



Experimental and numerical investigation of time-dependent consolidation coefficient using spectral-finite difference method with Chebyshev cardinal functions

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Abstract

In this study, modifications were made to the conventional oedometer apparatus by installing a pore water pressure sensor at the base of the soil specimen cell. This setup enabled the direct measurement of pore water pressure during consolidation at different times, providing valuable insight into the internal behavior of soil under loading. Based on the recorded data, a time-dependent consolidation coefficient was defined as a function of pore water pressure variation. To accurately represent the pore pressure dissipation trend, a double-exponential function was employed, ensuring high precision in curve fitting. Subsequently, the one-dimensional Terzaghi consolidation equation was solved in a nonlinear framework using the spectral-finite difference method in combination with Chebyshev cardinal functions. The obtained results demonstrated excellent accuracy, surpassing the conventional finite difference method and providing predictions closer to realistic soil behavior. This approach highlights the advantages of integrating advanced mathematical techniques with experimental modifications to enhance the understanding of soil consolidation processes.

Keywords. Consolidation coefficient, Pore water pressure, Nonlinear behavior, Modeling, Chebyshev cardinal functions.

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1. INTRODUCTION

Soil consolidation is one of the fundamental processes in geotechnical engineering, describing the gradual dissipation of excess pore water pressure under loading and the corresponding increase in effective stress. Terzaghi's classical one-dimensional consolidation theory (1943) provided the pioneering framework for analyzing this phenomenon, assuming constant soil properties and linear conditions [26]. Despite its simplicity and wide application, limitations arise in capturing the nonlinear and time-dependent behavior of soils, particularly when soil properties vary with drainage conditions and stress history. To overcome these limitations, researchers have investigated both experimental improvements and advanced numerical methods for consolidation analysis.

The foundation of consolidation theory was established by Terzaghi (1943), who introduced the one-dimensional consolidation equation to describe the dissipation of excess pore water pressure under external loading. His model assumed constant soil properties, small strains, and linearity between void ratio and effective stress. Although revolutionary, Terzaghi's formulation was limited in its ability to capture nonlinear and time-dependent soil behavior, particularly under conditions involving variable drainage, stress-dependent permeability, or heterogeneous layering. Early experimental studies primarily relied on settlement-time curves obtained from oedometer tests, which indirectly estimated the consolidation coefficient and pore pressure dissipation rate. These methods, however, lacked direct measurements of pore water pressure, restricting their ability to validate theoretical models.

Lee et al. [15] expanded on Terzaghi's framework by investigating one-dimensional consolidation in layered soil systems. Their study highlighted how stratification and variations in compressibility influence pore water pressure

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dissipation and settlement behavior. They developed analytical expressions to model different drainage conditions across layers, providing an important step toward representing real field conditions. Similarly, Aboshi and Yoshikuni [1] earlier explored consolidation in multi-layered clay deposits, emphasizing the influence of permeability contrasts. These studies showed that traditional single-layer assumptions underestimate consolidation times and do not reflect the complexity of natural soil deposits. Xie and Schiffman [29] introduced nonlinear formulations that considered the variation of soil compressibility and permeability with effective stress. Their approach allowed for more realistic predictions of pore pressure dissipation under large stress increments, bridging the gap between laboratory results and field observations. Building on this, Yin and Graham [31] applied elastoplastic models to describe soil consolidation, combining constitutive laws with nonlinear consolidation analysis. These studies confirmed that time-dependent soil properties play a significant role in settlement predictions, necessitating numerical methods beyond the classical theory. Liu and Lei [17] investigated one-dimensional consolidation of soils with exponentially time-growing drainage boundaries, reflecting field scenarios such as staged construction or gradual improvement of drainage conditions. Their model provided analytical solutions and highlighted the significance of boundary evolution on consolidation rates. Similarly, Chai et al. [7] studied vacuum preloading and vertical drains, showing how drainage patterns strongly influence pore pressure dissipation. These studies emphasized that consolidation is not only material-dependent but also sensitive to boundary condition evolution, underlining the need for flexible numerical formulations. The limitations of closed-form analytical solutions motivated the adoption of advanced numerical techniques. Schiffman and Stein [22] pioneered the application of finite element methods (FEM) to consolidation problems, enabling more accurate modeling of layered and nonlinear soils. Zienkiewicz and Shiomi [32] further advanced coupled consolidation analysis using FEM, integrating soil mechanics with groundwater flow equations. While FEM proved powerful, its implementation could be computationally demanding. To address this, finite difference methods (FDM) and spectral methods gained attention for their simplicity and efficiency. Among these, spectral approaches employing orthogonal functions, such as Chebyshev polynomials, provided improved stability and convergence compared to standard discretization schemes. Spectral methods gained prominence for solving differential equations with smooth solutions [5]. In consolidation studies, Chebyshev functions have been successfully employed for solving diffusion-type equations. Boyd [4] reviewed Chebyshev spectral methods and emphasized their ability to achieve exponential convergence with relatively few collocation points. More recently, researchers such as Khodaparast et al. [12] applied spectral techniques to soil mechanics, demonstrating higher accuracy than conventional FDM. The use of cardinal Chebyshev functions in solving the nonlinear Terzaghi consolidation equation represents an advancement in this direction, providing accurate solutions with reduced computational effort. Traditional oedometer tests relied solely on settlement data, which indirectly reflected pore pressure dissipation. Recognizing the limitations, researchers developed modified apparatuses to directly measure pore water pressure. Bishop and Henkel [3] were among the early pioneers in pore pressure measurement, developing laboratory techniques for triaxial tests. Later, Indraratna and Redana [11] used pore pressure transducers in large-scale consolidation tests, providing data for validating analytical models. More recent efforts, such as those by Leroueil and Marques [16], emphasized field instrumentation to monitor pore pressures during embankment loading. Despite these advances, the integration of direct pore pressure measurement into small-scale laboratory oedometer devices has been relatively limited, leaving opportunities for innovation. The accurate representation of pore pressure dissipation is essential for defining consolidation coefficients. While logarithmic and hyperbolic functions have been traditionally applied, their accuracy is often insufficient. To overcome this, double-exponential functions have been employed in various diffusion studies, offering improved fitting accuracy. For example, Tang and Onitsuka [25] used exponential models to describe settlement behavior in clays, while more recent applications have extended double-exponential functions to describe nonlinear diffusion in porous media. Applying such functions to pore water pressure-time data from modified oedometer tests ensures reliable estimation of consolidation parameters. Several contemporary studies have emphasized nonlinear and time-dependent approaches. Gong, Yin, and Gao [8] applied viscoplastic models to consolidation, highlighting the creep–consolidation interaction. Similarly, Wang et al. [28] explored consolidation under cyclic loading, showing that time-varying pore pressure coefficients better describe field performance. These works underline the need for advanced experimental and numerical approaches that capture complex soil behavior beyond classical assumptions.



