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# Some properties of a subclass of analytic functions involving a generalized differential operator

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**Abstract**

Here we introduce a new differintegral operator involving a family of special functions which combines Mittag-Leffler function with the generalized Gaussian hypergeometric function. Using subordination, we introduce a new Bazilevič functions of order $\alpha + i\beta$ involving the new differential operator. Since the defined class has a differential characterizations involving the differential operator, it helps in unifying the studies of starlike, convex, $\alpha$-convex function, close-to-convex, quasi-convex functions and so on. We have obtained some subordination results and the initial coefficient bounds of the functions belonging to the defined function class.

**Keywords.** Analytic functions, Bazilevič functions, differential operators, differential subordination, generalized $M$-series, Mittag-Leffler function, Gaussian hypergeometric function.

**2010 Mathematics Subject Classification.** 30C45, 30C50.

Uncorrected Proof

## 1. Introduction

Let $\mathcal{H}(a,n)$ denote the family consisting of functions of the form $\psi(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \cdots$, $\zeta \in \Theta = \{\zeta : | z | < 1\}$. Also, we let $\mathcal{Q}$ to be the collection of analytic functions which has a series expansion of the form

$$\psi(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots, \tag{1.1}$$

Let $\mathcal{S}$ be the class of functions $\psi \in \mathcal{Q}$ which are one-one in $\Theta$. The following are the well-known subclasses associated with $\mathcal{S}$:

(1) Carathéodory's function (see [8]): $\Xi = \{p \in \mathcal{H}(1,1);\ \mathrm{Re}\,(p(\zeta)) > 0;\ \forall\zeta \in \Theta\}$.
(2) Starlike functions class: $\mathcal{S}^* = \left\{\psi \in \mathcal{Q};\ \mathrm{Re}\left(\frac{\zeta\psi'(\zeta)}{\psi(\zeta)}\right) > 0;\ \forall\zeta \in \Theta\right\}$.
(3) Convex functions class: $\mathcal{C} = \left\{\psi \in \mathcal{Q};\ \mathrm{Re}\left(1 + \frac{\zeta\psi''(\zeta)}{\psi'(\zeta)}\right) > 0;\ \forall\zeta \in \Theta\right\}$.
(4) Close-to-convex functions class: $\mathcal{K} = \left\{\psi \in \mathcal{Q};\ \mathrm{Re}\left(\frac{\zeta\psi'(\zeta)}{\chi(\zeta)}\right) > 0;\ \chi(\zeta) \in \mathcal{S}^*\right\}$.
(5) Quasi-convex functions class: $\mathcal{C}^* = \left\{\psi \in \mathcal{Q};\ \mathrm{Re}\left(\frac{(\zeta\psi'(\zeta))'}{\chi'(\zeta)}\right) > 0;\ \chi(\zeta) \in \mathcal{C}\right\}$.

All those subclasses characterized above were geometrically defined, here we omit those details since it can be found in any standard text on *Univalent Function Theory*. Refer to [12, 43] for detailed study.

A function $\psi(\zeta) \in \mathcal{Q}$ is said to be in $\mathcal{B}(\alpha)$ if and only if it satisfies

$$\mathrm{Re}\left[\frac{\zeta\psi'(\zeta)}{\psi^{1-\alpha}(\zeta)\,\chi^{\alpha}(\zeta)}\right] > 0, \qquad (\zeta \in \Theta;\ \alpha \geq 0;\ \chi \in \mathcal{S}^*).$$

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The class $\mathcal{B}(\alpha)$ is popularly called as the class of Bazilevič functions. Note that functions in $\mathcal{B}(\alpha)$ are univalent. Further, it can be seen that $\mathcal{B}(1) = \mathcal{K}$(close-to-convex) and $\mathcal{B}(0) = \mathcal{S}^*$ (starlike). For detailed study, refer to [43, Chapter 10]. Recently, Niu et al. [32, Definition 1.1] extended $\mathcal{B}(\alpha)$ to $\mathcal{B}(\alpha, \beta; G, H)$, the so-called class of Bazilevič function of complex order and type $(G, H)$. Precisely, $\psi(\zeta) \in \mathcal{Q}$ is said to be in $\mathcal{B}(\alpha, \beta; G, H)$ if and only if satisfies

$$\frac{\zeta\psi'(\zeta)}{\psi(\zeta)}\left(\frac{\psi(\zeta)}{\chi(\zeta)}\right)^{\alpha+i\beta} \prec \frac{1+G\zeta}{1+H\zeta}, \quad (\alpha \geq 0,\ \beta \in \mathbb{R};\ -1 \leq H < G \leq 1),$$

where $\prec$ denotes the subordination and $\chi(\zeta) \in \mathcal{Q}$ satisfies $\frac{\zeta\psi'(\zeta)}{\psi(\zeta)} \prec \frac{1+G\zeta}{1+H\zeta}$. Ma and Minda in [29] extended the classes $\mathcal{S}^*$ and $\mathcal{C}$ by considering a general function $\phi \in \Xi$ starlike with respect to a fixed point 1 and they introduced

$$\mathcal{S}^*(\phi) = \left\{\psi \in \mathcal{Q} :\ \frac{\zeta\psi'(\zeta)}{\psi(\zeta)} \prec \phi(\zeta),\ \zeta \in \Theta\right\},$$

and

$$\mathcal{C}(\phi) = \left\{\psi \in \mathcal{Q} :\ 1 + \frac{\zeta\psi''(\zeta)}{\psi'(\zeta)} \prec \phi(\zeta),\ \zeta \in \Theta\right\}.$$

They further assumed $\phi$ to be symmetric with respect to real axis and found coefficient bounds which are very popular in this field of research.

