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Generalized Bernstein-like basis functions and their applications to fractional differential equations

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Abstract

This paper addresses the longstanding challenge of finding precise solutions for fractional differential equations by introducing and thoroughly examining Generalized Bernstein-like basis functions (GBFs). We propose two distinct subclasses of GBFs and elucidate their properties, which include recursive definitions, first and second derivatives, linear independence, and generalization of the partition of unity property. Delving deeper, we explore the definite integrals of the GBFs and introduce computational methodologies that prioritize both stability and efficiency. A significant achievement in our research is the development of computational schemes for Caputo fractional derivatives of Generalized Bernstein-like basis functions. Utilizing these approaches, we develop collocation techniques for a range of fractional differential equation types, including initial-value multi-order FDEs and two-point fractional boundary value problems. A central focus of this development is the comprehensive integration and understanding of both left-sided and right-sided fractional derivatives. The efficacy of our approach is further underscored by its successful application to intricate challenges like the Volterra-Fredholm fractional integro-differential equation. Our research reaches its zenith with an analytical evaluation of convergence rates using GBFs.

Keywords. Caputo fractional derivative, Collocation method, Convergence rates, Fractional equation, Generalized Bernstein-like basis functions.
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1. Introduction

Fractional calculus, although specifically niche within mathematical analysis, delves into the understanding and application of integrals and derivatives of often non-integer orders.

While the term "fractional" (referring to non-integer orders) might not be semantically precise, its use is pervasive and standard in the field. The rich history of this specialized branch of calculus can be traced back to early postulations by G.W. Leibniz in the late 17th century and L. Euler in the 18th century. Notable mathematicians, including P.S. Laplace, J.B.J. Fourier, and N.H. Abel, have furthered its foundations. In recent times, especially in the past two decades, fractional calculus has gained modern relevance. This resurgence can be measured by the rise in specialized conferences and academic publications on the subject. A turning point in its contemporary revival was the First Conference on Fractional Calculus and its Applications in 1974 at the University of New Haven, led by B. Ross [34]. The same year witnessed a pivotal publication on the topic by K.B. Oldham and J. Spanier [31]. A plethora of publications today encompass various facets of fractional calculus. Among them, the exhaustive work by Samko, Kilbas and Marichev is particularly noteworthy [23]. Additionally, notable publications by Davis, Erdélyi, Gel'fand, Shilov and others have delved deeply into its applications in the real world [8, 18, 28, 32]. Significantly, fractional calculus finds application in numerous sectors such as numerical analysis, physics, and engineering, particularly in describing fractal phenomena.

Fractional differential equations (FDEs) play pivotal roles in diverse scientific and engineering disciplines, they are

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crucial in signal processing, control theory, diffusion, thermodynamics, and many others [3, 11, 12]. Some numerical methods based on spline functions, wavelets, and multiwavelets have been proposed in the literature for solving optimal control problems, fractional differential equations, and integral equations. These approaches have shown computational efficiency and excellent accuracy by converting the original problems into systems of algebraic equations through suitable operational matrices. Guided by these successful developments, the present study employs a similar numerical approach for the problem under consideration [1, 21, 26, 27].

However, most FDEs lack exact solutions, thus analytical and numerical methodologies are instrumental in addressing them. Recent advancements have yielded several efficient methods for solving FDEs [5, 9]. While exact solutions can be complex, various numerical methods have been proposed, some of which are presented in [25, 49, 52]. In the domain of mathematical research, the use of fractional bases to handle diverse fractional equations has garnered increased attention. A review of specialized literature [6, 7, 22, 44–48] reveals a concerted effort to untangle the complexities of fractional differential equations. A central challenge that resonates throughout these studies is managing the singular kernel intrinsic to the Caputo derivative. Despite the plethora of proposed methodologies, many don't comprehensively address this issue. Against this backdrop, Bernstein polynomials emerge as a pivotal instrument, with their established proficiency in handling ordinary, partial differential, and integral equations [6, 10, 35]. However, applying them directly to fractional equations reveals obstacles. Notably, the gap between Bernstein polynomials' clear-cut formulation and the nuanced hurdles of fractional equations is evident. The fractional Bernstein series solution (FBSS) was first proposed by Alshbool et al. [6] as a numerical method for solving the fractional diffusion equation. Subsequent research [29] honed in on the Bernstein polynomial's potential to approximate solutions of fractional-integro differential equations (FIDE) employing the Caputo derivative.

Several numerical frameworks have been proposed for solving fractional models based on different functional bases and discretization strategies. A generalized form of Bernstein functions was recently introduced and employed within a collocation scheme to approximate linear and nonlinear fractional differential equations in the Caputo sense, where convergence and efficiency were rigorously demonstrated [36]. In a related direction, Chebyshev-based collocation and Galerkin methods were developed to solve fractional chemical kinetics models, showing high accuracy, stability, and reliable error behavior [37]. Furthermore, fractional backward differentiation formula methods were formulated for fractional differential matrix equations, providing a robust numerical treatment for matrix-valued fractional systems together with theoretical error analysis [38].

Further explorations such as [4, 17] emphasize the need for a breakthrough function imbued with a fractional essence skillfully designed to circumvent and correct these inherent challenges.

Building on this, we introduce an innovative approach by developing two types of fractional bases, drawing inspiration from the idea presented in [20]. These new bases are tailored specifically for the unique problems of fractional differential equations, representing a step forward in solving these difficult mathematical problems.

Drawing upon the foundational ideas from polyfractional and generalized Jacobi frameworks [13, 44], we adapt and extend the concept to Bernstein polynomials. This yields a new class of basis functions that inherit the stability and endpoint flexibility of Bernstein polynomials, providing a simpler and more effective tool for approximating functions with algebraic singularities.

Introduction of Generalized Bernstein-like basis functions (GBFs): We introduce two distinct subclasses of GBFs. These subclasses serve as foundational tools in our approach to fractional differential equations, allowing for a more nuanced and accurate understanding.

Properties of GBFs: The importance of any new mathematical tool lies in understanding its properties. We rigorously study and establish several key properties of the introduced GBFs:

- A recursive definition of the GBFs holds true.
- The first and second derivatives of the GBFs have been delineated using recursive formulas.
- Partition of unity property is extended for these polynomials.
- GBFs are shown to be linearly independent.
- Weierstrass' approximation theorem of GBFs is developed.

Computational methodology: Recognizing the potential growth of error when deploying the explicit form of Bernstein polynomials, we adopt a three-term recurrence formula associated with them for calculations. This choice

allows for stable and efficient computation of GBFs at collocation points. Notably, this contrasts with most prior works in the Bernstein domain, where explicit formulas were predominantly utilized.
Schemes for Caputo fractional derivative of GBFs: We have innovatively calculated fractional left and right derivatives, emphasizing stability and precision. By passing the pitfalls of conventional explicit forms, our approach integrates numerical rules and seamlessly extends to the Riemann-Liouville derivative, setting a new standard in mathematical methodologies.
Application to FDE Models: We construct collocation schemes tailored for numerically solving specific models of FDEs. Although our primary focus revolves around the left Caputo fractional derivative, it's imperative to highlight that the methodologies can be extended to cater to the right fractional derivative and other derivatives such as Riemann-Liouville fractional derivatives. Central to achieving high accuracy and efficiency is the judicious selection of collocation points. For the discretization, the Chebyshev-Gauss-Lobatto points defined on the interval $[0, 1]$ are selected.
Techniques for Systems with Left and Right Derivatives: We architect a collocation method adept at solving fractional systems that involve both left and right fractional derivatives. This is particularly pivotal for initial-valued multi-order FDEs and fractional Riccati differential equations.
Solution Technique for Complex Equations: A novel technique rooted in GBFs has been devised to numerically solve the Volterra-Fredholm fractional integro-differential equation. This equation, which features a combination of Caputo derivatives, showcases the versatility and applicability of our introduced methodologies.
Analysis of Convergence Order: Given the importance of convergence order in numerical methods, we have presented the theoretical convergence order for the errors, using GBFs, of the best approximation and the FDE. This convergence order is accurate regardless of the smoothness of the function and aligns well with numerical results.
This paper is structured as follows: Section 2 presents the necessary theoretical foundations and examines key features of Bernstein polynomials. Subsequently, Section 3 defines a generalized class of Bernstein-type basis functions, detailing their essential properties and their role within approximation theory. Section 4 presents schemes for obtaining the Caputo fractional derivative of GBFs. Application of the collocation method using GBFs to solve fractional problems is discussed in Section 5. Section 6 provides theoretical convergence rates for errors our proposed method. Section 7 includes numerical experiments validating our methodologies and theoretical assertions. The final section reports conclusions drawn from the study and discusses potential implications for future research.

2. Preliminaries

In this section, we present several preliminary concepts and essential results that will be employed in the forthcoming analysis. We begin with a brief review of fractional calculus, followed by a summary of key properties and relations associated with the Bernstein polynomials.

2.1. Fractional integrals and derivatives.

In this subsection, we restrict our attention to the interval $\Lambda = (a, b)$, more detailed information may be found in [23, 31, 32, 34].

Definition 2.1. For $\mu > 0$, the left and right Riemann-Liouville fractional integrals of order μ are defined by

$$
\begin{aligned}
{}_aI_t^{\mu}u(t) &= \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1}u(s)ds, \\
{}_tI_b^{\mu}u(t) &= \frac{1}{\Gamma(\mu)} \int_t^b (s-t)^{\mu-1}u(s)ds.
\end{aligned}
$$

Definition 2.2. For $\mu \in (k-1, k]$, $k \in \mathbb{N}$, the left and right Caputo fractional derivative of order μ, are defined by

$$
{}_a^cD_t^{\mu}u(t) = {}_aI_t^{k-\mu}\left(D^k u(t)\right) = \frac{1}{\Gamma(k-\mu)} \int_a^t (t-s)^{k-\mu-1}u^{(k)}(s)ds, \tag{2.1}
$$

$$
{}_t^cD_b^{\mu}u(t) = (-1)^k \, {}_tI_b^{k-\mu}\left(D^k u(t)\right) = \frac{(-1)^k}{\Gamma(k-\mu)} \int_t^b (s-t)^{k-\mu-1}u^{(k)}(s)ds, \tag{2.2}
$$

and the left and right Riemann-Liouville fractional derivative of order μ are defined by

$$ {}_aD_t^\mu u(t) = D^k \left({}_aI_t^{k-\mu} u(t) \right) = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dx^k} \int_a^t (t-s)^{k-\mu-1} u(s)ds, $$
$$ {}_tD_b^\mu u(t) = (-1)^k D^k \left({}_tI_b^{k-\mu} u(t) \right) = \frac{(-1)^k}{\Gamma(k-\mu)} \frac{d^k}{dx^k} \int_t^b (s-t)^{k-\mu-1} u(s)ds. $$

Lemma 2.3. *The Riemann-Liouville fractional integral and the Caputo fractional derivative are correlated by*

$$ {}_aI_t^\mu ({}_a^cD_t^\mu u(t)) = u(t) - \sum_{i=0}^{k-1} \frac{u^{(i)}(a)}{i!} (t-a)^i, $$
$$ {}_tI_b^\mu ({}_t^cD_b^\mu u(t)) = u(t) - \sum_{i=0}^{k-1} \frac{u^{(i)}(b)}{i!} (b-t)^i. $$

Lemma 2.4. *The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected by formula*

$$ {}_aD_t^\mu u(t) = {}_a^cD_t^\mu u(t) + \sum_{i=0}^{k-1} \frac{u^{(i)}(a)}{\Gamma(1+i-\mu)} (t-a)^{i-\mu}, $$
$$ {}_tD_b^\mu u(t) = {}_t^cD_b^\mu u(t) + \sum_{i=0}^{k-1} \frac{(-1)^i u^{(i)}(b)}{\Gamma(1+i-\mu)} (b-t)^{i-\mu}, $$

moreover, there holds

$$ {}_aD_t^\mu ({}_aI_t^\mu u(t)) = u(t), \qquad {}_tD_b^\mu ({}_tI_b^\mu u(t)) = u(t). $$

One can derive exact analytical formulas for the fractional integral and derivative corresponding to power functions.

$$ {}_aI_t^\mu (t-a)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\mu+1)} (t-a)^{\alpha+\mu}, \tag{2.3} $$

and for $\mu \in (k-1, k]$

$$ {}_a^cD_t^\mu (t-a)^\alpha = \begin{cases} 0, & \alpha = 0, 1, \dots, n-1, \\ \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} (t-a)^{\alpha-\mu}, & \alpha > n-1. \end{cases} \tag{2.4} $$

Also, for $\alpha > -1$, we have

$$ {}_aD_t^\mu (t-a)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} (t-a)^{\alpha-\mu}. \tag{2.5} $$

Apparently, similar formulas can be derived for right fractional integral and derivative of $(b-t)^\alpha$.