In recent decades, fractional differential equations have attracted considerable interest because of their ability to accurately model nonlinear and complex physical phenomena including damping behavior, electromagnetic effects, acoustic waves, viscoelastic materials, electroanalytical chemistry, neuron dynamics, diffusion processes, and various applications in material science [2, 6]. Recently, there have been multiple efforts to develop both analytical and numerical solutions for fractional problems. Approaches employed in these studies include the finite difference method, the Adomian decomposition method [19, 20], the variational iteration method [21], the homotopy perturbation method [21], the homotopy analysis method [9], along with several other techniques [6, 10, 14, 18, 21, 23, 24, 27, 30].

1.1. Summary of research gaps. From the reviewed studies, three key gaps are evident:

- **Experimental measurement:** While field and large-scale studies have applied pore pressure sensors, their integration into oedometer devices remains limited. Direct measurement of pore pressure in small-scale laboratory tests is underexplored.
- **Curve representation:** Conventional settlement-based methods for estimating consolidation coefficients are less accurate than direct pore pressure fitting, particularly when nonlinear effects are considered.
- **Numerical approaches:** Although FEM and FDM are widely used, spectral methods employing Chebyshev functions remain underutilized in consolidation problems, despite their superior accuracy.

The present study addresses these gaps by modifying an oedometer device to include a pore water pressure sensor, applying a double-exponential function to fit pore pressure dissipation curves, and solving the nonlinear Terzaghi consolidation equation with the spectral-finite difference method using Chebyshev cardinal functions. This combination provides both experimental innovation and numerical advancement, leading to more realistic modeling of consolidation processes.

2. THEORETICAL CONSIDERATIONS

2.1. Pore water pressure dissipation as a diffusion process. The dissipation of excess pore water pressure during soil consolidation can be regarded as a transient diffusion process, governed by Terzaghi's one-dimensional consolidation equation. In its simplest form, this partial differential equation is written as:

$$\frac{\partial u(z, t)}{\partial t} = C_v(u_{base}(t)) \frac{\partial^2 u(z, t)}{\partial z^2}. \quad (2.1)$$

It should be emphasized that the dependence of the consolidation coefficient on pore water pressure adopted in this study is phenomenological in nature. The function $C_v(u_{base}(t))$ is not derived from first-principle constitutive relationships but is instead obtained through experimental fitting of measured pore water pressure dissipation data. This approach allows the effective consolidation behavior observed in laboratory tests to be incorporated into the governing equation, while avoiding assumptions regarding explicit stress-strain-permeability relationships. Accordingly, the resulting formulation should be interpreted as an experimentally informed model rather than a fully constitutive description of soil behavior.

2.2. Double-exponential function for pore pressure variation. The general form of the double-exponential function used to describe pore pressure dissipation can be written as:

$$u_{base}(t) = Ae^{-at} + Be^{-bt}, \quad (2.2)$$

where A and B are weighting coefficients, and a and b are dissipation rate constants. Without loss of generality, it is common to require that $A + B = 1$ in normalized form, ensuring that the initial condition at $t = 0$ yields $u_{base}(0) = 1$. This normalization removes explicit dependence on the initial excess pore pressure and instead focuses on the relative dissipation profile. It should be noted that the condition $A + B = 1$ applies only to the normalized form of the double-exponential function, in which the pore water pressure is scaled by its initial value so that $u_{base}(0) = 1$. In the present study, the coefficients A and B are obtained by fitting the non-normalized experimental pore water pressure data expressed in physical units (kPa). Consequently, in this case A and B represent pressure amplitudes rather than dimensionless weights, and their sum is not required to be unity. The normalized form can be recovered by dividing both coefficients by $A + B$, without affecting the dissipation rates a and b .



