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Analytical approach to extended beta and hypergeometric functions and their applications

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Abstract

In this paper, we introduce a new generalization of beta and Gauss hypergeometric functions in terms of $k > 0$ parameter, also investigate numerical and graphical representations for different values of k-parameter. For numerical verification of the analytical results, we implemented a 20-point Gauss-Legendre quadrature scheme in Python. Particularly, the k-parameter has a suppressive effect due to its role in scaling the exponent and argument of the hypergeometric function. This makes the function highly tunable, which is important in applications requiring shape control and heavy-tailed behavior modeling. The plots illustrate these relationships vividly, supporting the mathematical formulation through visual interpretation. In this work, we examine several fundamental properties of the newly introduced generalized extended beta k-function, including its integral representations, Mellin transforms, inverse Mellin transforms, and associated summation formulas. Furthermore, we establish Laplace transforms, the Riemann–Liouville fractional integral, and a variety of integrals involving the Gauss hypergeometric k-function, along with results concerning the derivatives of the generalized extended Gauss hypergeometric k-function.

Keywords. Gamma k-function, Beta k-function, Hypergeometric k-functions, Extended gamma k-function, Extended beta k-function, Mellin transform, Laplace transform.

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1. Introduction and preliminaries

The generalized extended beta and Gauss hypergeometric k-functions are extension of classical special functions, often used in fractional calculus, statistical distributions and mathematical physics. The k-parameter generalizations provide more control and adaptability in analytical and numerical modelling. Such generalizations has valuable role in solving fractional differential equations, series expansions and asymptotic analysis. The generalized hypergeometric and beta k-functions have emerged as significant tools in the evaluation of Feynman integrals. The classical Gauss hypergeometric function is well known for its extensive applications in diverse fields, including probability theory, statistics, approximation theory, the analytic study of differential equations, and theoretical physics. Motivated by these wide-ranging applications, several scholars have developed generalizations, extensions, and integral representations of such functions, both for the case $k > 0$ and for formulations independent of the k-parameter(see[5, 8, 12, 14, 16, 17, 20–22]). Abdalla *et al.* [1], investigated k-hypergeometric function and associated fractional integral. Chand *et al.* [9], investigated a generalized beta function and its applications.

The generalized k-functions can also be used to construct analytical solutions to fractional differential equations (FDEs), especially in problems modeled by Riemann-Liouville or Caputo derivatives where memory effects are present.

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The k-beta and k-gamma functions can be used to define new generalized probability distributions, which are useful in reliability theory, life data analysis, and modeling skewed data. The k-parameter provides flexibility in approximation schemes involving series expansions of functions. These generalized forms may be used in orthogonal polynomials, Chebyshev or Jacobi approximations. The extended k-hypergeometric functions provide exact or approximate solutions to second-order linear differential equations that appear in mathematical physics.

In quantum mechanics, extended special functions often appear in the solution of Schrödinger equations with complex potentials. The k-parameter allows tuning solutions under non-classical potentials. The generalized k-functions model anomalous diffusion, where traditional Gaussian assumptions break down. They help describe systems with nonlinear or memory-dependent thermal conductivity. Integral representations using k-parameters can be applied in solving Volterra-type integral equations, common in population dynamics. In field theory, generalized beta and gamma functions are useful in regularizing divergences in Feynman integrals. k-parameter can provide new tools for asymptotic behavior of integrals. By adapting k-extended beta distributions, one can develop more flexible models of biological growth, infection rates, or immune response, especially in systems with fractional kinetics.

In [13], the researchers introduced the following functions

$$\Gamma_k(w) = \int_0^\infty v^{w-1} e^{-\frac{v^k}{k}} \, dv. \tag{1.1}$$

If $\Re(l_1) > 0$, $\Re(l_2) > 0$, and $k > 0$, the beta k-function can be expressed as follows:

$$\beta_k(l_1, l_2) = \frac{\Gamma_k(l_1)\,\Gamma_k(l_2)}{\Gamma_k(l_1 + l_2)} \tag{1.2}$$

$$= \frac{1}{k}\int_0^1 t^{\frac{l_1}{k}-1}(1-t)^{\frac{l_2}{k}-1}\, dt, \tag{1.3}$$

$$(\tau)_{u,k} = \frac{\Gamma_k(\tau + uk)}{\Gamma_k(\tau)} \quad (\tau \in \mathbb{C} \setminus \{0\}). \tag{1.4}$$

The Gauss hypergeometric k-function is given by

$$_2F_{1,k}(\lambda_1, \lambda_2; \lambda_3; z) = \sum_{n=0}^{\infty} \frac{(\lambda_1)_{n,k}\,(\lambda_2)_{n,k}}{(\lambda_3)_{n,k}} \frac{z^n}{n!}, \tag{1.5}$$

where $\lambda_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $|z| < 1$, and $k \in \mathbb{R}^+$.

Moreover, in [6], Arshad *et al.* proposed the generalized k-gamma function, which is defined as

$$\Gamma_{w,k}^{(p,q)}(v) = \int_0^\infty t^{v-1}\, {}_1F_{1,k}\left(p; q; -\frac{t^k}{k} - \frac{w^k}{kt^k}\right) dt, \tag{1.6}$$

where $\Re(w) > 0$, $\Re(q) > 0$, $\Re(v) > 0$, $\Re(p) > 0$, and $k > 0$.

If $\Re(\delta_1) > 0$, $\Re(\delta_2) > 0$, $\Re(\xi) \geq 0$, $\Re(p) \geq 0$, $\Re(q) \geq 0$, $k > 0$, then generalized extended beta k-function in [6], defined as follow:

$$\beta_{\xi,k}^{(\delta_1,\delta_2)}(p,q) = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1}\, {}_1F_{1,k}(\delta_1; \delta_2; \frac{-\xi^k}{ks(1-s)}) ds. \tag{1.7}$$

In [3], (α, k)-beta function defined as; if $\Re(p), \Re(q) > 0$, $s \in C$, $Q \geq 0$, then

$$\beta_{k,\varpi_1,p_2}^{\alpha,Q}(p,q) = \frac{1}{\alpha k}\int_0^1 s^{\frac{p}{\alpha k}-1}(1-s)^{\frac{q}{\alpha k}-1} E_{(k,\varpi_1,p_2)}\left(\frac{-Q^k}{ks(1-s)}\right) d_\alpha s. \tag{1.8}$$

In [3], (α, k)- hypergeometric function is defined as follows:

$$F^{\alpha,Q}_{k,\varpi_1,p_2}(v_1, v_2, v_3; x^\alpha) = \sum_{m=0}^{\infty} \frac{(v_1)_{m,k}\beta^{\alpha,Q}_{k,\varpi_1,p_2}(v_2 + mk\alpha, v_3 - v_2)}{\beta^\alpha_k(v_2, v_3 - v_2)} \frac{x^{\alpha m}}{m!}, \tag{1.9}$$

where $\Re(v_1, \Re(v_2), \Re(v_3 > 0)$, $\alpha \in (0, 1)$, $k > 0, |x^\alpha| < 1, Q \geq 0$.

