \chi_1.$$

Also, we know that $\hat{\zeta}_0, \tilde{\zeta}_0 \in S_\Upsilon$. Using Lemma 3.2, we construct two sequences $\{\hat{\zeta}_n\}$ and $\{\tilde{\zeta}_n\}$ recursively by setting $\hat{\zeta}_n = \mathcal{G}(\hat{\zeta}_{n-1})$ and $\tilde{\zeta}_n = \mathcal{G}(\tilde{\zeta}_{n-1})$. Clearly, $\{\hat{\zeta}_n\}$ and $\{\tilde{\zeta}_n\}$ remain in S_Υ since $\mathcal{G}(S_\Upsilon) \subset S_\Upsilon$. We now assert that there exist elements $\hat{\zeta}^*$ and $\tilde{\zeta}^*$ such that $\lim_{n \rightarrow \infty} \hat{\zeta}_n = \hat{\zeta}^*$ and $\lim_{n \rightarrow \infty} \tilde{\zeta}_n = \tilde{\zeta}^*$, and these limits provide solutions to problem (1.2).

Let us define $\hat{\zeta}_n = \mathcal{G}(\hat{\zeta}_{n-1})$. So, in view of Lemma 2.8(iii), 3.2 and similar argument in (3.6), we have

$$\begin{aligned} \hat{\zeta}_1(t) &= \mathcal{G}\hat{\zeta}_0(t) = \int_0^\infty \Xi(t, s) e^{-\lambda s} \int_0^\infty G(s, r) \mathcal{V}(r, \hat{\zeta}_0(r)) dr ds \\ &\leq \frac{(1+t^{\sigma-1})}{\mathfrak{A}_1} \int_0^\infty e^{-\lambda s} \int_0^\infty G(s, r) \mathcal{V}(r, \hat{\zeta}_0(r)) dr ds \end{aligned}$$



$$\begin{aligned}
&\leq \frac{1}{2}(1+t^{\sigma-1})\mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \chi_1 \\
&\leq (1+t^{\sigma-1})\mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \chi_1 = \widehat{\zeta}_0(t).
\end{aligned}$$

Taken together with (H2), this yields the inequality $\widehat{\zeta}_2(t) \leq \widehat{\zeta}_1(t)$. Therefore, by the inductive argument, we conclude that $\widehat{\xi}_n(t) \leq \widehat{\xi}_{n-1}(t)$. From $\widehat{\xi}_n = \mathcal{G}(\widehat{\xi}_{n-1})$ and complete continuity of $\mathcal{G}$, we have $\widehat{\xi}_n \rightarrow \widehat{\xi}^*$ as $n \rightarrow \infty$ and $\mathcal{G}(\widehat{\xi}^*) = \widehat{\xi}^*$. Hence, $\widehat{\xi}^*$ is a solution of (1.2).

Define $\widetilde{\zeta}_n = \mathcal{G}(\widetilde{\zeta}_{n-1})$ with $\widetilde{\zeta}_0(t) = 0$. So, for every $t \in (0, +\infty)$, we obtain

$$\widetilde{\zeta}_1(t) = \mathcal{G}(\widetilde{\zeta}_0(t)) \geq 0 = \widetilde{\zeta}_0(t).$$

Accordingly, the inductive argument ensures that $\widetilde{\zeta}_n(t) \geq \widetilde{\zeta}_{n-1}(t)$. Since $\widetilde{\zeta}_n = \mathcal{G}(\widetilde{\zeta}_{n-1})$ and $\mathcal{G}$ is completely continuous, the sequence $\{\widetilde{\zeta}_n\}$ converges to $\widetilde{\zeta}^*$ as $n \rightarrow \infty$. Moreover, $\mathcal{G}(\widetilde{\zeta}^*) = \widetilde{\zeta}^*$, and thus $\widetilde{\zeta}^*$ satisfies Equation (1.2).