Motivated by the class $\mathcal{B}(\alpha)$, Goyal and Goswami [11] introduced the following: For $0 \leq \alpha < \infty$, a function $\psi(\zeta) \in \mathcal{Q}$ is said to be in $\mathcal{B}(\alpha;\ \phi)$ if and only if it satisfies the condition

$$\frac{\zeta\psi'(\zeta)}{[\psi(\zeta)]^{1-\alpha}\,[\chi(\zeta)]^{\alpha}} \prec \phi(\zeta), \quad (\zeta \in \Theta;\ \chi \in \mathcal{S}^*(\phi)), \tag{1.2}$$

where $\phi \in \Xi$ has a series expansion of the form

$$\phi(\zeta) = 1 + L_1\zeta + L_2\zeta^2 + L_3\zeta^3 + \cdots, \qquad (L_1 > 0;\ \zeta \in \Theta). \tag{1.3}$$

In [11], authors obtained the sufficient conditions for functions in $\mathcal{B}(\alpha;\ \phi)$ to be starlike and close-to-convex. They obtained their results using the principle of differential subordination.

In view of extending our study to a broad audience, we will study our new class involving a new differintegral operator. We will begin with a short introduction to the theory of special functions that will be required for our study. *Mittag-Leffler* function appears naturally in the kernel of the fractional derivative operators. Indeed, most the derivatives which is applied to the natural phenomena involves the *Mittag-Leffler* function. Whereas the generalized *Gaussian hypergeometric* function which is a solution of the differential equations, has been an important tool in both theoretical analysis and practical applications. The classical Mittag-Leffler function and its extension are as follows:

$$E_{\gamma}(\zeta) = \sum_{n=0}^{\infty}\frac{\zeta^n}{\Gamma(\gamma n+1)}, \quad E_{\gamma,\,\delta}(\zeta) = \sum_{n=0}^{\infty}\frac{\zeta^n}{\Gamma(\gamma n+\delta)}, \quad \text{and} \quad E^{\nu,\,\varepsilon}_{\gamma,\,\delta}(\zeta) = \sum_{n=0}^{\infty}\frac{(\nu)_{n\varepsilon}\zeta^n}{\Gamma(\gamma n+\delta)\,n!},$$

where $\zeta, \gamma, \delta, \nu, \varepsilon \in \mathbb{C}, Re(\gamma) > 0, Re(\varepsilon) > 0, \zeta \in \Theta$, $\mathbb{C}$ denotes the set of complex numbers and $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & if\, n = 0, \\ x(x+1)(x+2)\ \ldots\ (x+n-1), & if\, n \in \mathbb{N}. \end{cases}$$

The classical Mittag-Leffler function $E_{\gamma}(\zeta)$ was introduced by Gosta Mittag-Leffler [31] and it generalization $E_{\gamma,\,\delta}(\zeta)$ by Wiman [46]. Whereas, the extension $E^{\nu,\,\varepsilon}_{\gamma,\,\delta}(\zeta)$ is called the Srivastava–Tomovski generalization of the Mittag-Leffler function (see [41, 44]). Note that the function $E^{\nu,\,1}_{\gamma,\,\delta}(\zeta)$ is popularly known as Prabhakar function or generalized Mittag-Leffler three-parameter function (see [2]). For general reference of Mittag-Leffler function pertaining to this duality theory, refer to Kiryakova [24–26] and Srivastava [38, 39].

Uncorrected Proof

For complex parameters $\eta_1, \ldots, \eta_r$ and $\theta_1, \ldots, \theta_s$ ($\theta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$; $\mathbb{Z}_0^- = 0, -1, -2, \ldots$; $j = 1, \ldots, s$), the generalized hypergeometric function ${}_rF_s(\eta_1, \ldots, \eta_r; \theta_1, \ldots, \theta_s; \zeta)$ is defined by

$$ {}_rF_s(\eta_1, \eta_2, \ldots, \eta_r; \theta_1, \theta_2, \ldots, \theta_s; \zeta) = \sum_{n=0}^{\infty} \frac{(\eta_1)_n \ \ldots\ (\eta_r)_n}{(\theta_1)_n \ \ldots\ (\theta_s)_n} \frac{\zeta^n}{n!}, \qquad (r, s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\};\ \zeta \in \Theta). $$

Recently, Karthikeyan et al. in [20] studied classes of analytic functions involving the generalized $M$-series[40, Eq. 1] (also see [42]). The generalized $M$-series was defined to unify the two well-known functions namely Mittag-Leffler function and Gaussian hypergeometric function. The generalized $M$-series is given by

$$ {}_r\mathcal{M}_s^{\gamma,\delta}(\zeta) = {}_r\mathcal{M}_s^{\gamma,\delta}(\eta_1, \ldots, \eta_r; \theta_1, \ldots, \theta_s;\, \zeta) = \sum_{n=0}^{\infty} \frac{(\eta_1)_n \ldots (\eta_r)_n}{(\theta_1)_n \ldots (\theta_s)_n} \frac{\zeta^n}{\Gamma(n\gamma + \delta)}. \tag{1.4} $$

For discussions on special cases and convergence of the series (1.4), refer to [40, 42].

1.1. **New Family of Differential Operator.** Corresponding to the function $g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta)$ defined by

$$ g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta) = {}_rg_s^{\gamma,\delta}(\eta_1, \ldots, \eta_r; \theta_1, \ldots, \theta_s;\, \zeta) = \zeta\Gamma(\delta)\, {}_rF_{s+1}(\eta_1, \eta_2, \ldots, \eta_r; \theta_1, \theta_2, \ldots, \theta_s, 1; \zeta), $$

we now introduce an operator $\mathcal{D}_{\lambda,\mu}^m(\eta_1, \theta_1)\psi(\zeta) : \mathcal{Q} \to \mathcal{Q}$ which is defined by

$$ \begin{aligned} \mathcal{D}_{\lambda,\mu}^0(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) &= \psi(\zeta) * g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta), \\ \mathcal{D}_{\lambda,\mu}^1(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) &= (1 - \lambda + \mu)(\psi(\zeta) * g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta)) + (\lambda - \mu)\zeta(\psi(\zeta) * g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta))' \\ &\quad + \lambda\mu\zeta^2(\psi(\zeta) * g_{r,s}^{\gamma,\delta}(\eta_1, \theta_1; \zeta)^{''}, \\ \mathcal{D}_{\lambda,\mu}^m(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) &= \mathcal{D}_{\lambda,\mu}^1(\mathcal{D}_{\lambda,\mu}^{m-1}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)), \end{aligned} $$