By putting Equation (1.8) into (1.9), we get these integral representations:

$$F^{\alpha,Q}_{k,\varpi_1,p_2}(v_1, v_2, v_3; x^\alpha) = \frac{1}{\alpha k \beta^\alpha_k(v_2, v_3 - v_2)} \int_0^1 s^{\frac{v_2}{\alpha k}-1}(1 - s)^{\frac{v_3-v_2}{\alpha k}-1}(1 - kx^\alpha s)^{\frac{-v_1}{k}} \times E_{(k,\varpi_1,p_2)}\left(\frac{-Q^k}{ks(1 - s)}\right) d_\alpha s, \tag{1.10}$$

where $\Re(v_1) > 0, \Re(v_2), \Re(v_3) > 0$, $\alpha \in (0, 1)$, $k > 0, |x^\alpha| < 1, Q \geq 0$. Ghayasuddin *et al.*[15], presented the concept of generalized extended beta functions as follow:

$$\beta^{(\delta_1,\delta_2,l_1,l_2)}_\xi(u, v) = \int_0^1 z^{u-1}(1 - z)^{v-1} {}_1F_1(\delta_1; \delta_2; \frac{-\xi}{z^{l_1}(1 - z)^{l_2}})dz. \tag{1.11}$$

A new extension of the Gauss hypergeometric function was described by Agarwal *et al.* [2], in the following manner:

$$F^{(\delta_1,\delta_2,l_1:l_2)}_\xi(\sigma_1, \sigma_2, \sigma_3; t) = \sum_{m=0}^{\infty} (\sigma_1)_m \frac{\beta^{(\delta_1,\delta_2,l_1,l_2)}_\xi(\sigma_2 + m, \sigma_3 - \sigma_2)}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \frac{t^m}{m!}, \tag{1.12}$$

where $|t| < 1, \min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \Re(\sigma_3) > \Re(\sigma_2) > 0, \Re(\sigma_1) > 0, \tau \geq 0, l_1, l_2 \geq 1$.

Utilizing Equation (1.11), in Equation(1.12), the generalized extended hypergeometric function can be expressed in the following integral form:

$$F^{(\delta_1,\delta_2,l_1:l_2)}_\xi(\sigma_1, \sigma_2, \sigma_3; t) = \frac{1}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-1}(1 - t)^{\sigma_3-\sigma_2-1} \times (1 - zt)^{-\sigma_1} {}_1F_1(\delta_1; \delta_2; \frac{-\xi}{t^{l_1}(1 - t)^{l_2}})dt. \tag{1.13}$$

Remark 1.1. Several known extensions of the hypergeometric and beta functions can be recovered from (1.11)–(1.13) through suitable parameter choices:

- Setting $l_1 = l_2$ yields the extended hypergeometric and beta functions, together with their integral representations, as proposed by Parmar [23].
- Choosing $\sigma_1 = \sigma_2$ and $l_1 = l_2 = 1$ leads to the extended hypergeometric and beta functions, (see[10, 11]).
- Imposing $l_1 = l_2 = 1$ gives the extensions of the hypergeometric and beta functions, together with their integral forms, defined by Altin *et al.* [4].

In 1897, Mellin obtained the Mellin and laplace transform for some $r > 0$, as (see [8])

$$M[f(s); r] = F(r) = \int_0^\infty s^{r-1}f(s)ds, \tag{1.14}$$

$$F(r) = L\{f(y)\} = \int_0^\infty e^{-ry}f(y)dy. \tag{1.15}$$

In [20], the Riemann-Liouville k-fractional integral of order α is defined as

$$\mathfrak{I}^{\alpha,k}_w(f(w)) = \frac{1}{k\,\Gamma_k(\alpha)} \int_0^w (w - t)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad \big(k \in \mathbb{R}^+, \Re(\alpha) > 0\big). \tag{1.16}$$

1.1. Numerical Calculation and graphical Representation of Generalized Extended Beta k-Function.

$$\beta^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(p, q) = \frac{1}{k} \int_0^1 s^{\frac{p}{k}-1}(1 - s)^{\frac{q}{k}-1} {}_1F_{1,k}(\varpi_1; \varpi_2; \frac{-\tau^k}{ks^{\xi_1}(1 - s)^{\xi_2}})ds. \tag{1.17}$$

Remark 1.2. The generalized extended beta k-function defined in above Equation (1.17) serves as a unified framework that systematically reduces to several well-known classical and extended functions under specific parameter restrictions:

(i) Setting $\xi_1 = \xi_2 = 1$ and $k = 1$ simplifies to the generalized extended beta function involving the confluent hypergeometric core, as investigated by Altin *et al.* [4].
(ii) Setting $\varpi_1 = \varpi_2$ and $\xi_1 = \xi_2 = 1$ reduces to the extended beta k-function studied by Mubeen *et al.* or Parmar [23].
(iii) Setting $\varpi_1 = \varpi_2$, $\xi_1 = \xi_2 = 1$, and $k = 1$ yields the foundational extended beta function introduced by Chaudhry *et al.* [11].
(iv) Setting $\tau = 0$ reduces the equation directly to the classical k-beta function $\beta_k(p, q)$, and further setting $k = 1$ recovers the classical Euler Beta function $B(p, q)$.

If we take the fix parameters $\tau = 0.5$, $\varpi_1 = 1.5$, $\varpi_2 = 2.0$, $\xi_1 = 1$, $\xi_2 = 1$, $p = q = 0.50$, and different values of $k > 0$, then the Equation (2.17), can be written as

$$\beta_{0.5,k}^{(1.5,2.0;1,1)}(0.5, 0.5) = \frac{1}{k} \int_0^1 s^{\frac{0.5}{k}-1}(1-s)^{\frac{0.5}{k}-1} {}_1F_{1,k}\left(1.5; 2.0; \frac{-0.5^k}{ks(1-s)}\right) ds. \tag{1.18}$$

First we convert the confluent hypergeometric k-function into the standard confluent hypergeometric function ${}_1F_1$ as follow:

$${}_1F_{1,k}(\alpha; \beta; z) = {}_1F_1\left(\frac{\alpha}{k}; \frac{\beta}{k}; z\right).$$

Now we apply Gauss-Legendre quadrature method. Since Standard Gauss-Legendre quadrature integrates over the interval $[-1, 1]$. To evaluate over $[0, 1]$, we transform the standard nodes (t_i) and weights (w_i) to our specific interval $(s_i$ and $w_i^*)$:

$$s_i = \frac{t_i + 1}{2}, \quad w_i^* = \frac{w_i}{2}.$$

The numerical approximation of the integral I_k becomes:

$$I_k \approx \sum_{i=1}^{20} w_i^* f_k(s_i),$$

where the integrand is:

$$f_k(s_i) = \frac{1}{k} s_i^{\frac{0.5}{k}-1}(1-s_i)^{\frac{0.5}{k}-1} {}_1F_1\left(\frac{1.5}{k}; \frac{2.0}{k}; \frac{-0.5^k}{ks_i(1-s_i)}\right). \tag{1.19}$$

By substituting $k = 0.5$, the exponent terms simplify as $\frac{0.5}{0.5} - 1 = 0$, so $s_i^0(1-s_i)^0 = 1$. The integrand heavily simplifies to:

$$f_{0.5}(s_i) = 2 \cdot {}_1F_1\left(3; 4; \frac{-\sqrt{0.5}}{0.5s_i(1-s_i)}\right).$$

Node $i = 1$ (Near the lower boundary):