2.2. **Bernstein polynomials.** In the following, we present definition and some properties of the Bernstein polynomials [7, 10].

Definition 2.5. The set of Bernstein polynomials of order m, defined on the interval $[0, 1]$, is given by

$$ B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad i = 0, 1, \dots, m. \tag{2.6} $$

Lemma 2.6. *The Bernstein polynomials have the following properties:*

(1) *They are non-negative.*
(2) *The Bernstein basis polynomials $B_{i,m}$ and $B_{m-i,m}$ are symmetric, i.e., $B_{i,m}(t) = B_{m-i,m}(1-t)$.*

(3) *A recursive definition of the Bernstein basis polynomials is derived as follows*

$$B_{i,m}(t) = (1-t)B_{i,m-1}(t) + tB_{i-1,m-1}(t),$$
$$B_{0,k}(t) = (1-t)^k, \quad B_{k,k}(t) = t^k, \quad k = 0, 1, \ldots, m.$$

(4) *The first and second derivatives of the Bernstein polynomials can be expressed by*

$$B'_{i,m}(t) = m\left(B_{i-1,m-1}(t) - B_{i,m-1}(t)\right),$$
$$B''_{i,m}(t) = m(m-1)\left(B_{i-2,m-2}(t) - 2B_{i-1,m-2}(t) + B_{i,m-2}(t)\right).$$

(5) *They possess the partition of unity property, i.e.,*

$$\sum_{k=0}^{m} B_{k,m}(t) = 1.$$

(6) *The set of Bernstein basis polynomials of degree m are linearly independent, i.e.,*

$$\sum_{k=0}^{m} c_k B_{k,m}(t) = 0 \Longrightarrow c_k = 0, \quad k = 0, 1, \ldots, m.$$

(7) *The following integral relations hold:*

$$\int_0^1 B_{i,m}(t)dt = \frac{1}{m+1},$$

and

$$\int_0^1 B_{i_1,m_1}(t)B_{i_2,m_2}(t)dt = \frac{\binom{m_1}{i_1}\binom{m_2}{i_2}}{(m_1+m_2+1)\binom{m_1+m_2}{i_1+i_2}}.$$

Definition 2.7. For a continuous function u on the interval $[0,1]$, the Bernstein operator of degree m associated with u is defined by

$$\mathfrak{B}_m(u;t) = \sum_{k=0}^{m} u\left(\frac{k}{m}\right) B_{k,m}(t).$$

Lemma 2.8. *[14] For any function u that is continuous on $[0,1]$, the Bernstein operator $\mathfrak{B}_m$ converges uniformly to u as m tends to infinity.*

Theorem 2.9. *Let u be a function that is $m+1$ times continuously differentiable on $[0,1]$. Then, for any $t \in [0,1]$, the approximation error of the Bernstein operator $\mathfrak{B}_m$ satisfies:*

$$|u - \mathfrak{B}_m u| \leq \frac{M}{2^{2m} m!},$$

where $M = \|u^{(m+1)}\|_\infty$.

Lemma 2.10. *The Bernstein operators have the following properties:*

$$\mathfrak{B}_m(1;t) = 1, \quad \mathfrak{B}_m(t;t) = t, \quad \mathfrak{B}_m(t(1-t);t) = \left(1 - \frac{1}{m}\right)t(1-t).$$

Lemma 2.11. *Given a function $u \in C[0,1]$, it follows that*

$$\|u - \mathfrak{B}_m u\|_\infty \underset{m\to\infty}{\longrightarrow} 0.$$

Lemma 2.12. *If u is bounded on $[0,1]$, then*

$$\|u - \mathfrak{B}_m u\|_\infty < \frac{3}{2}\omega\left(\frac{1}{\sqrt{m}}\right),$$

where ω is the modulus of continuity and defined for $t_1, t_2 \in [0,1]$ as

$$\omega(\delta) = \sup_{|t_1 - t_2| \le \delta} |u(t_1) - u(t_2)|.$$

Theorem 2.13. *[14] Let $u \in C^2(0,1)$ be a bounded function, then for any arbitrary point $t_0 \in [0,1]$, we have*

$$\lim_{m\to\infty} m\left[\mathfrak{B}_m(u;t_0) - u(t_0)\right] = \frac{1}{2} t_0(1-t_0)u''(t_0).$$

3. Generalized Bernstein-like basis functions

In this section, we introduce the generalized Bernstein-like basis functions (GBFs) as two subclasses of basis functions.

Definition 3.1. For $\alpha > 0$, we define the GBFs as

$$^1\mathrm{B}^{\alpha}_{i,m}(t) = \binom{m}{i} t^{\alpha+i}(1-t)^{m-i}, \quad i = 0,1,\ldots,m, \tag{3.1}$$

and

$$^2\mathrm{B}^{\alpha}_{i,m}(t) = \binom{m}{i} t^{i}(1-t)^{m-i+\alpha}, \quad i = 0,1,\ldots,m. \tag{3.2}$$

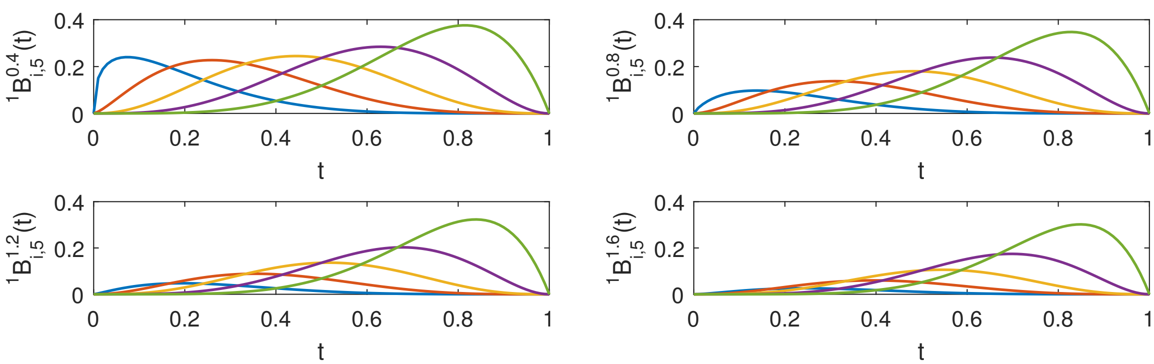

Figure 1. Plot of the functions $^1\mathrm{B}^{\alpha}_{i,m}(t)$ for various values of α.

The functions of $^1\mathrm{B}^{\alpha}_{i,5}(t)$ for $i = 0,1,2,3,4$ are depicted for different values of α in Figure 1. It is worth mentioning that selecting the optimal value for the parameter in function approximation requires further discussion. It appears that the best choice for function approximation is the same as the singular index of the functions, while for solving fractional differential equations, the optimal corresponds to the highest fractional order in the equation.
The optimal parameter for function approximation typically matches the function's singularity index, whereas for solving FDEs, it is often chosen based on the largest fractional order in the equation, provided the right-hand side is sufficiently smooth [50]. This principle is reflected in existing methods. For example, works such as [42] construct approximation bases—like polyfractonomials or generalized Jacobi functions—using either the singularity index of the solution or the fractional derivative order. In a related approach, [43] proposes piecewise fractional interpolation (PFI) for solving FDEs, which requires prior knowledge of the solution's singular behavior at the initial time. However, for Bernstein-based fractional bases, the additional requirement that the fractional derivative of the basis remains polynomial forces α to exactly equal the fractional derivative order, unlike generalized Jacobi bases where other choices are possible.

We now study some properties of GBFs. From Lemma 2.6 and Definition 3.1 the following formulas can be derived straightforwardly

Lemma 3.2. *The GBFs have the following properties:*

- *The GBFs are non-negative.*
- *The GBFs* ${}^{j}\mathrm{B}^{\alpha}_{i,m}(t)$ *and* ${}^{j}\mathrm{B}^{\alpha}_{m-i,m}(t)$ *are symmetry, i.e.,* ${}^{j}\mathrm{B}^{\alpha}_{i,m}(1-t) = {}^{j}\mathrm{B}^{\alpha}_{m-i,m}(t)$.
- *A recursive definition of the GBF is derived as follows*

$${}^{j}\mathrm{B}^{\alpha}_{i,m}(t) = (1-t)\,{}^{j}\mathrm{B}^{\alpha}_{i,m-1}(t) + t\,{}^{j}\mathrm{B}^{\alpha}_{i-1,m-1}(t), \qquad j = 1, 2,$$

where

$${}^{1}\mathrm{B}^{\alpha}_{0,i}(t) = t^{\alpha}(1-t)^{i}, \quad {}^{1}\mathrm{B}^{\alpha}_{i,i}(t) = t^{\alpha+i}, \quad i = 0, 1, \ldots, m,$$

and

$${}^{2}\mathrm{B}^{\alpha}_{0,k}(t) = (1-t)^{k+\alpha}, \quad {}^{2}\mathrm{B}^{\alpha}_{k,k}(t) = t^{k}(1-t)^{\alpha}, \quad k = 0, 1, \ldots, m-1.$$

- *The first and second derivatives of the GBFs can be expressed by*

$$\frac{d}{dt}\left({}^{1}\mathrm{B}^{\alpha}_{i,m}(t)\right) = \frac{\alpha}{t}\,{}^{1}\mathrm{B}^{\alpha}_{i,m}(t) + m\left({}^{1}\mathrm{B}^{\alpha}_{i-1,m-1}(t) - {}^{1}\mathrm{B}^{\alpha}_{i,m-1}(t)\right), \tag{3.3}$$

$$\frac{d}{dt}\left({}^{2}\mathrm{B}^{\alpha}_{i,m}(t)\right) = -\frac{\alpha}{1-t}\,{}^{2}\mathrm{B}^{\alpha}_{i,m}(t) + m\left({}^{2}\mathrm{B}^{\alpha}_{i-1,m-1}(t) - {}^{2}\mathrm{B}^{\alpha}_{i,m-1}(t)\right), \tag{3.4}$$

and

$$\frac{d^2}{dt^2}\left({}^{1}\mathrm{B}^{\alpha}_{i,m}(t)\right) = \frac{\alpha^2-\alpha}{t^2}\,{}^{1}\mathrm{B}^{\alpha}_{i,m}(t) + \frac{2m\alpha}{t}\left[{}^{1}\mathrm{B}^{\alpha}_{i-1,m-1}(t) - {}^{1}\mathrm{B}^{\alpha}_{i,m-1}(t)\right] \tag{3.5}$$

$$+ m(m-1)\left[{}^{1}\mathrm{B}^{\alpha}_{i-2,m-2}(t) - 2\,{}^{1}\mathrm{B}^{\alpha}_{i-1,m-2}(t) + {}^{1}\mathrm{B}^{\alpha}_{i,m-2}(t)\right], \tag{3.6}$$

$$\frac{d^2}{dt^2}\left({}^{2}\mathrm{B}^{\alpha}_{i,m}(t)\right) = \frac{\alpha^2-\alpha}{(1-t)^2}\,{}^{2}\mathrm{B}^{\alpha}_{i,m}(t) - \frac{2m\alpha}{1-t}\left[{}^{2}\mathrm{B}^{\alpha}_{i-1,m-1}(t) - {}^{2}\mathrm{B}^{\alpha}_{i,m-1}(t)\right] \tag{3.7}$$

$$+ m(m-1)\left[{}^{2}\mathrm{B}^{\alpha}_{i-2,m-2}(t) - 2\,{}^{2}\mathrm{B}^{\alpha}_{i-1,m-2}(t) + {}^{2}\mathrm{B}^{\alpha}_{i,m-2}(t)\right], \tag{3.8}$$

