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Existence and numerical results for 2D fractional Volterra integral equations via measure of noncompactness

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Abstract

The current research introduces a generalized Volterra type of fractional integral equations defined on two dimensions and aims to investigate the sufficient conditions for the existence of solutions of these equations by employing nonlinear analysis method such as fixed point theorem of Petryshyn connected with the measure of noncompactness in the space of continuous functions on two close intervals. In fact, we show that under weaker conditions, the existence results contain special cases achieved from similar studies. Further, to find the solutions of the problem of two variables, the successful iterative technique is formulated numerically. Of course, some provided examples confirm that our established results are effective.

Keywords. Nonlinear analysis, Measure of noncompactness, Fractional integral equation, Iterative technique.
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1. Introduction

The research of fractional calculus, nowadays, is a topic of importance for many researchers in the field of nonlinear analysis [18, 28]. Particularly, differential equations ($\mathbb{DE}$s) and integral equations ($\mathbb{IE}$s) appear in numerous practical problems including engineering [31, 36], physics [12], mechanics [13], modeling [1, 4, 34] and additional areas with the aid of fractional order [9, 29, 32]. These problems often are the theory of neutron transport, the theory of gases [21], radioactive transfer theory ([10]), and the theory of water wave [33]. While we are witnessing, there exist various powerful attempts to employ the theory of measure of noncompactness (MNC) in the recent investigation of the existence of solutions of fractional integral equation ($\mathbb{FIE}$s) [8, 15, 16, 23, 41, 42]. Abdi-mazraeh *et al.* introduced the operational matrices of integral and fractional integral using the flatlet oblique multiwavelets and utilized to reduce the solution of the Abel integral equations of the first and the second kinds to the solution of algebraic equations [2]. Chefnaj *et al.* studied the existence and uniqueness of solutions to fractional integro-differential equations involving the φ-Caputo fractional derivative with nonlocal conditions [11]. Nemati introduced a new generalization of fractional-order Bernoulli wavelets as new basis functions and used them to provide a numerical solution for Hammerstein-type fractional integro-differential equations with a weakly singular kernel [37].

In this research, we discussed theoretically the existence of solution for the nonlinear 2D-Volterra $\mathbb{FIE}$, which is formulated in terms of condensing operators on $\mathrm{J}^2_{cd} := [0, \tilde{c}] \times [0, \tilde{d}]$ with the tools of MNC, by

$$\mathbb{z}(r,h) = \Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\gimel(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big)\Big), \tag{1.1}$$

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where $(r,h) \in \mathrm{J}^2_{cd}$, $0 < s_1, s_2 \le 1$, $\Omega \in C(\mathrm{J}^2_{cd} \times \mathbb{R}^2)$, $\mathfrak{w} \in C(\mathrm{J}^2_{cd} \times \mathbb{R})$, $\beta, \phi : \mathrm{J}^2_{cd} \to \mathrm{J}^2_{cd}$, $\daleth \in C(\hat{\mathrm{J}}^2_{cd} \times \mathbb{R})$, with

$$\hat{\mathrm{J}}^2_{cd} := \Big\{(r,h,u,g) \in \mathrm{J}^2_{cd} \,:\, 0 \le u \le r \le \tilde{c},\, 0 \le g \le h \le \tilde{d}\Big\}.$$

Moreover, we apply the successive approximations method to solve numerically the nonlinear Fredholm $\mathbb{FIE}$ (6.1) via two variables and present an example to check our results.

In 2019, Das *et al.* studied the following 2D $\mathbb{IE}$,

$$\mathbb{z}(r,h) = A(r,h) + \Omega\bigg(r,h,\mathbb{z}(r,h), \int_0^r \int_0^h \daleth\big(r,h,u,g,\mathbb{z}(u,g)\big)\, \mathrm{d}g\, \mathrm{d}u\bigg), \qquad (r,h) \in \mathrm{J}^2_{cd}, \tag{1.2}$$

with $\tilde{c} = 1$, $\tilde{d} = 1$ [14]. Maleknejad *et al.* investigated the 2D Volterra $\mathbb{FIE}$ is given by,

$$\mathbb{z}(r,h) = A(r,h) + \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(u,g))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \qquad (r,h) \in \mathrm{J}^2_{cd}. \tag{1.3}$$

In [32], they discussed the existence result for the 2D Fredholm $\mathbb{FIE}$ of the form,

$$\mathbb{z}(r,h) = A(r,h) + \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(u,g))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \qquad (r,h) \in \mathrm{J}^2_{cd}. \tag{1.4}$$

Mishra *et al.* in [35], discussed the existence result for 2D Volterra $\mathbb{FIE}$ is expressed by,

$$\mathbb{z}(r,h) = A(r,h) + \tfrac{q(r,h,\mathbb{z}(r,h))}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(u,g))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \qquad (r,h) \in \mathbb{R}_+ \times [0,\tilde{c}]. \tag{1.5}$$

Further, famous 2D $\mathbb{IE}$s of Fredholm and Hammerstein type in [39], have the forms,

$$\mathbb{z}(r,h) = A(r,h) + \int_0^{\tilde{c}} \int_0^{\tilde{d}} \daleth\big(r,h,u,g,\mathbb{z}(u,g)\big)\, \mathrm{d}g\, \mathrm{d}u, \tag{1.6}$$

$$\mathbb{z}(r,h) = A(r,h) + \int_0^r \int_0^h p_1(r,h,u,g)\, p_2\big(u,g,\mathbb{z}(u,g)\big)\, \mathrm{d}g\, \mathrm{d}u, \tag{1.7}$$

for $(r,h) \in \mathrm{J}^2_{cd}$ with $\tilde{c} = \tilde{d} = 1$. It should see that Volterra $\mathbb{FIE}$ (1.1) is more general than Eqs. (1.2)–(1.7). In fact, if consider $\mathfrak{w}(r,h,\mathbb{z}) = \mathbb{z}(r,h)$, $\phi(r,h) = \beta(r,h) = (r,h)$, $s_1 = s_2 = 1$, and $\Omega(r,h,\mathbb{z},\mathbb{w}) = A(r,h) + \Omega_1(r,h,\mathbb{w},\mathbb{z})$, then $\mathbb{FIE}$ (1.1) can be reduced to 2D $\mathbb{IE}$ (1.2). If $\Omega(r,h,\mathbb{z},\mathbb{w}) = A(r,h) + \mathbb{z}$, and $\beta(r,h) = (r,h)$, then 2D Volterra $\mathbb{FIE}$ (1.3) is obtained from $\mathbb{FIE}$ (1.1). Further, $\mathbb{FIE}$ (1.1) can be converted to Volterra $\mathbb{FIE}$ (1.5) whenever $\Omega(r,h,\mathbb{z},\mathbb{w}) = A(r,h) + \mathbb{z} \cdot \mathbb{w}$, and $\beta(r,h) = (r,h)$. Again, if we put $\Omega(r,h,\mathbb{z},\mathbb{w}) = A(r,h) + \mathbb{z}$, $\beta(r,h) = (r,h)$, and $s_1 = s_2$, then $\mathbb{FIE}$ (1.1) can be reduced to Fredholm $\mathbb{IE}$ (1.6).

With the aforementioned research generalization approach, in the current article, we study the solvability of Volterra $\mathbb{FIE}$ (1.1) based on the fixed point theorem (FPT) of Petryshyn [40] which has been introduced as a extension of Darbo's FPT [8]. However, various authors have established excellent achievements to solve multiple functional $\mathbb{IE}$s by checking the conditions of Darbo's FPT [3, 17, 27], here, we have a special look to the weak conditions in Petryshyn's FPT for solving the proposed problem (1.1). Comparing the proposed FPT to Schauder and Darbo's FPTs, that it enables us to skip demonstrating closed, convex and compactness properties on the investigated operators. Also, the bounded condition (A2) of Theorem 3.1 shows that the "sub-linear condition" that has been discussed by different researchers [32, 39] (see e.g. condition (H3) in [35] and condition (2) in [14]) does have not a important role.

In addition to, the successfully approximation solution, by mixing quadrature formula with the FP method to solve $\mathbb{FIE}$ (1.1), is presented. We have always seen that, finding a numerical solution for most $\mathbb{FIE}$s is impossible. Hence, it is why more emphasis is placed on valuable researches using numerical methods for the solutions and approximate solutions of fractional-order $\mathbb{IE}$s [6, 7, 22, 24, 26]. We utilize the midpoint rule to find the approximate solutions of $\mathbb{FIE}$ (1.1).

In the sequel, Section 2 contains some auxiliary concepts and preliminary definitions that are applied in the sequel. The main discussion in the Sections 3 and 5 are related to state and to prove an existence theorem and uniqueness of the solution for the mentioned 2D-Volterra $\mathbb{FIE}$ (1.1), respectively well done, relating to condensing operators using

FPT of Petryshyn. In Section 4, we introduce some examples that prove the importance of this kind of nonlinear $\mathbb{FIE}$. In Sections 6 and 7, we introduce numerical method and convergence analysis with applications that prove the importance of our study. Finally, we conclude the article in Section 8.

2. Preliminaries

For a subset Y of Real Banach space $\mathfrak{E}$, the MNC of Hausdorff [8] and Kuratowski [30] are given by,

$$\mathfrak{M}_H(\mathsf{Y}) = \inf\Big\{\xi > 0 \,:\, \text{there is a finite } \xi\text{-net for } \mathsf{Y} \text{ in } \mathfrak{E}\Big\},$$
$$\mathfrak{M}_K(\mathsf{Y}) = \inf\Big\{\xi > 0 \,:\, \mathsf{Y} = \bigcup_{i=1}^{m} A_i,\ \mathrm{diam}(A_i) \leq \xi,\ i = 1, 2, \dots, n\Big\},$$

in which, a finite δ-net for Y means that a set $\{z_1, z_2, \dots, z_n\} \subset \mathfrak{E}$ such that $B_\sigma(\mathfrak{E}, z_1)$, $B_\xi(\mathfrak{E}, z_2)$, $\dots$, $B_\xi(\mathfrak{E}, z_n)$ over Y. These MNC satisfy for any bounded set $\mathsf{Y} \subset \mathfrak{E}$ in $\mathfrak{M}_H(\mathsf{Y}) \leq \mathfrak{M}_K(\mathsf{Y}) \leq 2\,\mathfrak{M}_K(\mathsf{Y})$. We denote open ball having z as a center with radius τ, and closed convex hull of a set Y, by $B(z, \tau)$ and $\mathrm{co}\,\bar{\mathsf{Y}}$, respectively.

Theorem 2.1 ([40])**.** *For $\mathsf{Y}, \grave{\mathsf{Y}} \subset \mathfrak{E}$, we have,*

a) $\mathfrak{M}_H(\mathsf{Y}) = 0$ *if and only if* Y *is pre-compact;*
b) $\mathsf{Y} \subseteq \grave{\mathsf{Y}} \implies \mathfrak{M}_H(\mathsf{Y}) \leq \mathfrak{M}_H(\grave{\mathsf{Y}})$;
c) $\mathfrak{M}_H(\overline{\mathrm{co}}\mathsf{Y}) = \mathfrak{M}_H(\mathsf{Y})$;
d) $\mathfrak{M}_H\big(\mathsf{Y} \cup \grave{\mathsf{Y}}\big) = \max\big\{\mathfrak{M}_H(\mathsf{Y}), \mathfrak{M}_H(\grave{\mathsf{Y}})\big\}$;
e) $\mathfrak{M}_H(\theta\mathsf{Y}) = |\theta|\mathfrak{M}_H(\mathsf{Y})$, $\theta \in \mathbb{R}$;
f) $\mathfrak{M}_H\big(\mathsf{Y} + \grave{\mathsf{Y}}\big) \leq \mathfrak{M}_H(\mathsf{Y}) + \mathfrak{M}_H(\grave{\mathsf{Y}})$.

