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Spectral collocation-based procedure with shifted third-kind chebyshev polynomials for nonlinear time-fractional integro-differential equations

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Abstract

In our study, we present a new algorithm for solving nonlinear time-fractional partial integro-differential equations *(NLTFPIDEs)* with weakly singular kernels using modified shifted Chebyshev polynomials of the third kind *(MSCP3K)* as basis functions. We then employ the collocation method to transform the *NLTFPIDEs* into a system of equations that can be solved with appropriate algebraic techniques to obtain the unknowns. We obtain exact algebraic representations for every element of the associated matrices. These matrices have a distinct pattern that facilitates the algebraic problem and streamlines the inversion process. The novelty lies in deriving new operational matrices for MSCP3K, enabling a structured algebraic system with reduced complexity. The work advances our understanding of the approach's reliability by carefully analyzing convergence and error behavior. The suggested technique's accuracy, efficiency, and applicability are illustrated by numerical examples, complemented by comparisons with prior studies, achieving reasonable accuracy with only a few retained modes. The results demonstrate the effectiveness of this approach in solving time-fractional partial integro-differential equations, highlighting its contribution to the field's body of knowledge.

Keywords. Time-fractional partial integro-differential equations, Collocation method, Shifted ultraspherical polynomials, Convergence analyses.
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1. Introduction

Spectral methods play a crucial role in solving various differential equations, with the tau, collocation, and Galerkin formulations being the most widely applied. Collocation strategies have recently gained significant attention due to their reliable accuracy for nonlinear models. For instance, generalized Chebyshev-based collocation was shown to be highly effective in [37], while modified schemes with enhanced stability were presented in [38]. Further alleviated structures were developed in [29], and improved Legendre formulations were examined in [30]. Applications to generalized multi-term systems were provided in [36], compact forms were analyzed in [27], and multi-operator spectral models were explored in [28]. Early foundations involving new collocation constructions are documented in [9], while integral-integration hybrid approaches for nonlinear dynamics were investigated in [24].

Research in numerical analysis has yielded diverse strategies for solving complex differential models. For instance, Lakestani et al. [20] addressed parabolic partial differential equations with time-dependent coefficients and auxiliary measurements by expanding the approximate solution through Chebyshev cardinal functions. A similar basis was employed by Lakestani and Dehghan [19] to solve fourth-order integro-differential equations, utilizing an operational matrix of derivatives to streamline the computation. Recent advancements have expanded these techniques into

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fractional calculus and mathematical biology. Kammappa et al. [18] introduced a numerical approach using Cubic Trigonometric B-spline (CuTBS) interpolation to solve time-fractional diffusion equations (TFDEs) under the Caputo-Fabrizio derivative. Furthermore, Thirumalai et al. [33] applied a spectral collocation technique based on truncated Chebyshev polynomials to analyze a modified five-compartment smoking model, which tracks the dynamics between potential, irregular, and regular smokers, as well as those in the snuffing and quitter classes.

The spectral collocation method is particularly powerful for fractional models because it converts the governing equations into algebraic systems enforced at collocation points. This allows the method to combine the fast convergence of spectral approximations with the simplicity of direct pointwise enforcement. When integrated with Chebyshev polynomials, this framework becomes especially effective for fractional systems, as demonstrated in [14]. Additional progress includes shifted Chebyshev schemes designed for fractional Volterra–Fredholm models in [17], polynomial matrix collocation structures summarized in [10], and refined spectral corrections for nonsmooth fractional behavior described in [39]. New orthogonal bases were constructed for delayed fractional dynamics in [22], and kernel-driven approaches for fractional differential–algebraic systems were introduced in [32]. Other recent advances include Laguerre-perturbed Galerkin formulations for high-order integro-differential systems [25], comparative analyses of transfer- and Fibonacci-type collocation methods [5], fractional Vieta–Fibonacci polynomial families for variable-order reaction–diffusion structures [16], and updated Ritz strategies for intricate fractional systems [11].

Chebyshev polynomials remain among the most influential orthogonal families in approximation theory and numerical analysis. Defined over $[-1, 1]$, the first-kind functions $T_n(x)$ and second-kind functions $U_n(x)$ possess orthogonality with respect to classical weight functions and satisfy simple recurrence identities, including the well-known relation $T_n(\cos\theta) = \cos(n\theta)$. Owing to these properties, Chebyshev polynomials support stable interpolation, quadrature, and spectral discretization. Their recent use in fractional models continues to expand: a Chebyshev spectral treatment of fractional population dynamics was proposed in [6], a Chebyshev–tau formulation for fractional diffusion equations was implemented in [3], and an improved second-kind Chebyshev strategy for nonlinear equations was developed in [23]. These contributions highlight the continuing value of Chebyshev structures in numerical computation.

Fractional calculus remains one of the major fields of modern mathematical analysis due to its broad applicability across physics, biology, fluid dynamics, and related areas. Because analytical solutions are rare for most fractional models, numerical approximation becomes indispensable. Several significant contributions have been made toward numerical solutions of such equations [1, 12–14], with fifth-order strategies reported in [2]. These developments underscore the necessity of robust computational tools for fractional integro-differential structures.

The next sections describe the development and implementation of the proposed Galerkin algorithm, including the construction of third-kind Chebyshev polynomials and their specific computational advantages. The analysis focuses on nonlinear time-fractional partial integro-differential equations and evaluates the efficiency of the method through several numerical examples and comparisons with earlier approaches. The following sections therefore provide a detailed presentation of the methodology, its theoretical basis, and its effectiveness in producing accurate numerical solutions for complex fractional models.

The following is a summary of the primary objectives of this study:

- Section 2 focuses on presenting an overview of the Caputo fractional derivative and outlining the essential properties and relationships of MSCP3K.
- In spectral techniques, define the basis function of *MSCP3K* and emphasize how flexible they are in improving the precision and effectiveness of numerical solutions in Section 3.
- We provide an innovative approach to solve the specified class of equations by integrating *MSCP3K* using the Galerkin algorithm to circumvent inherent difficulties in Section 4.
- Carefully, we examine the recommended approach using error analysis to clarify its accuracy and reliability in Section 5.
- Using numerical examples and comparisons with previous research, demonstrate the suggested method's effectiveness, precision, and applicability in Section 6.
- Finally, in Section 7, we discuss the overall results of the paper.

Our purpose is to solve *NLTFPIDE* that follows [4]:

$$D_t^{\lambda} u(z,\,t) + u(z,\,t)\, u_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\, u_{zz}(z,\,r) dr + f(z,\,t), \qquad 0<\lambda,\,\mu<1, \tag{1.1}$$

considering the following initially and boundary conditions:

$$\begin{aligned} &u(z,\,0) = q, \qquad 0 < z \leq 1, \\ &u(0,\,t) = p, \qquad u(1,\,t) = l, \qquad 0 < t \leq 1, \end{aligned} \tag{1.2}$$

where $f(z,t)$ is the source term.
We demonstrate the useful relations and power forms of the proposed polynomials, which are illustrated in the next section.

2. Preliminaries of the fractional calculus and the proposed polynomials

An overview of the Caputo fractional derivative is the main goal of this section. Furthermore, certain fundamental relationships and characteristics of *MSCP3K*.

2.1. Caputo fractional derivatives.

Definition 2.1. *[34] The definition of the Caputo fractional derivative of order m is:*

$$D_z^m f(z) = \frac{1}{\Gamma(d-m)} \int_0^z (z-t)^{d-m-1}\, f^{(d)}(t)\, dt, \quad m>0, \quad z>0, \tag{2.1}$$

where $d-1 \leqslant m < d$, $d = \lceil m \rceil$, $d \in \mathbb{N} = \{1,2,3,...\}$.

D_z^m is an operator that satisfies the following conditions for $d-1 \leqslant m < d$, $d \in \mathbb{N}$,

$$D_z^m k = 0, \;\; (k\, is\, a\, constant) \tag{2.2}$$

$$D_z^m\, z^d = \begin{cases} 0, & \text{if}\, d \in \mathbb{N}_0 \quad and \quad d < \lceil m \rceil, \\ \frac{\Gamma(d+1)}{\Gamma(d-m+1)}\, z^{-m}, & \text{if } d \in \mathbb{N}_0 \quad and \quad d \geq \lceil m \rceil, \end{cases} \tag{2.3}$$

where $\lceil m \rceil$ is the ceiling function and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.2. A brief discussion of the third-kind Chebyshev polynomials.

This subsection introduces power forms and a few useful relations of *CP3K*, *SCP3K*, and *MSCP3K*.