- *The GBFs of degree m satisfy*

$$\sum_{k=0}^{m} {}^{1}\mathrm{B}^{\alpha}_{k,m}(t) = t^{\alpha}, \quad \sum_{k=0}^{m} {}^{2}\mathrm{B}^{\alpha}_{k,m}(t) = (1-t)^{\alpha}.$$

- *The $m+1$ GBFs of degree m are linearly independent, i.e.,*

$$\sum_{k=0}^{m} c_{k,j}\,{}^{j}\mathrm{B}^{\alpha}_{k,m}(t) = 0 \Longrightarrow c_{k,j} = 0, \quad k = 0, 1, \ldots, m, \quad j = 1, 2.$$

- *Definite integrals of the GBFs of degree m reads:*

$$\int_0^1 {}^{1}\mathrm{B}^{\alpha}_{i,m}(t)dt = \binom{m}{i}\frac{\Gamma(\alpha+i+1)\Gamma(m-i+1)}{\Gamma(m+\alpha+2)},$$

$$\int_0^1 {}^{2}\mathrm{B}^{\alpha}_{i,m}(t)dt = \binom{m}{i}\frac{\Gamma(i+1)\Gamma(m-i+\alpha+1)}{\Gamma(m+\alpha+2)},$$

and

$$\int_0^1 {}^{1}\mathrm{B}^{\alpha}_{i_1,m_1}(t)\,{}^{1}\mathrm{B}^{\alpha}_{i_2,m_2}(t)dt = \binom{m_1}{i_1}\binom{m_2}{i_2}\frac{\Gamma(2\alpha+i_1+i_2+1)\Gamma(m_1+m_2-i_1-i_2+1)}{\Gamma(2\alpha+m_1+m_2+2)},$$

$$\int_0^1 {}^{2}\mathrm{B}^{\alpha}_{i_1,m_1}(t)\,{}^{2}\mathrm{B}^{\alpha}_{i_2,m_2}(t)dt = \binom{m_1}{i_1}\binom{m_2}{i_2}\frac{\Gamma(i_1+i_2+1)\Gamma(m_1+m_2-i_1-i_2+2\alpha+1)}{\Gamma(2\alpha+m_1+m_2+2)}.$$

where Γ represents the well-known Gamma function.

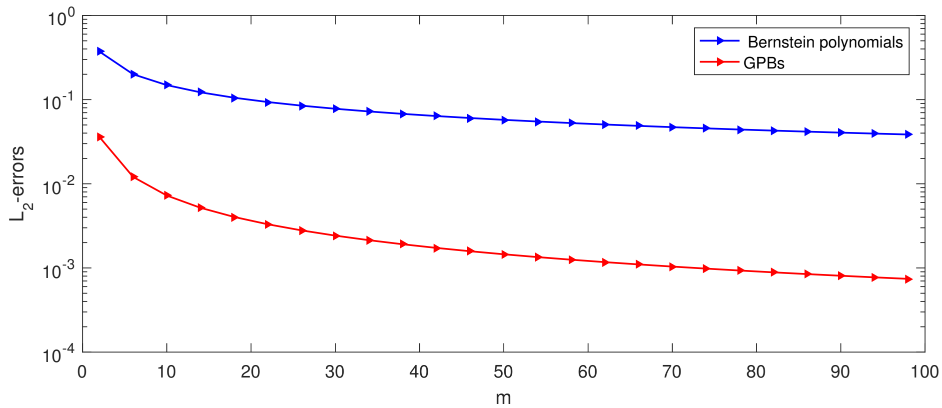

FIGURE 2. The L_2-norm erorrs of approximation with classical Bernstein basis and GBFs.

To illustrate the superior efficiency and accuracy of the introduced functions (i.e., GBFs) compared to classical Bernstein polynomials, we approximated the function $u(t) = t^{0.7}\cos(t)$ using (2.6) and (3.1). For this goal, we approximate $u(t)$ as

$$u(t) \approx \sum_{i=0}^{m} c_i \ {}^1B_{i,m}^{\alpha}(t).$$

Then

$$\sum_{i=0}^{m} c_i \ {}^1B_{i,m}^{\alpha}(t) \approx t^{0.7}\cos(t).$$

Now, by using collocation points $\frac{i}{m}$ for $i = 0, \dots, m$, we can obtain the unknown coefficients $\{c_i\}_{i=0}^{m}$.

For comparison of the obtained results, we use the L^2 norm, defined by

$$\|u\|_{L^2} = \left(\int_0^1 |u(t)|^2 dt\right)^{1/2}.$$

We measure the L^2-errors and show the results in Figure 2, where the red line represents the approximation using GBFs, and the blue line corresponds to the regular Bernstein polynomials, represented by Equation (2.6).

The results in the figure show that the functions defined in this work are more suitable for approximating non-smooth functions.

For a clearer understanding of how the GBFs approximate functions, we define the following spaces on the interval $[0, 1]$:

$$\mathbf{F}_1^{\alpha} = \{u \mid u(t) = t^{\alpha}v(t), v \in C[0,1]\},$$
$$\mathbf{F}_2^{\alpha} = \{u \mid u(t) = (1-t)^{\alpha}v(t), v \in C[0,1]\}.$$

To approximate functions that exhibit singularities at the boundaries of the interval $[0, 1]$, we define two generalized Bernstein operators based on the type of singularity. The operator ${}^1\mathcal{B}_m^{\alpha}$ is designed for functions with a singularity of type t^{α} at the left endpoint $t = 0$, while the operator ${}^2\mathcal{B}_m^{\alpha}$ is for functions with a singularity of type $(1-t)^{\alpha}$ at the right endpoint $t = 1$. The core idea behind these operators is to retain the desirable properties of the classical Bernstein operator for smooth functions and extend them to the class of non-smooth functions. Intuitively, each operator works by first factoring out the expected singular term (t^{α} or $(1-t)^{\alpha}$) from the original function $u(t)$. The classical Bernstein operator $\mathcal{B}_m$ is then applied to the remaining smooth part, defined as $v(t) = u(t)/t^{\alpha}$ for a singularity at $t = 0$, or $v(t) = u(t)/(1-t)^{\alpha}$ for a singularity at $t = 1$. Finally, the result is multiplied back by the extracted singular factor.

This process ensures that the approximation structure of the Bernstein operator is effectively preserved for functions with singular boundary behavior.

Definition 3.3. For a given function $u \in \mathbf{F}_1^\alpha \cup \mathbf{F}_2^\alpha$ on the interval $[0,1]$, the GBFs of degree m associated with u is defined by

$$ {}^1\mathfrak{B}_m^\alpha(u;t) = {}^1\mathrm{B}_{0,m}^\alpha(t) \lim_{k\longrightarrow 0} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) + \sum_{k=1}^{m} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) {}^1\mathrm{B}_{k,m}^\alpha(t), \quad u \in \mathbf{F}_1^\alpha, $$

$$ {}^2\mathfrak{B}_m^\alpha(u;t) = {}^2\mathrm{B}_{m,m}^\alpha(t) \lim_{k\longrightarrow m} (1-\frac{k}{m})^{-\alpha} u\left(\frac{k}{m}\right) + \sum_{k=0}^{m-1} (1-\frac{k}{m})^{-\alpha} u\left(\frac{k}{m}\right) {}^2\mathrm{B}_{k,m}^\alpha(t), \quad u \in \mathbf{F}_2^\alpha. $$

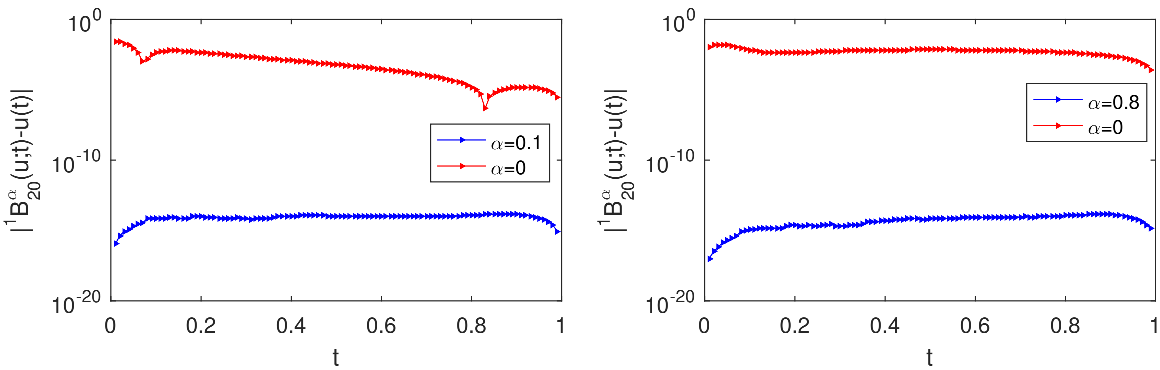

FIGURE 3. The errors $|{}^1\mathfrak{B}_{20}^\alpha(u;t) - u(t)|$ with classical Bernstein and Generalized Bernstein operators.

In Figure 3, we use the classical Bernstein and ${}^1\mathfrak{B}_{20}^\alpha(u;t)$ operators to approximate $u(t) = t^\alpha + t^{1+\alpha}$ and the errors $|{}^1\mathfrak{B}_{20}^\alpha(u;t) - u(t)|$ are depicted for $\alpha = 0.1, 0.8$.

The operators denoted by ${}^1\mathfrak{B}_m^\alpha$ and ${}^2\mathfrak{B}_m^\alpha$ are referred to as the generalized Bernstein-like operators (GBOs) of the first and second type, respectively.

To show that ${}^1\mathfrak{B}_m^\alpha(u;t)$ is well-defined, we verify the following:

(1) **Convergence of the limit:** The term $\lim_{k\to 0} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right)$ exists.
   - For $u \in \mathbf{F}_1^\alpha$, then there exists a function v such that $u(t) = t^\alpha v(t)$, $v \in C[0,1]$. This ensures that $\frac{m^\alpha u(\frac{k}{m})}{k^\alpha} = v(k/m)$ is finite as $k \to 0$.

(2) **Uniform boundedness of the weights:** The Bernstein weights ${}^1\mathrm{B}_{k,m}^\alpha(t)$ satisfy:

$$ \sum_{k=0}^{m} {}^1\mathrm{B}_{k,m}^\alpha(t) = t^\alpha, \quad \forall t \in [0,1], $$

which ensures the proper probabilistic structure of the operator.