where $\zeta, \gamma, \delta \in \mathbb{C}, \mathrm{Re}(\gamma) > 0$, $0 \le \mu \le \lambda \le 1$ and $m \in \mathbb{N}_0$. By the above definition, it is easy to note that

$$ \mathcal{D}_{\lambda,\mu}^m(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) = \zeta + \sum_{n=2}^{\infty} [1 + (n-1)(\lambda - \mu + n\mu\lambda)]^m B_n a_n \zeta^n, \tag{1.5} $$

where

$$ B_n = \frac{\Gamma(\delta)(\eta_1)_{n-1}(\eta_2)_{n-1} \ldots (\eta_r)_{n-1}}{\Gamma(\gamma(n-1) + \delta)(\theta_1)_{n-1}(\theta_2)_{n-1} \ldots (\theta_s)_{n-1}(n-1)!}. \tag{1.6} $$

The deviation in the operator $\mathcal{D}_{\lambda,\mu}^m(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)$ presented here is very minor when compared with the operator recently introduced by Karthikeyan et al. in [20] (also see [45]). However, we will present the results which will be of interest to the broad audience. For suitable values of $\eta_{i's}, \theta_{j's}, r, s, \lambda$ and $\mu$, we can deduce several operators as a special case of this operator. For example,

(1) $\mathcal{D}_{\lambda,\mu}^m(\eta_1, \theta_1;\, 0, \delta)\psi(\zeta) = \mathcal{I}_{\lambda,\mu}^m(\eta_1, \theta_1)\psi(\zeta)$, an operator introduced by Karthikeyan et al. [23].
(2) $\mathcal{D}_{\lambda,0}^m(\eta_1, \theta_1;\, 0, \delta)\psi(\zeta) = \mathcal{J}_{\lambda}^m(\eta_1, \theta_1)\psi(\zeta)$, an operator introduced by Karthikeyan et al. [20].
(3) $\mathcal{D}_{\lambda,0}^0(\eta_1, \theta_1;\, 0, \delta)\psi(\zeta) = \mathcal{J}(\eta_1, \theta_1)\psi(\zeta)$, an operator introduced by Dziok and Srivastava [9].
(4) For $r = 2$, $s = 1$; $\eta_1 = \theta_1$, $\eta_2 = 1$ and $\gamma = 0$, we get the operator introduced by Al-Oboudi [1].

Some of the studies in geometric function theory that was studied in dual with Mittag-Leffler function are Aouf and Mostafa [3], Attiya [4], Breaz et al. [6], Cang and Liu [7], and Liu [28].

The recent trend has been to study various subclasses of $\mathcal{S}$ after replacing the ordinary derivative with a non-Newtonian derivatives like multiplicative derivative (see [5, 21]), quantum derivative (see [15, 16, 36]), and so on. Here we will study the class involving quantum derivative. Jackson in [18] studied the $q$-derivative which is merely a ratio and is given by

$$ D_q\psi(\zeta) = \frac{\psi(q\zeta) - \psi(\zeta)}{(q-1)\zeta} = 1 + \sum_{n=2}^{\infty} [n]_q a_n \zeta^{n-1}, \quad (\psi \in \mathcal{Q};\ 0 < q < 1). $$

Uncorrected Proof

Note that $\lim_{q\to 1^-} D_q\psi(\zeta) = \psi'(\zeta)$. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C}),$$

and the $q$-shifted factorial by

$$(\eta; q)_n = \begin{cases} 1, & n = 0, \\ (1-\eta)(1-\eta q)\dots\left(1-\eta q^{n-1}\right), & n = 1, 2, \dots. \end{cases}$$

Srivastava in [37] initiated this duality study when he defined a class of starlike functions involving the quantum derivative. But the prominent study which had the high visibility and garnering the interest of the researchers was the study of Ismail, Merkes and Styer [17].

Now replacing the ordinary derivative with a quantum derivative in the definition of $\mathcal{D}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)$, we now introduce an operator $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) : \mathcal{Q} \to \mathcal{Q}$ is defined by

$$\begin{aligned} \mathcal{N}^0_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) &= \psi(\zeta) * g^{\gamma,\delta}_{r,s}(\eta_1, \theta_1; \zeta), \\ \mathcal{N}^1_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) &= (1-\lambda+\mu)(\psi(\zeta) * g^{\gamma,\delta}_{r,s}(\eta_1, \theta_1; \zeta)) + (\lambda-\mu)\zeta D_q\left[\psi(\zeta) * g^{\gamma,\delta}_{r,s}(\eta_1, \theta_1; \zeta)\right] \\ &\quad + \lambda\mu\zeta^2 D_q^2\left[\psi(\zeta) * g^{\gamma,\delta}_{r,s}(\eta_1, \theta_1; \zeta)\right], \\ &= \zeta + \sum_{n=2}^{\infty}\left[1 + (\lambda-\mu)([n]_q - 1) + [n]_q[n-1]_q\mu\lambda)\right] Q_n a_n \zeta^n \end{aligned}$$

Applying the above step recursively $m-1$ times, we get the following

$$\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) = \zeta + \sum_{n=2}^{\infty}\left[1 + (\lambda-\mu)([n]_q - 1) + [n]_q[n-1]_q\mu\lambda)\right]^m Q_n a_n \zeta^n, \tag{1.7}$$

where

$$Q_n = \frac{(\eta_1; q)_{n-1} \dots (\eta_r; q)_{n-1}}{(q; q)_{n-1}\,(\theta_1; q)_{n-1} \dots (\theta_s; q)_{n-1}} \frac{\Gamma_q(\delta)}{\Gamma_q(\gamma(n-1)+\delta)}. \tag{1.8}$$

Note that if we let $\mu = 0$ in (1.7), the operator $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)$ reduces to the operator $\mathcal{J}^m_\lambda(\eta_1, \theta_1;\, q;\, \zeta)\psi$, studied in [22, 34, 35]. Needless to mention that as $q \to 1-$, $\mathcal{J}^m_\lambda(\eta_1, \theta_1;\, q;\, \zeta)\psi(\zeta)$ reduces to the operator $\mathcal{D}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)$. Using the operators like one defined here, authors restudied the well-known subclasses of univalent functions. For example, several studies can be generalized or unified if we study a family $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta) \in \mathcal{S}^*(\phi)$ which would in-turn satisfy

$$\frac{\zeta\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)\right]'}{\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)} \prec \phi(\zeta).$$

Motivated by Goyal and Goswami [11], we now define the following class using subordination and the defined operator $\mathcal{J}^m_\lambda(\eta_1, \theta_1;\, q;\, \zeta)\psi(\zeta)$.