2.2.1. *Chebyshev Polynomials of 3rd-Kind (CP3K).*

The third-kind of Chebyshev polynomials are well-known in the $[-1,1]$ region and have the recurrence relation presented as follows [35]:

$$\omega_{k+1}(z) = 2\,z\,\omega_k(z) - \omega_{k-1}(z), \quad k \geq 1, \tag{2.4}$$

with

$$\begin{cases} \omega_0(z) = 1, \\ \omega_1(z) = 2\,z - 1. \end{cases} \tag{2.5}$$

The recurrence relation's (2.4) are provided as

$$\omega_k(z) = 0, \quad z_m = \cos\left(\frac{(2\,m-1)\pi}{2\,k+1}\right), \quad m = 1,2,\ldots,k. \tag{2.6}$$

The third-kind Chebyshev polynomials' Rodrigues representation is given in [35] by

$$\omega_k(z) = \frac{(-2)^k\, k!}{(2k)!} \sqrt{\frac{(1-z)}{(1+z)}} \frac{d^k}{dz^m} \left[(1-z)^{k-\frac{1}{2}}\, (1+z)^{k+\frac{1}{2}}\right]. \tag{2.7}$$

Furthermore, pointing out that the third-kind Chebyshev polynomials are a specific case of Jacobi polynomials, It is easy to show that

$$\omega_k(z) = \frac{(2k)!}{(2^k\, k!)^2}\, P_k^{(-\frac{1}{2},\frac{1}{2})}(z), \tag{2.8}$$

where $P_k^{(\alpha,\beta)}(z)$ is the hypergeometric function of order (α, β).

The power analytical form of *CP3K* in[35] is

$$\omega_k(z) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \binom{k-m}{m} \binom{2k-2m}{k} \left(\frac{z-1}{2}\right)^{k-2m}, \tag{2.9}$$

or

$$\omega_k(z) = 2^k \prod_{m=1}^{k} \left(z - \cos\left(\frac{2m-1}{2k+1}\right)\pi \right). \tag{2.10}$$

Another analytical form for *CP3K* is given in [35] by

$$\omega_k(z) = \sum_{m=0}^{\lceil k+\frac{1}{2} \rceil} (-1)^m\, (2\,k+1)\, \frac{(2k-m)!}{(2\,k-2\,m+1)!\, m!}\, (2\,z+2)^{k-m}, \quad k \in Z^+, \tag{2.11}$$

where $\lceil j \rceil = min\{n \in z | n \geq j\}$.

The inner product of the polynomials $\omega_k(z)$ with the weight function $W(z)$ indicates that they are orthogonal on $(-1, 1)$:

$$\langle \omega_j, \omega_k \rangle_W = \int_{-1}^{1} \omega_j(z)\, \omega_k(z)\, W(z)\, dz = \pi\, \delta_{jk}, \tag{2.12}$$

with

$$W(z) = \sqrt{\frac{1+z}{1-z}}, \quad \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{2.13}$$

2.2.2. *3rd-Kind of Shifted Chebyshev Polynomials (SCP3K).* In general, it is easier to use the range $[0, 1]$ than the range $[-1, 1]$. By using the transformation $z \to 2\,z - 1$, we may transform the independent variable $z \in [-1, 1]$ into the variable z in $[0, 1]$. This yields *SCP3K* $\omega_k^*(z) = \omega_k(2\,z - 1)$, $z \in [0, 1]$ of degree k.

The interval $[0, 1]$ is orthogonal to these polynomials:

$$\langle \omega_j^*(z), \omega_k^*(z) \rangle_{W^*} = \int_0^1 \omega_j^*(z)\, \omega_k^*(z)\, W^*(z) dz = \frac{\pi}{2}\, \delta_{j,k}, \tag{2.14}$$

using $W^*(z) = \sqrt{\frac{z}{1-z}}$ as the weight function. The recurrence relation can also be used to define them [26].

$$\omega_{k-1}^*(z) + \omega_{k+1}^*(z) = 2\,(2\,z-1)\,\omega_k^*(z), \quad k = 1, 2, \ldots, \tag{2.15}$$

with initial values

$$\begin{cases} \omega_0^*(z) = 1, \\ \omega_1^*(z) = 4\,z - 3. \end{cases} \tag{2.16}$$

The exact form of the *SCP3K* [26] is

$$\omega_k^*(z) = \sum_{m=0}^{k} \frac{(-1)^m\, 2^{2k-2m}\, (2k+1)\, (2k-m)!}{(2k-2m+1)!\, m!}\, z^{k-m}, \quad k \in \mathbb{Z}^+. \tag{2.17}$$

2.2.3. *3rd-Kind Modified Shifted Chebyshev Polynomials (MSCP3K).* The *MSCP3K* ($\varphi_i(z)$ and $\psi_j(t)$) for $z \in [0,1]$ and $t \geq 0$ are given as

$$\psi_j(z) = z\,(1-z)\,\omega_j^*(z), \tag{2.18}$$
$$\phi_i(t) = t^2\,\omega_i^*(t), \tag{2.19}$$

3. Basis Functions of the 3rd-Kind Modified Shifted Chebyshev Polynomials

This section describes how to construct basis functions with third-kind modified shifted Chebyshev polynomials. It determines their orthogonality with respect to a certain weight function and specifies the corresponding function spaces. Finally, it shows how to expand functions in terms of these fundamental functions, which makes them easier to use in numerical analysis.

Theorem 3.1. *[28] The orthogonality relation of the basis function $\psi_n(z)$ is given by:*

$$a_{ij} = \int_0^1 \psi_i(z)\,\psi_j(z)\,W_1(z)\,dz = \frac{\pi}{512}\begin{cases} 6, & j = i, \\ -4, & |i-j| = 2, \\ -4, & i+j = 1, \\ 0, & Otherwise. \end{cases} \tag{3.1}$$

For $N = 4$, the matrix $\mathbb{A}$ can be written as

$$\mathbb{A} = \begin{pmatrix} \frac{3\pi}{256} & \frac{-\pi}{128} & \frac{-\pi}{128} & \frac{\pi}{512} & \frac{\pi}{512} \\ \frac{-\pi}{128} & \frac{3\pi}{256} & \frac{\pi}{512} & \frac{-\pi}{128} & 0 \\ \frac{-\pi}{128} & \frac{\pi}{512} & \frac{3\pi}{256} & 0 & \frac{-\pi}{128} \\ \frac{\pi}{512} & \frac{-\pi}{128} & 0 & \frac{3\pi}{256} & 0 \\ \frac{\pi}{512} & 0 & \frac{-\pi}{128} & 0 & \frac{3\pi}{256} \end{pmatrix}. \tag{3.2}$$

Theorem 3.2. *[28] The following elements of the matrix represents the second derivative of MSCP3K*

$$b_{ij} = \int_0^1 \psi_i''(z)\,\psi_j(z)\,W_1(z)\,dz = \frac{\pi}{16}\begin{cases} -2 - i\,(i+1), & j = i, \\ i+1, & j-i = 1, \\ -i, & i-j = 1, \\ \frac{1}{4}\,(i^2+j^2+i+j) - \frac{1}{2}, & |i-j| = 2, \\ 0, & Otherwise. \end{cases} \tag{3.3}$$

For $N = 4$, we can express the matrix $\mathbb{B}$ as

$$\mathbb{B} = \begin{pmatrix} \frac{-\pi}{8} & \frac{\pi}{16} & \frac{\pi}{16} & 0 & 0 \\ \frac{-\pi}{16} & \frac{-\pi}{4} & \frac{\pi}{8} & \frac{3\pi}{16} & 0 \\ \frac{\pi}{16} & \frac{-\pi}{8} & \frac{-\pi}{2} & \frac{3\pi}{16} & \frac{3\pi}{8} \\ 0 & \frac{3\pi}{16} & \frac{-3\pi}{16} & \frac{-7\pi}{8} & \frac{\pi}{4} \\ 0 & 0 & \frac{3\pi}{8} & \frac{-\pi}{4} & \frac{-11\pi}{8} \end{pmatrix}. \tag{3.4}$$

Theorem 3.3. *$\phi_m(t)$ is orthogonal on $[0, 1]$. Consider the orthogonality relation with the weight function $W^*(t)$ is assigned as the following*

$$
\begin{aligned}
c_{ij} &= \int_0^1 D_t^{\lambda}\psi_i(z)\,\psi_j(z)\,W_1(z)\,dz = \sum_{k=0}^{i-1} \frac{(-1)^k\, 2^{2\,i-2\,k+1}\,(1+2\,i)\,(2\,i-k)!}{k!\,(1-2\,k+2\,i)!}\,(A-B),\\
d_{ij} &= \int_0^1 \phi_i(t)\,\phi_j(t)\,W_2(t)\,dt = \frac{\pi}{512}\begin{cases} \binom{8}{4-|j-i|}, & |j-i| \le 4,\\ 0, & Otherwise,\end{cases}\\
&\times \begin{cases} 9\,(\frac{1}{5+|j-i|}), & j=0, i<4,\\ 9\,\left(\frac{1}{|j-i|}\right)\left(\frac{\binom{|j-i|+4}{5}}{\binom{|j-i|+5}{45}}\right), & i=0, j<4,\\ \frac{39}{35}, & i=j=1,\\ \frac{57}{56}, & |i-j|=1,\ 0<j<2,\\ \frac{57}{56}, & |i-j|=1,\ 0<i<2,\end{cases}\\
e_{ij} &= \int_0^1 \psi_i'(z)\,\psi_j(z)\,W_1(z)\,dz = \frac{-\pi}{4}\begin{cases} \frac{j+1}{2}, & j-i=1,\\ \frac{1}{2}, & j=i, j>0,\\ \frac{-(i-1)}{2}, & i-j=1, j>0,\\ 1, & Otherwise,\end{cases}\\
h_{ij} &= \int_0^1\int_0^t (t-j)^{\mu-1}\,\phi_i(t)\,\phi_j(r)\,W_2(t)\,dr\,dt\\
&= \frac{1}{4\,\mu\,\Gamma(j+i+\mu+6)} \sum_{k=0}^{2\,(j+i+1)} \binom{2\,(j+i+1)}{k}\,\Gamma\left(\frac{k+5}{2}\right)\,\Gamma\left(j+i+\mu+\frac{7}{2}-\frac{k}{2}\right),
\end{aligned}
\tag{3.5}
$$