- **Standard Node & Weight:** $t_1 = -0.993128, \quad w_1 = 0.017614.$
- **Transformed:** $s_1 = 0.003436, \quad w_1^* = 0.008807.$
- **Argument z_1:** $\frac{-\sqrt{0.5}}{0.5(0.003436)(1-0.003436)} = -413.064.$
- **Function Evaluation:** ${}_1F_1(3; 4; -413.064) \approx 0.000000.$
- **Product:** $w_1^* \cdot f_{0.5}(s_1) \approx 0.000000.$

Node $i = 10$ (Near the center):

- **Standard Node & Weight:** $t_{10} = -0.076526, \quad w_{10} = 0.152753.$
- **Transformed:** $s_{10} = 0.461737, \quad w_{10}^* = 0.076377.$
- **Argument z_{10}:** $\frac{-\sqrt{0.5}}{0.5(0.461737)(1-0.461737)} = -5.690178.$
- **Function Evaluation:** ${}_1F_1(3; 4; -5.690178) = 0.030049.$
- **Integrand $f(s_{10})$:** $2 \cdot 0.030049 = 0.060098.$

TABLE 1. Final computed results.

Parameter k	$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)$
$k = 0.5$	0.028686
$k = 1.0$	0.349486
$k = 1.5$	0.750411
$k = 2.0$	1.049178

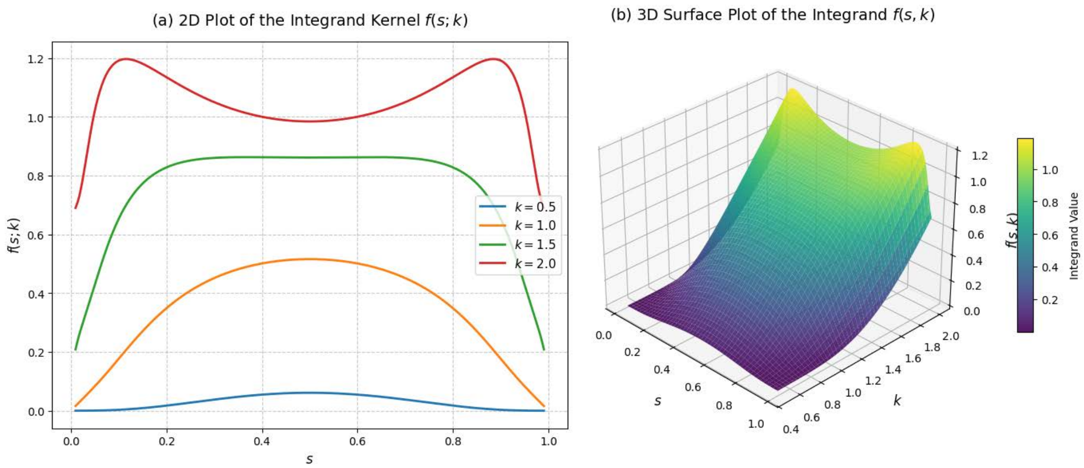

FIGURE 1. Visual representation of the integrand kernel $f(s;k)$ for the generalized extended Beta k-function. **(a)** Two-dimensional profiles illustrating the kernel's amplitude across the integration domain $s \in (0,1)$ for discrete values $k \in \{0.5, 1.0, 1.5, 2.0\}$. **(b)** A three-dimensional surface mapping showing the continuous topological evolution of the integrand across the (s,k) parameter space, highlighting the boundary singularities.

- **Product:** $w_{10}^* \cdot f_{0.5}(s_{10}) = 0.076377 \cdot 0.060098 = 0.004590.$

2. Application to Statistical Distribution Theory

In this section, we present a concrete physical application of our newly introduced generalized extended beta k-function by constructing a novel continuous probability distribution.

2.1. **Generalized Extended Beta k-Distribution.** We define the probability density function (PDF) of a continuous random variable X following the generalized extended beta k-distribution as follow :

$$f(x) = \begin{cases} \frac{1}{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)} x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\,{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{kx^{\xi_1}(1-x)^{\xi_2}}\right), & 0<x<1, \\ 0, & \text{Otherwise}, \end{cases} \tag{2.1}$$

where $p,q,k>0$, $\tau \geq 0$, $\varpi_1,\varpi_2>0$, and $\xi_1,\xi_2 \geq 1$. To confirm that $f(x)$ is a valid density function, it must satisfy $\int_0^1 f(x)\,dx = 1$.

Proof. Using definition (2.1), we write:

$$\int_0^1 f(x)\,dx = \frac{1}{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)} \int_0^1 x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\,{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{kx^{\xi_1}(1-x)^{\xi_2}}\right)dx.$$

By utilizing our main integral expression of the generalized beta k-function given in Equation (1.17), the integral evaluates precisely to $k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)$. Thus:

$$\int_0^1 f(x)\,dx = \frac{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)}{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)} = 1.$$

This completes the proof. □

2.2. Statistical Moments.

Theorem 2.1. *Let X be a random variable with the PDF given in (2.1). The r-th moment about the origin, $E[X^r]$, is given by:*

$$E[X^r] = \frac{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+rk,q)}{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)}. \tag{2.2}$$

Proof. By definition, the r-th statistical moment is expressed as

$$E[X^r] = \int_0^1 x^r\,f(x)\,dx.$$

Substituting (2.1) into the integral expression as

$$E[X^r] = \frac{1}{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)} \int_0^1 x^{\frac{p+rk}{k}-1}(1-x)^{\frac{q}{k}-1}\,{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{kx^{\xi_1}(1-x)^{\xi_2}}\right)dx.$$

Applying Equation (17) directly to the numerator with the updated parameter $p \to p+rk$, we easily get the following:

$$E[X^r] = \frac{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+rk,q)}{k\,\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)} = \frac{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+rk,q)}{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)}.$$

The theorem is verified. □

2.3. Mellin Transform.

Theorem 2.2. *If $k>0$, $\Re(p)>0$, $\Re(p+r\xi_1)>0$, $\Re(q+r\xi_2)>0$, $\Re(\varpi_1)>0$, $\Re(\varpi_2)>0$, and $\xi_1,\xi_2 \geq 1$, then the Mellin transform of the generalized extended beta k-function is given by*

$$\int_0^\infty \tau^{r-1}\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)\,d\tau = \Gamma_k^{(\varpi_1,\varpi_2)}(r)\beta_k(p+\xi_1 r, q+\xi_2 r), \tag{2.3}$$

where $\Gamma_k^{(\varpi_1,\varpi_2)}(r)$ is the Mellin transform of the confluent hypergeometric k-function.