**Definition 1.1.** For $\alpha \geq 0$ and $\beta \in \mathbb{R}$, we denote the class $\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\, \theta_1;\, \gamma,\, \delta; \alpha,\, \beta;\, \phi)$ consisting of functions in $\mathcal{Q}$ satisfying the subordination condition

$$\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)\right]}{\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)} \left(\frac{\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi(\zeta)}{\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\chi(\zeta)}\right)^{\alpha+i\beta} \prec \phi(\zeta), \tag{1.9}$$

where $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\chi(\zeta) = \zeta + \sum_{n=2}^{\infty}[1 + (\lambda-\mu)([n]_q - 1) + [n]_q[n-1]_q\mu\lambda)]^m Q_n b_n \zeta^n \in \mathcal{S}^*(\phi)$, the parameters range are as given in the operator $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\psi$, the powers in (1.9) are considered at the principal branch and $\phi(\zeta) \in \Xi$ is given by (1.3).

Uncorrected Proof

**Remark 1.2.** Note that unlike the class $\mathcal{B}(\alpha,\mu;\, A, B)$, the class $\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\, \theta_1;\, \gamma,\, \delta; \alpha,\, \beta;\, \phi)$ is more versatile because of the operator since it includes convex, close-to-convex, quasi-convex and $\alpha$-convex as its special class.

Now we will state some results, which we will be using to establish our main results.

**Lemma 1.3.** *[14] Let $h$ be convex in $\Theta$, with $h(0) = a$, $d \neq 0$ and $\mathrm{Re}(d) > 0$. Suppose that $\vartheta(\zeta)$ is analytic $\Theta$, which is given by*

$$\vartheta(\zeta) = a + \vartheta_n\zeta^n + \vartheta_{n+1}\zeta^{n+1} + \cdots, \quad \zeta \in \Theta. \tag{1.10}$$

*If*

$$\vartheta(\zeta) + \frac{\zeta\vartheta^{'}(\zeta)}{d} \prec h(\zeta),$$

*then,*

$$\vartheta(\zeta) \prec q(\zeta) \prec h(\zeta),$$

*where*

$$q(\zeta) = \frac{d}{n\,\zeta^{d/n}} \int_0^\zeta h(t)\, t^{(d/n)-1} dt.$$

*The function $q$ is convex and is the best $(a,\, n)$-dominant.*

**Lemma 1.4.** *[30] Let $\psi \in \mathcal{Q}$ and $h$ belongs to $\mathcal{S}$. If $\frac{\zeta h'(\zeta)}{h(\zeta)}$ is starlike, then*

$$\frac{\zeta\psi'(\zeta)}{\psi(\zeta)} \prec \frac{\zeta h'(\zeta)}{h(\zeta)} \quad \Longrightarrow \quad \psi(\zeta) \prec h(\zeta),$$

*and $h$ is the best dominant.*

## 2. Main results

We begin with the following

**Theorem 2.1.** *Let $h$ be convex univalent in $\Theta$ with $h(0) = 0$ and $Re\, h(\zeta) > 0$. Also let $\psi(\zeta), \chi(\zeta) \in \mathcal{Q}$ with $\chi(\zeta) \neq 0$ for all $\zeta \in \Theta \setminus \{0\}$. Suppose that $\mathcal{N}^m_{\lambda,\mu}(\eta_1, \theta_1;\, \gamma, \delta)\chi(\zeta) \in \mathcal{S}(\phi)$ and*

$$\begin{aligned}&\left(\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}{\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]^{1-\alpha-i\beta}\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)\right]^{\alpha+i\beta}}\right)^2\left[3+2\left\{\frac{\zeta\left(D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]\right)'}{D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}\right.\right.\\&\left.\left.-(1-\alpha-i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right)'}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)}-(\alpha+i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)\right)'}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)}\right\}\right]\prec h(\zeta).\end{aligned} \tag{2.1}$$

*Then*

$$\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}{\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]^{1-\alpha-i\beta}\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)\right]^{\alpha+i\beta}} \prec q(\zeta) = \sqrt{\Delta(\zeta)}, \tag{2.2}$$

*where $\Delta(\zeta) = \frac{1}{\zeta}\int_0^\zeta h(t)\, dt$ and $q$ is convex and is the best dominant.*

*Proof.* Let

$$p(\zeta) = \frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)}\left(\frac{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)}\right)^{\alpha+i\beta}, \qquad (\zeta \in \Theta;\, \alpha \geq 0, \beta \in \mathbb{R}).$$

Uncorrected Proof

Clearly, $p(\zeta) \neq 0$ in $\Theta$. Let $P(\zeta) = p^2(\zeta)$, then $P(\zeta) \in \mathcal{H}(1, 1)$ with $P(\zeta) \neq 0$ in $\Theta$. On computation, we have

$$\frac{\zeta P^{'}(\zeta)}{P(\zeta)} = 2\left[1 + \frac{\zeta\left(D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]\right)^{'}}{D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]} - (1-\alpha-i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right)^{'}}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)}\right.$$
$$\left. - (\alpha+i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)\right)^{'}}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)}\right].$$

Thus by (2.1), we have

$$P(\zeta) + \zeta P^{'}(\zeta) \prec h(\zeta) \quad (\zeta \in \Theta). \tag{2.3}$$

Now by Lemma 1.3, we deduce that $P(\zeta) \prec \Delta(\zeta) \prec h(\zeta)$. Since $Re\, h(\zeta) > 0$ and $\Delta(\zeta) \prec h(\zeta)$ we also have $Re\,\Delta(\zeta) > 0$. Hence the univalence of $\Delta$ implies the univalence of $\sqrt{\Delta(\zeta)}$ and $p^2(\zeta) \prec \Delta(\zeta) \Longrightarrow p(\zeta) \prec \sqrt{\Delta(\zeta)}$. Hence the proof of the Theorem. $\square$

**Remark 2.2.** Notice that the class $\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\,\theta_1;\,\gamma,\,\delta;\,\phi)$ involves only quantum derivative but we have used the Lemma which uses the classical derivative.