where

$$
A = \frac{\Gamma(2+i-k)\,\left(\frac{(-1)^j\,(1+2\,j)\,\sqrt{\pi}\,\Gamma(\frac{7}{2}+i-k-\alpha)}{2\,\Gamma(5+i-k-\alpha)} + 2^{2\,j-1}\,(2\,j+1)\,\sqrt{\pi}\,C_j\right)}{\Gamma(2+i-k-\alpha)},
\tag{3.6}
$$

and

$$
B = \frac{\Gamma(3+i-k)\,\left(\frac{(-1)^j\,(1+2\,j)\,\sqrt{\pi}\,\Gamma(\frac{9}{2}+i-k-\alpha)}{2\,\Gamma(6+i-k-\alpha)} + 2^{2\,j-1}\,(2\,j+1)\,\sqrt{\pi}\,D_j\right)}{\Gamma(3+i-k-\alpha)},
\tag{3.7}
$$

with C_j is the solution of the next recurrence relation for $n \in \mathbb{Z}$

$$
\begin{aligned}
&2\,(-n+j)\,(1-2\,n+2\,j)\,(3-2\,n+i+j-\alpha)\,(4-2\,n+i+j-\alpha)\,Y_n\\
&+\big(9\,n-59\,n^2+80\,n^3-32\,n^4-2\,n\,i-36\,n^2 i+32\,n^3 i-2\,n\,i^2-8\,n^2 i^2\\
&+6\,j+88\,n\,j-188\,n^2\,j+96\,n^3\,j+18\,ij+60\,n\,i\,j-80n^2 ij+6i^2 j+16\,n\,i^2\,j\\
&-30\,j^2+118\,n\,j^2-88\,n^2\,j^2-16\,i\,j^2+48\,n\,i\,j^2-4\,i^2\,j^2-22\,j^3+32\,n\,j^3\\
&-8\,i\,j^3-4\,j^4+2\,n\,\alpha+36\,n^2\,\alpha-32\,n^3\,\alpha+4\,n\,i\,\alpha+16\,n^2\,i\,\alpha\\
&-18\,j\,\alpha-60\,n\,j\,\alpha+80\,n^2\,j\,\alpha-12\,i\,j\,\alpha-32\,n\,i\,j\,\alpha+16\,j^2\alpha\\
&-48\,n\,j^2\,\alpha+8\,i\,j^2\,\alpha+8j^3\,\alpha-2\,n\,\alpha^2-8\,n^2\,\alpha^2+6\,j\,\alpha^2\\
&+16\,n\,j\,\alpha^2-4\,j^2\,\alpha^2\big)\,Y_{n+1}\\
&-(1+n)\,(-n+2\,j)\,(3-4\,n+2\,i+2\,j-2\,\alpha)(5-4\,n+2\,i+2\,j-2\,\alpha)\,Y_{n+2}=0,
\end{aligned}
\tag{3.8}
$$

initial conditions:

$$Y_0 = 0, \tag{3.9}$$

$$Y_1 = \frac{(2\,j)!\,\Gamma\big(\frac{7}{2} + i + j - \alpha\big)}{(1+2\,j)!\,\Gamma\big(5 + i + j - \alpha\big)}, \tag{3.10}$$

and D_j is the solution of the next recurrence relation for $n \in \mathbb{Z}$

$$\begin{aligned}
&2\,(-n + j)\,(1 - 2\,n + 2\,j)\,(4 - 2\,n + i + j - \alpha)(5 - 2\,n + i + j - \alpha)\,X_n \\
&+ \big(5\,n - 103\,n^2 + 112\,n^3 - 32\,n^4 - 6\,n\,i - 52\,n^2\,i + 32\,n^3\,i - 2\,n\,i^2 - 8\,n^2\,i^2 \\
&+ 30\,j + 164\,n\,j - 268\,n^2\,j + 96\,n^3\,j + 30\,i\,j + 92\,n\,i\,j - 80\,n^2\,i\,j + 6\,i^2\,j + 16\,n\,i^2\,j \\
&- 50\,j^2 + 166\,n\,j^2 - 88\,n^2\,j^2 - 24\,i\,j^2 + 48\,n\,i\,j^2 - 4\,i^2\,j^2 - 30\,j^3 + 32\,n\,j^3 - 8\,i\,j^3 - 4\,j^4 \\
&+ 6\,n\,\alpha + 52\,n^2\,\alpha - 32\,n^3\,\alpha + 4\,n\,i\,\alpha + 16\,n^2\,i\,\alpha - 30\,j\,\alpha - 92\,n\,j\,\alpha + 80\,n^2\,j\,\alpha \\
&- 12\,i\,j\,\alpha - 32\,n\,i\,j\,\alpha + 24\,j^2\alpha - 48\,n\,j^2\,\alpha + 8\,i\,j^2\,\alpha + 8j^3\alpha \\
&- 2\,n\,\alpha^2 - 8\,n^2\,\alpha^2 + 6\,j\,\alpha^2 + 16\,n\,j\alpha^2 - 4j^2\alpha^2\big)\,X_{n+1} \\
&- (1+n)\,(-n + 2\,j)\,(5 - 4\,n + 2\,i + 2\,j - 2\,\alpha)\,(7 - 4\,n + 2\,i + 2\,j - 2\,\alpha)\,X_{n+2} = 0,
\end{aligned} \tag{3.11}$$

initial conditions:

$$X_0 = 0, \tag{3.12}$$

$$X_1 = \frac{(2\,j)!\,\Gamma\big(\frac{9}{2} + i + j - \alpha\big)}{(1+2\,j)!\,\Gamma\big(6 + i + j - \alpha\big)}. \tag{3.13}$$

Proof. The procedures for proving this theorem are identical to those used to prove Theorem 1 in [28]. □

Now, define

$$P_N = \operatorname{span}\{\psi_p(z)\,\phi_q(t) : p, q = 0, 1, \ldots, N\}, \tag{3.14}$$

$$\tilde{P}_N = \{\psi \in P_N : \psi(0,t) = \psi(1,t) = \psi(z,0) = 0,\ 0 < z < 1,\ 0 < t \leq 1\}. \tag{3.15}$$

Then, any function $\psi_N(z,t) \in \tilde{P}_N$ can be expanded as

$$\zeta_N(z,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\,\psi_i(z)\,\phi_j(t) = \hat{v}\,\mathbf{\Psi}(z)\,\mathbf{\Phi}(t), \tag{3.16}$$

where

$$\mathbf{\Psi}(z) = [\psi_0(z),\, \psi_1(z),\, \cdots,\, \psi_N(z)]^T \quad \textit{and} \quad \mathbf{\Phi}(t) = [\phi_0(t),\, \phi_1(t),\, \cdots,\, \phi_N(t)]^T, \tag{3.17}$$

and $\hat{v} = (v_{ij})_{0 \leq i,j \leq N}$ is the matrix of unknown coefficients of order $(N+1) \times (N+1)$.

4. Numerical structure of our method

This section describes the numerical approach used to solve the nonlinear time-fractional partial integro-differential equation *(TFPIDE)*. The spectral collocation approach is used as the basic numerical strategy, resulting in great accuracy and efficiency. The problem is defined with proper initial and boundary conditions.

In the first step, we convert the non-homogeneous boundary and initial conditions to homogeneous ones that satisfy the specified basis functions by the next relation:

$$V(z,\,t) = u(z,\,t) - (1-z)\,u(0,\,t) - z\,u(1,\,t), \tag{4.1}$$

then the nonlinear TFPIDE in (1.1) transforms into

$$D_t^{\lambda} V(z,\,t) + V(z,\,t)\,V_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\,V_{zz}(z,\,r)dr + g(z,\,t), \qquad 0 < \lambda,\, \mu < 1, \tag{4.2}$$

with the homogeneous ICs and BCs:

$$\begin{aligned} &V(z,\,0)=0, \qquad 0<z\le 1,\\ &v(0,\,t)=0, \qquad V(1,\,t)=0, \qquad 0<t\le 1, \end{aligned} \tag{4.3}$$

where

$$f(z,\,t)=g(z,\,t)-\left((1-z)\,D_t^{\lambda}u(0,\,t)+z\,D_t^{\lambda}u(1,\,t)\right). \tag{4.4}$$

Using the formula (3.16) in Eq. (4.2), we obtain the residual given as

$$\begin{aligned} R(z,\,t)=&\sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\,D_t^{\lambda}\,\psi_i(z)\,\phi_j(t)+\sum_{k=0}^{N}\sum_{l=0}^{N} v_{kl}\,\psi_k(z)\,\phi_l(t)\,\sum_{m=0}^{N}\sum_{n=0}^{N} v_{mn}\,\psi_m^{'}(z)\,\phi_n(t)\\ &-\int_0^t (t-r)^{\mu-1}\sum_{o=0}^{N}\sum_{q=0}^{N} v_{oq}\,\psi_o^{''}(z)\,\phi_q(r)\,dr-f(z,\,t), \end{aligned} \tag{4.5}$$

we can simplify the last relation to the next to save time and effort during programming

$$R(z,\,t)=\sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\,c_{ij}\,d_{ij}+\sum_{k=0}^{N}\sum_{l=0}^{N} v_{kl}\,a_{kl}\,d_{kl}\,\sum_{m=0}^{N}\sum_{n=0}^{N} v_{mn}\,e_{mn}\,d_{mn}-\sum_{o=0}^{N}\sum_{q=0}^{N} v_{oq}\,h_{oq}\,b_{oq}-f_{ij}, \tag{4.6}$$

where

$$f_{ij}=\int_0^1\int_0^1 \psi_i(z)\,\phi_j(t)\,W_1(z)\,W_2(t)\,dz\,dt. \tag{4.7}$$

Then applying the collocation method to the residual (4.6)

$$R\left(\frac{1+\cos(\frac{(i+1)\,\pi}{N+3})}{2},\,\frac{1+\cos(\frac{(j+1)\,\pi}{N+3})}{2}\right)=0, \qquad 0\le i,\,j\le N, \tag{4.8}$$

the relation (4.8) produces $(N+1)$ equations into a linear system with unknowns $\hat{v}$. Finally, we use any algebraic method to solve the system.