Proof. Using the integral representation of the extended generalized beta k-function from Equation (1.17), we have

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q) = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds. \tag{2.4}$$

Substituting (2.4) into the left-hand side of (2.3) and interchanging the order of integration, we obtain

$$\int_0^\infty \tau^{r-1}\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)\,d\tau = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1} \times \left[\int_0^\infty \tau^{r-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)d\tau\right]ds. \tag{2.5}$$

To evaluate the inner integral, we employ the substitution

$$\lambda = \frac{\tau}{s^{\xi_1/k}(1-s)^{\xi_2/k}}, \quad \text{which implies} \quad d\tau = s^{\xi_1/k}(1-s)^{\xi_2/k}\,d\lambda.$$

The inner integral over τ transforms as follows:

$$\int_0^\infty \tau^{r-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)d\tau = \int_0^\infty \left(\lambda s^{\frac{\xi_1}{k}}(1-s)^{\frac{\xi_2}{k}}\right)^{r-1}{}_1F_{1,k}\left(\varpi_1,\varpi_2;\frac{-\lambda^k}{k}\right)s^{\frac{\xi_1}{k}}(1-s)^{\frac{\xi_2}{k}}\,d\lambda$$
$$= s^{\frac{r\xi_1}{k}}(1-s)^{\frac{r\xi_2}{k}}\int_0^\infty \lambda^{r-1}{}_1F_{1,k}\left(\varpi_1,\varpi_2;\frac{-\lambda^k}{k}\right)d\lambda. \tag{2.6}$$

The remaining integral in (2.6) is recognized as the Mellin transform of the confluent hypergeometric k-function, denoted as $\Gamma_k^{(\varpi_1,\varpi_2)}(r)$. Substituting this result back into (2.5), we have

$$\begin{aligned}\text{L.H.S.} &= \Gamma_k^{(\varpi_1,\varpi_2)}(r)\left[\frac{1}{k}\int_0^1 s^{\frac{p+r\xi_1}{k}-1}(1-s)^{\frac{q+r\xi_2}{k}-1}\,ds\right]\\ &= \Gamma_k^{(\varpi_1,\varpi_2)}(r)\beta_k(p+\xi_1 r, q+\xi_2 r),\end{aligned}$$

which completes the proof. □

Corollary 2.3. *If $k>0$, $\Re(p)>0$, $\Re(q)>0$, $\tau\geq 0$, $\Re(\varpi_1)>0$, $\Re(\varpi_2)>0$, and $\xi_1,\xi_2\geq 1$, then*

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\Gamma_k^{(\varpi_1,\varpi_2)}(r)\Gamma_k(p+\xi_1 r)\Gamma_k(q+\xi_2 r)}{\Gamma_k(p+q+(\xi_1+\xi_2)r)}\tau^{-r}\,dr. \tag{2.7}$$

Proof. From theorem 1, the Mellin transform of the generalized extended beta k-function with respect to τ is established as

$$\mathcal{M}\left\{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)\right\}(r) = \Gamma_k^{(\varpi_1,\varpi_2)}(r)\beta_k(p+\xi_1 r, q+\xi_2 r). \tag{2.8}$$

Using the fundamental relationship between the beta and gamma k-functions, namely $\beta_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$, we can expand the right-hand side of (2.8) to obtain:

$$\mathcal{M}\left\{\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q)\right\}(r) = \frac{\Gamma_k^{(\varpi_1,\varpi_2)}(r)\Gamma_k(p+\xi_1 r)\Gamma_k(q+\xi_2 r)}{\Gamma_k(p+q+(\xi_1+\xi_2)r)}. \tag{2.9}$$

Applying the inverse Mellin transform operator, defined generally as $f(\tau)=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\mathcal{M}\{f\}(r)\tau^{-r}dr$, to both sides of (2.9) yields the desired contour integral as

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\Gamma_k^{(\varpi_1,\varpi_2)}(r)\Gamma_k(p+\xi_1 r)\Gamma_k(q+\xi_2 r)}{\Gamma_k(p+q+(\xi_1+\xi_2)r)}\tau^{-r}\,dr. \tag{2.10}$$

We note that the right-hand side takes the form of a Mellin-Barnes type contour integral. In the literature of special functions, such integrals over ratios of gamma functions serve as the defining generating representations for generalized hypergeometric functions, thereby justifying the connection to the ${}_2F_{1,k}$ function class. □

2.4. **Integral Representations.**

Theorem 2.4. *Let $k > 0$, $\tau \geq 0$, $\xi_1, \xi_2 \geq 1$. If $q > p$ and $\min\{\Re(x), \Re(y), \Re(\varpi_1), \Re(\varpi_2)\} > 0$, then the following integral representations hold true:*

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(x,y) = \frac{1}{k}\int_0^{\infty} \frac{s^{\frac{x}{k}-1}}{(s+1)^{\frac{x+y}{k}}} \times {}_1F_{1,k}\left(\varpi_1, \varpi_2; \frac{-\tau^k (s+1)^{\xi_1+\xi_2}}{k s^{\xi_1}}\right) ds \tag{2.11}$$

$$= \frac{2^{1-\frac{x+y}{k}}}{k}\int_{-1}^{1} (1+s)^{\frac{x}{k}-1}(1-s)^{\frac{y}{k}-1} \times {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k 2^{\xi_1+\xi_2}}{k(s+1)^{\xi_1}(1-s)^{\xi_2}}\right) ds \tag{2.12}$$

$$= \frac{(q-p)^{1-\frac{x+y}{k}}}{k}\int_{p}^{q} (u-p)^{\frac{x}{k}-1}(q-u)^{\frac{y}{k}-1} \times {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k (q-p)^{\xi_1+\xi_2}}{k(u-p)^{\xi_1}(q-u)^{\xi_2}}\right) du \tag{2.13}$$

$$= \frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2x}{k}-1}(\sin\theta)^{\frac{2y}{k}-1} \times {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k \sec^{2\xi_1}\theta \csc^{2\xi_2}\theta}{k}\right) d\theta \tag{2.14}$$

$$= \frac{2}{k}\int_0^{\frac{\pi}{4}} (\tanh\theta)^{\frac{2x}{k}-1}(\operatorname{sech}\theta)^{\frac{2y}{k}-1} \times {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k \coth^{2\xi_1}\theta \cosh^{2\xi_2}\theta}{k}\right) d\theta. \tag{2.15}$$

Proof. By using $t = \frac{s}{s+1}$, $t = \frac{s+1}{2}$, $t = \frac{u-p}{q-p}$, $t = \cos^2\theta$, $t = \tanh^2\theta$ into (1.17), we get the above integral representations. □

Theorem 2.5. *If $k > 0$, $\Re(p) > 0$, $\Re(q) > 0$, $\tau \geq 0$, $\Re(\varpi_1) > 0$, $\Re(\varpi_2) > 0$, $\xi_1, \xi_2 \geq 1$, then we have*

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q+k) + \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+k, q) = \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q). \tag{2.16}$$

Proof. Consider left hand side of (2.16), we have

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q+k) + \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+k, q) = \frac{1}{k}\int_0^1 t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}}\, {}_1F_{1,k}(\varpi_1; \varpi_2; \frac{-\tau^k}{kt^{\xi_1}(1-t)^{\xi_2}})dt$$
$$+\frac{1}{k}\int_0^1 t^{\frac{p}{k}}(1-t)^{\frac{q}{k}-1}\, {}_1F_{1,k}(\varpi_1; \varpi_2; \frac{-\tau^k}{kt^{\xi_1}(1-t)^{\xi_2}})dt.$$

After simplification we get the required result. □

Corollary 2.6. *The following result holds*

$$\beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q+1) + \beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+1, q) = \beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q). \tag{2.17}$$

Proof. If we take $k = 1$ in (2.16), then we get required result. □

2.5. Summation Formulas.

Theorem 2.7. *If $k > 0$, $\Re(p) > 0$, $\Re(q) > 0$, $\tau \geq 0$, $\Re(\varpi_1) > 0$, $\Re(\varpi_2) > 0$, and $\xi_1, \xi_2 \geq 1$, then the following summation formula holds:*