**Corollary 2.3.** *If $\psi \in \mathcal{Q}$ and $\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta) \in \mathcal{S}^*$ satisfies $Re\,[\Omega(\zeta)] > \nu \quad (0 \le \nu < 1)$, where*

$$\Omega(\zeta) = \left(\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}{[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)]^{1-\alpha-i\beta}\,[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)]^{\alpha+i\beta}}\right)^2$$
$$\left[3 + 2\left\{\frac{\zeta\left(D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]\right)^{'}}{D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]} - (1-\alpha-i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right)^{'}}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)}\right.\right.$$
$$\left.\left. - (\alpha+i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)\right)^{'}}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)}\right\}\right],$$

*then*

$$Re\left[\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)\right]}{[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\psi(\zeta)]^{1-\alpha-i\beta}\,[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\,\gamma,\delta)\chi(\zeta)]^{\alpha+i\beta}}\right] > Z(\nu),$$

*where $Z(\nu) = [2(1-\nu)\cdot\log 2 + (2\nu-1)]^{\frac{1}{2}}$. This result is sharp*

*Proof.* If we let $h(\zeta) = \frac{1+(2\nu-1)\zeta}{1+\zeta}$ $0 \le \nu < 1$ in Theorem 2.1 and retracing the steps, we can get the assertion of the Theorem. $\square$

If we let $\alpha = 0$, $m = \beta = \mu = \lambda = \gamma = 0$, $r = 2$, $s = 1$, $\eta_1 = \theta_1$, $\eta_2 = q$ and $q \to 1-$ in the Corollary 2.3, then we have the following

**Corollary 2.4.** *If $\psi \in \mathcal{Q}$ satisfies*

$$Re\left\{\left(\frac{\zeta\psi^{'}(\zeta)}{\psi(\zeta)}\right)^2\left[3 + \frac{2\,\zeta\psi^{''}(\zeta)}{\psi^{'}(\zeta)} - \frac{2\,\zeta\psi^{'}(\zeta)}{\psi(\zeta)}\right]\right\} > \nu,$$

*then $Re\,\frac{\zeta\psi^{'}(\zeta)}{\psi(\zeta)} > Z(\nu)$, where $Z(\nu) = [2(1-\nu)\cdot\log 2 + (2\nu-1)]^{\frac{1}{2}}$.*

If we let $\alpha = 1$, $m = \beta = \mu = \lambda = \gamma = 0$, $r = 2$, $s = 1$, $\eta_1 = \theta_1$, $\eta_2 = q$ and $q \to 1-$ in the Corollary 2.3, we get

Uncorrected Proof

**Corollary 2.5.** *If $\psi \in \mathcal{Q}$ and $\chi \in \mathcal{S}^*$ satisfies the condition*

$$Re\left\{\left(\frac{\zeta\psi'(\zeta)}{\chi(\zeta)}\right)^2\left[3+\frac{2\zeta\psi''(\zeta)}{\psi'(\zeta)}-\frac{2\zeta\chi'(\zeta)}{\chi(\zeta)}\right]\right\}>\nu,$$

*then* $Re\,\frac{\zeta\psi'(\zeta)}{\chi(\zeta)} > Z(\nu)$*, where* $Z(\nu) = [2(1-\nu)\cdot\log 2 + (2\nu-1)]^{\frac{1}{2}}$.

**Theorem 2.6.** *If $\psi(\zeta) \in \mathcal{Q}$ and $\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta) \in \mathcal{S}^*$ satisfies the condition*

$$\left[1+\frac{\zeta\left(D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right]\right)'}{D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right]}-(1-\alpha-i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right)'}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}\right.$$
$$\left.-(\alpha+i\beta)\frac{\zeta\left(\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta)\right)'}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta)}\right]\prec\frac{2\zeta}{1-\zeta^2}. \tag{2.4}$$

*then we have*

$$\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right]}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}\left(\frac{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta)}\right)^{\alpha+i\beta}\prec\frac{1+\zeta}{1-\zeta}.$$

*Proof.* Let

$$M(\zeta):=\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right]}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}\left(\frac{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta)}\right)^{\alpha+i\beta},\quad \Delta(\zeta)=\frac{1+\zeta}{1-\zeta}.$$

The function $M(\zeta)$ is analytic in $\Theta$ with $M(0)=\Delta(0)\neq 0$. Therefore

$$\frac{\zeta\Delta'(\zeta)}{\Delta(\zeta)}=\frac{\zeta}{1+\zeta}+\frac{\zeta}{1-\zeta}=\frac{2\zeta}{1-\zeta^2}.$$

Then the function $\frac{2\zeta}{1-\zeta^2}$ starlike in $\Theta$ (see [43, Lemma 10.4.2.]). The condition (2.4) is equivalent to

$$\frac{\zeta M'(\zeta)}{M(\zeta)}\prec\frac{\zeta\Delta'(\zeta)}{\Delta(\zeta)}=\frac{2\zeta}{1-\zeta^2}.$$

Hence by applying the Lemma 1.4, we can establish the theorem. □

Letting $\alpha=1$, $m=\beta=\mu=\lambda=\gamma=0$, $r=2$, $s=1$, $\eta_1=\theta_1$, $\eta_2=q$ and $q\to 1-$ in Theorem 2.6, we get the following.