(3) **Summability of the series:** The series

$$ \sum_{k=1}^{m} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) {}^1\mathrm{B}_{k,m}^\alpha(t), $$

converges for all $t \in [0,1]$. This follows from the fact that $u \in \mathbf{F}_1^\alpha$ implies that $u(x)$ is bounded and

$$ \sum_{k=1}^{m} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) {}^1\mathrm{B}_{k,m}^\alpha(t) = t^\alpha \sum_{k=1}^{m} v(\frac{k}{m}) \mathrm{B}_{k,m}(t) \qquad v(t) = t^{-\alpha} u(t) $$

$$\leq M \sum_{k=1}^{m} \mathrm{B}_{k,m}(t) = M(1 - \mathrm{B}_{0,m}(t)), \quad M = \sup_{0 \leq k \leq m} \left\{ u(\frac{k}{m}) \right\}.$$

(4) **Continuity of the operator:** Since $u(x)$ is continuous on $[0,1]$, and both the limit and series are uniformly bounded, the operator ${}^1\mathfrak{B}_m^\alpha(u;t)$ inherits continuity with respect to t.

Thus, ${}^1\mathfrak{B}_m^\alpha(u;t)$ is well-defined for all $t \in [0,1]$ and $u \in \mathbf{F}_1^\alpha$.
We now prove that if ${}^1\mathfrak{B}_m^\alpha(u_1;t) = {}^1\mathfrak{B}_m^\alpha(u_2;t)$ for all $t \in [0,1]$, then $u_1 = u_2$.
Assume ${}^1\mathfrak{B}_m^\alpha(u_1;t) ={}^1 \mathfrak{B}_m^\alpha(u_2;t)$ for all $t \in [0,1]$. From the definition of ${}^1\mathfrak{B}_m^\alpha(u;t)$, we can write:

$${}^1\mathrm{B}_{0,m}^\alpha(t) \lim_{k \to 0} \frac{m^\alpha}{k^\alpha} (u_1 - u_2) \left(\frac{k}{m} \right) + \sum_{k=1}^{m} \frac{m^\alpha}{k^\alpha} (u_1 - u_2) \left(\frac{k}{m} \right) \ {}^1\mathrm{B}_{k,m}^\alpha(t) = 0.$$

Since the Bernstein weights ${}^1\mathrm{B}_{k,m}^\alpha(t)$ are linearly independent for different k, the above equation implies that:

$$(u_1 - u_2) \left(\frac{k}{m} \right) = 0, \quad \forall k = 0, 1, \ldots, m.$$

Due to the density of $\mathbb{Q}$ in $(0,1)$ and continuity of u_1 and u_2, it follows that:

$$u_1(x) = u_2(x), \quad \forall x \in [0,1].$$

Thus, ${}^1\mathfrak{B}_m^\alpha(u;t)$ uniquely determines u, and the operator is injective.
By Using Lemma 2.10, we have the following Lemma:

Lemma 3.4. *The GBOs have the following properties:*

$${}^1\mathfrak{B}_m^\alpha(t^\alpha;t) = t^\alpha, \quad {}^1\mathfrak{B}_m^\alpha(t^{\alpha+1};t) = t^{\alpha+1}, \quad {}^1\mathfrak{B}_m^\alpha(t^{\alpha+1}(1-t);t) = \left(1 - \frac{1}{m}\right) t^{\alpha+1}(1-t).$$

and

$${}^2\mathfrak{B}_m^\alpha((1-t)^\alpha;t) = (1-t)^\alpha, \quad {}^2\mathfrak{B}_m^\alpha((1-t)^{\alpha+1};t) = (1-t)^{\alpha+1},$$
$${}^2\mathfrak{B}_m^\alpha(t(1-t)^{\alpha+1};t) = \left(1 - \frac{1}{m}\right) t(1-t)^{\alpha+1}.$$

Now, we are ready to state the approximation property of the GBOs:

Lemma 3.5. *Let $u \in \mathbf{F}_1^\alpha \cup \mathbf{F}_2^\alpha$, it follows that*

$$\| u - {}^j\mathfrak{B}_m^\alpha u \|_\infty \underset{m \to \infty}{\longrightarrow} 0, \quad j = 1, 2.$$

Proof. We prove the lemma for the case $j = 1$. From Lemma 2.11, for $v \in C[0,1]$, we have

$$\|v - \mathfrak{B}_m v\|_\infty \xrightarrow[m \to \infty]{} 0.$$

Since $u \in \mathbf{F}_1^\alpha$, there exists a function v such that $u(t) = t^\alpha v(t)$. Therefore, we can write:

$$\begin{aligned}
\Big| u(t) - {}^1\mathfrak{B}_m^\alpha(u;t) \Big| &= \Big| u(t) - {}^1\mathrm{B}_{0,m}^\alpha(t) \lim_{k \to \infty} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) - \sum_{k=1}^{m} \frac{m^\alpha}{k^\alpha} u\left(\frac{k}{m}\right) \ {}^1\mathrm{B}_{k,m}^\alpha(t) \Big| \\
&= \Big| t^\alpha v(t) - t^\alpha \sum_{k=0}^{m} v\left(\frac{k}{m}\right) B_{k,m}(t) \Big| \\
&= t^\alpha \Big| v(t) - \mathfrak{B}_m(v;t) \Big|.
\end{aligned}$$

Using the fact that $v(t) \in C[0,1]$ and applying Lemma 2.11, it follows that

$$\Big| u(t) - {}^1\mathfrak{B}_m^\alpha(u;t) \Big| \xrightarrow[m \to \infty]{} 0.$$

Hence, the desired result is achieved for the case $j = 1$.
By following a similar procedure as above, the case $j = 2$ can be proven. □

Proof. We prove the lemma for the case $j=1$. From Lemma 2.11, for $v \in C[0,1]$, we have

$$\|v - \mathfrak{B}_m v\|_\infty \xrightarrow[m\to\infty]{} 0. \tag{3.9}$$

Since $u \in \mathbf{F}_1^\alpha$, there exists a function v such that $u(t) = t^\alpha v(t)$. Therefore, we can write:

$$\begin{aligned}
\left|u(t) - {}^1\mathfrak{B}_m^\alpha(u;t)\right| &= \left|u(t) - \Big({}^1\mathrm{B}_{0,m}^\alpha(t)\lim_{k\to 0}\frac{m^\alpha}{k^\alpha}u\Big(\frac{k}{m}\Big) + \sum_{k=1}^m \frac{m^\alpha}{k^\alpha}u\Big(\frac{k}{m}\Big){}^1\mathrm{B}_{k,m}^\alpha(t)\Big)\right| \\
&= \left|t^\alpha v(t) - \Big(t^\alpha v(0)(1-t)^m + \sum_{k=1}^m v\Big(\frac{k}{m}\Big)\binom{m}{k}t^{\alpha+k}(1-t)^{m-k}\Big)\right| \\
&= t^\alpha\left|v(t) - \Big(v(0)(1-t)^m + \sum_{k=1}^m v\Big(\frac{k}{m}\Big)\binom{m}{k}t^{k}(1-t)^{m-k}\Big)\right| \\
&= t^\alpha\left|v(t) - \mathfrak{B}_m(v;t)\right|.
\end{aligned}$$

Using the fact that $v \in C[0,1]$ and applying (3.9), it follows that

$$\left|u(t) - {}^1\mathfrak{B}_m^\alpha(u;t)\right| \xrightarrow[m\to\infty]{} 0.$$

Hence, the desired result is achieved for the case $j=1$.

By following a similar procedure as above, the case $j=2$ can be proven. □

Finally, the following lemma can be obtained using Lemma 2.12 and Lemma 3.5:

Lemma 3.6. *If $u \in \mathbf{F}_1^\alpha \cup \mathbf{F}_2^\alpha$ and it is bounded on $[0,1]$, then*

$$\| u - {}^j\mathfrak{B}_m^\alpha u \|_\infty < \frac{3}{2}\omega\left(\frac{1}{\sqrt{m}}\right), \quad j=1,2, \quad m\in\mathbb{N},$$

where for any $t_1, t_2 \in [0,1]$ and $\delta > 0$

$$\omega(\delta) = \sup_{|t_1-t_2|\le\delta} |t_1^{-\alpha}u(t_1) - t_2^{-\alpha}u(t_2)|.$$

4. Calculating Caputo fractional derivative of GBFs

In this section, we present efficient and stable techniques for computing the Caputo fractional derivative of introduced GBFs. Consider an arbitrary set of collocation points on the interval $[0,1]$, denoted as $\{t_k\}_{k=0}^m$. First, we suppose that $\mu \in (0,1)$, substituting (3.1) in (2.1) and utilizing (3.3), we can derive

$$\begin{aligned}
\mathbf{D}_{k,i}^{(\mu,\alpha)} &= {}_0^cD_t^\mu\, {}^1\mathrm{B}_{i,m}^\alpha(t_k) = \frac{1}{\Gamma(1-\mu)}\int_0^{t_k}(t_k-s)^{-\mu}\frac{d}{ds}\left({}^1\mathrm{B}_{i,m}^\alpha(s)\right)ds \\
&= \frac{1}{\Gamma(1-\mu)}\int_0^{t_k}(t_k-s)^{-\mu}\left[\frac{\alpha}{s}\,{}^1\mathrm{B}_{i,m}^\alpha(s) + m\left({}^1\mathrm{B}_{i-1,m-1}^\alpha(s) - {}^1\mathrm{B}_{i,m-1}^\alpha(s)\right)\right]ds \\
&= \frac{\alpha}{\Gamma(1-\mu)}\Big(\frac{t_k}{2}\Big)^{\alpha-\mu}\int_{-1}^1(1-z)^{-\mu}(1+z)^{\alpha-1}B_{i,m}\Big(\frac{1+z}{2}t_k\Big)dz \\
&+ \frac{m}{\Gamma(1-\mu)}\Big(\frac{t_k}{2}\Big)^{\alpha-\mu+1}\int_{-1}^1(1-z)^{-\mu}(1+z)^{\alpha}\left(B_{i-1,m-1}\Big(\frac{1+z}{2}t_k\Big) - B_{i,m-1}\Big(\frac{1+z}{2}t_k\Big)\right)dz,
\end{aligned} \tag{4.1}$$

where $s = \frac{1+z}{2}t_k$. The above integrals can be computed with the Jacobi-Gauss-type quadrature formulas (see [39]). Assume that $\left\{z_k^{\nu,\beta}, w_k^{\nu,\beta}\right\}$ be the Jacobi-Gauss-type points and weights with respect to the Jacobi weight function

$w^{\nu,\beta}(z) = (1-z)^{\nu}(1+z)^{\beta}$. Then, we have

$$\begin{aligned}
\mathbf{D}_{k,i}^{(\mu,\alpha)} &= \frac{\alpha}{\Gamma(1-\mu)}(\frac{t_k}{2})^{\alpha-\mu}\sum_{j=1}^{\lceil\frac{m+1}{2}\rceil} w_j^{-\mu,\alpha-1} B_{i,m}(\frac{1+z_j^{-\mu,\alpha-1}}{2}t_k) \\
&+ \frac{m}{\Gamma(1-\mu)}(\frac{t_k}{2})^{\alpha-\mu+1}\sum_{j=1}^{\lceil\frac{m}{2}\rceil} w_j^{-\mu,\alpha}\,(B_{i-1,m-1}-B_{i,m-1})\,(\frac{1+z_j^{-\mu,\alpha}}{2}t_k), \quad k=1,\dots,m.
\end{aligned} \tag{4.2}$$

Here, it is emphasized that, owing to the increasing errors associated with using the explicit form of Bernstein polynomials, we opt for employing the three-term recurrence formula linked to them for calculations. The efficient and stable computation of ${}^{j}\mathrm{B}_{i,m}^{\alpha}(t)$ at collocation points can be achieved through the application of three-term recurrence formulas as defined in Lemma 3.4.