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, q) = \sum_{m=0}^{\infty} \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+mk, q+k). \tag{2.18}$$

Proof. Let us initiate the proof from the right-hand side of identity (2.18), we have

$$\text{R.H.S.} = \sum_{m=0}^{\infty} \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+mk, q+k).$$

By employing the integral definition of the generalized extended beta k-function given in Equation (1.17), the series can be expressed in its integral equivalent as follow:

$$\text{R.H.S.} = \sum_{m=0}^{\infty}\left[\frac{1}{k}\int_0^1 s^{\frac{p+mk}{k}-1}(1-s)^{\frac{q+k}{k}-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds\right].$$

By expanding and rewriting the fractional exponents in the integrands ($\frac{p+mk}{k}-1=\frac{p}{k}+m-1$ and $\frac{q+k}{k}-1=\frac{q}{k}$), we obtain

$$\text{R.H.S.} = \sum_{m=0}^{\infty}\frac{1}{k}\int_0^1 s^{\frac{p}{k}+m-1}(1-s)^{\frac{q}{k}}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds.$$

Assuming the uniform convergence of the infinite summation over the open interval $(0,1)$, we can validly interchange the order of the summation and the integration operators, which yields:

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}}\left[\sum_{m=0}^{\infty}s^m\right]{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds.$$

By invoking the classical geometric series expansion, $\sum_{m=0}^{\infty}s^m=(1-s)^{-1}$, which is strictly convergent for $|s|<1$, the integral reduces to

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}}(1-s)^{-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds.$$

Combining the identical bases $(1-s)^{\frac{q}{k}}$ and $(1-s)^{-1}$ leads directly to the simplified format:

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds.$$

Recognizing that the final integral representation matches precisely the definition of our primary function, we immediately get following:

$$\text{R.H.S.} = \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q) = \text{L.H.S.}$$

This completes the proof of theorem. □

Corollary 2.8. *The following result holds*

$$\beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,q) = \sum_{m=0}^{\infty}\beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+m,q+1). \tag{2.19}$$

Proof. If we take $k=1$ in (2.18), then we get required result. □

Theorem 2.9. *If $k>0$, $\Re(p)>0$, $\Re(q)>0$, $\tau\geq 0$, $\Re(\varpi_1)>0$, $\Re(\varpi_2)>0$, and $\xi_1,\xi_2\geq 1$, then the following identity holds:*

$$\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p,k-q) = \sum_{m=0}^{\infty}\frac{(q)_{m,k}}{k^{m+1}\,m!}\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+mk,k). \tag{2.20}$$

Proof. Let us initiate the verification by considering the right-hand side of identity (2.20), we have

$$\text{R.H.S.} = \sum_{m=0}^{\infty}\frac{(q)_{m,k}}{k^{m+1}\,m!}\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+mk,k).$$

By substituting the integral definition of the generalized extended beta k-function given in Equation (1.17) into each term of the infinite series, we have

$$\text{R.H.S.} = \sum_{m=0}^{\infty}\frac{(q)_{m,k}}{k^{m+1}\,m!}\left[\int_0^1 s^{\frac{p+mk}{k}-1}(1-s)^{\frac{k}{k}-1}{}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right)ds\right].$$

Observing that $(1-s)^{\frac{k}{k}-1} = (1-s)^0 = 1$, and factoring out the constants from the series, the expression simplifies to

$$\text{R.H.S.} = \sum_{m=0}^{\infty} \frac{(q)_{m,k}}{k^{m+1}\, m!} \int_0^1 s^{\frac{p}{k}+m-1}\, {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right) ds.$$

Assuming the uniform convergence of the expansion within the valid interval of integration, we can switch the order of the summation and the integration operators

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}\left[\sum_{m=0}^{\infty} \frac{(q)_{m,k}}{k^m}\frac{s^m}{m!}\right] {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right) ds.$$

By employing the generalized binomial series expansion variant for the Pochhammer k-symbol as

$$(1-s)^{-\frac{q}{k}} = \sum_{m=0}^{\infty} \frac{(q)_{m,k}}{k^m}\frac{s^m}{m!}, \quad (|s| < 1).$$

Substituting this summation equivalent back into our integral yielding

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{-\frac{q}{k}}\, {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right) ds.$$

By algebraically modifying the exponent of the second term as $-\frac{q}{k} = \frac{k-q}{k} - 1$, the equation can be explicitly structured as:

$$\text{R.H.S.} = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{k-q}{k}-1}\, {}_1F_{1,k}\left(\varpi_1; \varpi_2; \frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right) ds.$$

This expression is the exact integral representation for $\beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, k-q)$. Thus, we arrive at

$$\text{R.H.S.} = \beta_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, k-q) = \text{L.H.S.}$$

This finishes the proof. □

Corollary 2.10. *The following result holds*

$$\beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p, 1-q) = \sum_{m=0}^{\infty} \frac{(q)_m}{m!}\beta_{\tau}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(p+m, 1). \tag{2.21}$$

Proof. If we take $k = 1$ in (2.20), then we get required result. □

3. Generalized extended Gauss hypergeometric k-function

In this section, we present the generalized extended Gauss hypergeometric k-function and explore several of its properties, including integral representations, various integral transforms, Mellin transforms, and the Hankel transforms, Riemann-Liouville, Riemann-Liouville k-fractional integral and derivative of generalized extended Gauss hypergeometric k-function. We define the generalized extended Gauss hypergeometric k-function as follows:

Definition 3.1. If $(\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\xi_1), \Re(\xi_2)\} > 0, \Re(\lambda_1) > 0, \Re(\lambda_3) > \Re(\lambda_2) > 0, \Re(\tau) \geq 0, \xi_1, \xi_2 \geq 1, |t| < 1)$, then

$$F_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1;\xi_2)}(\lambda_1, \lambda_2, \lambda_3; z) = \sum_{m=0}^{\infty} \frac{(\lambda_1)_{m,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+mk, \lambda_3-\lambda_2)}{\beta_k(\lambda_2, \lambda_3-\lambda_2)}\frac{z^m}{m!}. \tag{3.1}$$

Remark 3.2. By considering special parameter choices in (3.1), we recover several previously studied extensions of the Gauss hypergeometric function.

- For $k = 1$,(3.1) reduces to the generalized extended Gauss hypergeometric functions defined by Ghayasuddin *et al.* [15].
- For $k = 1$ and $\xi_1 = \xi_2$, we obtain the extended Gauss hypergeometric functions introduced by Parmar [23].

- For $\varpi_1 = \varpi_2$, $\xi_1 = \xi_2 = 1$, and $k = 1$, we recover the extended Gauss hypergeometric functions studied by Chaudhry *et al.* [10, 11].
- For $k = 1$ and $\xi_1 = \xi_2 = 1$, we obtain the extension of the Gauss hypergeometric function defined by Altin *et al.* [4].