We now establish that $\widehat{\xi}^*$ and $\widetilde{\zeta}^*$ serve as the maximal and minimal solutions of problem (1.2). Take an arbitrary solution $\zeta(t)$ of (1.2). Then $\mathcal{G}(\zeta(t)) = \xi(t)$, and for each $t \in (0, +\infty)$ one obtains

$$\widetilde{\zeta}_0(t) = 0 \leq \zeta(t) \leq (1+t^{\sigma-1})\mathfrak{A}_1^{-1} \left(\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right) \chi_1 = \widehat{\zeta}_0(t).$$

Thus, using the monotonicity of $\mathcal{G}$ together with the inductive argument, we deduce that

$$\widetilde{\zeta}_n(t) \leq \zeta(t) \leq \widehat{\xi}_n(t), \quad \forall 0 < t < +\infty.$$

Because $\widehat{\xi}_n \rightarrow \widehat{\xi}^*$ and $\widetilde{\zeta}_n \rightarrow \widetilde{\zeta}^*$ as $n \rightarrow \infty$, the desired assertion follows. $\square$

3.2. Existence result by partially ordered set. In this section, we work within the Banach space X , endowed with the partial order defined as follows:

$$\eta_1, \eta_2 \in X, \quad \eta_1 \leq \eta_2 \Leftrightarrow \eta_1(z) \leq \eta_2(z), \quad \text{for all } z \in (0, +\infty), \quad (3.7)$$

and we define the following metric by

$$d(\eta_1, \eta_2) = \sup_{z \in (0, +\infty)} \frac{|\eta_1(z) - \eta_2(z)|}{1 + z^{\sigma-1}}. \quad (3.8)$$

The next two lemmas will be instrumental in establishing the main theorem of this section.

Lemma 3.4. ([16, Theorem 2]) *Let $(X, \leq)$ be a partially ordered set, and suppose that X is equipped with a metric d making (X, d) a complete metric space. Assume further that X satisfies the property that for any nondecreasing sequence $\{\zeta_n\} \subset X$ converging to some $\zeta \in X$, we have $\zeta_n \leq \zeta$ for every $n \in \mathbb{N}$. Let $\mathcal{G} : X \rightarrow X$ be a mapping that preserves the order, i.e., it is nondecreasing, and suppose that*

$$d(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) \leq d(\zeta_1, \zeta_2) - \mathfrak{E}(d(\zeta_1, \zeta_2)), \quad \text{for all } \zeta_1 \geq \zeta_2, \quad (3.9)$$

where $\mathfrak{E} : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, nondecreasing, and strictly positive on $(0, +\infty)$, with $\mathfrak{E}(0) = 0$. If there exists an element $\zeta_0 \in X$ such that $\zeta_0 \leq \mathcal{G}(\zeta_0)$, then the mapping $\mathcal{G}$ admits at least one fixed point.

Lemma 3.5. ([16]) *Let the assumptions of Lemma 3.4 be satisfied, and suppose furthermore that*

$$\text{for any } \zeta_1, \zeta_2 \in X, \text{ there exists an element } \zeta_3 \in X \text{ that is comparable to both } \zeta_1 \text{ and } \zeta_2.$$

Then the mapping $\mathcal{G}$ possesses a unique fixed point.

We now present our main result in this section:



Theorem 3.6. *Suppose that*

$$0 < \mu \leq \frac{\mathfrak{A}_1}{\frac{1}{12\lambda^4\Theta} + \frac{3}{4\lambda^3} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i}}. \quad (3.10)$$

Furthermore, there exist $\zeta_1, \zeta_2 \in [0, +\infty)$ with $\zeta_1 \geq \zeta_2$ and $0 < t < +\infty$, such that

$$0 \leq \mathcal{V}(t, (1+t^{\sigma-1})\zeta_1) - \mathcal{V}(t, (1+t^{\sigma-1})\zeta_2) \leq \mu \mathfrak{B}(\zeta_1 - \zeta_2), \quad (3.11)$$

where $\mathfrak{B} \in C([0, +\infty); [0, +\infty))$ is a nondecreasing function, and the function $\mathfrak{E}(\zeta) = \zeta - \mathfrak{B}(\zeta)$ fulfills the following conditions:

- (a) $\mathfrak{E} : [0, +\infty) \rightarrow [0, +\infty)$ is nondecreasing,
- (b) $\mathfrak{E}(0) = 0$,
- (c) $\mathfrak{E}(\zeta) > 0$ for all $\zeta \in (0, +\infty)$.