The indicator modulus of continuity of a function $\mathbb{z} \in C\big(\mathrm{J}^2_{cd}\big)$ is formulated as

$$\omega(\mathbb{z}, \sigma) = \sup\Big\{\big|\mathbb{z}(r, h) - \mathbb{z}\big(\grave{r}, \grave{h}\big)\big| \,:\, r, \grave{r} \in [0, \tilde{c}],\ h, \grave{h} \in [0, \tilde{d}],\ \big|r - \grave{r}\big|, \big|h - \grave{h}\big| \leq \sigma\Big\},$$

in Banach space $\big(C(\mathrm{J}^2_{cd}), \|\cdot\|\big)$ with $\|\mathbb{z}\| = \sup\big\{|\mathbb{z}(r, h)| \,:\, r \in [0, \tilde{c}],\ h \in [0, \tilde{d}]\big\}$. The general characteristics are also defined as $\omega(\mathsf{Y}, \sigma) = \sup\big\{\omega(\mathbb{z}, \sigma) \,:\, \mathbb{z} \in \mathsf{Y}\big\}$, and $\omega_0(\mathsf{Y}) = \lim_{\sigma \to 0} \omega(\mathsf{Y}, \sigma)$. One can found that $\omega_0(\mathsf{Y})$ is a regular MNC in $C\big(\mathrm{J}^2_{cd}\big)$ [8].

Definition 2.2 ([38])**.** Consider $G \in C(\mathfrak{E}, \mathfrak{E})$ such that it satisfies the following condition: if for every bounded subset $\mathsf{Y} \subset \mathfrak{E}$, $G(\mathsf{Y})$ is bounded and $\mathfrak{M}_K(G\mathsf{Y}) \leq k\mathfrak{M}_K(\mathsf{Y})$, $0 < k < 1$. If $\mathfrak{M}_K(G\mathsf{Y}) < \mathfrak{M}_K(\mathsf{Y})$, for every $\mathfrak{M}_K(\mathsf{Y}) > 0$, then G is called condensing or densifying mapping.

Theorem 2.3 ([40])**.** *Assume that $G : B_\tau \to E$ is a condensing mapping which for some $\mathbb{z} \in \partial B_\sigma$ fulfill the boundary condition $G(\mathbb{z}) = k\mathbb{z}$, then $k \leq 1$. Then $\varnothing \neq \mathrm{Fix}(G) \subset B_\sigma$.*

3. Main results

In this part, we seriously study the existence of solution for $\mathbb{FIE}$ (1.1) with the following assumptions,

A1: There exist $c_1, c_2, c_3 > 0$ withj $c_2 c_3 < 1$ such that,

$$\big|\Omega(r, h, \mathbb{z}, \mathbb{w}) - \Omega(r, h, \grave{\mathbb{z}}, \grave{\mathbb{w}})\big| \leq c_1\big|\mathbb{z} - \grave{\mathbb{z}})\big| + c_2\big|\mathbb{w} - \grave{\mathbb{w}}\big|,$$
$$\big|\mathfrak{w}(r, h, \mathbb{w}) - \mathfrak{w}(r, h, \grave{\mathbb{w}})\big| \leq c_3\big|\mathbb{w} - \grave{\mathbb{w}}\big|;$$

A2: There exists $\tau > 0$ such that Ω satisfies the bounded condition as form,

$$\sup\bigg\{\big|\Omega(r, h, \mathbb{z}, \mathbb{w})\big| \,:\, (r, h) \in \mathrm{J}^2_{cd},\ \mathbb{w} \in [-\tau, \tau],\ \mathbb{z} \in \Big[-\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}, \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\bigg\} \leq \tau,$$

where $\beth^* = \sup\big\{\big|\beth(r, h, u, g, \mathbb{z})\big| \,:\, \forall\,(r, h, u, g) \in \hat{\mathrm{J}}^2_{cd},\ \mathbb{z} \in [-\tau, \tau]\big\}$.

Theorem 3.1. *Under the assumptions* (A1)-(A2), *for $\mathbb{z} \in \mathrm{J}^2_{cd}$, $\mathbb{FIE}$ (1.1) has at least one solution in $\mathfrak{E}$.*

Proof. The proof is examined in several steps.

`Step 1.` First, we show that an operator $G : B_\tau \to \mathfrak{E}$ is formulated by

$$(G\mathbb{z})(r,h) = \Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big)\Big).$$

is continuous on the ball B_τ. For $\sigma > 0$ and $\mathbb{z}, \grave{\mathbb{z}} \in B_\tau$ with $\|\mathbb{z} - \grave{\mathbb{z}}\| < \sigma$, we have,

$$\begin{aligned}
\big|(G\mathbb{z})(r,h) - (G\grave{\mathbb{z}})(r,h)\big| &= \Big|\Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,z(\phi(r,h))\big)\Big) \\
&\quad - \Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big)\Big| \\
&\le \Big|\Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big)\Big) \\
&\quad - \Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big)\Big| \\
&\quad + \Big|\Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big) \\
&\quad - \Omega\Big(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big)\Big| \\
&\le c_2\big|\mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big) - \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\big| \\
&\quad + c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{|\daleth(r,h,u,g,\mathbb{z}(\beta(u,g))) - \daleth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))|}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u \\
&\le c_2 c_3 \big|\mathbb{z}\big(\phi(r,h)\big) - \grave{\mathbb{z}}\big(\phi(r,h)\big)\big| + c_1 \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\omega(\daleth, \sigma) \\
&\le (c_2 c_3)\|\mathbb{z} - \grave{\mathbb{z}}\| + c_1 \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\omega(\daleth, \sigma),
\end{aligned}$$

where

$$\omega(\daleth, \sigma) = \sup\Big\{\big|\daleth(r,h,u,g,\mathbb{z}) - \daleth(r,h,u,g,\grave{\mathbb{z}})\big| \,:\, (r,h,u,g) \in \hat{\mathrm{J}}^2_{cd},\ \mathbb{z}, \grave{\mathbb{z}} \in [-\tau, \tau],\ \|\mathbb{z} - \grave{\mathbb{z}}\| \le \sigma\Big\}.$$

From uniform continuity of $\daleth(r,h,u,g,\mathbb{z})$ on the set $\hat{\mathrm{J}}^2_{cd} \times [-\tau, \tau]$, respectively, we infer that $\omega(\daleth, \sigma) \to 0$ as $\sigma \to 0$. Hence, from above estimate $G \in C(B_\tau)$.

`Step 2.` Now, we examine the condensing mapping G. Select a fixed $\sigma > 0$ and $\mathbb{z} \in \mathsf{Y}$. For $(r_1,h_1), (r_2,h_2) \in \mathrm{J}^2_{cd}$ with $r_1 - r_2 \le \sigma$, $h_1 - h_2 \le \sigma$, we get

$$\begin{aligned}
\big|(G\mathbb{z})(r_2,h_2) - (G\mathbb{z})(r_1,h_1)\big| &= \Big|\Omega\Big(r_2, h_2, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_2} \int_0^{h_2} \tfrac{\daleth(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_2,h_2))\big)\Big) \\
&\quad - \Omega\Big(r_1, h_1, \tfrac{1}{\Gamma s_1 \Gamma s_2} \int_0^{r_1} \int_0^{h_1} \tfrac{\daleth(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r_1,h_1,\mathbb{z}(\phi(r_1,h_1))\big)\Big)\Big| \\
&\le \Big|\Omega\Big(r_2, h_2, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_2} \int_0^{h_2} \tfrac{\daleth(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_2,h_2))\big)\Big) \\
&\quad - \Omega\Big(r_2, h_2, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1} \int_0^{h_1} \tfrac{\daleth(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_2,h_2))\big)\Big)\Big| \\
&\quad + \Big|\Omega\Big(r_2, h_2, \tfrac{1}{\Gamma s_1 \Gamma s_2} \int_0^{r_1} \int_0^{h_1} \tfrac{\daleth(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\, \mathrm{d}g\, \mathrm{d}u, \mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_2,h_2))\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \Omega\Big(r_2, h_2, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} \tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r_1,h_1,\mathbb{z}(\phi(r_1,h_1))\big)\Big)\Big| \\
&+ \Big|\Omega\Big(r_2, h_2, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} \tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r_1,h_1,\mathbb{z}(\phi(r_1,h_1))\big)\Big) \\
&- \Omega\Big(r_1, h_1, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} \tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r_1,h_1,\mathbb{z}(\phi(r_1,h_1))\big)\Big)\Big| \\
&\leq c_1\Big|\tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_2}\int_0^{h_2} \tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&- \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} \tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\Big| \\
&+ c_2\big|\mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_2,h_2))\big) - \mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_1,h_1))\big)\big| \\
&+ c_2\big|\mathfrak{w}\big(r_2,h_2,\mathbb{z}(\phi(r_1,h_1))\big) - \mathfrak{w}\big(r_1,h_1,\mathbb{z}(\phi(r_1,h_1))\big)\big| + \omega_1(\Omega,\sigma) \\
&\leq c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\Big| \int_0^{r_1}\int_0^{h_1} \Big[\tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}} - \tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\Big]\,\mathrm{d}g\,\mathrm{d}u \\
&+ \int_{r_1}^{r_2}\int_{h_1}^{h_2} \tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u + \int_0^{r_1}\int_{h_1}^{h_2} \tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&+ \int_{r_1}^{r_2}\int_0^{h_1} \tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\Big| \\
&+ c_2c_3\big|\mathbb{z}\big(\phi(r_2,h_2)\big) - \mathbb{z}\big(\phi(r_1,h_1)\big)\big| + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma) \\
&\leq c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} \tfrac{|\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g))) - \gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))|}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&+ c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_0^{h_1} |\gimel(r_1,h_1,u,g,\mathbb{z}(\beta(u,g)))|\, \Big|\tfrac{1}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}} \\
&- \tfrac{1}{(r_1-u)^{1-s_1}(h_1-g)^{1-s_2}}\Big|\,\mathrm{d}g\,\mathrm{d}u + c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_{r_1}^{r_2}\int_{h_1}^{h_2} \tfrac{|\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))|}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&+ c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{r_1}\int_{h_1}^{h_2} \tfrac{|\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))|}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&+ c_1 \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_{r_1}^{r_2}\int_0^{h_1} \tfrac{|\gimel(r_2,h_2,u,g,\mathbb{z}(\beta(u,g)))|}{(r_2-u)^{1-s_1}(h_2-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u \\
&+ c_2c_3\,\big|\mathbb{z}\big(\phi(r_2,h_2)\big) - \mathbb{z}\big(\phi(r_1,h_1)\big)\big| + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma) \\
&\leq \tfrac{c_1\gimel^*\omega_1(\gimel,\sigma)}{\Gamma(s_1)\Gamma(s_2)}\Big[\tfrac{(r_2-r_1)^{s_1}}{s_1} - \tfrac{r_2^{s_1}}{s_1}\Big]\Big[\tfrac{(h_2-h_1)^{s_2}}{s_2} - \tfrac{h_2^{s_2}}{s_2}\Big] \\
&+ \tfrac{c_1\gimel^*}{\Gamma(s_1)\Gamma(s_2)}\Big[\tfrac{r_1^{s_1}h_1^{s_2}}{s_1s_2} - \Big(\tfrac{(r_2-r_1)^{s_1}-r_2^{s_1}}{s_1}\Big)\Big(\tfrac{(h_2-h_1)^{s_2}-h_2^{s_2}}{s_2}\Big)\Big] \\
&+ \tfrac{c_1\gimel^*}{\Gamma s_1\Gamma s_2}\Big[\tfrac{(r_2-r_1)^{s_1}(h_2-h_1)^{s_2}}{s_1s_2} + \tfrac{(h_2-h_1)^{s_2}\left[r_2^{s_1}-(r_2-r_1)^{s_1}\right]}{s_1s_2} \\
&+ \tfrac{(r_2-r_1)^{s_1}\left[h_2^{s_2}-(h_2-h_1)^{s_2}\right]}{s_1s_2}\Big] + c_2c_3\omega_1(\mathbb{z},\omega_1(\phi,\sigma)) + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma) \\
&\leq \tfrac{c_1\gimel^*\omega_1(\gimel,\sigma)r_1^{s_1}h_1^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)} + \tfrac{c_1\gimel^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[r_1^{s_1}h_1^{s_2} + (r_2-r_1)^{s_1}(h_2^{s_2} - (h_2-h_1)^{s_2}) \\
&+ r_2^{s_1}(h_2-h_1)^{s_2} - r_2^{s_1}h_2s_2\Big] + \tfrac{c_1\gimel^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[r_2^{s_1}(s_2-s_1)^{s_2} + (r_2-r_1)^{s_1}(h_2^{s_2}
\end{aligned}
$$