This approach represents a departure from many previous studies on Bernstein polynomials, in which explicit formulas were typically utilized for computational purposes.

From Eq. (3.5), a similar result can be provided for $\mu \in (1,2)$.

$$\begin{aligned}
\mathbf{D}_{k,i}^{(\mu,\alpha)} &= \frac{\alpha^2-\alpha}{\Gamma(2-\mu)}(\frac{t_k}{2})^{\alpha-\mu}\sum_{j=1}^{\lceil\frac{m+1}{2}\rceil} w_j^{1-\mu,\alpha-2} B_{i,m}(\frac{1+z_j^{1-\mu,\alpha-2}}{2}t_k) \\
&+ \frac{2m\alpha}{\Gamma(2-\mu)}(\frac{t_k}{2})^{\alpha-\mu+1}\sum_{j=1}^{\lceil\frac{m}{2}\rceil} w_j^{1-\mu,\alpha-1}\,(B_{i-1,m-1}-B_{i,m-1})\,(\frac{1+z_j^{1-\mu,\alpha-1}}{2}t_k) \\
&+ \frac{m(m-1)}{\Gamma(2-\mu)}(\frac{t_k}{2})^{\alpha-\mu+2}\sum_{j=1}^{\lceil\frac{m-1}{2}\rceil} w_j^{1-\mu,\alpha}\,(B_{i-2,m-2}-2B_{i-1,m-2}+B_{i,m-2})\,(\frac{1+z_j^{1-\mu,\alpha}}{2}t_k), \quad k=1,\dots,m.
\end{aligned} \tag{4.3}$$

Now, we consider ${}^{2}\mathrm{B}_{i,m}^{\alpha}$, from Definition 2.2 and Eq. (3.3) for $\mu \in (0,1)$, we can obtain

$$\begin{aligned}
\mathcal{D}_{k,i}^{(\mu,\alpha)} &= {}_{t}^{c}D_{1}^{\mu}\;{}^{2}\mathrm{B}_{i,m}^{\alpha}(t_k) = \frac{-1}{\Gamma(1-\mu)}\int_{t_k}^{1}(s-t_k)^{-\mu}\frac{d}{ds}\,({}^{2}\mathrm{B}_{i,m}^{\alpha}(s))ds \\
&= \frac{-1}{\Gamma(1-\mu)}\int_{t_k}^{1}(s-t_k)^{-\mu}\left[\frac{-\alpha}{1-s}\;{}^{2}\mathrm{B}_{i,m}^{\alpha}(s) + m\left({}^{2}\mathrm{B}_{i-1,m-1}^{\alpha}(s) - {}^{2}\mathrm{B}_{i,m-1}^{\alpha}(s)\right)\right]ds \\
&= \frac{\alpha}{\Gamma(1-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu}\int_{-1}^{1}(1-z)^{\alpha-1}(1+z)^{-\mu}B_{i,m}(\frac{1-t_k}{2}z+\frac{1+t_k}{2})dz \\
&- \frac{m}{\Gamma(1-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu+1}\int_{-1}^{1}(1-z)^{\alpha}(1+z)^{-\mu}\,(B_{i-1,m-1}-B_{i,m-1}))\,(\frac{1-t_k}{2}z+\frac{1+t_k}{2})dz \\
&= \frac{\alpha}{\Gamma(1-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu}\sum_{j=1}^{\lceil\frac{m+1}{2}\rceil} w_j^{\alpha-1,-\mu}B_{i,m}(\frac{1-t_k}{2}z_j^{\alpha-1,-\mu}+\frac{1+t_k}{2}) \\
&- \frac{m}{\Gamma(1-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu+1}\sum_{j=1}^{\lceil\frac{m}{2}\rceil} w_j^{\alpha,-\mu}\,(B_{i-1,m-1}-B_{i,m-1})\,(\frac{1-t_k}{2}z_j^{\alpha,-\mu}+\frac{1+t_k}{2}),
\end{aligned} \tag{4.4}$$

where

$$s = \frac{1-t_k}{2}z + \frac{1+t_k}{2}, \quad k=0,1,\dots,m-1.$$

Through analogous computations as in the preceding scenario, for $\mu \in (1,2)$ and $k = 0,1,\ldots,m-1$, the following is obtained:

$$\mathcal{D}_{k,i}^{(\mu,\alpha)} = \frac{\alpha^2-\alpha}{\Gamma(2-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu}\sum_{j=1}^{\lceil\frac{m+1}{2}\rceil} w_j^{\alpha-2,1-\mu}B_{i,m}(\frac{1-t_k}{2}z_j^{\alpha-2,1-\mu}+\frac{1+t_k}{2})$$
$$-\frac{2m\alpha}{\Gamma(2-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu+1}\sum_{j=1}^{\lceil\frac{m}{2}\rceil} w_j^{\alpha-1,1-\mu}\left(B_{i-1,m-1}-B_{i,m-1}\right)(\frac{1-t_k}{2}z_j^{\alpha-1,1-\mu}+\frac{1+t_k}{2})$$
$$+\frac{m(m-1)}{\Gamma(2-\mu)}(\frac{1-t_k}{2})^{\alpha-\mu+2}\sum_{j=1}^{\lceil\frac{m-1}{2}\rceil} w_j^{\alpha,1-\mu}\left(B_{i-2,m-2}-2B_{i-1,m-2}+B_{i,m-2}\right)(\frac{1-t_k}{2}z_j^{\alpha-1,1-\mu}+\frac{1+t_k}{2}). \quad (4.5)$$

By employing the explicit form of the GBFs, it is possible to derive the Caputo fractional derivative of GBFs at boundary points:

$$\mathbf{D}_{0,i}^{(\mu,\alpha)} = [\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\mu)}t_0^{\alpha-\mu},0,\ldots,0], \quad (4.6)$$

$$\mathcal{D}_{m,i}^{(\mu,\alpha)} = [0,\ldots,0,\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\mu)}(1-t_m)^{\alpha-\mu}]. \quad (4.7)$$

5. The collocation method

In this section, the tools introduced in previous sections are applied to construct collocation schemes for numerical solution of some models of fractional differential equations (FDEs). It needs to be mentioned that although this section is mostly focused on left Caputo fractional derivative, the existing methods can be extended to the right fractional derivative and other derivatives such as Riemann-Liouville fractional derivatives. Although the collocation points may be selected freely, their proper choice plays a crucial role in ensuring both efficiency and high accuracy. In this work, we employ the Chebyshev-Gauss-Lobatto (CGL) points $\{t_k\}_{k=0}^m$ on $[0,1]$ which satisfy

$$t_k = \frac{1}{2}(1-cos\frac{k\pi}{m}), \quad k=0,1,\ldots,m.$$

5.1. Initial-valued multi-order FDEs.

We first consider the linear multi-order FDE of order $\mu \in (0,2)$ as

$${}_0^cD_t^{\mu}u(t)+\sum_{r=1}^{l-1}\lambda_r(t)\ {}_0^cD_t^{\mu_r}u(t)+\lambda_l(t)u(t)=f(t), \quad t\in(0,1], \quad (5.1)$$

with initial conditions

$$u(0)=u'(0)=0,$$

where $0 \le \mu_1 < \mu_2 < \ldots < \mu_l < \mu \le 2$. It is worth mentioning that the condition $u'(0)=0$ is needed only for the case $\mu \in (1,2)$. The initial conditions imply to use the GBFs in Eq. (3.1), so in the spectral collocation method, we seek for a solution

$$u_m(t) \in span\left\{t^{\alpha},t^{\alpha+1},\ldots,t^{\alpha+m}\right\}$$

which can be represented as

$$u_m(t)=\sum_{i=0}^{m}c_i\ {}^1\mathrm{B}_{i,m}^{\alpha}(t). \quad (5.2)$$

with unknown coefficients $\{c_i\}_{i=0}^m$. Substituting Eq. (5.2) in Eq. (5.1) and utilizing the collocation method, we have

$$\sum_{i=0}^{m}c_i\ {}_0^cD_{t_k}^{\mu}\ {}^1\mathrm{B}_{i,m}^{\alpha}(t)+\sum_{r=1}^{l-1}\lambda_r(t_k)\sum_{i=0}^{m}c_i\ {}_0^cD_{t_k}^{\mu_r}\ {}^1\mathrm{B}_{i,m}^{\alpha}(t)+\lambda_l(t_k)\sum_{i=0}^{m}c_i\ {}^1\mathrm{B}_{i,m}^{\alpha}(t_k)=f(t_k), \quad k=1,2,\ldots,m.$$

Using Eq. (4.6) for $k = 0$, the corresponding linear system becomes

$$(\mathbf{D}^{(\mu,\alpha)} + \sum_{r=1}^{l-1} \Lambda_r \mathbf{D}^{(\mu_r,\alpha)} + \Lambda_l \hat{B}^{\alpha})\mathbf{c} = \mathbf{f},$$

where $\mathbf{D}^{(\mu,\alpha)}$ and $\mathbf{D}^{(\mu_r,\alpha)}$ are computed in Eqs. (4.2)-(4.3), $\hat{B}^{\alpha}_{k,i} = {}^{1}\mathrm{B}^{\alpha}_{im}(t_k)$ and

$$\Lambda_r = diag(\lambda_r(t_0), \lambda_r(t_1), \ldots, \lambda_r(t_m)),$$
$$\mathbf{c} = (c_0, c_1, \ldots, c_m)^T, \quad \mathbf{f} = (f(t_0), f(t_1), \ldots, f(t_m))^T.$$

Figure 4 illustrates the condition numbers corresponding to the fractional matrices. $\mathbf{D}^{\mu,\alpha} + \hat{B}^{\alpha}$ for different values

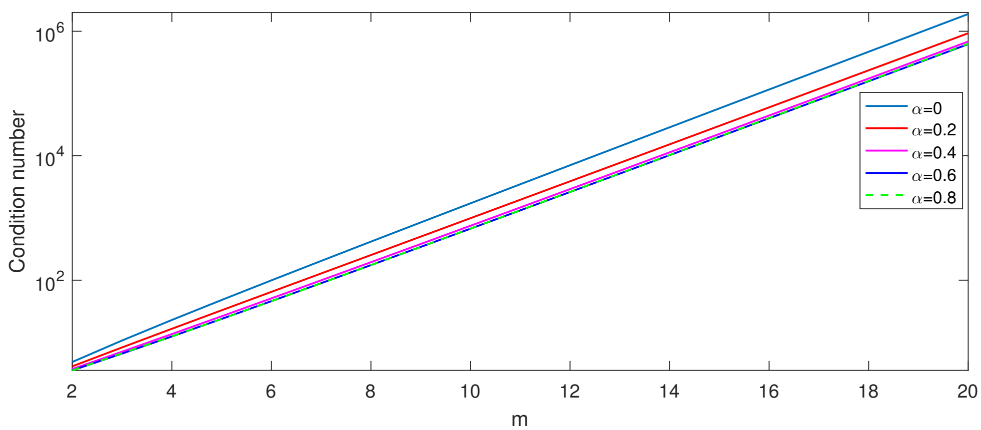

FIGURE 4. Condition numbers of $\mathbf{D}^{\mu,\alpha} + \hat{B}^{\alpha}$ for different values of α.

of α. It should be noted that in the above linear system, the matrices $\mathbf{D}^{(\mu,\alpha)}$ and $\hat{B}^{\alpha}$ influence the magnitude of the condition number. Furthermore, it can be observed that the values of α do not contribute significantly to increasing the condition number. In fact, the condition number when using GBFs (for $\alpha > 0$) is somewhat smaller than the condition number in the case of classical Bernstein polynomials (when $\alpha = 0$).