3.1. **Integral Representations.**

Theorem 3.3. *Let $\tau \geq 0$ and $\xi_1, \xi_2 \geq 1$. If $\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\xi_1), \Re(\xi_2)\} > 0$, $\Re(\lambda_1) > 0$, and $\Re(\lambda_3) > \Re(\lambda_2) > 0$, then the following integral representations hold true:*

$$
\begin{aligned}
F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z) &= \frac{1}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^1 s^{\frac{\lambda_2}{k}-1}(1-s)^{\frac{\lambda_3-\lambda_2}{k}-1}(1-kzs)^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}}\right) ds \qquad (3.2)\\
&= \frac{1}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^\infty \frac{u^{\frac{\lambda_2}{k}-1}}{(1+u)^{\frac{\lambda_3}{k}}} \left(1-\frac{kzu}{1+u}\right)^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k(1+u)^{\xi_1+\xi_2}}{ku^{\xi_1}}\right) du \qquad (3.3)\\
&= \frac{2}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2\lambda_2}{k}-1}(\sin\theta)^{\frac{2(\lambda_3-\lambda_2)}{k}-1}(1-kz\cos^2\theta)^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k\sec^{2\xi_1}\theta\csc^{2\xi_2}\theta}{k}\right) d\theta \qquad (3.4)\\
&= \frac{2}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^{\frac{\pi}{2}} (\tanh\theta)^{\frac{2\lambda_2}{k}-1}(\operatorname{sech}\theta)^{\frac{2(\lambda_3-\lambda_2)}{k}}(1-kz\tanh^2\theta)^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k\coth^{2\xi_1}\theta\cosh^{2\xi_2}\theta}{k}\right) d\theta \qquad (3.5)\\
&= \frac{(q-p)^{1-\frac{\lambda_3}{k}}}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_p^q (u-p)^{\frac{\lambda_2}{k}-1}(q-u)^{\frac{\lambda_3-\lambda_2}{k}-1}\left[1-kz\left(\frac{u-p}{q-p}\right)\right]^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k(q-p)^{\xi_1+\xi_2}}{k(u-p)^{\xi_1}(q-u)^{\xi_2}}\right) du \qquad (3.6)\\
&= \frac{2^{1-\frac{\lambda_3}{k}}}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_{-1}^1 (1+u)^{\frac{\lambda_2}{k}-1}(1-u)^{\frac{\lambda_3-\lambda_2}{k}-1}\left[1-kz\left(\frac{u+1}{2}\right)\right]^{-\frac{\lambda_1}{k}} \\
&\times {}_1F_{1,k}\left(\varpi_1;\varpi_2;\frac{-\tau^k 2^{\xi_1+\xi_2}}{k(u+1)^{\xi_1}(1-u)^{\xi_2}}\right) du. \qquad (3.7)
\end{aligned}
$$

Proof. By using Equation (1.17) into (3.1), we get (3.2), further by substituting $t = \frac{u}{u+1}$, $t = \frac{u+1}{2}$, $t = \frac{u-p}{q-p}$, $t = \cos^2\theta$, $t = \tanh^2\theta$ into (3.2), we get other remaining integral representations. □

3.2. **Mellin Transforms.**

Theorem 3.4. *Let $k > 0$, $\tau \geq 0$, and $\xi_1, \xi_2 \geq 1$. If $\Re(r) > 0$ and $\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\varpi_1 + r), \Re(\varpi_2 + r)\} > 0$, then the following Mellin transform identity holds true:*

$$
\begin{aligned}
\int_0^\infty \tau^{r-1} F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z)\, d\tau &= \frac{\Gamma_k^{(\varpi_1,\varpi_2)}(r)\beta_k(\lambda_2+\xi_1 r,\lambda_3-\lambda_2+\xi_2 r)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \\
&\times {}_2F_{1,k}(\lambda_1,\lambda_2+\xi_1 r,\lambda_3+(\xi_1+\xi_2)r;z). \qquad (3.8)
\end{aligned}
$$

Proof. Consider

$$F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;z) = \frac{1}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^1 s^{\frac{\lambda_2}{k}-1}(1-s)^{\frac{\lambda_3-\lambda_2}{k}-1}(1-kzs)^{\frac{-\lambda_1}{k}} \times {}_1F_{1,k}(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}})ds. \tag{3.9}$$

By taking left hand side we obtain

$$\begin{aligned}\int_0^\infty \tau^{r-1}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;z)d\tau &= \frac{1}{k\beta_k(\lambda_2,\lambda_3-\lambda_2)} \int_0^1 s^{\frac{\lambda_2}{k}-1}(1-s)^{\frac{\lambda_3-\lambda_2}{k}-1}(1-kzs)^{\frac{-\lambda_1}{k}}ds \\ &\times \int_0^\infty \tau^{r-1}{}_1F_{1,k}(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}})d\tau. \end{aligned} \tag{3.10}$$

By substituting $\lambda = \frac{\tau}{s^{\frac{\xi_1}{k}}(1-s)^{\frac{\xi_2}{k}}}$ into (3.10), and after simplification we get

$$\int_0^\infty \tau^{r-1}{}_1F_{1,k}(\varpi_1;\varpi_2;\frac{-\tau^k}{ks^{\xi_1}(1-s)^{\xi_2}})d\tau = s^{\frac{\xi_1 r}{k}}(1-s)^{\frac{\xi_2 r}{k}}\Gamma^{(\varpi_1,\varpi_2)}_k(r). \tag{3.11}$$

By substituting Equation (3.11) in Equation (3.10), and after simplification we get required result. □

Corollary 3.5. *The following inverse Mellin transforms holds truly*

$$\begin{aligned}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;z) &= \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{\Gamma^{(\varpi_1,\varpi_2)}_k(r)\beta_k(\lambda_2+\xi_1 r,\lambda_3-\lambda_2+\xi_2 r)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \\ &\times {}_2F_{1,k}(\lambda_1,\lambda_2+\xi_1 r,\lambda_3+(\xi_1+\xi_2)r;z)\tau^{-r}d\tau. \end{aligned} \tag{3.12}$$

Proof. By taking Mellin inverse on both sides of Equation (3.8), we get required result. □

3.3. Laplace Transformation.

Theorem 3.6. *If* $\Re(\lambda_3) > \Re(\lambda_2) > 0,\ \tau \geq 0,\ \xi_1,\xi_2 \geq 1,\ \Re(d) > 0,$ *then*

$$\int_0^\infty e^{-st}t^{\frac{d}{k}-1}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;vt)dt = \sum_{m=0}^\infty \frac{(\lambda_1)_{m,k}\Gamma_k(d+mk)}{s^{\frac{d}{k}+m}k^{\frac{d}{k}+m-1}} \times \frac{\beta^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}_{\tau,k}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^m}{m!}. \tag{3.13}$$

Proof. By taking left hand side of (3.13), then

$$\int_0^\infty e^{-st}t^{\frac{d}{k}-1}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;vt)dt = \int_0^\infty e^{-st}t^{\frac{d}{k}-1}\sum_{m=0}^\infty \frac{(\lambda_1)_{m,k}\beta^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}_{\tau,k}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{v^m t^m}{m!}dt. \tag{3.14}$$

The result follows upon interchanging the order of integration and summation as

$$\begin{aligned}\int_0^\infty e^{-st}t^{\frac{d}{k}-1}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;vt)dt &= \sum_{m=0}^\infty \frac{(\lambda_1)_{m,k}\beta^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}_{\tau,k}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{(\frac{d}{k}+m-1)!}{s^{\frac{d}{k}+m}}\frac{v^m}{m!} \qquad (3.15) \\ &= \sum_{m=0}^\infty \frac{(\lambda_1)_{m,k}\Gamma_k(d+mk)}{s^{\frac{d}{k}+m}k^{\frac{d}{k}+m-1}} \times \frac{\beta^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}_{\tau,k}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^m}{m!}. \qquad (3.16)\end{aligned}$$