$$
\begin{aligned}
&- (h_2 - h_1)^{s_2}\Big)\Big] + c_2c_3\omega_1\big(\mathbb{z}, \omega_1(\phi,\sigma)\big) + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma)\\
&\le \tfrac{c_1\beth^*\omega_1(\beth,\sigma)r_1^{s_1}h_1^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)} + \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[(r_1-r_2)^{s_1}h_2^{s_2} + h_1^{s_2}(r_2-r_1)^{s_1}\\
&+ r_2^{s_1}(h_2-h_1)^{s_2}\Big] + \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[r_2^{s_1}(h_2-h_1)^{s_2} + h_1^{s_2}(r_2-r_1)^{s_1}\Big]\\
&+ c_2c_3\omega_1\big(\mathbb{z}, \omega_1(\phi,\sigma)\big) + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma)\\
&\le \tfrac{c_1\beth^*\omega_1(\beth,\sigma)\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)} + \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[\sigma^{s_1}\tilde{d}^{s_2} + \tilde{d}^{s_2}\sigma^{s_1} + \tilde{c}^{s_1}\sigma^{s_2}\Big]\\
&+ \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[c^{s_1}\sigma^{s_2} + \tilde{d}^{s_2}\sigma^{s_1}\Big] + c_2c_3\omega_1\big(\mathbb{z}, \omega_1(\phi,\sigma)\big) + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma).
\end{aligned}
$$

By choosing,

$$
\begin{aligned}
&\omega_1(\mathfrak{w},\sigma) = \sup\Big\{\big|\mathfrak{w}(r,h,z) - \mathfrak{w}(\grave{r},\grave{h},z)\big| \,:\, \big|r-\grave{r}\big| \le \sigma,\ \big|h-\grave{h}\big| \le \sigma, \mathbb{z} \in [-\tau,\tau]\Big\},\\
&\omega_1(\beth,\sigma) = \sup\Big\{\big|\beth(r,h,u,g,\mathbb{z}) - \beth(\grave{r},\grave{h},u,g,\mathbb{z})\big| \,:\, \big|r-\grave{r}\big| \le \sigma, \big|h-\grave{h}\big| \le \sigma, (r,h,u,g) \in \hat{\mathrm{J}}^2_{cd},\ z \in [-\tau,\tau]\Big\},\\
&\omega_1(\Omega,\sigma) = \sup\Big\{\big|\Omega(r,h,y,\mathbb{z},\mathbb{w}) - \Omega(\grave{r},\grave{h},y,\mathbb{z},\mathbb{w})\big| \,:\, \big|r-\grave{r}\big| \le \sigma,\ \big|h-\grave{h}\big| \le \sigma,\\
&\qquad \mathbb{w} \in [-\tau,\tau],\ \mathbb{z} \in \Big[-\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}, \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\Big\},
\end{aligned}
$$

the above relations deduce that,

$$
\begin{aligned}
\omega_1(G\mathsf{Y},\sigma) &\le \tfrac{c_1\beth^*\omega_1(\beth,\sigma)\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)} + \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[\sigma^{s_1}\tilde{d}^{s_2} + \tilde{d}^{s_2}\sigma^{s_1} + \tilde{c}^{s_1}\sigma^{s_2}\Big]\\
&+ \tfrac{c_1\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big[\tilde{c}^{s_1}\sigma^{s_2} + \tilde{d}^{s_2}\sigma^{s_1}\Big] + c_2c_3\omega_1(\mathsf{Y}, \omega_1(\phi,\sigma)) + \omega_1(\Omega,\sigma) + c_2\omega_1(\mathfrak{w},\sigma).
\end{aligned}
$$

Taking limit as $\sigma \to 0$, $\omega_1(G\mathsf{Y},\sigma) \le c_2c_3\,\omega_1(\mathsf{Y})$, i.e. $\mathfrak{M}_H(G\mathsf{Y}) \le c_2c_3\,\mathfrak{M}_H(\mathsf{Y})$, which explains that G is a densifying mapping.

`Step 3`. Now, let $\mathbb{z} \in \partial B_\tau$ and if $G\mathbb{z} = k\mathbb{z}$ then, we get $\|G\mathbb{z}\| = k\|\mathbb{z}\| = k\tau$ and by assumption (A2),

$$
|G\mathbb{z}(r,h)| = \left|\Omega\left(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{\beth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big)\right)\right| \le \tau,
$$

for each $(r,h) \in \mathrm{J}^2_{cd}$, hence $\|G\mathbb{z}\| \le \tau$ i.e., $k \le 1$. □

Corollary 3.2. *Assume that $\Omega \in C\big(\mathrm{J}^2_{cd} \times \mathbb{R}^2\big)$, $\beth \in C\big(\hat{\mathrm{J}}^2_{cd} \times \mathbb{R}\big)$ with $\hat{\mathrm{J}}^2_{cd}$ which is defined in $\mathbb{FIE}$ (1.1), $\phi, \beta : \mathrm{J}^2_{cd} \to \mathrm{J}^2_{cd}$ and let the following hypothesis hold,*

- **A3:** *The exist c_1, c_2 with $c_2 < 1$ such that $\big|\Omega(r,h,\mathbb{z},\mathbb{w}) - \Omega(r,h,\grave{\mathbb{z}},\grave{\mathbb{w}})\big| \le c_1\big|\mathbb{z}-\grave{\mathbb{z}}\big| + c_2\big|\mathbb{w}-\grave{\mathbb{w}}\big|$;*
- **A4:** *There exists $\tau > 0$ such that Ω satisfies the following bounded condition*

$$
\sup\Big\{\big|\Omega(r,h,\mathbb{z},\mathbb{w})\big| \,:\, (r,h) \in \mathrm{J}^2_{cd}, \mathbb{w} \in [-\tau,\tau],\ \mathbb{z} \in \Big[-\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}, \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\beth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\Big\} \le \tau.
$$

If $\Omega(r,h,\mathbb{z},\mathbb{w}) = \hat{\Omega}(r,h,\mathbb{z},\mathbb{w})$ and $\mathfrak{w}(r,h,\mathbb{w}) = \mathbb{z}(r,h)$, then the following $\mathbb{IE}$,

$$
\mathbb{z}(r,h) = \Omega\left(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{\beth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathbb{z}\big(\phi(r,h)\big)\right), \tag{3.1}
$$

under (A3) and (A4) has at least one solution in $C\big(\mathrm{J}^2_{cd}\big)$,

Proof. The proof is similar to Theorem 3.1, so we omit the proof. □

Corollary 3.3. *Assume that $\Omega \in C\big(\mathrm{J}^2_{cd} \times \mathbb{R}\big)$, $\mathfrak{w} \in C\big(\mathrm{J}^2_{cd} \times \mathbb{R}\big)$, $\beth \in C\big(\hat{\mathrm{J}}^2_{cd} \times \mathbb{R}\big)$ with $\hat{\mathrm{J}}^2_{cd}$ which is defined in $\mathbb{FIE}$ (1.1), $\phi, \beta : \mathrm{J}^2_{cd} \to \mathrm{J}^2_{cd}$ and let the following hypothesis hold,*

- **A5:** *There exist non-negative constants c_1, c_2 with $c_1 < 1$ such that*

$$
\big|\Omega(r,h,\mathbb{z}) - \Omega(r,h,\grave{\mathbb{z}}\big| \le c_1\big|\mathbb{z}-\grave{\mathbb{z}}\big|, \qquad \big|\mathfrak{w}(r,h,\mathbb{w}) - \mathfrak{w}(r,h,\grave{\mathbb{w}})\big| \le c_2\big|\mathbb{w}-\grave{\mathbb{w}}\big|;
$$

A6: *There exists* $\tau > 0$ *such that* Ω *satisfies the following bounded condition*

$$\sup\left\{\left|\mathfrak{w}(r,h,\mathbb{w})+\Omega(r,h,\mathbb{z})\right| \,:\, (r,h)\in \mathrm{J}^2_{cd},\ \mathbb{z}\in\left[-\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\daleth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}, \tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}\daleth^*}{\Gamma(s_1+1)\Gamma(s_2+1)}\right]\right\}\leq \tau.$$

If $\Omega(r,h,\mathbb{z},\mathbb{w}) = \mathfrak{w}(r,h,\mathbb{w}) + \tilde{\Omega}(r,h,z)$ then the following $\mathbb{IE}$,

$$\mathbb{z}(r,h) = \mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big) + \Omega\bigg(r,h,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\bigg), \tag{3.2}$$

under (A5) and (A6) has at least one solution in $C\big(\mathrm{J}^2_{cd}\big)$.

Proof. The proof is similar to Theorem 3.1, so we leave it to the readers. □

4. Illustrative examples

In the first example, we check Theorem 3.1 when fractional order s_1 of $\mathbb{FIE}$ changes.

Example 4.1. In view of the proposed problem (1.1), consider the following 2D Volterra $\mathbb{FIE}$,

$$\begin{aligned}\mathbb{z}(r,h) &= \tfrac{r}{2}h\exp\left(-\tfrac{r+h}{2}\right) + \tfrac{r+h}{4\Gamma(s_1)\Gamma(1/2)}\\ &\times\int_0^r\int_0^h \frac{\frac{gu}{2}\sin(\mathbb{z}(u,g))+\frac{4u+g}{10}\tan^{-1}\left(\frac{|\mathbb{z}(u,g)|}{1+|\mathbb{z}(u,g)|}\right)}{(r-u)^{1-s_1}(h-g)^{1/2}}\,\mathrm{d}g\,\mathrm{d}u + \tfrac{rh}{2+h^2}\mathbb{z}\left(\sqrt{r},\sqrt{h}\right),\end{aligned} \tag{4.1}$$

for $s_1 \in \left\{\frac{2}{3},\frac{3}{4},\frac{8}{9}\right\}$. Here, we study the solution in $C\big(\mathrm{J}^2_{cd}\big)$ with $\tilde{c} = \tilde{d} = 1$. We have $s_2 = \frac{1}{2}$, $\phi(r,h) = \big(\sqrt{r},\sqrt{h}\big)$, $\beta(u,g) = (u,g)$ and for each $(r,h),(u,g)\in \mathrm{J}^2_{cd}$, $\Omega(r,h,\mathbb{z},\mathbb{w}) = \frac{r}{2}h\exp\left(-\frac{r+h}{2}\right) + \mathbb{z} + \mathbb{w}$,

$$\begin{aligned}&\mathbb{z} = \tfrac{1}{\Gamma(s_1)\Gamma(1/2)}\int_0^r\int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1/2}}\,\mathrm{d}g\,\mathrm{d}u, \quad \mathbb{w} = \mathfrak{w}(r,h,\mathbb{z}) = \tfrac{rh}{2+h^2}\mathbb{z},\\ &\daleth\big(r,h,u,g,\mathbb{z}(\beta(u,g))\big) = \tfrac{r+h}{4}\left(\tfrac{g}{2}u\sin\left(\mathbb{z}(u,g)\right) + \tfrac{4u+g}{10}\tan^{-1}\left(\tfrac{|\mathbb{z}(u,g)|}{1+|\mathbb{z}(u,g)|}\right)\right),\end{aligned}$$