Algorithm 1 Numerical Solution Procedure for the Transformed Time-Fractional Partial Integro-Differential Equation (NLTFPIDE).

Input Specify the truncation parameter (N).
Step 1. Transformation of Boundary and Initial Conditions

- Convert the non-homogeneous initial and boundary conditions (ICs and BCs) into homogeneous ones.
- Define the transformation function as described in Equation (4.1).
- Use this transformation to express the NLTFPIDE ($u(z,\, t)$) with homogeneous ICs and BCs.

Step 2. Construction of Basis Functions
Define the basis functions as:

$$\psi_n(z) = z\,(1-z)\,\omega_n^*(z),$$
$$\phi_m(t) = t^2\,\omega_m^*(z),$$

where ($\omega_n^*(z)$) and ($\omega_m^*(t)$) are the chosen orthogonal basis functions.
Step 3. Series Representation of the Approximate Solution

- Represent the approximate solution ($\psi_N(z,t)$) as a finite double series expansion:

$$\zeta_N(z,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\,\psi_i(z)\,\phi_j(t) = \hat{v}\,\mathbf{\Psi}(z)\,\mathbf{\Phi}(t),$$

  where (v_{ij}) are unknown coefficients to be determined.

Step 4. Application of the Collocation Method

- Substitute the series expansion into the transformed NLTFPIDE.
- Impose the collocation conditions by enforcing orthogonality of the residual ($R(z,t)$) with respect to the basis functions:

$$\langle R(z,t), \psi_i(z)\,\phi_j(t)\rangle = 0, \quad i,j = 0,1,\ldots,N.$$

- This process yields an algebraic system of equations for the coefficient matrix $\hat{v}$.

Step 5. Solution of the Algebraic System

- Solve the resulting algebraic system using an appropriate numerical method (e.g., Gaussian elimination or an iterative solver) to obtain the unknown coefficients ($\hat{v}$).

Output Compute and present the approximate solution $\zeta_N(z,t)$.

- Evaluate and report the absolute errors corresponding to the solution defined in ***Step* 3**.

5. Convergence analysis

To validate the reliability and efficiency of the proposed NLTFPIDE method, it is essential to establish its convergence behavior. The convergence analysis provides a theoretical guarantee that the approximate solution approaches the exact solution as the number of basis functions N increases. By employing the properties of the chosen orthogonal basis and the boundedness of the associated operators, the error estimate is derived to confirm the stability and accuracy of the method. This analysis ensures the robustness of the numerical scheme for various fractional orders and kernel functions.

Lemma 5.1. *For the modified shifted third-kind Chebyshev polynomials $\psi_j(z)$ and $\phi_i(t)$, the following inequality applies:*

$$\begin{aligned} |\psi_j(z)| &\le 2^{j+1}; \quad j>0, \quad z\in[0,\,1], \\ |\phi_i(t)| &\le 2^{i+1}; \quad i>0, \quad t\in[0,\,1]. \end{aligned} \tag{5.1}$$

Proof. From the relations in (2.17) and (2.19), we get the next

$$\begin{aligned}|\psi_j(z)| &= \left| z\,(1-z)\sum_{j=0}^{k}\frac{(-1)^j\,2^{2k-2j}\,(2k+1)\,(2k-j)!}{(2k-2j+1)!\,j!}\,z^{k-j}\right| \\ &= \frac{1}{2}\left|(1-z)\,z^{1+j}\left(1+\sqrt{\frac{z-1}{z}}\right)^{1+2\,j}\right| \\ &\leq 2^{j+1},\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}|\phi_i(t)| &= \left| t^2\sum_{i=0}^{k}\frac{(-1)^i\,2^{2k-2i}\,(2k+1)\,(2k-i)!}{(2\,k-2\,i+1)!\,i!}\,t^{k-i}\right| \\ &= \frac{1}{2}\left|t^{2+i}\left(1+\sqrt{\frac{t-1}{t}}\right)^{1+2\,i}\right| \\ &\leq 2^{i+1}.\end{aligned} \tag{5.3}$$

□

Theorem 5.2. *The expansion coefficients for* $\zeta(z,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_{ij}\,\psi_i(z)\,\phi_j(t)$ *satisfy:*

$$v_{ij}\leq\frac{1}{(i+1)!\,(j+1)!}, \tag{5.4}$$

with any function $\zeta(z,t)=f_1(z)\,f_2(t)\in P_N$.

Proof. The expansion of $\zeta(z,t)$ can be represented as follows:

$$\zeta(z,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_{ij}\,\psi_i(z)\,\phi_j(t), \tag{5.5}$$

the orthogonality connection enables us to obtain

$$\begin{aligned}v_{ij} &= \frac{1}{a_{ij}\,d_{ij}}\int_0^1\int_0^1\zeta(z,t)\,\psi_i(z)\,\phi_j(t)\,W_1(z)\,W_2(t)\,dz\,dt \\ &\leq \frac{1}{a_{ij}\,d_{ij}}\,2^{j+1}\,2^{i+1}\int_0^1\int_0^1\zeta(z,t)\sqrt{\frac{z}{1-z}}\sqrt{\frac{t}{1-t}}\,dz\,dt \\ &\leq \frac{1}{(i+1)!\,(j+1)!}.\end{aligned} \tag{5.6}$$

Now, Theorem 5.2 is proven. □

Theorem 5.3. *If* $u(z,\,t)$ *satisfies the conditions of Theorem 5.2 and* $u_N(z,\,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}v_{ij}\,\psi_i(z)\,\phi_j(t)$*, then the following truncation error estimate is applicable*

$$|E_N(z,\,t)|\leq 2\left(1-\frac{\Gamma(N+1,\,2)}{\Gamma(N+1)}\right). \tag{5.7}$$

Proof. One way to express the truncation error is:

$$|E_N(z,\,t)|=|u(z,\,t)-u_N(z,\,t)|, \tag{5.8}$$

next, we apply Lemma 5.1 and Theorem 5.2 to get the next

$$
\begin{aligned}
|E_N(z,\,t)| &= \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{ij}\,\psi_i(z)\,\phi_j(t) - \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\,\psi_i(z)\,\phi_j(t)\right| \\
&\le \left|\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} v_{ij}\,\psi_i(z)\,\phi_j(t)\right| + \left|\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} v_{ij}\,\psi_i(z)\,\phi_j(t)\right| \\
&\le \sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{2^{i+1}\,2^{j+1}}{(i+1)!\,(j+1)!} + \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{2^{i+1}\,2^{j+1}}{(i+1)!\,(j+1)!} \\
&\le \sum_{i=0}^{N} \frac{2^{i+1}}{(i+1)!} \sum_{j=N+1}^{\infty} \frac{2^{j+1}}{(j+1)!} + \sum_{i=N+1}^{\infty} \frac{2^{i+1}}{(i+1)!} \sum_{j=0}^{\infty} \frac{2^{j+1}}{(j+1)!} \\
&\le \frac{\Gamma(N+1,\,2)}{\Gamma(N+1)}\,\frac{\gamma(N+1,\,2)}{\Gamma(N+1)} + \frac{\gamma(N+1,\,2)}{\Gamma(N+1)} \\
&\le 2\,\frac{\gamma(N+1,\,2)}{\Gamma(N+1)} \\
&\le \frac{2}{\Gamma(N+1)} \int_0^2 z^N\, e^{-z} dz \\
&\le 2\left(1 - \frac{\Gamma(N+1,\,2)}{\Gamma(N+1)}\right),
\end{aligned}
\tag{5.9}
$$

where $\Gamma(n,\,m)$ the upper incomplete gamma function given by the next integration

$$
\Gamma(n,\,m) = \int_m^{\infty} t^{n-1}\, e^{-1}\, dt, \tag{5.10}
$$

and $\gamma(n,\,m)$ the lower incomplete gamma function given by

$$
\gamma(n,\,m) = \int_0^{m} t^{n-1}\, e^{-t}\, dt, \tag{5.11}
$$

Further information and foundational definitions regarding this relationship can be found in [21]. The theorem is proven. □

6. Examine our method

To demonstrate the effectiveness and application of the suggested method, we will now look at *(NLTFPIDE)*. This example demonstrates how well the technique handles the combined effects of nonlinearity, fractional dynamics, and integral operators. Using our algorithm, we demonstrate the correctness and reliability of the generated solutions. All examples were solved using Mathematica 15.