In this formulation, the first exponential term Ae^{-at} represents the rapid dissipation stage, which dominates at early times due to higher hydraulic gradients and greater pore connectivity. The second exponential term Be^{-bt} , with a much smaller rate constant ($b < a$), reflects the slower dissipation stage, associated with decreasing hydraulic gradients and more tortuous flow paths. The superposition of these two processes provides a realistic and flexible description of pore pressure decay across the entire consolidation period.

2.3. Derivation of the time-dependent consolidation coefficient. The consolidation coefficient is a fundamental parameter that controls the rate of pore pressure dissipation in Terzaghi's equation. While traditionally considered constant, it is more realistic to treat it as time-dependent, reflecting changes in soil permeability and compressibility during the loading process. The double-exponential function provides a natural way to define such a dynamic coefficient.

Differentiating the pore pressure function with respect to time gives the dissipation rate:

$$\frac{du_{\text{base}}(t)}{dt} = -aAe^{-at} - bBe^{-bt}.$$

The relative dissipation rate is then obtained as:

$$\frac{1}{u_{\text{base}}(t)} \frac{du_{\text{base}}(t)}{dt} = -\frac{aAe^{-at} + bBe^{-bt}}{Ae^{-at} + Be^{-bt}}. \quad (2.3)$$

Equation (2.3) represents the instantaneous relative dissipation rate of excess pore water pressure. The negative sign explicitly reflects the monotonic decay of pore pressure with time, while the ratio of exponential terms captures the combined influence of the fast and slow dissipation mechanisms.

In other words, this expression represents the instantaneous decay rate of pore pressure at time t . Since Terzaghi's equation relates the rate of dissipation to the diffusion operator scaled by the consolidation coefficient, this relative rate can be used as the basis for defining a time-dependent consolidation coefficient:

$$C_v(u_{\text{base}}(t)) \propto -\frac{1}{u_{\text{base}}(t)} \frac{du_{\text{base}}(t)}{dt} = \frac{aAe^{-at} + bBe^{-bt}}{Ae^{-at} + Be^{-bt}}. \quad (2.4)$$

Here, the time-dependent consolidation coefficient $C_v(u_{\text{base}}(t))$ captures the combined influence of the rapid and slow pore pressure dissipation mechanisms. The parameters A , B , a , and b are determined by fitting a double-exponential function to the experimentally measured pore water pressure at the base of the soil specimen. Once identified, these parameters allow direct evaluation of $C_v(u_{\text{base}}(t))$ at any stage of the consolidation process.

Equation (2.4) represents the relative rate of pore pressure dissipation and is therefore proportional to the consolidation coefficient appearing in Terzaghi's one-dimensional consolidation equation. To obtain a physically meaningful and dimensionally consistent expression for C_v , this proportionality must be scaled appropriately. The required scaling factor is derived from the fundamental eigenvalue of Terzaghi's consolidation equation under one-way drainage conditions.

In the present study, one-way drainage is assumed, with a drained boundary at the top surface and an impermeable boundary at the base of the soil specimen. Under these boundary conditions, the fundamental eigenvalue is given by:

$$\lambda_1 = \frac{4\pi^2}{H^2},$$

and its incorporation transforms the proportional relationship in Equation (2.4) into an equality, ensuring consistency between the experimentally derived dissipation rate and the governing diffusion equation.

$$C_v(u_{\text{base}}(t)) = -\frac{4H^2}{\pi^2} \left(\frac{1}{u(t)} \frac{du(t)}{dt} \right).$$

Substituting the double-exponential form $u_{\text{base}}(t) = Ae^{-at} + Be^{-bt}$ yields:

$$C_v(u_{\text{base}}(t)) = \frac{4H^2}{\pi^2} \cdot \frac{aAe^{-at} + bBe^{-bt}}{Ae^{-at} + Be^{-bt}}. \quad (2.5)$$



This expression provides a direct way to compute the time-dependent consolidation coefficient from the pore-water pressure dissipation function fitted to the experimental data.

It should be noted that the consolidation coefficient defined in this study represents an effective, spatially averaged parameter for the soil specimen. Although pore water pressure is measured at the base of the sample, the formulation assumes spatial homogeneity of soil properties, as is customary in standard oedometer analysis. Consequently, $C_v(t)$ should be interpreted as an equivalent consolidation coefficient that captures the overall time-dependent response of the specimen, rather than a strictly local material property. Possible spatial variations of permeability and compressibility are therefore not explicitly modeled and are acknowledged as a limitation of the present approach.

2.4. Interpretation and significance. This formulation highlights several theoretical advantages:

- **Multi-stage representation of dissipation:** The double-exponential model explicitly accounts for the presence of both rapid and slow pore pressure decay mechanisms. This dual-process interpretation aligns well with experimental observations, where initial dissipation is significantly faster than long-term behavior.
- **Elimination of initial pressure dependency:** By normalizing the function ($u_{\text{base}}(0) = 1$), the analysis focuses on relative dissipation rather than absolute magnitudes of pore pressure. This improves the generality of the model and makes it directly applicable across different loading conditions.
- **Dynamic consolidation coefficient:** Unlike the classical constant C_v , the time-dependent formulation captures soil nonlinearity, evolving drainage paths, and changing stress states. This results in predictions that are closer to actual soil behavior, particularly in nonlinear or layered systems.
- **Analytical convenience:** The double-exponential function is simple yet flexible. Its derivative yields a closed-form expression for $C_v(u_{\text{base}}(t))$, which can be easily incorporated into numerical schemes such as the spectral-finite difference method with Chebyshev functions.