such that $\left|\daleth(r,h,u,g,\mathbb{z})\right| \leq \frac{1}{4}\left[1+|\mathbb{z}|\right]$ and $\left|\mathfrak{w}(r,h,\mathbb{z})\right| \leq \frac{1}{3}|\mathbb{z}|$. One can easy understand that (A1) holds. We review the correctness (A2)d. Let $\|\mathbb{z}\| \leq \tau$, $\tau > 0$, then by choosing $\tau = 0.65$, we can easily verify

$$\begin{aligned}|\mathbb{z}(r,h)| &= \bigg|\tfrac{r}{2}h\exp\left(-\tfrac{r+h}{2}\right) + \tfrac{rh}{2+h^2}\mathbb{z}\left(\sqrt{r},\sqrt{h}\right)\\ &\quad + \tfrac{r+h}{4\Gamma(s_1)\Gamma(1/2)}\int_0^r\int_0^h \frac{\frac{g}{2}u\sin(\mathbb{z}(u,g))+\frac{4u+g}{10}\tan^{-1}\left(\frac{|\mathbb{z}(u,g)|}{1+|\mathbb{z}(u,g)|}\right)}{(r-u)^{1-s_1}(h-g)^{1/2}}\,\mathrm{d}g\,\mathrm{d}u\bigg|\\ &\quad \leq \tfrac{1}{2} + \tfrac{1}{3}\tau + \tfrac{1}{4\Gamma(5/3)\Gamma(3/2)}(1+\tau) \approx \left\{\begin{array}{ll} 0.5481, & s_1 = 2/3,\\ 0.5393, & s_1 = 3/4,\\ 0.5250, & s_1 = 8/9,\end{array}\right\} \leq \tau.\end{aligned} \tag{4.2}$$

Hence (A2) holds. Figs. 1a, 1b and 1c show the 3D plot of Ineq. (4.2) for $s_1 = \frac{2}{3}, \frac{3}{4}, \frac{8}{9}$ whenever $\tau = 0.65$, respectively. These numerical results are shown in Table 1. Thus, Theorem 3.1 implies $\mathbb{FIE}$ (4.1) has at least one solution in $C\big(\mathrm{J}^2_{cd}\big)$.

In the next example, Theorem 3.1 is examined when fractional order s_2 of $\mathbb{FIE}$ changes.

Example 4.2. Consider the following nonlinear 2D Volterra $\mathbb{FIE}$,

$$\begin{aligned}\mathbb{z}(r,h) &= \tfrac{1}{4}\sin\left(\tfrac{1}{1+r+h}\right) + \tfrac{\exp(rh)}{(r+h+4)\Gamma(2/5)\Gamma(s_2)}\int_0^r\int_0^h \frac{\frac{g}{2}u\exp\left(-\frac{g+u}{2}\right)\sin\left(\sqrt{g+u}\right)+\sqrt{1+\mathbb{z}(u,g)}}{(r-u)^{3/5}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\ &\quad + \tfrac{\exp(r)}{4+\sqrt{r+h}}\sin\big(\mathbb{z}(r,h)\big) + \tfrac{rh}{1+r^2+h}\ln\big(1+|\mathbb{z}(r,h)|\big),\end{aligned} \tag{4.3}$$

TABLE 1. Numerical results in Example 4.1 for different values of $s_1 = \frac{2}{3}, \frac{3}{4}, \frac{8}{9}$.

(r,h)	Ineq. (4.2)		
	$s_1 = 2/3$	$s_1 = 3/4$	$s_1 = 8/9$
$(0.00, 0.00)$	0.0000	0.0000	0.0000
$(0.00, 0.10)$	0.0000	0.0000	0.0000
$(0.00, 0.20)$	0.0000	0.0000	0.0000
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.10, 0.30)$	0.0219	0.0218	0.0218
$(0.10, 0.40)$	0.0281	0.0280	0.0279
$(0.10, 0.50)$	0.0338	0.0336	0.0334
$(0.10, 0.60)$	0.0388	0.0386	0.0383
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.10, 0.70)$	0.0433	0.0430	0.0426
$(0.10, 0.80)$	0.0473	0.0469	0.0463
$(0.10, 0.90)$	0.0508	0.0503	0.0496
$(0.10, 1.00)$	0.0539	0.0532	0.0523
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.40, 0.20)$	0.0574	0.0571	0.0567
$(0.40, 0.30)$	0.0834	0.0829	0.0822
$(0.40, 0.40)$	0.1074	0.1067	0.1057
$(0.40, 0.50)$	0.1294	0.1284	0.1270
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1.00, 0.70)$	0.4178	0.4126	0.4043
$(1.00, 0.80)$	0.4637	0.4574	0.4473
$(1.00, 0.90)$	0.5070	0.4995	0.4874
$(1.00, 1.00)$	0.5482	0.5394	0.5250

for $s_2 \in \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$. Here, we study the solution in $C\big(\mathrm{J}^2_{cd}\big)$, with $\tilde{c} = \tilde{d} = 1$. These data show $s_1 = \frac{2}{5}$, and for every $(r,h) \in \mathrm{J}^2_{cd}$, $\phi(r,h) = \beta(r,h) = (r,h)$, $\Omega(r,h,\mathbb{z},\mathbb{w}) = \frac{1}{4}\sin\left(\frac{1}{1+r+h}\right) + \mathbb{z} + \mathbb{w}$,

$$\begin{aligned}
\mathbb{z} &= \tfrac{1}{\Gamma(2/5)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{3/5}(h-g)^{1-s_2}} \,\mathrm{d}g\,\mathrm{d}u,\\
\mathbb{w} &= \mathfrak{w}(r,h,\mathbb{z}) = \tfrac{\exp(r)}{4+\sqrt{r+h}} \sin(\mathbb{z}) + \tfrac{rh}{1+r^2+h} \ln(1+|\mathbb{z}|),
\end{aligned}$$

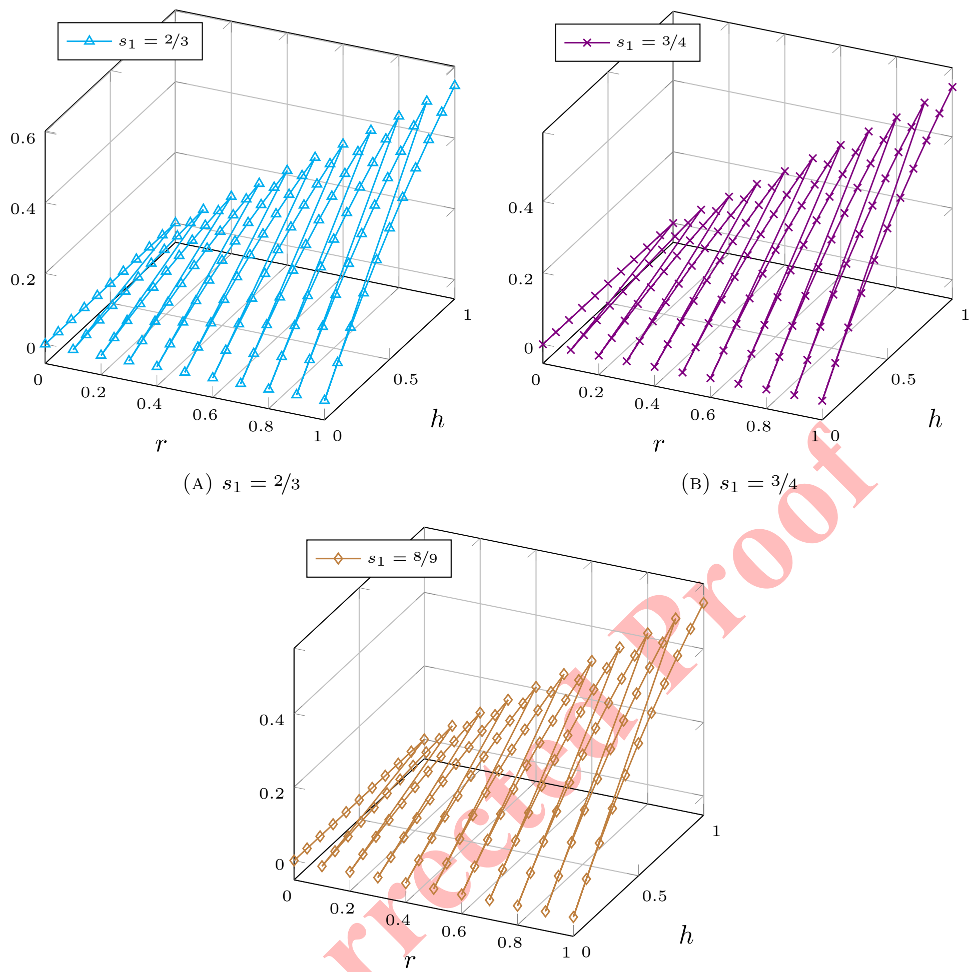
and

$$\daleth\big(r,h,u,g,\mathbb{z}(\beta(u,g))\big) = \tfrac{\exp(rh)}{(r+h+4)} \left(\tfrac{g}{2} u \exp\left(-\tfrac{g+u}{2}\right)\sin\left(\sqrt{g+u}\right) + \sqrt{1+\mathbb{z}(u,g)}\right),$$

such that $\big|\daleth(r,h,u,g,\mathbb{z})\big| \le \frac{1}{4}\left[\frac{1}{2} + \sqrt{1+|\mathbb{z}|}\,\right]$ and $\big|\mathfrak{w}(r,h,z)\big| \le \frac{1}{4} + \frac{1}{3}|\mathbb{z}|$. Clearly, assumption (A1) holds. We check the assumption (A2). Suppose that $\|\mathbb{z}\| \le \tau$, $\tau > 0$. Then by choosing $\tau = 2.73$, we obtain

$$\begin{aligned}
|\mathbb{z}(r,h)| &= \bigg|\tfrac{1}{4}\sin\left(\tfrac{1}{1+r+h}\right) + \tfrac{\exp(rh)}{(r+h+4)\Gamma(2/5)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\frac{g}{2}u\exp\left(-\frac{g+u}{2}\right)\sin\left(\sqrt{g+u}\right)+\sqrt{1+\mathbb{z}(u,g)}}{(r-u)^{3/5}(h-g)^{1-s_2}} \,\mathrm{d}g\,\mathrm{d}u\\
&\qquad + \tfrac{\exp(r)}{4+\sqrt{r+h}} \sin\big(\mathbb{z}(r,h)\big) + \tfrac{rh}{1+r^2+h} \ln\big(1+|\mathbb{z}(r,h)|\big)\bigg|\\
&\le \tfrac{1}{2} + \tfrac{1}{3}\tau + \tfrac{1}{4\Gamma(7/5)\Gamma(s_2)}\left(\tfrac{1}{2} + \sqrt{1+\tau}\right) \approx \left\{\begin{array}{ll} 0.5481, & s_2 = 1/4,\\ 0.5393, & s_2 = 1/3,\\ 0.5250, & s_2 = 1/2, \end{array}\right\} \le \tau.
\end{aligned} \tag{4.4}$$

Indeed (A2) holds. Figs. 2a, 2b and 2c show the 3D plot of Ineq. (4.4) for $s_2 = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ whenever $\tau = 2.73$, respectively. These numerical results are shown in Table 2. By satisfying all assumption of Theorem 3.1, one can conclude that $\mathbb{FIE}$ (4.3) has at least one solution in $C\big(\mathrm{J}^2_{cd}\big)$.



(A) $s_1 = 2/3$

(B) $s_1 = 3/4$

(C) $s_1 = 8/9$

FIGURE 1. Representaion of Ineq. (4.2) for Volterra $\mathbb{FIE}$ (4.1) with three different values of s_1 in Example 4.1 when $\tau = 0.65$.