As previously indicated, these methods can be extended to the nonlinear case. For instance, let's consider the following non-linear FDE with homogeneous initial conditions:

$${}_0^cD_t^{\mu} u(t) + \lambda_1(t)u(t) = f(t, u(t)), \quad \alpha \in (0, 2). \tag{5.3}$$

Using GBFs, we have

$$\left(\mathbf{D}^{(\mu,\alpha)} + \Lambda_1 \hat{B}^{\alpha}\right) \mathbf{c} = \mathbf{f}(t, \hat{B}^{\alpha}\mathbf{c}),$$

where

$$\mathbf{f}(t, \hat{B}^{\alpha}c) = [f(t_0, \hat{B}^{\alpha}c), \ldots, f(t_m, \hat{B}^{\alpha}\mathbf{c})]^T.$$

Consequently, the unknown coefficients can be calculated using Newton's method or any standard iterative method. We now consider the following nonlinear multi-order fractional differential equation:

$${}_0^cD_t^{\mu} u(t) = f\big(t, u(t), {}_0^c D_t^{\mu_1} u(t), \ldots, {}_0^c D_t^{\mu_l} u(t)\big), \quad t \in (0, 1], \tag{5.4}$$

with initial conditions

$$u(0) = u'(0) = 0,$$

where $0 \leq \mu_1 < \mu_2 < \ldots < \mu_l < \mu \leq 2$. The function f is assumed to be sufficiently smooth.

To solve Eq. (5.4) numerically substituting the approximation (5.2) into the governing equation (5.4) and enforcing it at the collocation points t_k (excluding $k=0$ if necessary, depending on the initial conditions), we obtain:

$$\sum_{i=0}^{m} c_i \ {}_0^cD_{t_k}^{\mu} \ {}^1\mathrm{B}_{i,m}^{\alpha}(t_k) = f\left(t_k,\ \sum_{i=0}^{m} c_i \ {}^1\mathrm{B}_{i,m}^{\alpha}(t_k),\ \sum_{i=0}^{m} c_i \ {}_0^cD_{t_k}^{\mu_1} \ {}^1\mathrm{B}_{i,m}^{\alpha}(t_k),\ \dots,\ \sum_{i=0}^{m} c_i \ {}_0^cD_{t_k}^{\mu_l} \ {}^1\mathrm{B}_{i,m}^{\alpha}(t_k)\right),$$

for $k = 1, 2, \dots, m$.

Then the collocation system can be written compactly as:

$$\mathbf{D}^{(\mu,\alpha)}\mathbf{c} = \mathbf{F}\big(t, \hat{B}^{\alpha}\mathbf{c}, \mathbf{D}^{(\mu_1,\alpha)}\mathbf{c}, \dots, \mathbf{D}^{(\mu_l,\alpha)}\mathbf{c}\big), \tag{5.5}$$

where $\mathbf{c} = (c_0, c_1, \dots, c_m)^T$ and the vector $\mathbf{F}$ is defined componentwise by:

$$\big[\mathbf{F}(t, \mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_l)\big]_k = f\big(t_k,\ \mathbf{u}_k,\ (\mathbf{v}_1)_k,\ \dots,\ (\mathbf{v}_l)_k\big), \quad k = 1, \dots, m.$$

Equation (5.5) represents a system of m nonlinear algebraic equations (since the equation at $k=0$ is replaced by initial conditions).

5.2. **Fractional two-point boundary value problem.** In this subsection, we construct a collocation scheme to solve fractional systems that include both left- and right-sided fractional derivatives. It is worth mentioning that these systems usually can be generated by necessary optimality conditions of fractional optimal control problems (see [2]). We consider the following fractional systems with $\mu_1, \mu_2 \in (0,2)$ on the interval $[0,1]$

$$\begin{cases} {}_0^cD_t^{\mu_1}u(t) + \lambda_1(t)u(t) + \lambda_2(t)v(t) = f(t), \\ {}_t^cD_1^{\mu_2}v(t) + \lambda_3(t)v(t) + \lambda_4(t)u(t) = g(t), \end{cases} \tag{5.6}$$

subject to the homogeneous initial conditions

$$\begin{cases} u(0) = u'(0) = 0, \quad \mu_1, \mu_2 \in (0,1), \\ v(1) = v'(1) = 0, \quad \mu_1, \mu_2 \in (1,2). \end{cases}$$

To develop the spectral collocation method, the unknown functions $u(t)$ and $v(t)$ are approximated by GBFs as

$$\begin{cases} u_m(t) = \sum_{i=0}^{m} c_i \ {}^1\mathrm{B}_{i,m}^{\alpha_1}(t), \\ v_m(t) = \sum_{i=0}^{m} d_i \ {}^2\mathrm{B}_{i,m}^{\alpha_2}(t). \end{cases} \tag{5.7}$$

Plugging Eq. (5.7) in Eq. (5.6) and using CGL collocation points, we have

$$\begin{cases} \sum_{i=0}^{m} c_i \ {}_0^cD_{t_k}^{\mu_1} \ {}^1B_{i,m}^{\alpha_1}(t) + \lambda_1(t_k)\sum_{i=0}^{m} c_i \ {}^1\mathrm{B}_{i,m}^{\alpha_1}(t_k) + \lambda_2(t_k)\sum_{i=0}^{m} d_i \ {}^2\mathrm{B}_{i,m}^{\alpha_2}(t_k) = f(t_k), \\ \\ \sum_{i=0}^{m} d_i \ {}_{t_k}^cD_1^{\mu_2} \ {}^2B_{i,m}^{\alpha_2}(t) + \lambda_3(t_k)\sum_{i=0}^{m} d_i \ {}^2\mathrm{B}_{i,m}^{\alpha_2}(t_k) + \lambda_3(t_k)\sum_{i=0}^{m} c_i \ {}^1\mathrm{B}_{i,m}^{\alpha_1}(t_k) = g(t_k). \end{cases} \tag{5.8}$$

Similar to the previous subsection, the system (5.8) can be represented in an equivalent form as

$$\begin{cases} (\mathbf{D}^{\mu_1,\alpha_1} + \Lambda_1\hat{B}^{\alpha_1})\mathbf{c} + \Lambda_2\check{B}^{\alpha_2})\mathbf{d} = \mathbf{f}, \\ (\mathcal{D}^{(\mu_2,\alpha_2)} + \Lambda_3\check{B}^{\alpha_2})\mathbf{d} + \Lambda_4\hat{B}^{\alpha_1})\mathbf{c} = \mathbf{g}. \end{cases}$$

The notation $\mathbf{d}$, Λ_i and $\check{B}^{\alpha}$ is defined similar to $\mathbf{c}$, Λ_r and $\hat{B}^{\alpha}$ in previous subsection, respectively. Rewriting the above equations, we have

$$\begin{bmatrix} \mathbf{D}^{(\mu_1,\alpha_1)} + \Lambda_1\hat{B}^{\alpha_1} & \Lambda_2\check{B}^{\alpha_2} \\ \Lambda_4\hat{B}^{\alpha_1} & \mathcal{D}^{(\mu_2,\alpha_2)} + \Lambda_3\check{B}^{\alpha_2} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \tag{5.9}$$

where

$$\mathbf{d} = (d_0, d_1, \dots, d_m)^T, \quad \mathbf{g} = (g(t_0), g(t_1), \dots, g(t_m))^T.$$

Despite the previous case, here we have a linear system of equations that can be easily handled with numerical linear system solvers like Gauss or Gauss-Jordan elimination.

5.3. **Volterra-Fredholm fractional integro-differential equation.** In this subsection, technique based GBFs is developed for the numerical solution of Volterra-Fredholm fractional integro-differential equation with combination of Caputo derivatives. The Volterra-Fredholm fractional integro-differential equation is given by

$$ {}_0^cD_t^{\mu}u(t) = f(t) + \lambda(t)u(t) + \int_0^t l_1(t,s)u(s)ds + \int_0^1 l_2(t,s)u(s)ds, \quad \mu \in (0,1), \tag{5.10}$$

with homogeneous initial conditions. We approximate $u(t)$ in terms of GBFs as Eq. (5.2), this leads to $u_m(0) = 0$. Substituting (5.2) in (5.10) and employing the collocation points $\{t_k\}_{k=0}^m$, we obtain

$$\sum_{i=0}^m c_i \, {}_0^cD_t^{\mu,\alpha} \, {}^1\mathrm{B}_{i,m}^{\alpha}(t_k) = f(t_k) + \lambda(t_k)\sum_{i=0}^m c_i \, {}^1\mathrm{B}_{i,m}^{\alpha}(t_k) + \sum_{i=0}^m c_i \int_0^{t_k} l_1(t_k,s) \, {}^1\mathrm{B}_{i,m}^{\alpha}(s)ds + \sum_{i=0}^m c_i \int_0^1 l_2(t_k,s) \, {}^1\mathrm{B}_{i,m}^{\alpha}(s)ds.$$

Based on the notations presented in the previous subsections, we obtain the following system of algebraic equations

$$(\mathbf{D}^{(\mu,\alpha)} - \Lambda_1\hat{B}^{\alpha} - \hat{H}^{\alpha} - \hat{F}^{\alpha})\mathbf{c} = \mathbf{f},$$

where the elements of the matrices $\hat{H}_{k,i}^{\alpha}$ and $\hat{F}_{k,i}^{\alpha}$ could be represented as:

$$\hat{H}_{k,i}^{\alpha} = \int_0^{t_k} l_1(t_k,s) \, {}^1\mathrm{B}_{i,m}^{\alpha}(s)ds, \quad \hat{F}_{k,i}^{\alpha} = \int_0^1 l_2(t_k,s) \, {}^1\mathrm{B}_{i,m}^{\alpha}(s)ds,$$

Here the above integrals can be computed using the explicit form of GBFs and Jacobi-Gauss-type quadrature formula, depending on the forms of $l_1(t,s)$ and $l_2(t,s)$.

Remark 5.1. It is worth mentioning that by substituting the left fractional derivative with the right fractional derivative with corresponding initial conditions in Equations (5.1) and (5.10), we can apply the same approach used to solve Equation (5.6). This involves utilizing the bases defined in the second type (3.2) to address them effectively. Furthermore, we assumed that the initial conditions of the equations under consideration are homogeneous to align with the introduced bases. However, it is straightforward to extend the presented methods for solving these equations to the non-homogeneous case.

Here, we present the algorithm for solving the nonlinear FDEs. Similar algorithms can be used to solve other equations.

Algorithm

Step 1: Choose the parameters $\mu, \alpha, n = 0$ and functions λ_1, f, and the convergence criterion ϵ.

Step 2: Define the GBF ${}^1\mathcal{B}_{i,m}^{\alpha}(t)$.

Step 3: Compute the collocation points $\{t_k\}_{k=0}^m$ and the Jacobi-Gauss-type points $\{z_k^{\nu,\beta}, w_k^{\nu,\beta}\}$.

Step 4: Compute the matrices $\hat{B}_{\alpha} = {}^1\mathrm{B}_{im}^{\alpha}(t_k), \Lambda_1 = diag(\lambda_1(t_0), \lambda_1(t_1), \dots, \lambda_1(t_m))$ and $\mathbf{D}_{\mu,\alpha}$ using Equations (4.4) and (4.5).

Step 5: Set the initial vector c_0.

Step 6: Solve the resulting system depending on the type of the equation:

Case 1: For the linear multi-order FDE (5.1), solve the linear system directly:

$$\left(\mathbf{D}^{(\mu,\alpha)} + \sum_{r=1}^{l-1} \Lambda_r \mathbf{D}^{(\mu_r,\alpha)} + \Lambda_l \hat{B}^{\alpha}\right)\mathbf{c} = \mathbf{f},$$

where $\mathbf{f} = (f(t_0), f(t_1), \dots, f(t_m))^T$.