By employing a double-exponential representation of pore water pressure dissipation, the consolidation process can be described in a manner that is both mathematically tractable and physically meaningful. The definition of a time-dependent consolidation coefficient derived from this function bridges the gap between theoretical modeling and experimental observations. This approach forms the foundation for the nonlinear solution of Terzaghi’s equation using advanced numerical methods, ensuring both higher accuracy and stronger correlation with real soil behavior.

3. TEST PROGRAM

Experimental tests were conducted to assess the dissipation of pore water pressure during consolidation using a modified one-dimensional consolidation apparatus. To monitor the excess pore water pressure throughout the consolidation process, a highly sensitive pore water pressure sensor was installed at the base of the test cell. Special care was taken in selecting the sensor to minimize inconsistencies in results and reduce measurement errors.

Figure 1 illustrates the apparatus utilized in these experiments. The soil samples used in this study were obtained from the Baghmeisheh area of Tabriz city and were classified as clay with a low plasticity index (CL). Figure 2 illustrates the location of the sampling site along with the samples collected. The characteristics of these samples, including specific gravity, plasticity limit, and liquid limit, are detailed in Table 1.

TABLE 1. Physical characteristics of the samples.

Sample Number	Grains Distribution percent			Atterberg Limits			Density (gr/cm ³)		Soil Type
	Sand	Silt	Clay	LL(%)	PL(%)	PI(%)	Wet	Dry	
1	0.83	52.11	47.06	39	24	15	1.84	1.43	CL
2	0.90	54.78	44.32	40	24.5	15.5	1.84	1.44	CL
3	0.81	53.72	45.47	38	24	14	1.85	1.41	CL

A sensitive pore water pressure sensor (20 mA output, 0.25% accuracy, and 0.5 ms response time) was installed at the base of the test cell, enabling it to effectively measure rapid changes in pore water pressure. This sensor is connected to a data acquisition system, allowing for data storage during loading. Soil samples were positioned within a consolidation ring measuring 75 mm in diameter and 30 mm in height. To minimize friction against the ring





FIGURE 1. Image of the modified test device.

wall, silicone grease was applied. The samples used were intact and selected from one-third of the distance between the sampling points to ensure minimal disturbance. Porous rock (as depicted in Figure 3) and the top cap of the sample were placed, and loads of 50, 100, and 200 kPa were subsequently applied. The deformation and excess pore water pressure were measured simultaneously over time during the consolidation process. The drainage of the sample occurred through a single path from the top, which could be regulated using a valve installed on the top cap.

Figure 4 illustrates the changes in pore water pressure at the bottom of the samples.

Using the results of the graph 4, the values of the coefficients of Equation (2.2) are obtained, which are shown in Table 2.

TABLE 2. Coefficients of the double-exponential function obtained from fitting the non-normalized experimental pore water pressure data (in kPa) for different samples.

Sample	A	B	a	b
1	35	14	0.02	0.005
2	6.7	93.3	0.44	0.0062
3	19.5	178.5	0.47	0.0061

For practical implementation in the numerical scheme, the time-dependent consolidation coefficient given by Equation (2.5) is further approximated by a simplified exponential form. This expression is empirical and obtained through curve fitting of the values calculated from Equation (2.5), rather than being analytically equivalent to it. The approximation provides a computationally efficient representation of the experimentally derived consolidation coefficient while preserving its essential time-dependent characteristics.

$$C_v(u_{\text{base}}(t)) \approx \alpha + \beta e^{-\kappa t}. \quad (3.1)$$

In this study, $H = 0.03$ m and the α , β , and κ coefficients obtained from fitting the consolidation coefficient-time variation curve are shown in Table 3. R^2 represents the correlation coefficient, and values close to 1 indicate good correlation.





FIGURE 2. Location of the research borehole (Fereshte North Alley, Tabriz, Iran), and image of samples.

TABLE 3. Simplification parameter values for different examples.

Sample	α	β	κ	R^2
1	1.78×10^{-6}	4.43×10^{-6}	0.008	0.991
2	1.16×10^{-6}	2.83×10^{-5}	0.012	0.999
3	6.28×10^{-6}	2.08×10^{-5}	0.017	0.999

Figure 5 shows the time-varying consolidation coefficient plots for different samples. The values for the plots $C_v(t)$ are based on the original functions (Equation (2.5)), and the fitted plots are based on the simplified functions (Equation (3.1)). These plots show a high correlation between the original functions and the simplified functions. To compare the research results with classical methods, the consolidation coefficient diagrams obtained from the Taylor and Casagrande methods are also shown in Figure 5.

The graphs in Figure 5 indicate that the classical linear Taylor and Casagrande methods show the value of the consolidation coefficient at intermediate and final times unrealistically higher.



FIGURE 3. Positioning porous rock on top of the sample.