□

Theorem 3.7. *If* $\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\xi_1), \Re(\xi_2)\} > 0$, $\Re(\lambda_1) > 0$, $\Re(\lambda_3) > \Re(\lambda_2) > 0$, $\tau \geq 0$, $\xi_1, \xi_2 \geq 1$, *and* $\Re(d) > 0$, *then*

$$\int_0^\infty e^{-st}t^{d-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;vt)\,dt = \sum_{m=0}^{\infty}\frac{(\lambda_1)_{m,k}\Gamma(m+d)}{s^{m+d}} \times \frac{\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^m}{m!}. \tag{3.17}$$

Proof. By taking left hand side of (3.17), then

$$\int_0^\infty e^{-st}t^{d-1}F_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1:\xi_2)}(\lambda_1,\lambda_2,\lambda_3;vt)dt = \sum_{m=0}^{\infty}\frac{(\lambda_1)_{m,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^m}{m!} \times \int_0^\infty e^{-st}t^{m+d-1}dt$$

$$= \sum_{m=0}^{\infty}\frac{(\lambda_1)_{m,k}\Gamma(m+d)}{s^{m+d}} \times \frac{\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+mk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^m}{m!}.$$

□

3.4. Some Integrals Involving Generalized Extended Gauss Hypergeometric k-Function.

Theorem 3.8. *If* $k > 0$,$\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\xi_1), \Re(\xi_2)\} > 0$, $\Re(p) \geq 0$,$\Re(\lambda_3) > \Re(\lambda_2) > 0$, $\Re(\tau) \geq 0$,$\xi_1, \xi_2 \geq 1$, $|arg(1-qx)| < \pi$, *then*

$$\int_1^\infty x^{-p}(x-1)^{\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \beta(\lambda,p-\lambda)\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{(p)_{-l}}{(p-\lambda)_{-l}}\frac{q^l}{l!}. \tag{3.18}$$

Proof. Consider left hand side of (3.18), we have

$$\int_1^\infty x^{-p}(x-1)^{\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \int_1^\infty x^{-p}(x-1)^{\lambda-1} \times \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{(qx)^l}{l!}dx.$$

The result follows upon interchanging the order of integration and summation as

$$\int_1^\infty x^{-p}(x-1)^{\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}$$

$$\times \frac{(q)^l}{l!}\int_1^\infty x^{-p+l}(x-1)^{\lambda-1}dx. \tag{3.19}$$

By substituting $x-1=u$ and $dx=du$ in Equation (3.19), when $x=1$, $u=0$, $x=\infty$, $u=\infty$, we get

$$\int_1^\infty x^{-p}(x-1)^{\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}$$

$$= \frac{(q)^l}{l!}\int_0^\infty (1+u)^{-(p-l)}(u)^{\lambda-1}du. \tag{3.20}$$

By using the following formula

$$\Gamma(y)\Gamma(z) = \Gamma(y+z)\int_0^\infty x^{y-1}(x+1)^{-(y+z)}dx. \tag{3.21}$$

The Equation(3.19) will become

$$\int_1^\infty x^{-p}(x-1)^{\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{(q)^l}{l!}\times\frac{\Gamma(\lambda)\Gamma(p-\lambda-l)}{\Gamma(p-l)}$$
$$=\beta(\lambda,p-\lambda)\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\times\frac{(p)_{-l}}{(p-\lambda)_{-l}}\frac{q^l}{l!}.$$

□

Theorem 3.9. *If $k>0$, $\min\{\Re(\varpi_1),\Re(\varpi_2),\Re(\xi_1),\Re(\xi_2)\}>0$, $\Re(p)\geq 0, \Re(\lambda_3)>\Re(\lambda_2)>0$, $\Re(\tau)\geq 0, \xi_1,\xi_2\geq 1, |arg(1-qx)|<\pi$,*

$$\int_0^1 x^{-p}(1-x)^{p-\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx=\beta(1-p,p-\lambda)$$
$$\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\times\frac{(1-p)_l}{(1-\lambda)_l}\frac{q^l}{l!}. \tag{3.22}$$

Proof. Consider left hand side of (3.22), we have

$$\int_0^1 x^{-p}(1-x)^{p-\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx=\int_0^1 x^{-p}(1-x)^{p-\lambda-1}$$
$$\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{(qx)^l}{l!}dx. \tag{3.23}$$

The result follows upon interchanging the order of integration and summation as

$$\int_0^1 x^{-p}(1-x)^{p-\lambda-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx=\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{q^l}{l!}$$
$$=\beta(1-p,p-\lambda)$$
$$\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\times\frac{(1-p)_l}{(1-\lambda)_l}\frac{q^l}{l!}.$$

□

Theorem 3.10. *For $k>0$, $\min\{\Re(\varpi_1),\Re(\varpi_2),\Re(\xi_1),\Re(\xi_2)\}>0$, $\Re(p)\geq 0, \Re(\lambda_3)>\Re(\lambda_2)>0$, $\Re(\tau)\geq 0$, $\xi_1,\xi_2\geq 1$, $|arg(1-qx)|<\pi$, we have*

$$\int_0^\infty x^p J_\eta^\lambda(x)F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx=\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}$$
$$\times\frac{\Gamma(p+l+1)}{\Gamma(1+\eta-\lambda-\lambda(p+l))}\frac{q^l}{l!}, \tag{3.24}$$

where

$$J_\eta^\lambda(x)=\phi(\lambda,\eta+1;x)\sum_{l=0}^{\infty}\frac{1}{\Gamma(\lambda l+\eta+1)}\frac{(-x)^l}{l!},$$

is Bessel maitland function.

Proof. Consider left hand side of Equation (3.24), we have

$$\int_0^\infty x^p J_\eta^\lambda(x) F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \int_0^\infty x^p J_\eta^\lambda(x) \times \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \frac{(qx)^l}{l!}dx.$$

By changing the order of integration and summation, we have

$$\int_0^\infty x^p J_\eta^\lambda(x) F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \frac{(q)^l}{l!} \times \int_1^\infty x^{p+l} J_\eta^\lambda(x)dx. \tag{3.25}$$

By using the following formula

$$\int_1^\infty x^{p+l} J_\eta^\lambda(x)dx = \frac{\Gamma(p+1)}{\Gamma(1+\eta-\lambda-\lambda p)}, \quad where \quad (Re(p) > -1, 0 < \lambda < 1). \tag{3.26}$$

The Equation (3.25), become

$$\int_0^\infty x^p J_\eta^\lambda(x) F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;qx)dx = \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{\Gamma(p+l+1)}{\Gamma(1+\eta-\lambda-\lambda(p+l))} \frac{q^l}{l!}.$$