Example 6.1. *[7, 31] Consider the following nonlinear TFPIDE*

$$
D_t^{\lambda}\psi(z,\,t) + \psi(z,\,t)\,\psi_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\,\psi_{zz}(z,\,r)dr + f(z,\,t), \qquad 0<\lambda,\,\mu<1, \tag{6.1}
$$

with the next IC:

$$
\psi(z,\,0) = 0, \qquad 0 < z \le 1, \tag{6.2}
$$

and the BCs:

$$
\psi(0,\,t) = 0, \qquad \psi(1,\,t) = 0, \qquad 0 < t \le 1, \tag{6.3}
$$

TABLE 1. Comparison of AEs for Example 6.1.

(z,t)	Our method $N = 3$	$\lambda = 0.5,\ \mu = 0.1$ Method in [31] $N = 12$	Method in [7] $N = 12$	Our method $N = 6$	$\lambda = 0.5,\ \mu = 0.3$ Method in [31] $N = 12$	Method in [7] $N = 12$
(0.1, 0.1)	2.92735×10^{-18}	5.2665×10^{-11}	5.20417×10^{-18}	6.72205×10^{-18}	7.5026×10^{-11}	5.90891×10^{-18}
(0.2, 0.2)	7.37257×10^{-18}	8.8713×10^{-10}	7.02563×10^{-17}	8.89046×10^{-18}	7.0142×10^{-10}	6.33174×10^{-17}
(0.3, 0.3)	7.37257×10^{-18}	4.6062×10^{-9}	1.31839×10^{-16}	1.12757×10^{-17}	3.6713×10^{-9}	1.45717×10^{-16}
(0.4, 0.4)	6.07153×10^{-18}	1.4766×10^{-8}	5.55112×10^{-17}	1.38778×10^{-17}	1.2353×10^{-8}	6.93889×10^{-18}
(0.5, 0.5)	8.89046×10^{-18}	3.6426×10^{-8}	4.16334×10^{-16}	1.40946×10^{-17}	3.1853×10^{-8}	5.55112×10^{-16}
(0.6, 0.6)	7.80626×10^{-18}	7.6243×10^{-8}	1.38778×10^{-16}	1.47451×10^{-17}	6.9345×10^{-8}	1.11022×10^{-16}
(0.7, 0.7)	7.37257×10^{-18}	1.4258×10^{-7}	1.99814×10^{-15}	1.36609×10^{-17}	1.3432×10^{-7}	1.83187×10^{-15}
(0.8, 0.8)	7.37257×10^{-18}	2.4555×10^{-7}	3.94129×10^{-15}	9.54098×10^{-18}	2.3833×10^{-7}	3.77476×10^{-15}
(0.9, 0.9)	4.33681×10^{-18}	3.8165×10^{-7}	3.52496×10^{-15}	6.28837×10^{-18}	3.7801×10^{-7}	4.13558×10^{-15}
(1.0, 1.0)	0	7.3344×10^{-15}	9.77076×10^{-16}	0	1.1944×10^{-16}	1.88391×10^{-15}

TABLE 2. Running times with $\lambda = 0.5$,and $\mu = 0.1$ for Example 6.1 in seconds.

N	3	4	5
CPU	41.466	50.8808	106.43

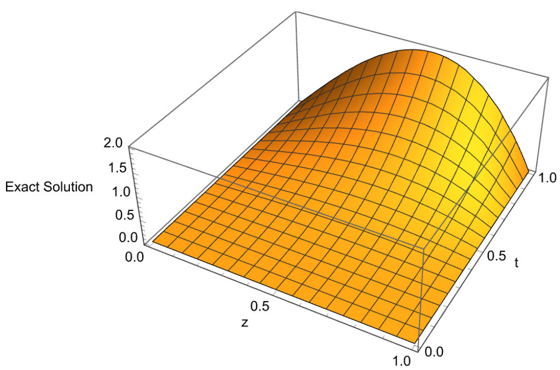

FIGURE 1. Exact solution via $\lambda = 0.7$ and $\mu = 0.4$ for Example 6.1.

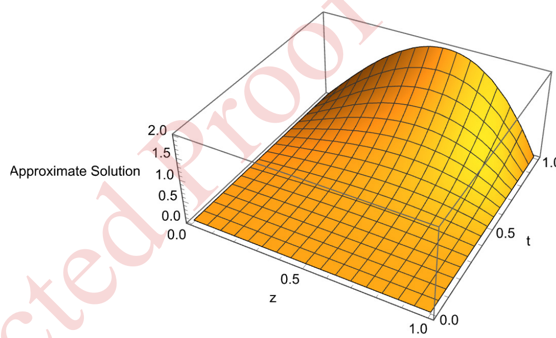

FIGURE 2. Approximate solution via $N = 3$, $\lambda = 0.7$, and $\mu = 0.4$ for Example 6.1.

where

$$f(z,\,t) = \sin(\pi\, z)\left(\frac{6\,t^{3-\lambda}}{\Gamma(4-\lambda)} + \frac{(6\,\pi^2\,\Gamma(\mu))\,t^{\mu+3}}{\Gamma(\mu+4)} + \pi\,t^6\,\cos(\pi\, z)\right). \tag{6.4}$$

The analytic solution is $\psi(z,\,t) = t^3\,\sin(\pi\, z)$. Table 1 shows the comparison of the absolute errors (AEs) between our method at $N = 3,\,6$ and $\lambda = 0.5$ with the method developed in Taghipour and Aminikhah [31] and the method in [7] at $N = 12$ and $\lambda = 0.5$ at different values of μ. Table 2 evaluates the computational efficiency of the suggested spectral collocation method by reporting the corresponding CPU time (in seconds) for various values of N with $\lambda = 0.5$,and $\mu = 0.1$. Figures 1 and 2 depict the exact and approximate solutions when $N = 3$, $\lambda = 0.7$, and $\mu = 0.4$, with no difference between them. Fig. 3 illustrates the absolute error for $N = 3$, $\lambda = 0.7$, and $\mu = 0.4$.

Example 6.2. *[31] Consider the following nonlinear TFPIDE*

$$D_t^{\lambda}\psi(z,\,t) + \psi(z,\,t)\,\psi_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\,\psi_{zz}(z,\,r)dr + f(z,\,t), \qquad 0 < \lambda,\,\mu < 1, \tag{6.5}$$

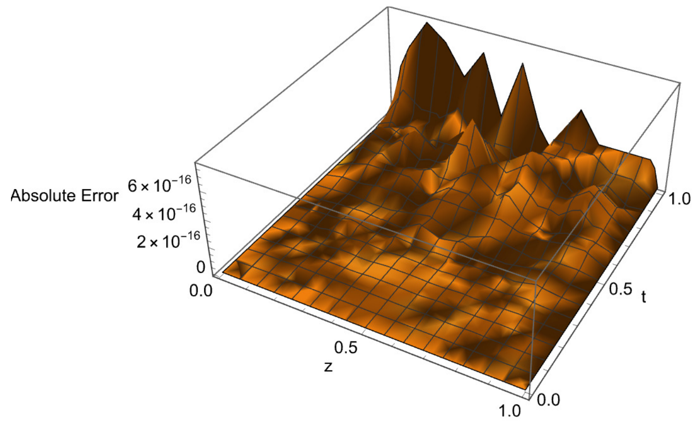

FIGURE 3. AE via $N = 3$, $\lambda = 0.7$, and $\mu = 0.4$ for Example 6.1.

TABLE 3. Comparison of absolute errors for Example 6.2.