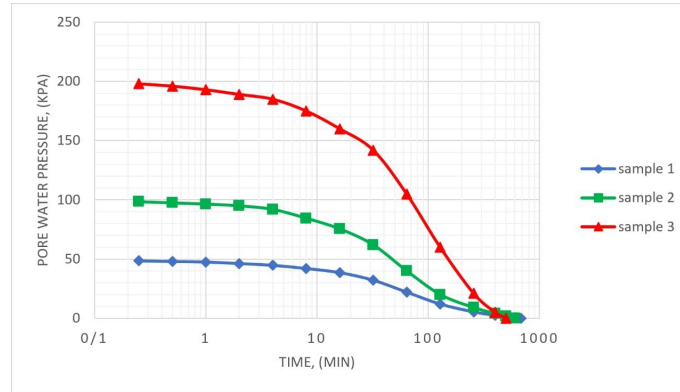


FIGURE 4. Typical experimental variation of pore water pressure with time at the base of samples.

4. SOLVING THE GOVERNING EQUATION

By substituting Equation (3.1) into Equation (2.1), we have:

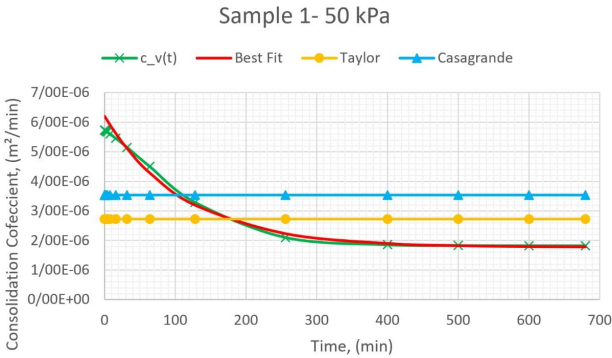
$$\frac{\partial u(z, t)}{\partial t} = (\alpha + \beta e^{-\kappa t}) \frac{\partial^2 u(z, t)}{\partial z^2}. \quad (4.1)$$

The boundary and initial conditions in the one-way drainage case are defined as follows:

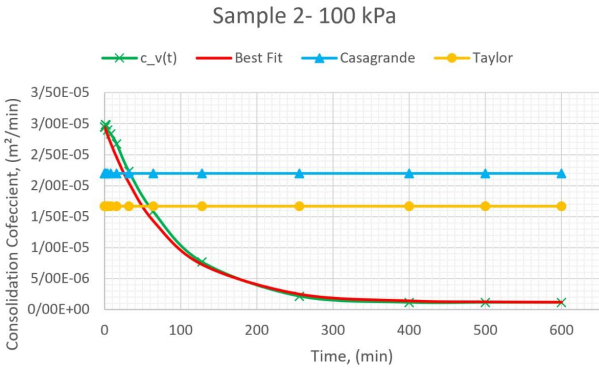
$$u(z, 0) = u_0 = \sigma_{\text{Total}}, \quad (4.2)$$

$$u(0, t) = 0, \quad \left. \frac{\partial u}{\partial z} \right|_{z=H} = 0, \quad (4.3)$$

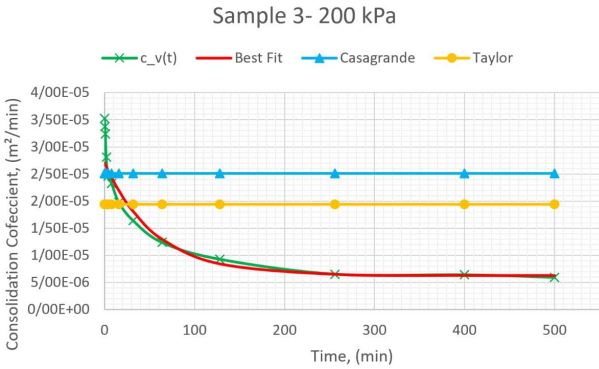




(A) Soil Sample 1 ($\alpha = 1.78 \times 10^{-6}$, $\beta = 4.43 \times 10^{-6}$, $\kappa = 0.008$) – applied pressure 50 kPa.



(B) Soil Sample 2 ($\alpha = 1.16 \times 10^{-6}$, $\beta = 2.83 \times 10^{-5}$, $\kappa = 0.012$) – applied pressure 100 kPa.



(C) Soil Sample 3 ($\alpha = 6.28 \times 10^{-6}$, $\beta = 2.08 \times 10^{-5}$, $\kappa = 0.017$) – applied pressure 200 kPa.

FIGURE 5. Fitting curves of consolidation coefficient in samples.



where σ_{Total} represents the initial excess pore water pressure generated by the applied vertical load, consistent with classical one-dimensional consolidation theory.

4.1. Chebyshev Cardinal Functions. The N 'th order Chebyshev cardinal functions over the range of $[-1, 1]$ are formally defined as per [2]:

$$C_j(x) = \frac{T_{N+1}(x)}{T_{N+1,x}(x_j)(x - x_j)}, \quad j = 1, 2, \dots, N+1,$$

where $T_{N+1}(x)$ is the second kind Chebyshev function of order $N+1$ in domain of $[-1, 1]$, and defined as:

$$T_{N+1}(x) = \cos((N+1) \arccos(x)).$$

Here the subscript x denote x -differentiation and $x_j, j = 1, 2, \dots, N+1$ are the zeros of $T_{N+1}(x)$ defined by $\cos\left(\frac{2j-1}{2N+2}\right)$, $j = 1, 2, \dots, N+1$.