Under these conditions, (1.2) admits a unique solution.

Proof. We first show that $\mathcal{G}$ is a nondecreasing mapping. To see this, from (3.11), for $x \geq y$, we obtain

$$\begin{aligned} \mathcal{G}x(t) &= \int_0^{+\infty} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, x(r)) dr ds \\ &\geq \int_0^{+\infty} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \mathcal{V}(r, y(r)) dr ds = \mathcal{G}y(t). \end{aligned}$$

For $\zeta_1 \geq \zeta_2$, by Lemma 2.8(iii) and in view of (3.8), the above inequality and (3.11), we obtain

$$\begin{aligned} d(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) &= \sup_{t \in (0, +\infty)} \frac{|\mathcal{G}\zeta_1(t) - \mathcal{G}\zeta_2(t)|}{1 + t^{\sigma-1}} \\ &= \sup_{t \in (0, +\infty)} \left\{ \int_0^{+\infty} \frac{1}{1 + t^{\sigma-1}} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \left(\mathcal{V}\left(r, (1 + r^{\sigma-1}) \frac{\zeta_1(r)}{1 + r^{\sigma-1}}\right) \right. \right. \\ &\quad \left. \left. - \mathcal{V}\left(r, (1 + r^{\sigma-1}) \frac{\zeta_2(r)}{1 + r^{\sigma-1}}\right) \right) dr ds \right\} \\ &\leq \sup_{t \in (0, +\infty)} \left\{ \int_0^{+\infty} \frac{1}{1 + t^{\sigma-1}} \Xi(t, s) e^{-\lambda s} \int_0^{+\infty} G(s, r) \mu \mathfrak{B}\left(\frac{\zeta_1(r) - \zeta_2(r)}{1 + r^{\sigma-1}}\right) dr ds \right\}. \end{aligned}$$

Since $\mathfrak{B}$ is nondecreasing, then, for $\zeta_1 \geq \zeta_2$, we get

$$\mathfrak{B}\left(\frac{\zeta_1(r) - \zeta_2(r)}{1 + r^{\sigma-1}}\right) \leq \mathfrak{B}(\|\zeta_1 - \zeta_2\|),$$

and in view of (3.2), the above inequality and (3.10), we obtain

$$\begin{aligned} d(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) &\leq \frac{\mu}{\mathfrak{A}_1} \mathfrak{B}(\|\zeta_1 - \zeta_2\|) \sup_{z \in (0, +\infty)} \left\{ \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda s} G(s, r) dr ds \right\} \\ &\leq \frac{\mu}{\mathfrak{A}_1} \mathfrak{B}(\|\zeta_1 - \zeta_2\|) \left[\frac{1}{12\lambda^4\Theta} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda\mathfrak{R}_i} + \frac{3}{4\lambda^3} \right] \\ &\leq \mathfrak{B}(\|\zeta_1 - \zeta_2\|) = \|\zeta_1 - \zeta_2\| - (\|\zeta_1 - \zeta_2\| - \mathfrak{E}(\|\zeta_1 - \zeta_2\|)). \end{aligned}$$

Assume that $\mathfrak{E}(\zeta) = \zeta - \mathfrak{B}(\zeta)$ and $\mathfrak{E} : [0, +\infty) \rightarrow [0, +\infty)$ is nondecreasing, continuous, satisfies $\mathfrak{E}(0) = 0$, and is strictly positive for $\zeta \in (0, +\infty)$. Hence, for $\zeta_1 \geq \zeta_2$, $d(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) \leq d(\zeta_1, \zeta_2) - \mathfrak{E}(d(\zeta_1, \zeta_2))$. So, by Lemma 3.4, the problem (1.2) has at least one solution. Furthermore, this solution is unique since $(\mathcal{Q}, \leq)$ satisfies Lemma 3.5. Hence, we have the conclusion. $\square$