Case 2: For the simple nonlinear FDE (5.3) use a fixed-point iteration approach:

$$\mathbf{D}^{(\mu,\alpha)}\mathbf{c}^{n+1} = \mathbf{f}^n - \Lambda_1\hat{B}^{\alpha}\mathbf{c}^n,$$

where $\mathbf{f}^n = \left[f(t_0, \hat{B}^{\alpha}\mathbf{c}^n), f(t_1, \hat{B}^{\alpha}\mathbf{c}^n), \dots, f(t_m, \hat{B}^{\alpha}\mathbf{c}^n)\right]^T$. Newton's method may be employed as an alternative for faster convergence.

Case 3: For the general nonlinear multi-order FDE (5.4), solve the nonlinear system:

$$\mathbf{D}^{(\mu,\alpha)}\mathbf{c} = \mathbf{F}\big(t, \hat{B}^{\alpha}\mathbf{c}, \mathbf{D}^{(\mu_1,\alpha)}\mathbf{c}, \dots, \mathbf{D}^{(\mu_l,\alpha)}\mathbf{c}\big),$$

using Newton's method or a fixed-point iteration. In the fixed-point approach, one may iterate as:

$$\mathbf{D}^{(\mu,\alpha)}\mathbf{c}^{n+1} = \mathbf{F}\big(t, \hat{B}^{\alpha}\mathbf{c}^{n}, \mathbf{D}^{(\mu_1,\alpha)}\mathbf{c}^{n}, \dots, \mathbf{D}^{(\mu_l,\alpha)}\mathbf{c}^{n}\big).$$

Step 7: Repeat the Step 6 (**Case 2-3**) until achieving the convergence criterion $\epsilon = \| \mathbf{c}^{n+1} - \mathbf{c}^{n} \|_2$.

Step 8: Set $\mathbf{c} = \mathbf{c}^{n+1}$ and approximate the solution using:

$$u_m(t) = \sum_{i=0}^{m} c_i \ {}^{1}\mathcal{B}^{\alpha}_{i,m}(t).$$

6. Convergence analysis

In this section, we study the convergence properties of the proposed method based on the modified Bernstein operator. Our main goal is to determine the rate at which the approximate solution converges to the exact solution and to provide explicit error estimates.

6.1. **Convergence rates.** We now delve into the theoretical underpinnings of convergence rates of the proposed method. The analytical framework leverages the application of generalized Bernstein-like operators (GBOs). As we explore the convergence rates, our focus centers on GBOs of the first type, with a subsequent acknowledgment that parallel developments can be extended to operators of the second type. The validation of these theoretical convergence rates will be expounded upon through illustrative numerical examples in the subsequent section. The analysis of the convergence is based on the discussions presented in [19, 42].

To state the results, we consider the space of *poly-fractionals* $\mathcal{P}^{\alpha}_m$ as follows

$$\mathcal{P}^{\alpha}_m = \{u \mid u = t^{\alpha}v, v \in \mathbb{P}_m(0,1)\} = span\left\{{}^{1}\mathrm{B}^{\alpha}_{i,m}\right\}_{i=0}^{m},$$

which $\mathbb{P}_m(0,1)$ is the set of polynomials of degree less than or equal to m. According to the orthogonal projection of π_m on $\mathbb{P}_m(0,1)$, the orthogonal projection $\pi^{\alpha}_m : L^2 \longrightarrow \mathcal{P}^{\alpha}_m$ is defined as

$$(\pi^{\alpha}_m u - u, v)_{L^2} = 0, \qquad \forall v \in \mathcal{P}^{\alpha}_m,$$

then, we can write

$$\pi^{\alpha}_m u(t) = \sum_{i=0}^{m} c_i \ {}^{1}\mathrm{B}^{\alpha}_{i,m}(t).$$

Now we assume that the singularity behavior of the function $u(t)$ near $t = 0$ is as follows

$$u(t) \sim t^{\alpha} + t^{\beta_1} + t^{\beta_2}, \quad \alpha \le \beta_1 < \beta_2.$$

From discussions presented in [19, 42], the convergence rate of $\| \pi^{\alpha}_m - u \|_{\hat{w}^{0,\alpha}}$ for weight function $\hat{w}^{0,\alpha} = t^{\alpha}$ is obtained as

$$\| \pi^{\alpha}_m u - u \|_{\hat{w}^{0,\alpha}} = \begin{cases} O(N^{-(1+2\beta_1+\alpha)}), & \beta_1 \neq 1+\alpha, \\ O(N^{-(1+2\beta_2+\alpha)}), & \beta_1 = 1+\alpha, \beta_2 \neq 2+\alpha \\ \textit{Exponential convergence} & \beta_1 = 1+\alpha, \quad \beta_2 = 2+\alpha, \end{cases} \tag{6.1}$$

Through a straightforward examination, we ascertain that the convergence order of a nonsmooth function, when employing classical Bernstein polynomials, is given by $1+2\alpha$. Illustrating the convergence order of our proposed method involves considering the fractional differential equation ${}^{c}_{0}D^{\alpha}_{t}u(t) = f(t)$ with an order $\alpha \in (0,2)$ and homogeneous initial conditions. Let u_m represent the approximate solution derived from the proposed method. By defining $v(t) = t^{-\alpha}u(t)$ and $v_m(t) = t^{-\alpha}u_m(t)$, we transform v_m into a polynomial of degree m

$$v(t) \sim 1 + t^{\beta_1-\alpha} + t^{\beta_2-\alpha}.$$

Utilizing (6.1) and through straightforward calculations, one has:

$$\| u - u_m \|_{L^2} \leq \| v - v_m \|_{\hat{w}^{0,\alpha}} \begin{cases} O(N^{-(1+2(\beta_1-\alpha)+\alpha)}), & \beta_1 \neq 1+\alpha, \\ O(N^{-(1+2(\beta_2-\alpha)+\alpha)}), & \beta_1 = 1+\alpha, \beta_2 \neq 2+\alpha \\ \textit{Exponential convergence} & \beta_1 = 1+\alpha, \quad \beta_2 = 2+\alpha. \end{cases}$$

The presented convergence rates of L_2 errors $r = 1 + 2\beta_i$ and $R = 1 + 2(\beta_i - \alpha) + \alpha = 1 + 2\beta_i - \alpha$ can be applied for other fractional equations as well.

6.2. **Error estimates.** This subsection is devoted to studying the error bounds associated with fractional derivatives and integrals. We also carry out an error analysis of the proposed method when applied to multi-order fractional derivative equations. In addition, some basic properties of Bernstein polynomials are recalled, as they play a key role in the subsequent discussion.

Theorem 6.1. *Let u be a continuous function on $[0,1]$. Then, for every $\varepsilon > 0$, one can find a positive integer M such that for all $xtin[0,1]$ and any $m \geq M$,*

$$\|u(t) - B_m u(t)\|_\infty < \varepsilon,$$

where $B_m u(t) = \sum_{i=0}^{m} c_i B_{i,m}(t)$ [24]. Moreover, if u is k times continuously differentiable on $[0,1]$ for some $k \geq 0$, then the derivatives of $B_m u$ converge uniformly to those of u [16]

$$\lim_{m\to\infty} (B_m u)^{(k)}(t) = u^{(k)}(t), \qquad t \in [0,1].$$

Theorem 6.2. *Let $u(t) = t^\alpha v(t)$ with $v \in C[0,1]$ and $0 < \alpha < 2$. Then, for every $\varepsilon > 0$, there exists a positive integer M such that for all $t \in [0,1]$ and $m \geq M$,*

$$\|u(t) - \pi_m^\alpha u(t)\|_\infty < \varepsilon.$$

Proof. From the definitions $u(t) = t^\alpha v(t)$ and $\pi_m^\alpha u(t) = t^\alpha B_m v(t)$, we obtain

$$\|u - \pi_m^\alpha u\|_\infty = \max_{t\in[0,1]} t^\alpha \big|v(t) - B_m v(t)\big| \leq \|v - B_m v\|_\infty.$$

Since v is continuous on $[0,1]$, the classical Bernstein approximation theorem (Theorem 6.1) implies that for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$,

$$\|v - B_m v\|_\infty < \varepsilon.$$

Hence $\|u - \pi_m^\alpha u\|_\infty < \varepsilon$ for $m \geq M$, which completes the proof. $\square$

We now examine the convergence of the fractional integral and the Caputo fractional derivative of $\pi_m^\alpha u$ to the fractional integral of u. Using the definition of the Riemann–Liouville fractional integral of order $\alpha > 0$, we have

$$\begin{aligned} \big|{}_0I_t^\alpha u - {}_0 I_t^\alpha \pi_m^\alpha u\big| &= \big|{}_0I_t^\alpha\big(u - \pi_m^\alpha u\big)\big| \\ &= \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(u(s) - \pi_m^\alpha u(s)\big) ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big|u(s) - \pi_m^\alpha u(s)\big| ds \\ &\leq \|u - \pi_m^\alpha u\|_\infty \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &= \|u - \pi_m^\alpha u\|_\infty \cdot \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\leq \frac{1}{\Gamma(\alpha+1)} \|u - \pi_m^\alpha u\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0,1]$ yields

$$\|{}_0I_t^{\alpha} u -{}_0 I_t^{\alpha} \pi_m^{\alpha} u\|_\infty \le \frac{1}{\Gamma(\alpha+1)} \|u - \pi_m^{\alpha} u\|_\infty.$$

Since $\|u - \pi_m^{\alpha} u\|_\infty \to 0$ as $m \to \infty$ (by the previous result for π_m^{α}), it follows that

$$\|{}_0I_t^{\alpha} u -{}_0 I_t^{\alpha} \pi_m^{\alpha} u\|_\infty \longrightarrow 0.$$

For the Caputo fractional derivative,two distinct ranges of α are considered.
First, suppose $0 < \alpha < 1$. In this range, using the relation between Caputo derivative and Riemann–Liouville integral, we have ${}_0^cD_t^{\alpha} u ={}_0 I_t^{1-\alpha} u'$. Therefore,

$$\|{}_0^cD_t^{\alpha} u -{}_0^c D_t^{\alpha} \pi_m^{\alpha} u\|_\infty = \|{}_0I_t^{1-\alpha}(u' - (\pi_m^{\alpha} u)')\|_\infty.$$

Writing the fractional integral as a convolution,

$${}_0I_t^{1-\alpha} w(t) = \frac{1}{\Gamma(1-\alpha)} (t^{-\alpha} * w)(t),$$

where $(t^{-\alpha} * w)(t) = \int_0^t (t-s)^{-\alpha} w(s) ds$. Using the convolution inequality $\|t^{-\alpha} * w\|_\infty \le \|t^{-\alpha}\|_1 \|w\|_\infty$, provided $t^{-\alpha}$ is integrable on $[0,1]$, we obtain

$$\|{}_0^cD_t^{\alpha} u -{}_0^c D_t^{\alpha} \pi_m^{\alpha} u\|_\infty \le \frac{1}{\Gamma(1-\alpha)} \|t^{-\alpha}\|_1 \|u' - (\pi_m^{\alpha} u)'\|_\infty.$$

For $0 < \alpha < 1$, the function $t^{-\alpha}$ is integrable on $[0,1]$ since $\int_0^1 t^{-\alpha} dt = \frac{1}{1-\alpha} < \infty$. Hence the factor $\|t^{-\alpha}\|_1$ is finite. Now, for $u(t) = t^{\alpha} v(t)$ and $\pi_m^{\alpha} u(t) = t^{\alpha} B_m v(t)$, we have

$$u'(t) - (\pi_m^{\alpha} u)'(t) = \alpha t^{\alpha-1} \big(v(t) - B_m v(t)\big) + t^{\alpha} \big(v'(t) - (B_m v)'(t)\big).$$

Taking the uniform norm and noting that $t^{\alpha-1}$ is unbounded near $t = 0$, we cannot directly bound it by $\|v - B_m v\|_\infty$ alone. However, we proceed differently: since the convolution with $t^{-\alpha}$ in the fractional integral already regularizes the singularity, we can instead use the property that ${}_0I_t^{1-\alpha}$ is a bounded operator from L^∞ to L^∞? More directly, from the earlier inequality we already have

$$\|{}_0^cD_t^{\alpha} u -{}_0^c D_t^{\alpha} \pi_m^{\alpha} u\|_\infty \le \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{1-\alpha} \cdot \|u' - (\pi_m^{\alpha} u)'\|_\infty.$$

Now, because v and v' are continuous on $[0,1]$ (assuming $v \in C^1[0,1]$), the classical Bernstein theorem (Theorem 6.1) implies $\|v - B_m v\|_\infty \to 0$ and $\|v' - (B_m v)'\|_\infty \to 0$. Moreover, the terms $t^{\alpha-1}$ and t^{α} are bounded in the L^1 norm, and a standard argument using the density of continuous functions shows that $\|u' - (\pi_m^{\alpha} u)'\|_\infty \to 0$. Consequently, the left-hand side tends to zero.