By introducing the transformed spatial variable $\xi = H(x+1)$, where $\xi \in [0, H]$, and employing Chebyshev cardinal functions over this interval, it is possible to approximate any spatial function $g(\xi)$ on the range $[0, H]$ as:

$$g(\xi) \approx \sum_{j=1}^{N+1} g(\xi_j) C_j(\xi) = G^T \psi_N(\xi), \quad (4.4)$$

where $\xi_j, j = 1, 2, \dots, N+1$ are the shifted points of $x_j, j = 1, 2, \dots, N+1$ by transform $\xi = H(x+1)$ (here we choose ξ_j so that, $\xi_1 < \xi_2 < \dots < \xi_{N+1}$),

$$G = [g(\xi_1), g(\xi_2), \dots, g(\xi_{N+1})]^T,$$

and

$$\psi_N(\xi) = [C_1(\xi), C_2(\xi), \dots, C_{N+1}(\xi)]^T. \quad (4.5)$$

4.2. The Operational Matrix of Derivative. The derivative of vectors ψ_N in Equation (4.5) can be represented as

$$\psi'_N = D\psi_N. \quad (4.6)$$

Here, D denotes an operational matrix of size $(N+1) \times (N+1)$ for computing derivatives with respect to Chebyshev cardinal functions. In [2,22] it shown that the entries of matrix D can be found as:

$$D = \begin{bmatrix} C'_1(\xi_1) & \cdots & C'_1(\xi_{N+1}) \\ \vdots & & \vdots \\ C'_{N+1}(\xi_1) & \cdots & C'_{N+1}(\xi_{N+1}) \end{bmatrix},$$

where

$$C'_j(\xi_k) = \begin{cases} C'_j(\xi_j) = \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{1}{\xi_j - \xi_i}, & k = j, \\ C'_j(\xi_k) = \frac{\beta}{T_{N+1,\xi}(\xi_j)} \prod_{\substack{l=1 \\ l \neq k,j}}^{N+1} (\xi_k - \xi_l), & k \neq j. \end{cases}$$

5. DISCUSSION

The governing equation considered in this study incorporates time-dependent consolidation behavior through an experimentally derived consolidation coefficient. Although the dependence of C_v originates from pore water pressure dissipation, the numerical formulation treats this dependence as an explicitly time-varying coefficient determined prior to solution. As a result, the problem reduces to a linear diffusion equation with variable (time-dependent) coefficients, rather than a fully nonlinear diffusion equation in which the coefficient depends directly on the evolving pore pressure field. This modeling choice allows experimental observations to be incorporated efficiently while maintaining numerical stability and accuracy.



5.1. Description of numerical method utilizing Chebyshev cardinal functions with Crank–Nicolson time stepping. This section focuses on solving Equation (4.1) accompanied by the boundary and initial conditions specified in (4.2), (4.3) using Chebyshev cardinal functions over the interval $[0, H]$.

By integrating Equation (4.1) with respect to t across the interval $[t, t + \delta t]$, the following expression is derived:

$$u(z, t + \delta t) - u(z, t) = \int_t^{t+\delta t} (\alpha + \beta e^{-\kappa t}) u_{zz}(z, t) dt. \quad (5.1)$$

By applying the trapezoidal integration method, we can approximate Eq. (5.1) as:

$$u(z, t + \delta t) - u(z, t) = \frac{\delta t}{2} \left\{ (\alpha + \beta e^{-\kappa t}) u_{zz}(z, t) + (\alpha + \beta e^{-\kappa(t+\delta t)}) u_{zz}(z, t + \delta t) \right\},$$

where δt is the time step size.

Using the notation $u^m = u(z, t^m)$ where $t^m = t^{m-1} + \delta t$, we obtain

$$u^{m+1} - u^m = \frac{\delta t}{2} \left\{ (\alpha + \beta e^{-\kappa t^m}) \nabla^2 u^m + (\alpha + \beta e^{-\kappa t^{m+1}}) \nabla^2 u^{m+1} \right\}. \quad (5.2)$$

The operator ∇ signifies the gradient.

By utilizing Eq. (4.4), the function $u(z, t^m)$ can be approximated as

$$u^m(z) = \sum_{j=1}^{N+1} u_j^m C_j(z) = U_m^T \psi_N(z), \quad (5.3)$$

where $U_m = [u_1^m, u_2^m, \dots, u_{N+1}^m]^T$, and ψ_N is defined as (4.4).

Also using (4.6) we can write

$$\nabla^2 u^m = U_m^T \frac{d^2}{dz^2} \psi_N(z) = U_m^T D \frac{d}{dz} \psi_N(z) = U_m^T D^2 \psi_N(z), \quad (5.4)$$

Replacing Eqs. (5.3) and (5.4) in Eq. (5.2) we get

$$U_{m+1}^T \psi_N(z) - U_m^T \psi_N(z) = \frac{\delta t}{2} \left\{ (\alpha + \beta e^{-\kappa t^m}) U_m^T D^2 \psi_N(z) + (\alpha + \beta e^{-\kappa t^{m+1}}) U_{m+1}^T D^2 \psi_N(z) \right\},$$

or

$$U_{m+1}^T \psi_N(z) \left\{ 1 - \frac{\delta t}{2} D^2 (\alpha + \beta e^{-\kappa t^{m+1}}) \right\} = U_m^T \psi_N(z) \left\{ 1 + \frac{\delta t}{2} D^2 (\alpha + \beta e^{-\kappa t^m}) \right\}. \quad (5.5)$$