**Corollary 2.7.** *If $\psi(\zeta) \in \mathcal{Q}$ satisfies the condition*

$$1+\frac{\zeta\psi''(\zeta)}{\psi'(\zeta)}-\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\prec\frac{2\zeta}{1-\zeta^2}.$$

*then we have* $\frac{\zeta\psi'(\zeta)}{\chi(\zeta)}\prec\frac{1+\zeta}{1-\zeta}, \quad \Longrightarrow \quad \psi\in\mathcal{K}.$

Letting $m=\alpha=\lambda=1$, $\beta=\mu=\gamma=0$, $r=2$, $s=1$, $\eta_1=\theta_1$, $\eta_2=q$ and $q\to 1-$ in Theorem 2.6, we get the following.

**Corollary 2.8.** *If $\psi(\zeta) \in \mathcal{Q}$ and $\chi\in\mathcal{S}^*$ satisfies the condition*

$$\frac{\zeta\left[\zeta\psi''(\zeta)+\psi'(\zeta)\right]'}{\zeta\psi''(\zeta)+\psi'(\zeta)}-\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\prec\frac{2\zeta}{1-\zeta^2}.$$

Uncorrected Proof

*then we have*

$$\frac{(\zeta\psi(\zeta))'}{\chi'(\zeta)} \prec \frac{1+\zeta}{1-\zeta}.$$

*That is, $\psi$ is quasi-starlike functions.*

Letting $m=\beta=\mu=\lambda=\gamma=0$, $r=2$, $s=1$, $\eta_1=\theta_1$, $\eta_2=q$ and $q\to 1-$ in Theorem 2.6, we get the following.

**Corollary 2.9.** *[43, Theorem 10.4.1.] Let $\psi\in\mathcal{Q}$ and $\alpha>0$. If*

$$1+\frac{\zeta\psi''(\zeta)}{\psi'(\zeta)}+(\alpha-1)\frac{\zeta\psi'(\zeta)}{\psi(\zeta)}-\alpha\frac{\zeta\chi'(\zeta)}{\chi(\zeta)} \prec \frac{2\zeta}{1-\zeta^2}, \quad (\zeta\in\Theta),$$

*then $\psi\in\mathcal{B}(\alpha)$. In particular, if $\psi\in\mathcal{Q}$ and $\alpha>0$, then $\psi\in\mathcal{B}(\alpha)$ provided* $\mathrm{Re}\left(1+\frac{\zeta\psi''(\zeta)}{\psi'(\zeta)}+(\alpha-1)\frac{\zeta\psi'(\zeta)}{\psi(\zeta)}\right)>0$, *for $\zeta\in\Theta$.*

## 3. Coefficient Inequalities

Initial coefficients bounds and Fekete-Szegő inequality for $\psi\in\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta;\alpha,\beta;\phi)$ will be found in this section.

Throughout this section, we denote

$$\Upsilon^m_n(\lambda,\mu)=\left[1+(\lambda-\mu)([n]_q-1)+[n]_q[n-1]_q\mu\lambda)\right]^m. \tag{3.1}$$

**Theorem 3.1.** *Let $\psi(\zeta)\in\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta;\alpha,\beta;\phi)$, then we have*

$$|a_2|\le\frac{1}{|\Upsilon^m_2(\lambda,\mu)Q_2|\sqrt{(\alpha+q)^2+\beta^2}}\left[|L_1|+2\sqrt{\alpha^2+\beta^2}\right], \tag{3.2}$$

$$\begin{aligned}|a_3|\le&\frac{1}{\sqrt{(\alpha+q+q^2)^2+\beta^2}\;|\Upsilon^m_3(\lambda,\mu)Q_3|}\left[|L_1|\max\left\{1,\left|\frac{L_2}{L_1}-\frac{L_1(\alpha+i\beta-1)\mathcal{K}_2}{2\mathcal{K}_1^2}\right|\right\}\right.\\&+2|L_1|\sqrt{\alpha^2+\beta^2}\left|1-\frac{(\alpha+i\beta-1)\mathcal{K}_2}{\mathcal{K}_1^2}\right|\\&\left.+\sqrt{\alpha^2+\beta^2}\max\left\{1,\left|3-2(1-\alpha-i\beta)\left(1+\frac{(\alpha+i\beta)\mathcal{K}_2}{\mathcal{K}_1^2}\right)\right|\right\}\right],\end{aligned} \tag{3.3}$$

*and for all $\rho\in\mathbb{C}$*

$$\begin{aligned}|a_3-\rho a_2^2|\le&\frac{1}{\sqrt{(\alpha+q+q^2)^2+\beta^2}\;|\Upsilon^m_3(\lambda,\mu)Q_3|}\left[|L_1|\max\left\{1,\left|\frac{L_2}{L_1}-\frac{L_1(\alpha+i\beta-1)\mathcal{K}_2}{2\mathcal{K}_1^2}\right.\right.\right.\\&\left.\left.+\frac{\rho L_1\mathcal{K}_3\Upsilon^m_3(\lambda,\mu)Q_3}{\mathcal{K}_1^2\left(\Upsilon^m_2(\lambda,\mu)\right)^2Q_2^2}\right|\right\}+2|L_1|\sqrt{\alpha^2+\beta^2}\left|1-\frac{(\alpha+i\beta-1)\mathcal{K}_2}{\mathcal{K}_1^2}-\frac{2\rho\mathcal{K}_3\Upsilon^m_3(\lambda,\mu)Q_3}{\mathcal{K}_1^2\left(\Upsilon^m_2(\lambda,\mu)\right)^2Q_2^2}\right|\\&\left.+\sqrt{\alpha^2+\beta^2}\max\left\{1,\left|3-2(1-\alpha-i\beta)\left(1+\frac{(\alpha+i\beta)\mathcal{K}_2}{\mathcal{K}_1^2}+\frac{2\rho(\alpha+i\beta)\mathcal{K}_3\Upsilon^m_3(\lambda,\mu)Q_3}{(1-\alpha-i\beta)\mathcal{K}_1^2\left(\Upsilon^m_2(\lambda,\mu)\right)^2Q_2^2}\right)\right|\right\}\right],\end{aligned} \tag{3.4}$$