5. Uniqueness of the solution

Presently, we will address and prove the uniqueness of the solutions.

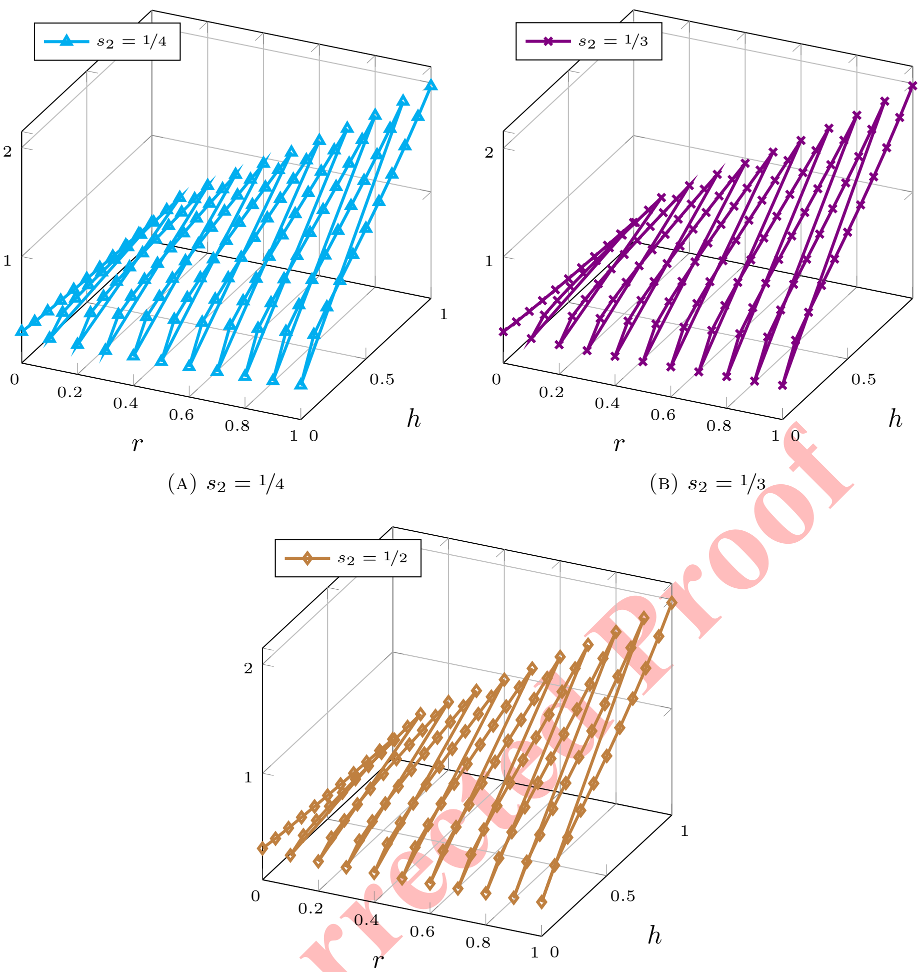
Theorem 5.1. *Consider the assumptions of Theorem 3.1 with the following hypothesis,*

A7: *There exists* $b_1 \geq 0$ *such that* $\left|\daleth(r,h,u,g,\mathbb{z}) - \daleth(r,h,u,g,\grave{\mathbb{z}})\right| \leq b_1 \left|\mathbb{z} - \grave{\mathbb{z}}\right|$ *for* $\mathbb{z}, \grave{\mathbb{z}} \in B_\tau$*, where* B_τ *is as in Theorem 3.1;*

A8: *With characterized* c_1, c_2, c_3 *and* τ *in hypothesis* (A1)*, assume that,*

$$\Lambda = c_2 c_3 + c_1 b_1 \tfrac{\tilde{c}^{s_1} \tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)} < 1.$$

Then $\mathbb{FIE}$ (1.1) *has a unique solution* $\mathbb{z}$ *in* B_τ*.*



(A) $s_2 = 1/4$

(B) $s_2 = 1/3$

(C) $s_2 = 1/2$

FIGURE 2. Representaion of Ineq. (4.4) for Volterra $\mathbb{FIE}$ (4.3) with three different values of s_2 in Example 4.2 when $\tau = 2.73$.

Proof. For two various solutions of $\mathbb{FIE}$ (1.1), $\mathbb{z}, \grave{\mathbb{z}} \in B_\tau$, we have

$$
\begin{aligned}
\big|\mathbb{z} - \grave{\mathbb{z}}\big| &= \big|(G\mathbb{z})(r,h) - (G\grave{\mathbb{z}})(r,h)\big| \\
&= \bigg|\Omega\bigg(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}} \,\mathrm{d}g \,\mathrm{d}u, \mathfrak{w}\big(r, h, \mathbb{z}(\phi(r,h))\big)\bigg) \\
&\quad - \Omega\bigg(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}} \,\mathrm{d}g \,\mathrm{d}u, \mathfrak{w}\big(r, h, \grave{\mathbb{z}}(\phi(r,h))\big)\bigg)\bigg| \\
&\leq \bigg|\Omega\bigg(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}} \,\mathrm{d}g \,\mathrm{d}u, \mathfrak{w}\big(r, h, \mathbb{z}(\phi(r,h))\big)\bigg) \\
&\quad - \Omega\bigg(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^r \int_0^h \tfrac{\daleth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}} \,\mathrm{d}g \,\mathrm{d}u, \mathfrak{w}\big(r, h, \grave{\mathbb{z}}(\phi(r,h))\big)\bigg)\bigg|
\end{aligned}
$$

TABLE 2. Numerical results in Example 4.2 for different values of $s_2 = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

(r,h)	Ineq. (4.2)		
	$s_2 = 1/4$	$s_2 = 1/3$	$s_2 = 1/2$
$(0.00, 0.00)$	0.3104	0.3104	0.3104
$(0.00, 0.10)$	0.2899	0.2899	0.2899
$(0.00, 0.20)$	0.2750	0.2750	0.2750
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.10, 0.10)$	0.4258	0.4048	0.3708
$(0.10, 0.20)$	0.4450	0.4279	0.3969
$(0.10, 0.30)$	0.4558	0.4421	0.4153
$(0.10, 0.40)$	0.4628	0.4522	0.4296
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.20, 0.40)$	0.5608	0.5464	0.5158
$(0.20, 0.50)$	0.5784	0.5675	0.5424
$(0.20, 0.60)$	0.5934	0.5859	0.5662
$(0.20, 0.70)$	0.6067	0.6023	0.5880
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0.70, 0.00)$	0.3053	0.3053	0.3053
$(0.70, 0.10)$	0.6187	0.5757	0.5062
$(0.70, 0.20)$	0.7311	0.6932	0.6249
$(0.70, 0.30)$	0.8275	0.7947	0.7307
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1.00, 0.70)$	1.4932	1.4757	1.4255
$(1.00, 0.80)$	1.6417	1.6317	1.5932
$(1.00, 0.90)$	1.8002	1.7988	1.7744
$(1.00, 1.00)$	1.9704	1.9790	1.9714

$$
\begin{aligned}
&+\Big|\Omega\Big(r,h,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{\beth(r,h,u,g,\mathbb{z}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big)\\
&-\Omega\Big(r,h,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{\beth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u, \mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\Big)\Big|\\
&\le c_2\big|\mathfrak{w}\big(r,h,\mathbb{z}(\phi(r,h))\big)-\mathfrak{w}\big(r,h,\grave{\mathbb{z}}(\phi(r,h))\big)\big|\\
&+c_1\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^r\int_0^h \tfrac{|\beth(r,h,u,g,\mathbb{z}(\beta(u,g)))-\beth(r,h,u,g,\grave{\mathbb{z}}(\beta(u,g)))|}{(r-u)^{1-s_1}(h-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&\le c_2c_3\big|\mathbb{z}\big(\phi(r,h)\big)-\grave{\mathbb{z}}\big(\phi(r,h)\big)\big|+c_1\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}b_1\|\mathbb{z}-\grave{\mathbb{z}}\|\\
&\le\Big[c_2c_3+c_1b_1\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\,\|\mathbb{z}-\grave{\mathbb{z}}\|.
\end{aligned}
$$

This implies $\|(G\mathbb{z})(r,h)-(G\grave{\mathbb{z}})(r,h)\|\le\Lambda\|\mathbb{z}-\grave{\mathbb{z}}\|$. Thus the defined operator G is a contraction, and hence Banach contraction principle guarantees that G has a unique FP which is the solution of $\mathbb{FIE}$ (1.1). □

6. Numerical method

At present, we apply the successive approximations method to solve the following nonlinear $\mathbb{FIE}$ of two variables,

$$
\mathbb{z}(r,h)=\Omega\Big(r,h,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde{c}}\int_0^{\tilde{d}}\tfrac{\beth(r,h,u,g,\mathbb{z}(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u,\mathfrak{w}\big(r,h,\mathbb{z}(r,h)\big)\Big). \tag{6.1}
$$

Consider a uniform partition $D=(\Delta_1,\Delta_2)$ of J^2_{cd} with

$$
\begin{aligned}
\Delta_1:&\quad 0=r_0<r_1<r_2<\cdots<r_{n-1}<r_n=\tilde{c},\\
\Delta_2:&\quad 0=h_0<h_1<h_2<\cdots<h_{n-1}<h_n=\tilde{d},
\end{aligned}
$$

with $r_i = i\delta_1$, $h_j = j\delta_2$, $i, j = 0, \dots, n$, where $\delta_1 = \frac{1}{n}\tilde{c}$, $\delta_2 = \frac{1}{n}\tilde{d}$. We proved the continuity G on the ball B_τ in proof of Theorem 3.1 with a FP, $\mathbb{z}$ in B_τ, that is,

$$\mathbb{z}(r,h) = (G\mathbb{z})(r,h) = \Omega\bigg(r, h, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{\beth(r,h,u,g,\mathbb{z}(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du, \mathfrak{w}\big(r,h,\mathbb{z}(r,h)\big)\bigg). \tag{6.2}$$

Now, we use a FP technique on G for $\mathbb{z}_0(r,h) = f(r,h) \in B_\tau$ and the points $r_i = i\delta_1$, $h_j = j\delta_2$,

$$\begin{aligned} \mathbb{z}_{M+1}(r_i,h_j) = (G\mathbb{z}_M)(r_i,h_j) = \Omega\bigg(& r_i, h_j, \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \\ & \times \int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{\beth(r_i,h_j,u,g,\mathbb{z}_M(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du, \mathfrak{w}\big(r_i,h_j,\mathbb{z}_M(r_i,h_j)\big)\bigg). \end{aligned} \tag{6.3}$$

Now, by using a composite midpoint rule to approximate the double integral on J^2_{cd} for n equally distant points in (6.3) numerically and assessment $\mathbb{z}_M(r_i,h_j)$. The suitable estimation of this words shows to the necessity to calculate the double integrals in J^2_{cd}. To do this, we utilize the combined midpoint quadrature rule to computation the integrals for points $u_p = p\delta_1$, $g_q = q\delta_2$, $p, q = 0, 1, \dots, n$.

$$\int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{\beth(r_i,h_j,u,g,\mathbb{z}_M(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du \tfrac{\tilde{c}\tilde{d}}{n^2} \sum_{q=0}^{n-1} \sum_{p=0}^{n-1} \tfrac{\beth\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\mathbb{z}_M\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}. \tag{6.4}$$