Using Eq. (5.3) in the boundary conditions (4.3) we have

$$U_m^T \psi_N(0) = 0, \quad U_m^T D \psi_N(H) = 0. \quad (5.6)$$

Applying Eq. (5.5) at $N - 1$ points t_j , $j = 2, 3, \dots, N$, and using the cardinal property $\psi_N(\xi_j) = e_j$, where e_j represents the j 'th column of the $N \times N$ identity matrix, we obtain:

$$U_{m+1}^T e_j \left(1 - \frac{\delta t}{2} (\alpha + \beta e^{-\kappa t^{m+1}}) D^2 \right) = U_m^T e_j \left(1 + \frac{\delta t}{2} (\alpha + \beta e^{-\kappa t^m}) D^2 \right), \quad j = 2, 3, \dots, N - 1. \quad (5.7)$$

Eq. (5.7) can be rewritten as

$$u_j^{m+1} - \frac{\delta t}{2} (\alpha + \beta e^{-\kappa t^{m+1}}) U_{m+1}^T D_j^2 = u_j^m + \frac{\delta t}{2} (\alpha + \beta e^{-\kappa t^m}) U_m^T D_j^2, \quad j = 2, 3, \dots, N, \quad (5.8)$$

where D_j^2 denotes the j 'th column of matrix D^2 .

Eq. (5.8), along with Eq. (5.6) and using the initial condition (4.2) as the starting point, forms a system of algebraic equations with $N + 1$ equations and unknowns. This system can be solved to determine the vector U_m at each time $m\delta t$, $m = 0, 1, \dots$. Consequently, the unknown functions $u^m(z)$ can be determined using the relation (5.7).



6. RESULTS

6.1. Pore water pressure variation. As the consolidation process starts, the pore water pressure gradually decreases due to drainage. Under one-path drainage conditions, changes in pore water pressure can be observed at various depths of the sample. Figure 6 illustrates these changes in Sample 1 at different time intervals.

6.2. Time–settlement relationship. In this study, we explore the relationship between the average degree of consolidation and real time, eliminating the necessity to calculate the consolidation coefficient and time factor. The graphs presented in Figure 7 illustrate the results obtained from our proposed method and are compared to both experimental findings and the conventional finite-difference method.

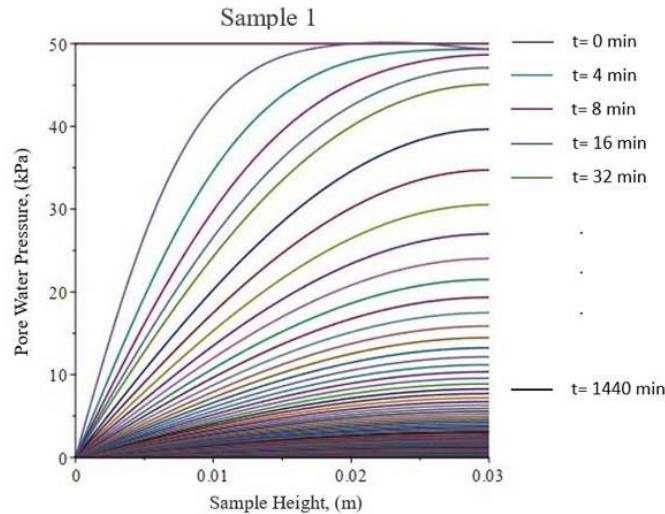
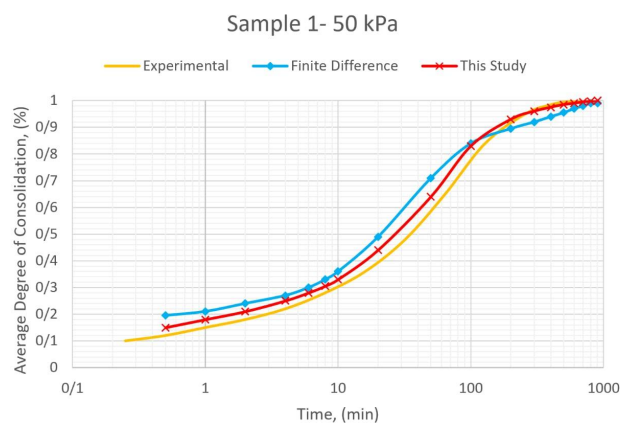


FIGURE 6. Typical variations in pore water pressure at the sample height over time.