4. APPLICATION

Example 1. Consider the problem of the form

$$\begin{cases} -(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta(t))'' + \frac{1}{4}(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta(t)) + \mathcal{V}(t, \zeta(t)) = 0, & t \in (0, +\infty), \\ D_{0+}^{\frac{3}{2}} \zeta(0) - 2 \left(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta \right)'(0) = \int_0^{+\infty} e^{-\frac{s}{2}} D_{0+}^{\frac{3}{2}} \zeta(s) ds - \sum_{i=1}^k \delta_i \left(e^{\frac{t}{2}} D_{0+}^{\sigma} \zeta \right)'(\mathfrak{R}_i), \\ \lim_{t \rightarrow +\infty} D_{0+}^{\frac{3}{2}} \zeta(t) = 0, \\ \zeta(0) = 0, \quad D_{0+}^{\frac{1}{2}} \zeta(+\infty) = \int_0^{+\infty} \mathfrak{g}(s) \zeta(s) ds + \delta \zeta(\mathfrak{R}) + \gamma I_{0+}^{\beta} \zeta(\kappa), \end{cases} \quad (4.1)$$

where $\sigma = \frac{3}{2}$, $\lambda = \frac{1}{2}$, $\mathfrak{g}(t) = \frac{1}{100} e^{-t} t^{-\frac{1}{2}}$, $\delta = \frac{1}{5}$, $\mathfrak{R} = e^2$, $\gamma = \frac{1}{30}$, $\beta = \frac{1}{2}$, $\kappa = e^{\frac{1}{7}}$, $\delta_1 = \delta_2 = \frac{1}{2}$, $\mathfrak{R}_1 = \frac{1}{2}$, $\mathfrak{R}_2 = \frac{1}{4}$ and

$$\mathcal{G}(t, x) = \begin{cases} \frac{1}{100(1+\sqrt{t})} \left(e^{-t} + \frac{(1+\sin(t))x}{7} \right), & 0 \leq x < 15, \\ \frac{1}{100(1+\sqrt{t})} \left(e^{-t} + \frac{15(1+\sin(t))}{7} \right), & x \geq 15. \end{cases}$$

Using the given values, we find that

$$\frac{6\lambda - 1}{6\lambda} + \sum_{i=1}^k \frac{\delta_i}{2\lambda} e^{-\lambda \mathfrak{R}_i} 1.497315 > 0,$$

$\Theta 0.6678621$, $\mathfrak{A} = 0.304118 > 0$ and $\mathfrak{A}_1 0.294118 > 0$. Choose $\chi_1 = 0.587268$, we get

$$\mathfrak{A}_1^{-1} \left[\frac{1}{12\lambda^4 \Theta} + \frac{1}{8\lambda^2 \Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right] \chi_1 \simeq 15,$$

and

$$\begin{aligned} & \sup_{t \in (0, +\infty)} \mathcal{V} \left(t, (1+t^{\sigma-1}) \mathfrak{A}_1^{-1} \left[\frac{1}{12\lambda^4 \Theta} + \frac{1}{8\lambda^2 \Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{R}_i} + \frac{3}{4\lambda^3} \right] \chi_1 \right) \\ & \leq \frac{1}{100} \left(1 + \frac{30}{7} \right) 0.052857 \leq \frac{\chi_1}{2} = 0.293634 \end{aligned}$$

for $t \in (0, \infty)$. So, by Theorem 3.3, problem (4.1) admits at least two solutions, denoted by $\widehat{\zeta}^*$ and $\widetilde{\zeta}^*$, which satisfy the bounds $\|\widehat{\zeta}^*\|_X, \|\widetilde{\zeta}^*\|_X \leq 15$.