Then the below mentioned iterative process, acquired by calculating the corresponding double integral with the quadrature equation provides the approximate solution of (6.3) in point $(r_i,h_j) \in \mathrm{J}^2_{cd}$, $i, j = 0, \dots, n$,

$$\begin{cases} \bar{\mathbb{z}}_0(r_i,h_j) = f(r_i,h_j), \\ \bar{\mathbb{z}}_M(r_i,h_j) = \Omega\Bigg(r_i, h_j, \tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)} \sum_{q=0}^{n-1} \sum_{p=0}^{n-1} \tfrac{\beth\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_{M-1}\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}, \\ \qquad \mathfrak{w}\big(r_i,h_j,\bar{\mathbb{z}}_{M-1}(r_i,h_j)\big)\Bigg), \qquad M \in \mathbb{N}. \end{cases} \tag{6.5}$$

Theorem 6.1. *Assume that all hypotheses of Theorem 3.1 satisfy. The error estimation between the approximate solutions $(\mathbb{z}_M)_{M\in\mathbb{N}}$ of (6.3) and the exact $\mathbb{z}^*$ is given by*

$$\|\mathbb{z}^* - \mathbb{z}_M\| \le \tfrac{\Lambda^M}{1-\Lambda}\|\mathbb{z}_1 - \mathbb{z}_0\|, \qquad \Lambda = c_2c_3 + c_1b_1\tfrac{\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}. \tag{6.6}$$

Proof. First, we see that

$$\begin{aligned} \big|\mathbb{z}_M(r,h) - \mathbb{z}^*(r,h)\big| &= \big|(G\mathbb{z}_{M-1})(r,h) - (G\mathbb{z}^*)(r,h)\big| \\ &\le c_1\bigg|\tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{\beth(r,h,u,g,\mathbb{z}_{M-1}(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du \\ &\quad - \tfrac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{\beth(r,h,u,g,\mathbb{z}^*(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du\bigg| \\ &\quad + c_2\big|\mathfrak{w}\big(r,h,z_{M-1}(r,h)\big) - \mathfrak{w}\big(r,h,z^*(r,h)\big)\big| \\ &\le \tfrac{c_1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{\tilde{c}} \int_0^{\tilde{d}} \tfrac{|\beth(r,h,u,g,\mathbb{z}_{M-1}(u,g)) - \beth(r,h,u,g,\mathbb{z}^*(u,g))|}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}} \, dg \, du \\ &\quad + c_2c_3\big|\mathbb{z}_{M-1}(r,h) - \mathbb{z}^*(r,h)\big| \\ &\le \tfrac{c_1b_1r^{s_1}h^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\big|\mathbb{z}_{M-1}(r,h) - \mathbb{z}^*(r,h)\big| + c_2c_3\big|\mathbb{z}_{M-1}(r,h) - \mathbb{z}^*(r,h)\big| \\ &\le \Lambda\big\|\mathbb{z}_{M-1} - \mathbb{z}^*\big\|. \end{aligned} \tag{6.7}$$

Also, we have,

$$\|\mathbb{z}^* - \mathbb{z}_{M-1}\| \leq \|\mathbb{z}^* - \mathbb{z}_M\| + \|\mathbb{z}_M - \mathbb{z}_{M-1}\|. \tag{6.8}$$

Combining (6.7) and (6.8), we have

$$\|\mathbb{z}^* - \mathbb{z}_M\| \leq \tfrac{\Lambda}{1-\Lambda}\|\mathbb{z}_M - \mathbb{z}_{M-1}\|. \tag{6.9}$$

Moreover,

$$\begin{aligned}\|\mathbb{z}_M - \mathbb{z}_{M-1}\| &= \left|(G\mathbb{z}_{M-1})(r,h) - G\mathbb{z}_{M-2})(r,h)\right| \\ &\leq \tfrac{c_1 b_1 \tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\left|\mathbb{z}_{M-1}(r,h) - \mathbb{z}_{M-2}(r,h)\right| + c_2c_3\left|\mathbb{z}_{M-1}(r,h) - \mathbb{z}_{M-2}(r,h)\right| \\ &= \Lambda\|\mathbb{z}_{M-1} - \mathbb{z}_{M-2}\|.\end{aligned}$$

By induction, we get

$$\|\mathbb{z}_M - \mathbb{z}_{M-1}\| \leq \Lambda\|\mathbb{z}_{M-1} - \mathbb{z}_{M-2}\| \leq \cdots \leq \Lambda^{M-1}\|\mathbb{z}_1 - \mathbb{z}_0\|. \tag{6.10}$$

From (6.9) and (6.10), we obtain $\|\mathbb{z}^* - \mathbb{z}_M\| \leq \frac{\Lambda^M}{1-\Lambda}\|\mathbb{z}_1 - \mathbb{z}_0\|$. □

7. Convergence analysis

Now, we investigate the convergence of the iterative proposed method to the solution of (6.2). Let K and $\daleth$, respectively on J^2_{cd} and $\hat{\mathrm{J}}^2_{cd} \times \mathbb{R}$ be bounded functions, for all $(r,h) \in \mathrm{J}^2_{cd}$ and $\mathbb{z} \in C(\mathrm{J}^2_{cd})$. We define

$$\begin{aligned}\omega_{\mathrm{J}^2_{cd}}(r,h)\,(K\daleth\mathbb{z},\varepsilon) = \sup_{(u_1,g_1),(u_2,g_2)\in \mathrm{J}^2_{cd}} &\Big\{\Big|K(u_1,g_1)\daleth\big(r,h,u_1,g_1,\mathbb{z}(u_1,g_1)\big) \\ &- K(u_2,g_2)\daleth\big(r,h,u_2,g_2,\mathbb{z}(u_2,g_2)\big)\Big| : |u_1-g_1|, |u_2-g_2| \leq \varepsilon\Big\},\\ \omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z},\varepsilon) = \sup_{(u_1,g_1),(u_2,g_2)\in \mathrm{J}^2_{cd}} &\Big\{\Big|\daleth\big(r_1,h_1,u_1,g_1,\mathbb{z}(u_1,g_1)\big) \\ &- \daleth\big(r_2,h_2,u_2,g_2,\mathbb{z}(u_2,g_2)\big)\Big| : .|u_1-g_1|, |u_2-g_2| \leq \varepsilon\Big\},\end{aligned}$$

and

$$\omega_2(\daleth,\varepsilon) = \sup_{(u_1,g_1),(u_2,g_2)\in \mathrm{J}^2_{cd}} \Big\{\Big|\daleth\big(r,h,u_1,g_1,\mathbb{z}(u,g)\big) - \daleth\big(r,h,u_2,g_2,\mathbb{z}(u,g)\big)\Big| : |u_1-g_1|, |u_2-g_2| \leq \varepsilon\Big\}.$$

The following corollary is the classical midpoint rule with a new error bound.

Lemma 7.1. *Let $\psi : \hat{\mathrm{J}}^2_{cd} \times \mathbb{R} \to \mathbb{R}$ be a 2D integrable bounded mapping. Then, the following inequality holds:*

$$\left|\int_0^{\tilde{c}}\int_0^{\tilde{d}} \psi\left(r,h,u,g\right) \mathrm{d}g\,\mathrm{d}u - \tilde{c}\,\tilde{d}\psi\left(r,h,\tfrac{\tilde{c}}{2},\tfrac{\tilde{d}}{2}\right)\right| \leq \tilde{c}\tilde{d}\omega_{[0,\tilde{c}][0,\tilde{d}]}\left(\psi,\tfrac{\tilde{c}}{2}+\tfrac{\tilde{d}}{2}\right),$$

where $\psi\big(r,h,u,g\big) = K(u,g)\daleth\big(r,h,u,g,\mathbb{z}(u,g)\big) = K\daleth\mathbb{z}$ and $K(u,g) = \frac{1}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}$, for $0 < u < \tilde{c}$ and $0 < g < \tilde{d}$.

Proof. The proof is similar to Corollary 2.5 in [25] and the proof is omitted. □

Lemma 7.2. *Let $z \in C(\mathrm{J}^2_{cd})$, $K \in C(\mathrm{J}^2_{cd})$ and $\daleth \in C\big(\hat{\mathrm{J}}^2_{cd} \times \mathbb{R}\big)$. Then*

$$\omega_{\mathrm{J}^2_{cd}}(r,h)\big(K\daleth\mathbb{z},\delta\big) \leq M_K\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z},\delta) + \daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta), \qquad \forall\ (r,h) \in \mathrm{J}^2_{cd}. \tag{7.1}$$

Let $z_M \in C(\mathrm{J}^2_{cd})$ be a sequence defined in 6.3, then

$$\omega_{\mathrm{J}^2_{cd}}(\daleth z_M,\delta) \leq \tfrac{c_1 b_1 \tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\omega_1(\daleth,\delta) + c_2 b_1\omega_1(\mathfrak{w},\delta) + \omega_2(\daleth,\delta), \qquad \delta := \delta_1 + \delta_2. \tag{7.2}$$

Proof. For all $(r,h),(u,g),(u',g')\in \mathrm{J}^2_{cd}$, we get

$$\begin{aligned}
&\big|K(u,g)\gimel\big(r,h,u,g,\mathbb{z}(u,g)\big)-K(u',g')\gimel\big(r,h,u',g',\mathbb{z}(u',g')\big)\big|\\
&\qquad\le \big|K(u,g)\gimel\big(r,h,u,g,\mathbb{z}(u,g)\big)-K(u,g)\gimel\big(r,h,u',g',\mathbb{z}(u',g')\big)\big|\\
&\qquad+\big|K(u,g)\gimel\big(r,h,u',g',\mathbb{z}(u',g')\big)-K(u',g')\gimel\big(r,h,u',g',\mathbb{z}(u',g')\big)\big|\\
&\qquad\le |K(u,g)|\,\big|\gimel\big(r,h,u,g,\mathbb{z}(u,g)\big)-\gimel\big(r,h,u',g',\mathbb{z}(u',g')\big)\big|\\
&\qquad+\big|\gimel\left(r,h,u',g',\mathbb{z}(u',g')\right)\big|\,\big|K(u,g)-K(u',g')\big|,
\end{aligned}$$

Taking supremum from the above inequality, we get the following result,

$$\omega_{\mathrm{J}^2_{cd}}(r,h)\big(K\gimel\mathbb{z},\delta\big)\le M_K\omega_{\mathrm{J}^2_{cd}}(\gimel\mathbb{z},\delta)+\gimel^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta), \tag{7.3}$$

where, $M_K=\max_{(u,g)\in \mathrm{J}^2_{cd}}\big\{K(u,g)\ :\ u=u_p+\frac{\delta_1}{2},\ g=g_q+\frac{\delta_2}{2},\ p,q=0,1,2,\dots,n-1\big\}$. Therefore, Ineq. (7.1), is obtained. For each $(r_1,h_1),(r_2,h_2)\in \mathrm{J}^2_{cd}$ and $M\in\mathbb{N}$, we have,

$$\begin{aligned}
\big|\mathbb{z}_{M+1}(r_1,h_1)-\mathbb{z}_{M+1}(r_2,h_2)\big| &= \big|G\mathbb{z}_{\mathbb{M}}(r_1,h_1)-G\mathbb{z}_{\mathbb{M}}(r_2,h_2)\big|\\
&\le c_1\bigg|\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde c}\int_0^{\tilde d}\tfrac{\gimel(r_1,h_1,u,g,\mathbb{z}_{M-1}(u,g))}{(\tilde c-u)^{1-s_1}(\tilde d-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&-\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde c}\int_0^{\tilde d}\tfrac{\gimel(r_2,h_2,u,g,\mathbb{z}_{M-1}(u,g))}{(\tilde c-u)^{1-s_1}(\tilde d-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\bigg|\\
&+c_2\big|\mathfrak{w}\big(r_1,h_1,z_{M-1}(r_1,h_1)\big)-\mathfrak{w}\big(r_2,h_2,\mathbb{z}_{M-1}(r_2,h_2)\big)\big|\\
&\le \tfrac{c_1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde c}\int_0^{\tilde d}\tfrac{|\gimel(r_1,h_1,u,g,\mathbb{z}_{M-1}(u,g))-\gimel(r_2,h_2,u,g,\mathbb{z}_{M-1}(u,g))|}{(\tilde c-u)^{1-s_1}(\tilde d-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u+c_2\omega_1(\mathfrak{w},\delta)\\
&\le \tfrac{c_1\tilde c^{s_1}\tilde d^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\omega_1(\gimel,\delta)+c_2\omega_1(\mathfrak{w},\delta),
\end{aligned}\tag{7.4}$$

and

$$\begin{aligned}
&\big|\gimel\big(r,h,u_1,g_1,\mathbb{z}_{M-1}(u_1,g_1)\big)-\gimel\big(r,h,u_2,g_2,\mathbb{z}_{M-1}(u_2,g_2)\big)\big|\\
&\qquad\le \big|\gimel\big(r,h,u_1,g_1,\mathbb{z}_{M-1}(u_1,g_1)\big)-\gimel\big(r,h,u_1,g_1,\mathbb{z}_{M-1}(u_2,g_2)\big)\big|\\
&\qquad+\big|\gimel\big(r,h,u_1,g_1,\mathbb{z}_{M-1}(u_2,g_2)\big)-\gimel\big(r,h,u_2,g_2,\mathbb{z}_{M-1}(u_2,g_2)\big)\big|\\
&\qquad\le b_1\big|\mathbb{z}_{M-1}(u_1,g_1)-\mathbb{z}_{M-1}(u_2,g_2)\big|+\omega_2(\gimel,\delta).
\end{aligned}\tag{7.5}$$