	$N=3 \quad \lambda=0.5 \quad \mu=0.5$		$N=3 \quad \lambda=0.7 \quad \mu=0.5$		$N=3 \quad \lambda=0.9 \quad \mu=0.5$		$N=3 \quad \lambda=1.0 \quad \mu=0.5$	
$z=t$	*Our method*	*Method in [31]*	*Our method*	*Method in [31]*	*Our method*	*Method in [31]*	*Our method*	*Method in [31]*
0.1	6.93889×10^{-18}	2.9677×10^{-15}	1.73472×10^{-18}	2.4715×10^{-15}	1.73472×10^{-18}	2.1732×10^{-15}	6.93889×10^{-18}	2.4820×10^{-15}
0.2	1.38778×10^{-17}	1.3601×10^{-14}	2.60209×10^{-18}	1.0819×10^{-14}	1.73472×10^{-18}	8.4386×10^{-15}	3.46945×10^{-18}	8.1749×10^{-15}
0.3	2.60209×10^{-18}	4.2369×10^{-14}	6.93889×10^{-18}	3.3196×10^{-14}	6.93889×10^{-18}	6.4282×10^{-14}	1.38778×10^{-17}	2.1178×10^{-14}
0.4	3.03577×10^{-18}	1.0748×10^{-14}	3.46945×10^{-18}	8.5140×10^{-14}	6.93889×10^{-18}	6.4282×10^{-14}	1.38778×10^{-17}	5.2874×10^{-14}
0.5	2.60209×10^{-18}	2.2699×10^{-13}	2.77556×10^{-17}	1.8235×10^{-13}	2.9677×10^{-15}	1.4061×10^{-13}	1.30104×10^{-18}	1.1730×10^{-13}
0.6	3.46945×10^{-18}	4.0491×10^{-13}	3.46945×10^{-18}	3.2892×10^{-13}	6.93889×10^{-18}	2.5840×10^{-13}	8.67362×10^{-19}	2.2102×10^{-13}
0.7	3.46945×10^{-18}	6.1083×10^{-13}	3.46945×10^{-18}	5.0086×10^{-13}	6.93889×10^{-18}	3.9917×10^{-13}	1.73472×10^{-18}	3.4946×10^{-13}
0.8	2.7756×10^{-18}	7.5499×10^{-13}	8.67362×10^{-19}	6.2282×10^{-13}	5.20417×10^{-18}	5.0181×10^{-13}	6.93889×10^{-18}	4.4802×10^{-13}
0.9	6.93889×10^{-18}	6.5503×10^{-13}	3.46945×10^{-18}	5.4298×10^{-13}	1.0842×10^{-18}	4.4097×10^{-13}	6.93889×10^{-18}	4.0014×10^{-13}
1.0	0	7.5493×10^{-16}	0	3.3867×10^{-16}	0	1.0991×10^{-15}	0	1.0437×10^{-15}

with the next IC:

$$\psi(z,\,0) = 0, \qquad 0 < z \le 1, \tag{6.6}$$

and the BCs:

$$\psi(0,\,t) = 0, \qquad \psi(1,\,t) = 0, \qquad 0 < t \le 1, \tag{6.7}$$

where

$$f(z,\,t) = \frac{4\,t^{\beta+2}}{\beta^3 + 3\,\beta^2 + 2\,\beta} + \frac{(2\,z\,(1-z))\,t^{2-\alpha}}{\Gamma(3-\alpha)} + t^4\,z\,(1-z)(1-2\,z) \tag{6.8}$$

The analytic solution is $\psi(z,\,t) = t^2\,z\,(1-z)$. Table 3 shows the comparison of the absolute errors between our method at $N = 3$, $\lambda = 0.5$, and $\mu = 0.5$ with the method developed in Taghipour and Aminikhah [31]. The maximum absolute error with (N = 7, 9) is shown in Tab. 4. Table 5 reports the associated CPU time (in seconds) for different values of N in order to assess the computational efficiency with $\lambda = 0.7$ and $\mu = 0.5$. Figure 4 shows the absolute error at N = 3, $\lambda = 0.7$, and $\mu = 0.5$. Figure 5 compares the exact and approximate solutions with various values of N.

Example 6.3. *[7, 8, 15] Consider the following nonlinear TFPIDE*

$$D_t^{\lambda}\psi(z,\,t) + \psi(z,\,t)\,\psi_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\,\psi_{zz}(z,\,r)dr + f(z,\,t), \qquad 0 < \lambda,\,\mu < 1, \tag{6.9}$$

with the next IC:

$$\psi(z,\,0) = (1-z)^2\,z^2, \qquad 0 < z \le 1, \tag{6.10}$$

TABLE 4. The maximum absolute error for Example 6.3.

	Our method ($\lambda = 1.0$, $\mu = 0.5$)	
$z = t$	$N = 7$	$N = 9$
0.1	1.73472×10^{-18}	1.0842×10^{-18}
0.2	3.46945×10^{-18}	1.73472×10^{-18}
0.3	4.33681×10^{-18}	3.03577×10^{-18}
0.4	4.77049×10^{-18}	3.90313×10^{-18}
0.5	3.90313×10^{-18}	3.90313×10^{-18}
0.6	3.46945×10^{-18}	3.90313×10^{-18}
0.7	3.03577×10^{-18}	3.46945×10^{-18}
0.8	3.03577×10^{-18}	3.90313×10^{-18}
0.9	1.51788×10^{-18}	2.60209×10^{-18}
1.0	2.98337×10^{-28}	2.22045×10^{-18}

TABLE 5. Running times for Ex. 6.2 in seconds with $\lambda = 0.7$ and $\mu = 0.5$.

N	*3*	*5*	*7*
CPU	*39.9127*	*109.237*	*130.921*

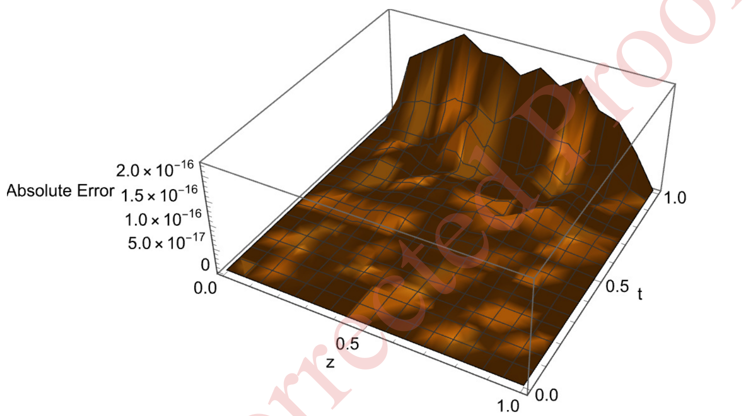

FIGURE 4. Absolute error via $N = 3$, $\lambda = 0.7$, and $\mu = 0.5$ for Example 6.2.

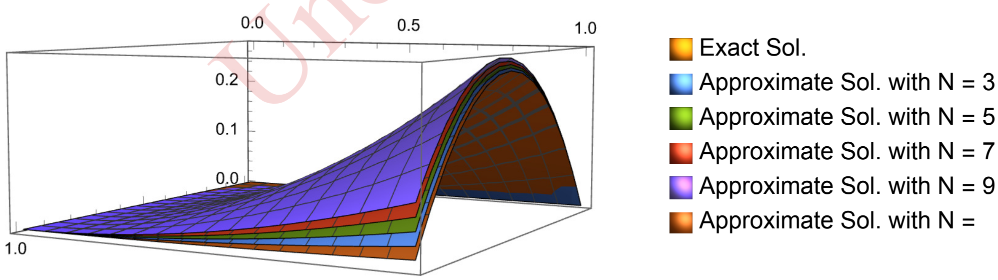

FIGURE 5. Comparison between exact and approximate solutions for various values of N via $\lambda = 0.7$ and $\mu = 0.5$ for Example 6.2.

Table 6. Comparison of absolute errors for Example 6.3.

$z=t$	$\lambda=0.5,\ \mu=0.1,\ N=3$	$\lambda=0.5,\ \mu=0.15,\ N=12$	$\lambda=0.75,\ \mu=0.15,\ N=12$
	Our method	*Method in [8]*	*Method in[7]*
0.1	6.93889×10^{-18}	2.278×10^{-4}	4.06707×10^{-8}
0.2	7.58942×10^{-19}	1.8783×10^{-3}	2.83869×10^{-8}
0.3	6.50521×10^{-19}	3.5717×10^{-3}	1.79009×10^{-9}
0.4	6.9388×10^{-18}	4.2866×10^{-3}	1.97597×10^{-8}
0.5	4.33681×10^{-19}	3.4757×10^{-3}	2.76727×10^{-8}
0.6	3.46945×10^{-18}	1.1975×10^{-3}	1.95891×10^{-8}
0.7	1.04083×10^{-17}	1.7931×10^{-3}	2.05927×10^{-9}
0.8	1.38778×10^{-17}	4.0788×10^{-3}	2.86461×10^{-8}
0.9	1.04083×10^{-17}	3.8964×10^{-3}	4.08221×10^{-8}

Table 7. Comparison of the exact, approximate solutions, and Max. absolute error for Example 6.3.