*where*

$$\mathcal{K}_1=\left([2]_q+\alpha+i\beta-1\right),\quad \mathcal{K}_2=\left(2[2]_q+\alpha+i\beta-2\right), \text{ and }\quad \mathcal{K}_3=\left([3]_q+\alpha+i\beta-1\right). \tag{3.5}$$

*Proof.* As $\psi(\zeta)\in\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta;\alpha,\beta;\phi)$, by (1.9) we have

$$\frac{\zeta D_q\left[\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)\right]}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}\left(\frac{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\psi(\zeta)}{\mathcal{N}^m_{\lambda,\mu}(\eta_1,\theta_1;\gamma,\delta)\chi(\zeta)}\right)^{\alpha+i\beta}=\Psi\left[w(\zeta)\right]. \tag{3.6}$$

Uncorrected Proof

The Schwartz function $w(\zeta)$ can be equivalently written in the form by $\vartheta(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)}$, $\quad \zeta \in \Theta$, for some arbitrary function $\vartheta(\zeta) = 1 + \sum_{k=1}^{\infty} \vartheta_n \zeta^n \in \Xi$. On computation,

$$\Psi[w(\zeta)] = 1 + \frac{\vartheta_1 L_1}{2}\zeta + \frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1}\right)\right]\zeta^2 + \cdots. \tag{3.7}$$

So, (3.6) can be equivalently rewritten in form

$$\begin{aligned}&\left[1 + \sum_{n=2}^{\infty}[n]_q \Upsilon_n^m(\lambda,\, \mu) Q_n a_n \zeta^{n-1}\right] \times \left[1 + \sum_{n=2}^{\infty}\Upsilon_n^m(\lambda,\, \mu) Q_n a_n \zeta^{n-1}\right]^{\alpha+i\beta-1}\\ &= \left[1 + \frac{\vartheta_1 L_1}{2}\zeta + \frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1}\right)\right]\zeta^2 + \cdots\right] \times \left[1 + \sum_{n=2}^{\infty} M_n \zeta^{n-1}\right]^{\alpha+i\beta},\end{aligned} \tag{3.8}$$

where $M_n = \Upsilon_n^m(\lambda,\, \mu) Q_n b_n$. On expanding and simplifying, we get

$$\begin{aligned}&1 + ([2]_q + \alpha + i\beta - 1)\,\Upsilon_2^m(\lambda,\, \mu) Q_2 a_2 \zeta + \{([3]_q + \alpha + i\beta - 1)\,\Upsilon_3^m(\lambda,\, \mu) Q_3 a_3\\ &\quad + \frac{(\alpha + i\beta - 1)}{2}\,(2[2]_q + \alpha + i\beta - 2)\left(\Upsilon_2^m(\lambda,\, \mu)\right)^2 Q_2^2 a_2^2\Big\}\zeta^2 + \cdots\\ &\quad = 1 + \left[\frac{\vartheta_1 L_1}{2} + (\alpha + i\beta) M_2\right]\zeta + \left\{\frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1}\right)\right] + \frac{\vartheta_1 L_1(\alpha + i\beta) M_2}{2}\right.\\ &\quad \left. + (\alpha + i\beta)\left[M_3 - \frac{(1 - \alpha - i\beta)}{2} M_2^2\right]\right\}\zeta^2 + \cdots.\end{aligned} \tag{3.9}$$

Equating the coefficients of $\zeta$ and $\zeta^2$ in (3.9), we obtain

$$a_2 = \frac{1}{\mathcal{K}_1 \Upsilon_2^m(\lambda,\, \mu) Q_2}\left[\frac{\vartheta_1 L_1}{2} + (\alpha + i\beta) M_2\right] \tag{3.10}$$

and

$$\begin{aligned}a_3 = \frac{1}{\mathcal{K}_3 \Upsilon_3^m(\lambda,\, \mu) Q_3}&\left\{\frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} + \frac{L_1(\alpha + i\beta - 1)\mathcal{K}_2}{2\mathcal{K}_1^2}\right)\right] + \frac{\vartheta_1 L_1(\alpha + i\beta) M_2}{2}\left[1 - \frac{(\alpha + i\beta - 1)\mathcal{K}_2}{\mathcal{K}_1^2}\right]\right.\\ &\left. + (\alpha + i\beta)\left[M_3 - \frac{(1 - \alpha - i\beta) M_2^2}{2}\left(1 + \frac{(\alpha + i\beta)\mathcal{K}_2}{\mathcal{K}_1^2}\right)\right]\right\},\end{aligned} \tag{3.11}$$

where $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are given as in (3.5). Equation (3.2) can be obtained by applying the known inequalities $|\vartheta_1| \leq 2$ ([33, p. 41]) and $|M_2| \leq 2$ ([43, Theorem 4.1.2.]) in (3.10). Using the respective Fekete-Szegő inequalities of the class $\Xi$ and $\mathcal{S}^*$ namely for $x \in \mathbb{C}$ $\left|\vartheta_2 - x\vartheta_1^2\right| \leq 2\max\{1, \mid 2x - 1 \mid\}$ (see [10]) and $\left|M_2 - xM_1^2\right| \leq \max\{1, \mid 3 - 4x \mid\}$ (see [27]) in (3.11), we get (3.3).