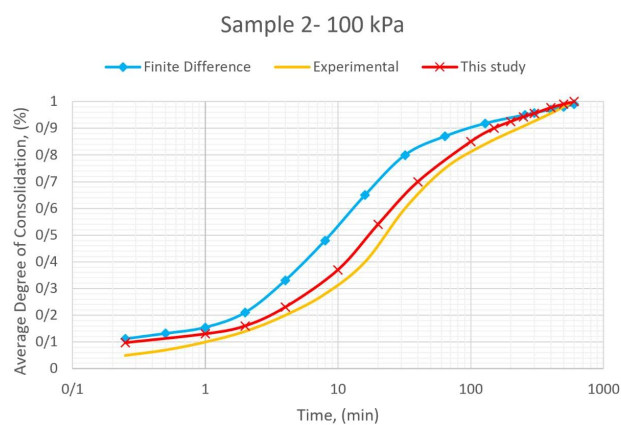
7. CONCLUSION

This study presents a comprehensive investigation of one-dimensional soil consolidation by integrating experimental measurements with advanced numerical modeling. Pore water pressure at the bottom of the sample was recorded at various times using a high-precision sensor, providing accurate temporal data for the dissipation process. By fitting these measurements with a double exponential function, a realistic and continuous relationship between pore water pressure and time was established, allowing for the definition of a new time-dependent consolidation coefficient that better reflects the true behavior of the soil. The introduction of this variable consolidation coefficient into Terzaghi's one-dimensional consolidation equation represents a significant improvement over the classical constant-coefficient model. This approach captures the temporal evolution of soil compressibility and permeability, which are often overlooked in traditional analyses. By accounting for these time-dependent properties, the model can more accurately predict the rate and extent of consolidation under different loading and drainage conditions, making it highly relevant for practical geotechnical engineering applications.

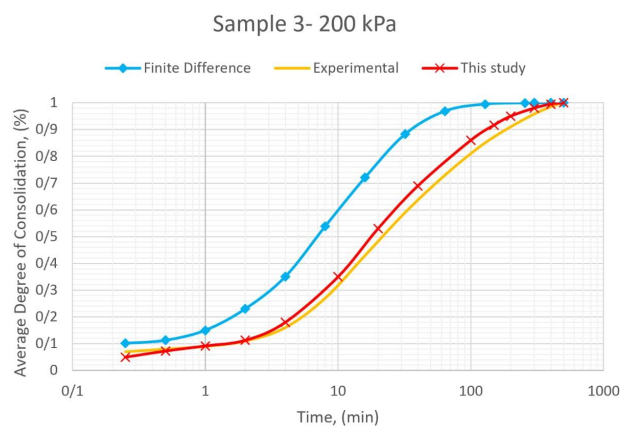
Numerical solutions were obtained using a combination of spectral-finite difference methods and Chebyshev cardinal functions. The results demonstrate excellent agreement between the two approaches, confirming the accuracy and efficiency of the spectral-based method in capturing the complex consolidation behavior. The high precision of the Chebyshev cardinal function method, coupled with the robustness of finite difference discretization, highlights the potential of these techniques for reliable and computationally efficient modeling of time-dependent consolidation phenomena. The comparison between experimental data and numerical predictions shows that the proposed methodology successfully integrates measured soil behavior with advanced analytical techniques. This combination ensures



(A) Sample 1 ($\alpha = 1.78 \times 10^{-6}$, $\beta = 4.43 \times 10^{-6}$, $\kappa = 0.008$), applied pressure 50 kPa.



(B) Sample 2 ($\alpha = 1.16 \times 10^{-6}$, $\beta = 2.83 \times 10^{-5}$, $\kappa = 0.012$), applied pressure 100 kPa.



(C) Sample 3 ($\alpha = 6.28 \times 10^{-6}$, $\beta = 2.08 \times 10^{-5}$, $\kappa = 0.017$), applied pressure 200 kPa.

FIGURE 7. Comparison of the average degree of consolidation over real time for different samples.



that both the physical characteristics of the soil and the mathematical rigor of the solution are respected. Such an approach not only enhances the predictive capability of consolidation models but also provides a framework for future studies where accurate representation of time-dependent soil properties is critical.

In summary, the study demonstrates the effectiveness of using experimentally derived, time-dependent consolidation coefficients in conjunction with spectral-finite difference and Chebyshev cardinal function methods for solving Terzaghi's equation. The methodology provides a high-accuracy, practical tool for predicting consolidation in engineering projects, with implications for foundation design, settlement estimation, and soil behavior analysis. Future work may extend this approach to layered soils, multi-dimensional problems, or nonlinear consolidation conditions to further improve predictive reliability and applicability in complex geotechnical scenarios.

The key advantages of this method include its high accuracy and its relevance to real-world applications. Based on the results of this study, the following conclusions can be made:

- A variable consolidation coefficient was defined using a double exponential function.
- Accurate determination of time steps in the numerical results is of utmost importance.
- Inadequate adjustment of these time steps can result in method instability. To mitigate this issue, experimental results were utilized to identify appropriate time steps.
- The proposed semi-analytical method, in conjunction with the combined numerical approach employed in this study, exhibits a high degree of accuracy. Consequently, it is capable of effectively simulating the behavior of discontinuous environments under practical conditions.
- The semi-analytical solution is a highly effective technique for tackling complex consolidation problems.

Despite the promising results, this study has several limitations that warrant consideration. The analysis was restricted to one-dimensional consolidation under idealized boundary conditions, which may not fully capture the complexity of field-scale soil behavior, including heterogeneity, anisotropy, and multi-directional drainage effects. Additionally, the experimental dataset was limited to a single soil type and sample height, which may influence the generalizability of the derived time-dependent consolidation coefficient. For future research, it is recommended to extend the methodology to two- and three-dimensional consolidation scenarios, investigate layered and heterogeneous soils, and incorporate nonlinear soil behavior under variable loading conditions. Further studies could also explore the integration of field monitoring data with numerical models to validate and refine time-dependent consolidation coefficients, enhancing the predictive capability and practical applicability of the proposed approach in real-world geotechnical projects.

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