Example 2. Consider the problem of the form:

$$\begin{cases} -(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta(t))'' + \frac{1}{4}(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta(t)) + \mathcal{V}(t, \zeta(t)) = 0, & t \in (0, +\infty), \\ D_{0+}^{\frac{3}{2}} \zeta(0) - 2 \left(e^{\frac{t}{2}} D_{0+}^{\frac{3}{2}} \zeta \right)'(0) = \int_0^{+\infty} e^{-\frac{s}{2}} D_{0+}^{\frac{3}{2}} \zeta(s) ds - \sum_{i=1}^k \delta_i \left(e^{\frac{t}{2}} D_{0+}^{\sigma} \zeta \right)'(\mathfrak{R}_i), \\ \lim_{t \rightarrow +\infty} D_{0+}^{\frac{3}{2}} \zeta(t) = 0, \\ \zeta(0) = 0, \quad D_{0+}^{\frac{1}{2}} \zeta(+\infty) = \int_0^{+\infty} \mathfrak{g}(s) \zeta(s) ds + \delta \zeta(\mathfrak{R}) + \gamma I_{0+}^{\beta} \zeta(\kappa), \end{cases} \quad (4.2)$$

where $\sigma = \frac{3}{2}$, $\lambda = \frac{1}{2}$, $\mathfrak{g}(t) = \frac{1}{100} e^{-t} t^{-\frac{1}{2}}$, $\delta = \frac{1}{5}$, $\mathfrak{R} = e^2$, $\gamma = \frac{1}{30}$, $\beta = \frac{1}{2}$, $\kappa = e^{\frac{1}{7}}$, $\delta_1 = \delta_2 = \frac{1}{2}$, $\mathfrak{R}_1 = \frac{1}{2}$, $\mathfrak{R}_2 = \frac{1}{4}$ and

$$\mathcal{V}(t, \zeta) = \frac{1}{300} \ln \left(2 + \frac{\zeta}{1+\sqrt{t}} \right).$$

Using the provided data, we determine that

$$\frac{6\lambda - 1}{6\lambda} + \sum_{i=1}^k \frac{\delta_i}{2\lambda} e^{-\lambda \mathfrak{R}_i} 1.497315 > 0,$$



$\Theta 0.6678621$, $\mathfrak{A} = 0.304118 > 0$ and $\mathfrak{A}_1 0.294118 > 0$.

On the other hand, since $\zeta_1 \geq \zeta_2 \geq 0$ then by straightforward calculation, we get

$$\frac{2 + \zeta_2}{2 + \zeta_1} \leq \zeta_2 - \zeta_1 + 1.$$

Consequently, by the fact that the function $\ln$ is strictly increasing function, we obtain

$$\ln\left(\frac{2 + \zeta_2}{2 + \zeta_1}\right) \leq \ln(\zeta_2 - \zeta_1 + 1).$$

Hence, we can choose $\mathfrak{B}(\zeta_2 - \zeta_1) = \ln(\zeta_2 - \zeta_1 + 1)$. Also, for $\zeta_2 \geq \zeta_1$, $t \in (0, +\infty)$ and since $(\zeta) = \ln(\zeta + 2)$ is nondecreasing on $[0, +\infty)$, we obtain

$$0 \leq \left[\frac{1}{300} \ln(2 + \zeta_2) - \frac{1}{300} \ln(2 + \zeta_1) \right],$$

and also

$$\frac{1}{300} [\ln(2 + \zeta_2) - \ln(2 + \zeta_1)] = \frac{1}{300} \ln \frac{2 + \zeta_2}{2 + \zeta_1} \leq \frac{1}{300} \ln(\zeta_2 - \zeta_1 + 1).$$

Hence, we get

$$\frac{\mathfrak{A}_1}{\frac{1}{12\lambda^4\Theta} + \frac{3}{4\lambda^3} + \frac{1}{8\lambda^2\Theta} \sum_{i=1}^k \delta_i e^{-\lambda \mathfrak{A}_i}} 0.0039151 > \frac{1}{300}.$$

So, in light of Theorem 3.6, problem (4.2) possesses a unique solution.

5. CONCLUSION

In this work, we established conditions ensuring the existence of one or two solutions for a newly introduced class of Riemann-Liouville fractional differential equations defined on an unbounded domain and subject to multi-point nonlocal boundary conditions. The approach relies on constructing iterative positive solutions within a partially ordered framework. The findings contribute to the ongoing development of theories involving Riemann-Liouville fractional integrals and derivatives and offer additional insight into such boundary value problems.

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