Now to prove (3.4), we consider

$$\begin{aligned}\left|a_3 - \rho a_2^2\right| &= \left|\frac{1}{\mathcal{K}_3 \Upsilon_3^m(\lambda,\, \mu) Q_3}\left\{\frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} + \frac{L_1(\alpha + i\beta - 1)\mathcal{K}_2}{2\mathcal{K}_1^2}\right)\right]\right.\right.\\ &\quad \left. + \frac{\vartheta_1 L_1(\alpha + i\beta) M_2}{2}\left[1 - \frac{(\alpha + i\beta - 1)\mathcal{K}_2}{\mathcal{K}_1^2}\right] + (\alpha + i\beta)\left[M_3 - \frac{(1 - \alpha - i\beta) M_2^2}{2}\left(1 + \frac{(\alpha + i\beta)\mathcal{K}_2}{\mathcal{K}_1^2}\right)\right]\right\}\\ &\quad \left. - \frac{\rho}{\mathcal{K}_1^2\left(\Upsilon_2^m(\lambda,\, \mu)\right)^2 Q_2^2}\left[\frac{\vartheta_1 L_1}{2} + (\alpha + i\beta) M_2\right]^2\right|\\ &= \left|\frac{1}{\mathcal{K}_3 \Upsilon_3^m(\lambda,\, \mu) Q_3}\left\{\frac{\vartheta_1 L_1(\alpha + i\beta) M_2}{2}\left[1 - \frac{(\alpha + i\beta - 1)\mathcal{K}_2}{\mathcal{K}_1^2} - \frac{2\rho \mathcal{K}_3 \Upsilon_3^m(\lambda,\, \mu) Q_3}{\mathcal{K}_1^2\left(\Upsilon_2^m(\lambda,\, \mu)\right)^2 Q_2^2}\right]\right.\right.\end{aligned}$$

Uncorrected Proof

$$+\frac{L_1}{2}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1(\alpha+i\beta-1)\mathcal{K}_2}{2\mathcal{K}_1^2}+\frac{\rho L_1\mathcal{K}_3\Upsilon_3^m(\lambda,\,\mu)Q_3}{\mathcal{K}_1^2\left(\Upsilon_2^m(\lambda,\,\mu)\right)^2Q_2^2}\right)\right]$$
$$+(\alpha+i\beta)\left[M_3-\frac{(1-\alpha-i\beta)M_2^2}{2}\left(1+\frac{(\alpha+i\beta)\mathcal{K}_2}{\mathcal{K}_1^2}+\frac{2\rho(\alpha+i\beta)\mathcal{K}_3\Upsilon_3^m(\lambda,\,\mu)Q_3}{(1-\alpha-i\beta)\mathcal{K}_1^2\left(\Upsilon_2^m(\lambda,\,\mu)\right)^2Q_2^2}\right)\right]\Bigg\}.$$

Using the inequalities and methods employed in obtaining the bounds for $a_3$, we can establish (3.4).

□

Setting $q=2$, $r=1$, $m=\alpha=\beta=\gamma=0$, $\eta_1=\theta_1$, $\eta_2=q$ and $q\to 1^-$ in (3.1), we get the following well-known result.

**Corollary 3.2.** *If $\psi\in\mathcal{S}^*(\phi)$, then,*

$$|a_2|\le L_1,\qquad |a_3|\le\frac{L_1}{2}\max\left\{1;\left|\frac{L_2}{L_1}+L_1\right|\right\},$$

*and for a complex number $\alpha$,*

$$\left|a_3-\alpha a_2^2\right|\le\frac{L_1}{2}\max\left\{1;\left|\frac{L_2}{L_1}+L_1\left(1-2\alpha\right)\right|\right\}.$$

Setting $q=2$, $r=1$, $m=1=\lambda$, $\mu=\alpha=\beta=\gamma=0$, $\eta_1=\theta_1$, $\eta_2=q$ and $q\to 1^-$ in Theorem (3.1), we get the following well-known result.

**Corollary 3.3.** *If $\psi\in\mathcal{C}(\phi)$, then,*

$$|a_2|\le\frac{L_1}{2},\qquad |a_3|\le\frac{L_1}{6}\max\left\{1;\left|\frac{L_2}{L_1}+L_1\right|\right\},$$

*and for a complex number $\alpha$,*

$$\left|a_3-\alpha a_2^2\right|\le\frac{L_1}{6}\max\left\{1;\left|\frac{L_2}{L_1}+L_1\left(1-\frac{3\alpha}{2}\right)\right|\right\}.$$

Letting $\phi(\zeta)=\frac{1+\zeta}{1-\zeta}$ in Corollaries 3.2 and 3.3, we get

**Corollary 3.4.** *([13, pg. 116],[43, Theorem 4.1.5]) If $\psi\in\mathcal{S}^*$, then,*

$$|a_2|\le 2,\qquad |a_3|\le 3$$

*and for a complex number $\alpha$,*

$$\left|a_3-\alpha a_2^2\right|\le\max\left\{1;\,|3-4\alpha|\right\}.$$

The Fekete-Szego inequality of $\mathcal{S}^*$ also appears in the proof of such an inequality for the class of close-to-convex function [43, pg.89].

**Corollary 3.5.** *([13, pg. 117],[43, Theorem 4.2.7]) If $\psi\in\mathcal{C}$, then,*

$$|a_2|\le 1,\qquad |a_3|\le 1,$$

*and for a complex number $\alpha$,*

$$\left|a_3-\alpha a_2^2\right|\le\frac{1}{3}\max\left\{1;\,3\left|1-\alpha\right|\right\}.$$

**Remark 3.6.** For an appropriate choice of the general function and specializing the parameters involved, we can get the bounds of the initial coefficients of starlike, convex, close-to-convex, quasi-convex and alpha convex functions.

Uncorrected Proof

## 4. Conclusion

Unifying two most prominent special function, we have defined a differential operator with a primary motive of generalizing and unifying the existing study of a class of analytic function. The definition needed some deviations and assumptions. A class of Bazilevič functions of the type $\alpha + i\beta$ was defined using the operator. This study generalizes or unifies the studies of various subclasses of starlike, convex and close-to-convex functions. Subordination results and initial coefficient bounds are our main results.

As a concluding remark, we will now enumerate the further scope of this paper:

(1) Radius problems which determine the largest disk $|\zeta| < r$ for which functions from defined function class $\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\, \theta_1;\, \gamma,\, \delta; \alpha,\, \beta;\, \phi)$ satisfy geometric and algebraic properties like symmetry, convexity, starlikeness and univalence.

(2) On replacing the quantum derivative with a multiplicative derivative, the defined function class $\mathcal{QB}^m_{\lambda,\mu}(\eta_1,\, \theta_1;\, \gamma,\, \delta; \alpha,\, \beta;\, \phi)$ can be redefined and studied. Defining such a class requires lots of adaptations (see [19]).

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Uncorrected Proof

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