□

3.5. **Riemann-Liouville k-Fractional Integral.**

Theorem 3.11. *If $k > 0$, $\min\{\Re(\varpi_1), \Re(\varpi_2), \Re(\xi_1), \Re(\xi_2)\} > 0$, $\Re(p) \geq 0$, $\Re(\lambda_3) > \Re(\lambda_2) > 0$, $\Re(\tau) \geq 0$, $\xi_1, \xi_2 \geq 1$, then*

$$\begin{aligned} {}_k\mathcal{B}_z^\mu\left[z^{\frac{n}{k}-1} F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z)\right] &= \frac{z^{\frac{n-\mu}{k}-1}}{k\Gamma_k(-\mu)} \\ &\times \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \beta(l+\frac{n}{k};\frac{-\mu}{k})\frac{z^l}{l!}. \end{aligned} \tag{3.27}$$

Proof. Consider left hand side, we have

$$\begin{aligned} {}_k\mathcal{B}_z^\mu\left[z^{\frac{n}{k}-1} F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z)\right] &= \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^{\frac{n}{k}-1} F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;t)(z-t)^{\frac{-\mu}{k}-1}dt \\ &= \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^{\frac{n}{k}-1} \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)} \times \frac{(t)^l}{l!}(z-t)^{\frac{-\mu}{k}-1}dt. \end{aligned}$$

The result follows upon interchanging the order of integration and summation as follow:

$$\begin{aligned} {}_k\mathcal{B}_z^\mu\left[z^{\frac{n}{k}-1} F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z)\right] &= \sum_{l=0}^\infty \frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)l!} \\ &\times \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^{\frac{n}{k}+l-1}(z-t)^{\frac{-\mu}{k}-1}dt. \end{aligned} \tag{3.28}$$

By substituting $t = vz$ and $dt = zdv$ in Equation (3.28), when $t = 0$, $z = 0$, $t = z$, $z = 1$, we have

$$\begin{aligned}{}_k\mathcal{B}_z^{\mu}[z^{\frac{n}{k}-1}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;z)] &= \frac{z^{\frac{n-\mu}{k}-1}}{k\Gamma_k(-\mu)}\int_0^z v^{\frac{n}{k}+l-1}(1-v)^{\frac{-\mu}{k}-1}dv\\ &\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)z^l}{\beta_k(\lambda_2,\lambda_3-\lambda_2)l!}\\ &= \frac{z^{\frac{n-\mu}{k}-1}}{k\Gamma_k(-\mu)}\times\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\beta(l+\frac{n}{k};\frac{-\mu}{k})\frac{z^l}{l!}.\end{aligned}$$

□

3.6. **Derivative.**

Theorem 3.12. *If $k > 0$, $\min\{\Re(\varpi_1),\Re(\varpi_2),\Re(\xi_1),\Re(\xi_2)\} > 0$, $\Re(\lambda_3) > \Re(\lambda_2) > 0$, $\Re(\tau) \geq 0$, $\xi_1,\xi_2 \geq 1$, then*

$$\frac{d^l}{dv^l}[F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;v)] = \frac{(\lambda_2\lambda_1)_{l,k}}{(\lambda_3)_{l,k}}F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1+lk,\lambda_2+lk,\lambda_3+lk;v). \tag{3.29}$$

Proof. By definition of generalized extended Gauss hypergeometric function

$$F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;v) = \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^l}{l!}. \tag{3.30}$$

By taking the derivative of Equation (3.30) on both sides, we have

$$\begin{aligned}\frac{d}{dv}[F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;v)] &= \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{lv^{l-1}}{l!}\\ &= \sum_{l=1}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{lv^{l-1}}{l(l-1)!}\\ &= \sum_{l=1}^{\infty}\frac{(\lambda_1)_{l,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+lk,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^{l-1}}{(l-1)!}.\end{aligned}$$

Replace l by *l+1*

$$\frac{d}{dv}[F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;v)] = \sum_{l=0}^{\infty}\frac{(\lambda_1)_{l+1,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+(l+1)k,\lambda_3-\lambda_2)}{\beta_k(\lambda_2,\lambda_3-\lambda_2)}\frac{v^l}{l!}.$$

We use the following property

$$\beta_k(\lambda_2,\lambda_3-\lambda_2) = \frac{(\lambda_3)_k}{(\lambda_2)_k}\beta_k(\lambda_2+k,\lambda_3-\lambda_2).$$

Then above equation can be written as

$$\frac{d}{dv}[F_{\tau,k}^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}(\lambda_1,\lambda_2,\lambda_3;v)] = \frac{(\lambda_2)_k}{(\lambda_3)_k}\sum_{l=0}^{\infty}\frac{(\lambda_1)_{l+1,k}\beta_{\tau,k}^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}(\lambda_2+(l+1)k,\lambda_3-\lambda_2)}{\beta_k(\lambda_2+k,\lambda_3-\lambda_2)}\frac{v^l}{l!}.$$

By using the following relation

$$(\lambda_1)_{l+1,k} = (\lambda_1)_k(\lambda_1+k)_{l,k},$$

$$\begin{aligned}\frac{d}{dv}[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;v)] &= \frac{(\lambda_2,\lambda_1)_k}{(\lambda_3)_k}\sum_{l=0}^{\infty}\frac{(\lambda_1+k)_{l,k}\beta^{(\varpi_1,\varpi_2,\xi_1,\xi_2)}_{\tau,k}(\lambda_2+(l+1)k,\lambda_3-\lambda_2)}{\beta_k(\lambda_2+k,\lambda_3-\lambda_2)}\frac{v^l}{l!}\\ &= \frac{(\lambda_2,\lambda_1)_k}{(\lambda_3)_k}[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1+k,\lambda_2+k,\lambda_3+k;v)]. \end{aligned} \tag{3.31}$$

Assume the result holds for *l-1*.

$$\begin{aligned}\frac{d^{l-1}}{dv^{l-1}}[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;v)] &= \frac{(\lambda_2,\lambda_1)_{l-1,k}}{(\lambda_3)_{l-1,k}}\\ &\quad\times[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1+(l-1)k,\lambda_2+(l-1)k,\lambda_3+(l-1)k;v)]. \end{aligned} \tag{3.32}$$

Then we get

$$\begin{aligned}\frac{d^{l}}{dv^{l}}[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1,\lambda_2,\lambda_3;v)] &= \frac{(\lambda_2,\lambda_1)_{l-1,k}(\lambda_2,\lambda_1+(l-1)k)}{(\lambda_3)_{l-1,k}(\lambda_3+(l-1)k)}\\ &\quad\times[F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1+lk,\lambda_2+lk,\lambda_3+lk;v)]\\ &= \frac{(\lambda_2\lambda_1)_{l,k}}{(\lambda_3)_{l,k}}F^{(\varpi_1,\varpi_2;\xi_1,\xi_2)}_{\tau,k}(\lambda_1+lk,\lambda_2+lk,\lambda_3+lk;v). \end{aligned}$$

□

4. Conclusion

In this work, we have presented generalized forms of the beta and Gauss hypergeometric functions by introducing a new parameter $k > 0$. Several properties of these generalized extended k-functions have been established, including their integral representations, Mellin and Laplace transforms, as well as Riemann–Liouville fractional and k-fractional integrals. Furthermore, we examined the derivatives of the generalized Gauss hypergeometric k-function and provided graphical illustrations to support and validate the theoretical findings. As a direction for future research, these functions may be further extended through the introduction of additional parameters. In particular, one may explore generalized beta functions associated with Mittag–Leffler functions and Bessel–Struve kernel functions.

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