$\lambda=0.5,\ \mu=0.1,\ N=5$			
$z=t$	*Exact Sol.*	*Approximate Sol.*	*Max. absolute error*
0.1	3.04875×10^{-4}	3.04875×10^{-4}	8.13152×10^{-19}
0.2	4.88×10^{-4}	4.88×10^{-4}	7.58942×10^{-19}
0.3	5.74875×10^{-4}	5.74875×10^{-4}	6.50521×10^{-19}
0.4	5.88×10^{-4}	5.88×10^{-4}	3.25261×10^{-19}
0.5	5.46875×10^{-4}	5.46875×10^{-4}	4.33681×10^{-19}
0.6	4.68×10^{-4}	4.68×10^{-4}	4.87891×10^{-19}
0.7	3.64875×10^{-4}	3.64875×10^{-4}	6.50521×10^{-19}
0.8	2.48×10^{-4}	2.48×10^{-4}	7.04731×10^{-19}
0.9	1.24875×10^{-4}	1.24875×10^{-4}	5.69206×10^{-19}

and the BCs:

$$\psi(0,\,t)=0, \qquad \psi(1,\,t)=0, \qquad 0<t\leq 1, \tag{6.11}$$

where

$$f(z,t)=2\left(t^{\frac{5}{2}}+1\right)^2(1-z\,\Gamma\left(\frac{7}{2}-\lambda\right)(1-z)^2\,z^2\,t^{\frac{5}{2}-\lambda}-2\,(6\,z^2-6\,z+1)\left(\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(\lambda)}{\Gamma\left(\mu+\frac{7}{2}\right)}\,t^{\mu+\frac{5}{2}}+\frac{t^{\mu}}{\mu}\right). \tag{6.12}$$

The analytic solution is $\psi(z,t)=\left(t^{\frac{5}{2}}+1\right)\,z^2\,(1-z)^2$. *Table 6 shows a comparison of the absolute errors between our method at* $N=3$, $\lambda=0.5$, *and* $\mu=0.1$ *with the method developed in Taghipour and Aminikhah [8] and [7] at different values of* λ *and* μ. *As shown in 7, the results obtained from our numerical solution are in excellent agreement with the exact solution in the case of* $N=5$. *This close correspondence validates the effectiveness and accuracy of the proposed approach for solving the problem. Table 8 shows the CPU time (in seconds) for* $N=(3,\,4,\,5)$ *and* $\lambda=0.5$, *and* $\mu=0.1$. *Figure 6 shows the exact solution via* $\lambda=0.5$ *and* $\mu=0.1$, *Figure 7 describes the approximate solution with the same status as the exact solution when* $N=3$, *and Figure 8 shows the absolute error at* $N=3$, $\lambda=0.5$, *and* $\mu=0.1$.

TABLE 8. Running times for Example 6.3 in seconds with $\lambda = 0.5$ and $\mu = 0.1$.

N	*3*	*4*	*5*
CPU	*39.9809*	*47.3133*	*102.768*

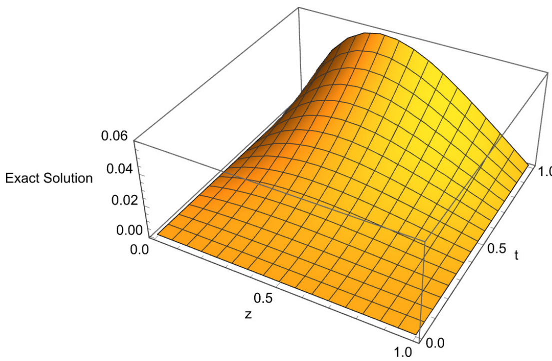

FIGURE 6. Exact solution via $\lambda =$ 0.5 and $\mu = 0.1$ for Example 6.3.

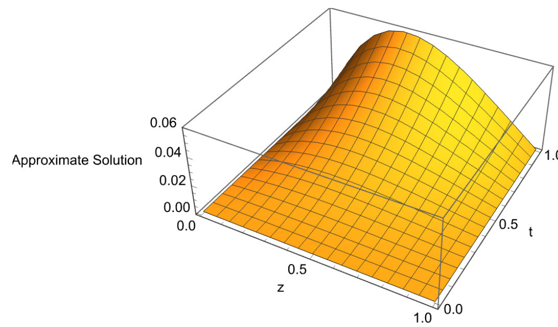

FIGURE 7. Approximate solution via $N = 3$, $\lambda = 0.5$, and $\mu = 0.1$ for Example 6.3.

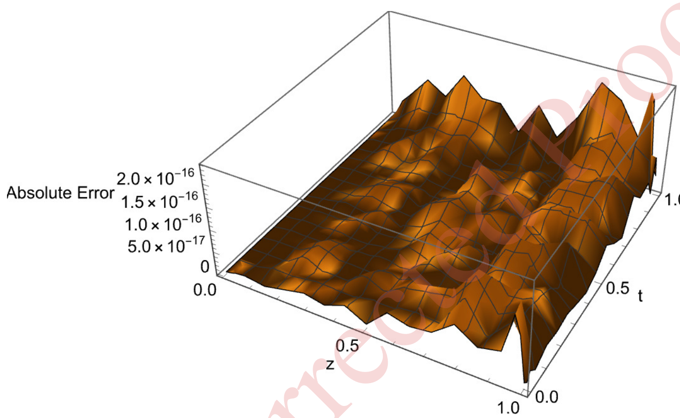

FIGURE 8. AE via $N = 3$, $\lambda = 0.5$, and $\mu = 0.1$ for Example 6.3.

Example 6.4. *[7, 31] Consider the following nonlinear TFPIDE*

$$D_t^{\lambda}\psi(z,\,t) + \psi(z,\,t)\,\psi_z(z,\,t) = \int_0^t (t-r)^{\mu-1}\,\psi_{zz}(z,\,r)\,dr + f(z,\,t), \qquad 0 < \lambda,\,\mu < 1, \tag{6.13}$$

with the next IC:

$$\psi(z,\,0) = 0, \qquad 0 < z \leq 1, \tag{6.14}$$

and the BCs:

$$\psi(0,\,t) = t^3, \qquad \psi(1,\,t) = e\,t^3, \qquad 0 < t \leq 1, \tag{6.15}$$

where

$$f(z,t) = \frac{(6\,e^z)\,t^{3-\lambda}}{\Gamma(4-\lambda)} - \frac{6\,e^z\,\Gamma(\mu)\,t^{\mu+3}}{\Gamma(\mu+4)} + t^6\,e^{2\,z}. \tag{6.16}$$

Table 9. Comparison of absolute errors for Example 6.4.

(z,t)	*Our method*	*Method in[7]*	*Method in [31]*
	$\lambda=0.5,\ \mu=0.5,\ N=3$	$\lambda=0.5,\ \mu=0.5,\ N=9$	$\lambda=0.5,\ \mu=0.5,\ N=9$
(0.1, 0.1)	5.55654×10^{-19}	5.28549×10^{-19}	1.1194×10^{-18}
(0.2, 0.2)	4.74338×10^{-19}	1.23599×10^{-17}	2.8112×10^{-12}
(0.3, 0.3)	3.93023×10^{-19}	3.03577×10^{-17}	8.1029×10^{-12}
(0.4, 0.4)	2.1684×10^{-19}	6.67153×10^{-17}	1.6060×10^{-11}
(0.5, 0.5)	1.35525×10^{-19}	2.94903×10^{-16}	3.5035×10^{-11}
(0.6, 0.6)	4.20128×10^{-19}	1.87351×10^{-16}	8.5052×10^{-11}
(0.7, 0.7)	5.69206×10^{-19}	3.88578×10^{-16}	1.9935×10^{-10}
(0.8, 0.8)	5.42101×10^{-19}	6.93889×10^{-16}	4.1556×10^{-10}
(0.9, 0.9)	4.67562×10^{-19}	5.13478×10^{-16}	7.3660×10^{-10}

Table 10. Comparison of the exact and approximate solutions with $\lambda=0.5,\ \mu=0.5,\ N=4$ for Example 6.4.

$z=t$	*Exact Sol.*	*Approximate Sol.*
0.1	1.10517×10^{-3}	1.10517×10^{-3}
0.2	1.2214×10^{-3}	1.2214×10^{-3}
0.3	1.34986×10^{-3}	1.34986×10^{-3}
0.4	1.49182×10^{-3}	1.49182×10^{-3}
0.5	1.64872×10^{-3}	1.64872×10^{-3}
0.6	1.82212×10^{-3}	1.82212×10^{-3}
0.7	2.01375×10^{-3}	2.01375×10^{-3}
0.8	2.22554×10^{-3}	2.22554×10^{-3}
0.9	2.4596×10^{-3}	2.4596×10^{-3}

Table 11. Running times with case $\lambda=0.5$, and $\mu=0.5$ for Example 6.4 in seconds.

N	*3*	*4*	*5*
CPU	*33.3055*	*46.7621*	*99.5842*

The analytic solution is $\psi(z,t)=e^z\,t^3$. Table 9 shows a comparison of the absolute errors between our method at $N=3$, $\lambda=0.5$, and $\mu=0.5$ with the methods developed in [7] and [31] at $N=9$, the same values of λ and μ. The results presented in the Table 10 show that the approximate solution is identical to the exact solution, This perfect match serves as strong validation for the proposed method, confirming its ability to accurately solve the problem. The corresponding CPU time (in seconds) for various values of N is displayed in Table 11. Figure 9 shows the exact solution, Figure 10 shows the approximate solution via $N=3$, and Figure 11 shows the absolute error at $N=5$, $\lambda=0.5$, and $\mu=0.5$.

Example 6.5. *[7, 31] Consider the following nonlinear TFPIDE*

$$D_t^{\lambda}\psi(z,\,t)+\psi(z,\,t)\,\psi_z(z,\,t)=\int_0^t(t-r)^{\mu-1}\,\psi_{zz}(z,\,r)\,dr+f(z,\,t),\qquad 0<\lambda,\,\mu<1,\tag{6.17}$$

with the next IC:

$$\psi(z,\,0)=0,\qquad 0<z\le1,\tag{6.18}$$

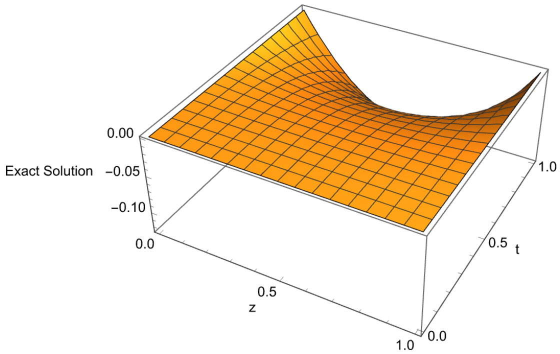

Figure 9. Exact solution via $\lambda = 0.5$ and $\mu = 0.5$ for Example 6.4.