Thanks to Eqs. (7.4) and (7.5), Ineq. (7.2) is obtained. □

Theorem 7.3. *Consider the Eq.* (1.1) *with the hypotheses of Theorem* 3.1 *and condition (A7). If*

$$\Upsilon=c_2c_3+\tfrac{\tilde c\tilde d}{\Gamma(s_1+1)\Gamma(s_2+1)}c_1b_1<1,$$

then the iterative procedure (6.3) *converges to the unique solution of Eq.* (1.1), $\mathbb{z}^*$, *and its error estimate is as follows,* $\|\mathbb{z}^*-\mathbb{z}_M\|\le \frac{\Lambda^M}{1-\Lambda}\|\mathbb{z}_1-\mathbb{z}_0\|+\frac{1}{1-\Upsilon}\eta(\delta)$, *where,*

$$\eta(\delta)=M_K\mathbb{A}(1-c_2c_3)\omega_1(\gimel,\delta)+M_K\mathbb{A}\omega_1(\varpi,\delta)+M_K\mathbb{A}\omega_2(\gimel,\delta)+\mathbb{A}\gimel^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta).$$

Proof. From (6.3) and (6.5), for $M=1$, one can get,

$$\begin{aligned}
\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big| \le \Bigg|&\Omega\bigg(r_i,h_j,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde c}\int_0^{\tilde d}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_0(u,g))}{(\tilde c-u)^{1-s_1}(\tilde d-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u,\mathfrak{w}\big(r_i,h_j,\mathbb{z}_0(r_i,h_j)\big)\bigg)\\
&-\Omega\Bigg(r_i,h_j,\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_0\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde c-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde d-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}},\mathfrak{w}\big(r_i,h_j,\bar{\mathbb{z}}_0(r_i,h_j)\big)\Bigg)\Bigg|.
\end{aligned}$$

Thus, there exist c_1, c_2 and c_3 such that

$$\begin{aligned}
|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)| &\le c_1\Big|\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde{c}}\int_0^{\tilde{d}}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_0(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&-\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_0\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\Big|\\
&+c_2|\mathrm{d}g\,\mathrm{d}u,\mathfrak{w}\big(r_i,h_j,\mathbb{z}_0(r_i,h_j)\big)-\mathfrak{w}\big(r_i,h_j,\bar{\mathbb{z}}_0(r_i,h_j)\big)|\\
&\le c_1\Big|\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_{r_p}^{r_{p-1}}\int_{h_q}^{h_{q-1}}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_0(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&-\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_0\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\Big|\\
&+c_2c_3|\mathbb{z}_0(r_i,h_j)-\bar{\mathbb{z}}_0(r_i,h_j)|\\
&\le c_1\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\Big|\int_{r_p}^{r_{p-1}}\int_{h_q}^{h_{q-1}}\tfrac{\gimel(r_i,h_j,u,g,f(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&-(r_p-r_{p-1})(h_q-h_{q-1})\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},f\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\Big|\\
&\le\tfrac{c_1(r_p-r_{p-1})(h_q-h_{q-1})}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\left(\omega_{[r_{p-1},r_p][h_{q-1},h_q]}\left(K\gimel\mathbb{z}_0,\tfrac{\delta_1}{2}+\tfrac{\delta_2}{2}\right)\right).
\end{aligned}$$

Hence,

$$\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big|\le\tfrac{3c_1\tilde{c}\tilde{d}}{2\Gamma(s_1)\Gamma(s_2)}\Big[\omega_{[0,\tilde{c}][0,\tilde{d}]}(K\gimel\mathbb{z}_0,\delta_1+\delta_2)\Big].$$

Now, from (6.3), (6.5) and by the triangle inequality, for $M=2$, we obtain,

$$\begin{aligned}
\big|\mathbb{z}_2(r_i,h_j)-\bar{\mathbb{z}}_2(r_i,h_j)\big| &\le\Bigg|\Omega\Bigg(r_i,h_j,\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde{c}}\int_0^{\tilde{d}}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_1(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u,\mathfrak{w}\big(r_i,h_j,\mathbb{z}_1(r_i,h_j)\big)\Bigg)\\
&-\Omega\Bigg(r_i,h_j,\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_1\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}},\mathfrak{w}\big(r_i,h_j,\bar{\mathbb{z}}_1(r_i,h_j)\big)\Bigg)\Bigg|\\
&\le c_1\Bigg|\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\tilde{c}}\int_0^{\tilde{d}}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_1(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u\\
&-\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\mathbb{z}_1\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\Bigg|\\
&+c_1\Bigg|\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\mathbb{z}_1\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\\
&-\tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\tfrac{\gimel\left(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_1\left(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\right)\right)}{\left(\tilde{c}-\left(u_p+\frac{\delta_1}{2}\right)\right)^{1-s_1}\left(\tilde{d}-\left(g_q+\frac{\delta_2}{2}\right)\right)^{1-s_2}}\Bigg|\\
&+c_2\big|\mathfrak{w}\big(r_i,h_j,\mathbb{z}_1(r_i,h_j)\big)-\mathfrak{w}\big(r_i,h_j,\bar{\mathbb{z}}_1(r_i,h_j)\big)\big|\\
&\le c_1\tfrac{1}{\Gamma(s_1)\Gamma(s_2)}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1}\Bigg|\int_{r_p}^{r_{p-1}}\int_{h_q}^{h_{q-1}}\tfrac{\gimel(r_i,h_j,u,g,\mathbb{z}_1(u,g))}{(\tilde{c}-u)^{1-s_1}(\tilde{d}-g)^{1-s_2}}\,\mathrm{d}g\,\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&- \delta_1\delta_2 \frac{\daleth\big(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\mathbb{z}_1\big(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\big)\big)}{\big(\tilde{c}-\big(u_p+\frac{\delta_1}{2}\big)\big)^{1-s_1}\big(\tilde{d}-\big(g_q+\frac{\delta_2}{2}\big)\big)^{1-s_2}}\Bigg| \\
&+ c_1 \tfrac{\delta_1\delta_2}{\Gamma(s_1)\Gamma(s_2)} \sum_{q=0}^{n-1}\sum_{p=0}^{n-1} \Bigg| \frac{\daleth\big(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\mathbb{z}_1\big(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\big)\big)}{\big(\tilde{c}-\big(u_p+\frac{\delta_1}{2}\big)\big)^{1-s_1}\big(\tilde{d}-\big(g_q+\frac{\delta_2}{2}\big)\big)^{1-s_2}} \\
&- \frac{\daleth\big(r_i,h_j,u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2},\bar{\mathbb{z}}_1\big(u_p+\frac{\delta_1}{2},g_q+\frac{\delta_2}{2}\big)\big)}{\big(\tilde{c}-\big(u_p+\frac{\delta_1}{2}\big)\big)^{1-s_1}\big(\tilde{d}-\big(g_q+\frac{\delta_2}{2}\big)\big)^{1-s_2}}\Bigg| + c_2c_3\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big| \\
&\le \tfrac{3c_1\tilde{c}\tilde{d}}{2\Gamma(s_1)\Gamma(s_2)}\Big[\omega_{[0,\tilde{c}][0,\tilde{d}]}(K\daleth\mathbb{z}_1,\delta)\Big] \\
&+ \tfrac{c_1b_1\tilde{c}\tilde{d}}{\Gamma(s_1)\Gamma(s_2)}M_K\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big| + c_2c_3\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big|,
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
\big|\mathbb{z}_2(r_i,h_j)-\bar{\mathbb{z}}_2(r_i,h_j)\big| &\le \tfrac{3c_1\tilde{c}\tilde{d}s_1s_2}{2\Gamma(s_1+1)\Gamma(s_2+1)}\Big[\omega_{[0,\tilde{c}][0,\tilde{d}]}(K\daleth\mathbb{z}_1,\delta)\Big] \\
&+ \Big[c_2c_3 + \tfrac{c_1b_1\tilde{c}\tilde{d}s_1s_2}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big| \\
&\le \tfrac{3c_1\tilde{c}\tilde{d}}{2\Gamma(s_1+1)\Gamma(s_2+1)}\Big[\omega_{[0,\tilde{c}][0,\tilde{d}]}(K\daleth\mathbb{z}_1,\delta)\Big] \\
&+ \Big[c_2c_3 + \tfrac{c_1b_1\tilde{c}\tilde{d}}{\Gamma(s_1+1)\Gamma(s_2+1)}\Big]\ \big|\mathbb{z}_1(r_i,h_j)-\bar{\mathbb{z}}_1(r_i,h_j)\big|.
\end{aligned}
$$

So, by (7.3), we have the following result

$$
\big|\mathbb{z}_2(r_i,h_j)-\bar{\mathbb{z}}_2(r_i,h_j)\big| \le M_K\mathbb{A}\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z}_1,\delta) + \mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta) + \Upsilon\|\mathbb{z}_1-\bar{\mathbb{z}}_1\|,
$$

where $\mathbb{A} = \frac{3c_1\tilde{c}\tilde{d}}{2\Gamma(s_1+1)\Gamma(s_2+1)}$. By induction, for $M\in\mathbb{N}$, $M\ge 3$, we obtain

$$
|\mathbb{z}_M(r_i,h_j)-\bar{\mathbb{z}}_M(r_i,h_j)| \le M_K\mathbb{A}\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z}_{M-1},\delta) + \mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta) + \Upsilon||\mathbb{z}_{M-1}-\bar{\mathbb{z}}_{M-1}|| \tag{7.6}
$$

Taking supremum for $(r,h)\in I_0$ from the above inequality, we have,

$$
\begin{aligned}
\big\|\mathbb{z}_M-\bar{\mathbb{z}}_M\ \| &\le M_K\mathbb{A}\Big(\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z}_{M-1},\delta)+\Upsilon\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z}_{M-2},\delta)+\cdots+\Upsilon^{m-1}\omega_{\mathrm{J}^2_{cd}}(\daleth\mathbb{z}_0,\delta)\Big) \\
&+ \mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta)\Big[1+\Upsilon+\Upsilon^2+\cdots+\Upsilon^{m-1}\Big].
\end{aligned} \tag{7.7}
$$