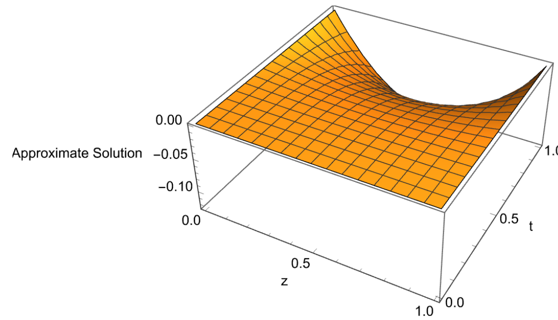

Figure 10. Approximate solution via $N = 3$, $\lambda = 0.5$, and $\mu = 0.5$ for Example 6.4.

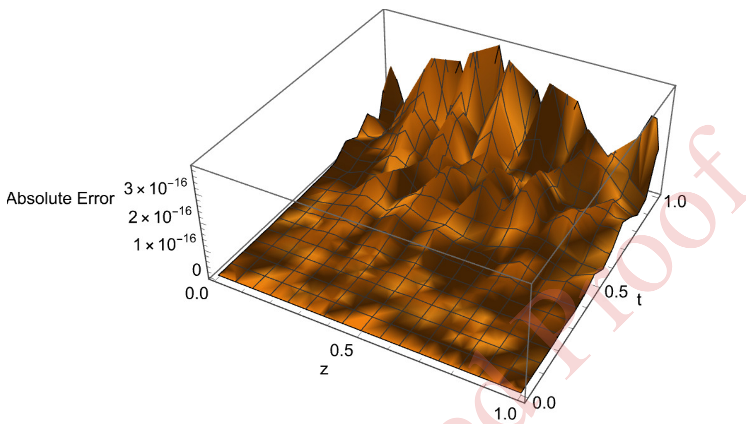

Figure 11. AE via $N = 5$, $\lambda = 0.5$, and $\mu = 0.5$ for Example 6.4.

and the BCs:

$$\psi(0,\, t) = t^2, \qquad \psi(1,\, t) = 0, \qquad 0 < t \le 1, \tag{6.19}$$

where

$$\begin{aligned} f(z,t) =& \frac{t^{2-\lambda}\,(2\,(1-z)\,\cos(z))}{\Gamma(3-\lambda)} - \frac{2\,\Gamma(\mu)\,t^{\mu+2}\,(2\,\sin(z)+(z-1)\,\cos(z))}{\Gamma(\mu+3)} \\ &+ t^4\,(1-z)\,\cos(z)\,(z\,\sin(z)-\sin(z)-\cos(z))\,. \end{aligned} \tag{6.20}$$

The analytic solution is $\psi(z,t) = t^2\,(1-z)\,\cos(z)$. Table 12 shows a comparison of the absolute errors between our method at $\lambda = 0.6, \mu = 0.9$, and $N = 3, 5$ with the method developed in [7] at $N = 8$, $\lambda = 0.3$ and $\mu = 0.8$. For different values of N and $\lambda = 0.6$, and $\mu = 0.9$, Table 13 shows the corresponding CPU time (in seconds). Figure 12 shows the exact solution with $\lambda = 0.6$ and $\mu = 0.9$, Figure 13 shows the approximate solution at $N = 3$, $\lambda = 0.6$, and $\mu = 0.9$. Figure 14 shows the absolute error at $N = 3$, $\lambda = 0.6$, and $\mu = 0.9$.

7. Conclusions

In this paper, we effectively developed and implemented a novel spectral collocation methodology built around MSCP3K for numerically solving NLTFPIDEs with weakly singular kernels. This technique shows extraordinary

TABLE 12. Comparison of the maximum absolute errors for Example 6.5.

(z,t)	*Our method* ($\lambda = 0.6,\ \mu = 0.9$)		*Method in*[7] ($\lambda = 0.3,\ \mu = 0.8$)
	$N = 3$	$N = 5$	$N = 8$
$(0.1, 0.1)$	3.25261×10^{-19}	5.6243×10^{-18}	2.73128×10^{-15}
$(0.2, 0.2)$	5.42101×10^{-20}	8.80914×10^{-18}	1.12764×10^{-14}
$(0.3, 0.3)$	4.33681×10^{-19}	1.04626×10^{-17}	2.03015×10^{-14}
$(0.4, 0.4)$	6.50521×10^{-19}	1.14383×10^{-17}	1.02121×10^{-13}
$(0.5, 0.5)$	7.58942×10^{-19}	1.1601×10^{-17}	1.57861×10^{-13}
$(0.6, 0.6)$	5.42101×10^{-19}	1.06252×10^{-17}	2.28061×10^{-13}
$(0.7, 0.7)$	2.1684×10^{-19}	9.1073×10^{-18}	1.07837×10^{-13}
$(0.8, 0.8)$	1.0842×10^{-19}	6.93889×10^{-18}	1.77199×10^{-13}
$(0.9, 0.9)$	1.6263×10^{-19}	*0*	2.15043×10^{-13}

TABLE 13. Running times for Example 6.5 in seconds.

N	*3*	*4*	*5*
CPU	*48.9138*	*59.0869*	*121.535*

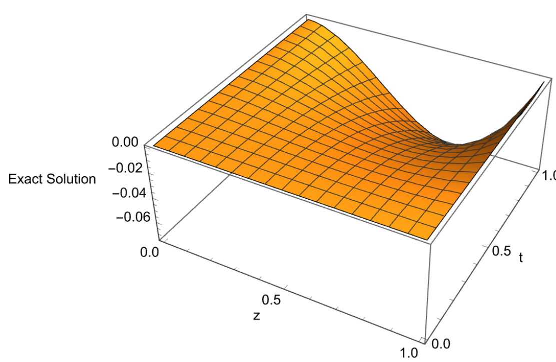

FIGURE 12. Exact solution via $\lambda = 0.6$ and $\mu = 0.9$ for Example 6.5.

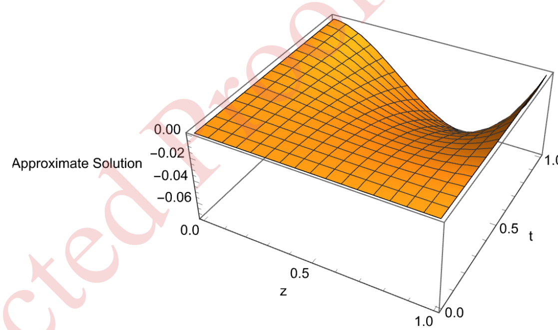

FIGURE 13. Approximate solution via $N = 3$, $\lambda = 0.6$, and $\mu = 0.9$ for Example 6.5.

skills in dealing with the fractional operator's inherent complexity as well as the system's nonlinearity. This study makes a significant theoretical addition by exactly deriving the operational matrices associated with the MSCP3K basis functions. This thorough design showed a distinct, highly structured matrix pattern, which is computationally advantageous since it considerably speeds the inversion process and lowers the overall computational cost of the algebraic system. Furthermore, the suggested framework's robustness was supported by a thorough theoretical study that rigorously verified the method's stability and confirmed a consistent rate of convergence. This research establishes the mathematical underpinning for the framework's widespread applicability. The MSCP3K-based spectrum method's practical usefulness, precision, and efficiency were persuasively demonstrated by a series of numerical tests. When compared to existing schemes in the literature, the obtained findings not only demonstrated great agreement with exact solutions but also superior performance, notably in terms of convergence rate and minimum necessary basis size. The overall framework marks a significant step forward in the numerical treatment of complex fractional integro-differential models. It provides a powerful, fast, and extremely accurate tool for researchers to simulate real-world occurrences. Future study will focus on expanding the MSCP3K technique to high-dimensional fractional differential equations and including adaptive time-stepping strategies to further maximize computational performance.

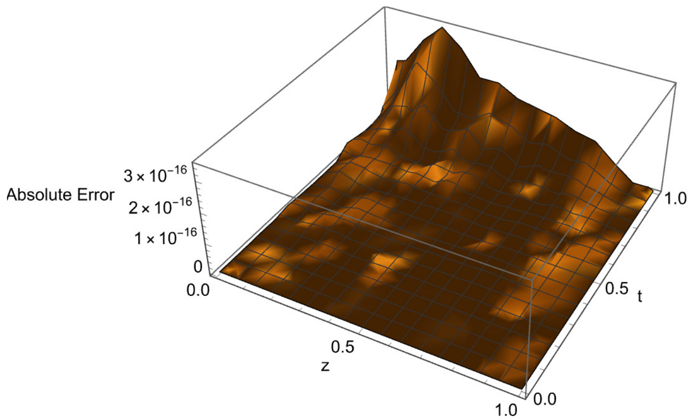

FIGURE 14. AE via $N = 3$, $\lambda = 0.6$, and $\mu = 0.9$ for Example 6.5.

Competing interests. The authors declare that they have no any competing interests.

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