By (7.6), (7.7) and Lemma 7.2, we see that,

$$
\begin{aligned}
\big\|\mathbb{z}_M-\bar{\mathbb{z}}_M\ \| &\le M_K\mathbb{A}\frac{c_1b_1\tilde{c}^{s_1}\tilde{d}^{s_2}}{\Gamma(s_1+1)\Gamma(s_2+1)}\omega_1(\daleth,\delta)\sum_{l=0}^{m-1}\Upsilon^l + M_K\mathbb{A}\omega_1(\varpi,\delta)\sum_{l=0}^{m-1}\Upsilon^l \\
&+ M_K\mathbb{A}\omega_2(\daleth,\delta)\sum_{l=0}^{m-1}\Upsilon^l + \mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta)\sum_{l=0}^{m-1}\Upsilon^l.
\end{aligned} \tag{7.8}
$$

By (7.8) since $\Upsilon<1$, we obtain,

$$
\begin{aligned}
\big\|\mathbb{z}_M-\bar{\mathbb{z}}_M\ \| &\le M_K\mathbb{A}(1-c_2c_3)\omega_1(\daleth,\delta)\tfrac{1}{1-\Upsilon} + M_K\mathbb{A}\omega_1(\varpi,\delta)\tfrac{1}{1-\Upsilon} \\
&+ M_K\mathbb{A}\omega_2(\daleth,\delta)\tfrac{1}{1-\Upsilon} + \mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta)\tfrac{1}{1-\Upsilon}.
\end{aligned} \tag{7.9}
$$

Finally,

$$
\big\|\mathbb{z}_M-\bar{\mathbb{z}}_M\big\| \le \tfrac{1}{1-\Upsilon}\Big[M_K\mathbb{A}(1-c_2c_3)\omega_1(\daleth,\delta)+M_K\mathbb{A}\omega_1(\varpi,\delta)+M_K\mathbb{A}\omega_2(\daleth,\delta)+\mathbb{A}\daleth^*\omega_{\mathrm{J}^2_{cd}}(r,h)(K,\delta)\Big]. \tag{7.10}
$$

Notice that $\Lambda\le\Upsilon<1$, so from Ineqs. (6.6), (7.10) and $\|\mathbb{z}^*-\bar{\mathbb{z}}_M\|\le\|\mathbb{z}^*-\mathbb{z}_M\|+\|\mathbb{z}_M-\bar{\mathbb{z}}_M\|$, it is concluded that $\bar{\mathbb{z}}_M$ converges to $\mathbb{z}^*$ and the desired error estimate is obtained. □

Algorithm of the approach. The iterative procedure (6.5) gives the following algorithm of computation for the solution of $\mathbb{FIE}$ (1.1). For checking the error of the method, we consider,

Algorithm 1 Iterative procedure (6.5) to estimate $\mathbb{FIE}$ (1.1).

Require: Input the values $\tilde{c}$, $\tilde{d}$, s_1, s_2, ε, n and the functions Ω, $\beth$, $\mathfrak{w}$.

```
δ1 ← c̃/n and δ2 ← d̃/n.
for i = 0 to n do
   for j = 0 to n do
      f ← z̄0(ri, hj) by (6.5).
      M ← 1.
   end for
end for
for i = 0 to n do
   for j = 0 to n do
      Compute z̄M(ri, hj) by (6.5).
      A ← |z̄M(ri, hj) − z̄M−1(ri, hj)|.
      if A < ε then
         Print M.
         Print z̄M(ri, hj) and i, j.
      end if
      M ← M + 1.
   end for
end for
```

Ensure: $\bar{\mathbb{z}}_M\big(r_i, h_j\big)$.

$$\|e_n\|_\infty := \max\Big\{\big|\mathbb{z}^*(r_i,h_j) - \bar{\mathbb{z}}_M(r_i,h_j)\big| \,:\, i,j = 1,2,\ldots,n\Big\},$$

with $\mathbb{z}^*$ and $\bar{\mathbb{z}}_M$ are is the exact and approximate solutions respectively, of $\mathbb{FIE}$ (1.1), which is obtained from Picard sequence (6.5). The experimental rate of convergence for the below mentioned example is also estimated according to [6, Chapter 2] which is introduced as,

$$\mathsf{Ratio} = \frac{\left\|\mathbb{z}^* - \bar{\mathbb{z}}_M^{(n)}\right\|_\infty}{\left\|\mathbb{z}^* - \bar{\mathbb{z}}_M^{(2n)}\right\|_\infty}, \qquad \rho_n = \log_2\left(\frac{\left\|\mathbb{z}^* - \bar{\mathbb{z}}_M^{(n)}\right\|_\infty}{\left\|\mathbb{z}^* - \bar{\mathbb{z}}_M^{(2n)}\right\|_\infty}\right),$$

where ρ_n estimates the convergence rate. The tolerance $\varepsilon = 10^{-15}$ is chose to stop the iterations, i.e., FP iterations stop when $\big\|\bar{\mathbb{z}}_M - \bar{\mathbb{z}}_{M-1}\big\| < \varepsilon$. All numerical results are computed by using Maple 17.

Example 7.4. The following nonlinear $\mathbb{FIE}$,

$$\mathbb{z}(r,h) = r^2 + h + 1 - \frac{290597(r+h)^2\Gamma(5/6)}{1403948(1+(r+h)^2)\Gamma(9/10)} + \frac{1}{20(1+(r+h)^2)\Gamma(2/5)\Gamma(2/3)} \int_0^1\int_0^1 \frac{\sqrt{ug}(r+h)^2\mathbb{z}^2(u,g)}{(1-u)^{3/5}(1-g)^{1/3}}\,\mathrm{d}g\,\mathrm{d}u, \tag{7.11}$$

for $(r,h) \in \mathrm{J}_{cd}^2$, with $\tilde{c} = 1$, $\tilde{d} = 1$, has unique exact solution $\mathbb{z}(r,h) = r^2 + h + 1$. $\mathbb{FIE}$ (7.11) is especial case of $\mathbb{FIE}$ (1.1). Clearly, (A1) of Theorem 3.1 holds. We need to check that assumption (A2) holds. Let $\|\mathbb{z}\| \le \tau$, $\tau > 0$, then

$$\begin{aligned}|\mathbb{z}(r,h)| &= \left|r^2 + h + 1 - \frac{290597(r+h)^2\Gamma\big(\frac{5}{6}\big)}{1403948(1+(r+h)^2)\Gamma\big(\frac{9}{10}\big)} + \frac{1}{20(1+(r+h)^2)\Gamma\big(\frac{2}{5}\big)\Gamma\big(\frac{2}{3}\big)}\int_0^1\int_0^1 \frac{\sqrt{ug}(r+h)^2\mathbb{z}^2(u,g)}{(1-u)^{3/5}(1-g)^{1/3}}\,\mathrm{d}g\,\mathrm{d}u\right| \\ &\le \left[3 + \frac{290597\Gamma\big(\frac{5}{6}\big)}{1403948\Gamma\big(\frac{9}{10}\big)} + \frac{\tau^2}{20\Gamma\big(\frac{7}{5}\big)\Gamma\big(\frac{5}{3}\big)}\right] \le \tau,\end{aligned}$$

for every J^2_{cd}. Hence (A2) holds if $\tau \in [4.4605, 11.558]$. Indeed, all essentials of Theorem 3.1 are fulfill. We choose

$$\bar{\mathbb{z}}_0(r,h) = r^2 + h + 1 - \frac{290597(r+h)^2\Gamma\left(\frac{5}{6}\right)}{1403948(1+(r+h)^2)\Gamma\left(\frac{9}{10}\right)} \in [-\tau, \tau].$$

In this level, for the grid points (r_i, h_j), for $i, j = 1, 2, \ldots, 5$, we compute the absolute error. These obtained numerical results (error between exact and approximate value of $\bar{\mathbb{z}}(r,h)$) with $n = 10, 20, 40$ are shown in Table 3. The results in Tables 3 and 4 confirm the convergence of Algorithm 1, that is, $e_{i,j} \to 0$ when δ_1, $\delta_2 \to 0$. One can see that $\|e_n\|_\infty$ is decreasing by factor of approximately 2 whenever n is doubled. Further, the curves in Fig. 3, show approximate value of $\bar{\mathbb{z}}(r,h)$ for $n = 10, 20, 40$ respectively.

TABLE 3. Numerical results in Example 7.4 for different values of $n = 10, 20, 40$.

(r,h)	Exact solution	$e_{i,j}$		
		$m = 10$	$m = 20$	$m = 40$
$(0.1, 0.1)$	1.11	2.956461×10^{-5}	1.479930×10^{-5}	7.404133×10^{-6}
$(0.3, 0.3)$	1.39	1.182585×10^{-4}	5.919716×10^{-5}	2.961631×10^{-5}
$(0.5, 0.5)$	1.75	2.660816×10^{-4}	1.331935×10^{-4}	6.663672×10^{-5}
$(0.7, 0.7)$	2.19	4.730339×10^{-4}	2.367884×10^{-4}	1.184652×10^{-4}
$(0.9, 0.9)$	2.71	7.391155×10^{-4}	3.699819×10^{-4}	1.851019×10^{-4}

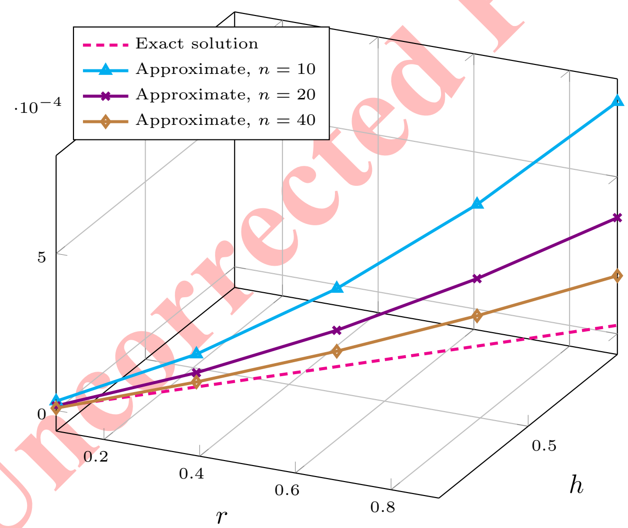


FIGURE 3. Exact and approximat solutions of nonlinear Volterra $\mathbb{FIE}$ (7.11) via error estimate e_m for $m = 10, 20, 40$ in Example 7.4.

TABLE 4. Summery of numerical results for $\mathbb{FIE}$ (7.11) when $n = 10, 20, 40$ in Example 7.4.

n	M	$\|e_m\|_\infty$	Ratio	ρ_n
10	11	2.956462×10^{-3}	–	–
20	12	1.479927×10^{-3}	1.9977	0.9983
40	12	7.404076×10^{-4}	1.9988	0.9991

8. Conclusion

We try to examine the existence results for the solution of $\mathbb{FIE}$ (1.1) in various Banach function spaces, e.g., Sobolev, Orlicz, Hölder, etc. The importance of defending the existence result in the research of this is one of the advantages of researchers. So far, several approaches have been devised for this idea. This research is based on a more general form of $\mathbb{FIE}$, which involves some other relevant works as well. In the suggested method, FPT of Petryshyn and the concept of MNC with more limited conditions were applied. The fact that, the Trapezoidal and Simpson rules can not be applied to approximate integrals that are not defined in the first and the endpoints of the integration. Therefore, in Section 6, we employed the midpoint rule to investigate the approximate solutions of nonlinear $\mathbb{FIE}$ (1.1) which it can consider in the future works to extend them numerically [5, 19, 20, 43].

Abbreviations

$\mathbb{DE}$: Differential equation; $\mathbb{IE}$: Integral equation; FPT: Fixed point theorem; MNC: Measure of noncompactness; $\mathbb{FIE}$: Fractional integral equation;

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

M. Singh: Methodology, validation, formal analysis, actualization, investigation, initial draft.

M. Kazemi: Methodology, validation, formal analysis, actualization, investigation, initial draft and was a major contributor in writing the manuscript.

M. E. Samei: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

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