Now consider the second range $1 < \alpha < 2$. For $1 < \alpha < 2$, the Caputo derivative is expressed via the second derivative: ${}_0^cD_t^{\alpha} u ={}_0 I_t^{2-\alpha} u''$. Hence,

$$\|{}_0^cD_t^{\alpha} u -{}_0^c D_t^{\alpha} \pi_m^{\alpha} u\|_\infty = \|{}_0I_t^{2-\alpha}(u'' - (\pi_m^{\alpha} u)'')\|_\infty.$$

Writing the fractional integral as a convolution,

$${}_0I_t^{2-\alpha} w(t) = \frac{1}{\Gamma(2-\alpha)} (t^{1-\alpha} * w)(t),$$

where $t^{1-\alpha}$ is integrable on $[0,1]$ because $1-\alpha > -1$ (since $\alpha < 2$). Applying the convolution inequality $\|t^{1-\alpha} * w\|_\infty \le \|t^{1-\alpha}\|_1 \|w\|_\infty$, we obtain

$$\|{}_0^cD_t^{\alpha} u -{}_0^c D_t^{\alpha} \pi_m^{\alpha} u\|_\infty \le \frac{1}{\Gamma(2-\alpha)} \|t^{1-\alpha}\|_1 \|u'' - (\pi_m^{\alpha} u)''\|_\infty.$$

For $1 < \alpha < 2$, we have $\int_0^1 t^{1-\alpha} dt = \frac{1}{2-\alpha} < \infty$, so the norm $\|t^{1-\alpha}\|_1$ is finite. Now compute $u''(t)$ for $u(t) = t^{\alpha} v(t)$:

$$u''(t) = \alpha(\alpha-1) t^{\alpha-2} v(t) + 2\alpha t^{\alpha-1} v'(t) + t^{\alpha} v''(t),$$

and similarly for $(\pi_m^\alpha u)''(t)$ with $B_m v$ in place of v. The difference $u'' - (\pi_m^\alpha u)''$ consists of three terms, each containing a factor $t^{\alpha-2}$, $t^{\alpha-1}$, or t^α multiplied by a difference of the form $v - B_m v$, $v' - (B_m v)'$, or $v'' - (B_m v)''$. Assuming $v \in C^2[0,1]$, Theorem 6.1 guarantees uniform convergence of $B_m v$ and its derivatives to v and its derivatives. Hence, each term tends to zero uniformly on any interval $[\delta, 1]$ for $\delta > 0$. The singularity at $t = 0$ in the $t^{\alpha-2}$ term is integrable (since $\alpha - 2 > -1$), and the convolution with $t^{1-\alpha}$ in the fractional integral regularizes it further. A standard density argument then yields $\|u'' - (\pi_m^\alpha u)''\|_\infty \to 0$, and consequently

$$\|{}_0^cD_t^\alpha u - {}_0^c D_t^\alpha \pi_m^\alpha u\|_\infty \longrightarrow 0.$$

This completes the convergence analysis for both ranges of α.

Theorem 6.3. *Let $u(t) = t^\mu v(t)$ with $v \in C^n[0,1]$, where $n-1 < \mu \leq n$ and $n = 1$ or 2 (i.e., $0 < \mu \leq 1$ or $1 < \mu \leq 2$). Consider the multi-order fractional differential equation*

$$ {}_0^cD_t^\mu u(t) + \sum_{r=1}^{l} \lambda_r(t)\, {}_0^cD_t^{\mu_r} u(t) + F(t, u(t)) = f(t),$$

with initial conditions $u(0) = u'(0) = 0$, where $0 \leq \mu_1 < \mu_2 < \ldots < \mu_l < \mu \leq 2$. Assume that $F(t, u(t))$ satisfies a Lipschitz condition in the second argument with constant η, i.e.,

$$|F(t, u_1) - F(t, u_2)| \leq \eta |u_1 - u_2|, \qquad \forall t \in [0,1],\ u_1, u_2 \in \mathbb{R}.$$

Furthermore, suppose the coefficient functions $\lambda_r(t)$ are continuous on $[0,1]$ for $r = 1, \ldots, l$. If $u_m(t) = \pi_m^\mu u(t) = t^\mu B_m v(t)$ denotes the approximate solution obtained by the modified Bernstein operator, then

$$\|u - u_m\|_\infty \longrightarrow 0 \quad \text{as } m \to \infty.$$

Proof. We know

$$\begin{aligned}
\|u - u_m\|_\infty &= \Big\| {}_0^cD_t^\mu \pi_m^\mu u + \sum_{r=1}^{l} \lambda_r(t)\, {}_0^cD_t^{\mu_r} \pi_m^\mu u + F(t, \pi_m^\mu u) - f(t) \Big\|_\infty \\
&= \Big\| {}_0^cD_t^\mu \pi_m^\mu u + \sum_{r=1}^{l} \lambda_r(t)\, {}_0^cD_t^{\mu_r} \pi_m^\mu u + F(t, \pi_m^\mu u) - {}_0^c D_t^\mu u - \sum_{r=1}^{l} \lambda_r(t)\, {}_0^cD_t^{\mu_r} u - F(t,u) \Big\|_\infty \\
&\leq \Big\| {}_0^cD_t^\mu \pi_m^\mu u - {}_0^c D_t^\mu u \Big\|_\infty + \Big\| \sum_{r=1}^{l} \lambda_r(t) \big({}_0^cD_t^{\mu_r} \pi_m^\mu u - {}_0^c D_t^{\mu_r} u \big) \Big\|_\infty + \Big\| F(t, \pi_m^\mu u) - F(t,u) \Big\|_\infty \\
&\leq \Big\| {}_0^cD_t^\mu \pi_m^\mu u - {}_0^c D_t^\mu u \Big\|_\infty + \sum_{r=1}^{l} \|\lambda_r\|_\infty \Big\| {}_0^cD_t^{\mu_r} \pi_m^\mu u - {}_0^c D_t^{\mu_r} u \Big\|_\infty + \eta \Big\| \pi_m^\mu u - u \Big\|_\infty .
\end{aligned}$$

Then, from the convergence results for $\pi_m^\mu u$ and its fractional derivatives established earlier (Theorems 6.2 and the subsequent analysis for fractional integrals and derivatives), we have

$$\Big\| \pi_m^\mu u - u \Big\|_\infty \longrightarrow 0, \qquad \Big\| {}_0^cD_t^\mu \pi_m^\mu u - {}_0^c D_t^\mu u \Big\|_\infty \longrightarrow 0, \qquad \Big\| {}_0^cD_t^{\mu_r} \pi_m^\mu u - {}_0^c D_t^{\mu_r} u \Big\|_\infty \longrightarrow 0,$$

as $m \to \infty$. Consequently,

$$\|u - u_m\|_\infty \longrightarrow 0.$$

□

7. Numerical experiments

This section is devoted to examples to assure the efficiency and applicability of the proposed techniques. In each case, $u(t)$ presents the exact solution and $u_m(t)$ is the mth order approximation obtained by the employed scheme. The L^2 norm, defined by $\| u \|_{L^2} = \left(\int_0^1 | u(t) |^2 \, dt \right)^{\frac{1}{2}}$, is utilized to measure the errors. In all examples, we consider

Table 1. The experimental and expected convergence rates, and L_2-errors $\| u - \pi_m^{\alpha} u \|_{L^2}$ and $\| u - u_m \|_{L^2}$.

m	$\| u - \pi_m^{\alpha} u \|_{L^2}$	r_m	r	$\| u - u_m \|_{L^2}$	R_m	R
5	3.4813e-04	2.6603		0.0112	2.1813	
15	3.1716e-05	2.7266		0.0014	2.1002	
25	8.9325e-06	2.7480	2.8	5.0718e-04	2.0874	2
30	5.5967e-06	2.7541		3.5147e-04	2.0835	
35	3.7506e-06	2.7588		2.5743e-04	2.0854	
40	1.2113e-06	2.7621		8.1896e-05	2.0502	
50	8.5213e-07	2.7712		4.1021e-05	2.0421	

Table 2. The experimental and expected convergence rates, and L_2-errors $\| u - \pi_m^{\alpha} u \|_{L^2}$ and $\| u - u_m \|_{L^2}$.

m	$\| u - \pi_m^{\alpha-1} u \|_{L^2}$	r_m	r	$\| u - u_m \|_{L^2}$	R_m	R
5	1.6883e-04	4.7312		6.5555e-04	3.3317	
15	2.8524e-06	4.5279		2.7345e-05	3.2400	
25	3.5571e-07	4.4806	4.4	5.6592e-06	3.2489	3.3
30	1.6636e-07	4.4681		3.1951e-06	3.2507	
35	8.7034e-08	4.4589		2.8815e-06	3.2601	
40	9.7212e-09	4.4421		9.1258e-07	3.2971	
50	1.1220e-09	4.4312		1.9712e-07	3.3219	

$\mu = \alpha$ and the L_2-norm is computed by Jacobi-Gauss quadrature with 200 points. Also, we measure the numerical convergence order of $\| u - \pi_m^{\alpha} u \|_{L^2}$ and $\| u - u_m \|_{L^2}$ as

$$r_m = \frac{\log\left(\frac{\| u - \pi_m^{\alpha} u \|_{L^2}}{\| u - \pi_{m+1}^{\alpha} u \|_{L^2}}\right)}{\log\left(1 + \frac{1}{m}\right)}, \quad R_m = \frac{\log\left(\frac{\| u - u_m \|_{L^2}}{\| u - u_{m+1} \|_{L^2}}\right)}{\log\left(1 + \frac{1}{m}\right)}.$$

Example 7.1. In order to investigate the expected order of convergence discussed in the previous section, first, we consider the equation ${}_0^c D_t^{\alpha} u(t) = f(t)$ with homogeneous initial conditions, with exact solution $u(t) = t^{\beta_1} + t^{\beta_2}$.

This problem is solved using the presented collocation method for some values of α and $\beta_1, \beta_2 \in \mathbb{R}$. In Table 1, we list the errors $\| u - \pi_m^{\alpha} u \|_{L^2}$, $\| u - u_m \|_{L^2}$ and expected convergence rates for $\alpha = \beta_1 = 0.8$, $\beta_2 = 0.9$. According to this table, it can be observed that the expected convergence of orders, i.e. $r = 1 + 2\beta_2 = 2.8$ and $R = 1 + 2\beta_2 - \alpha = 2$, are obtained by increasing the value of m. It should be noted that the order of convergence $1 + 2\beta_1 = 2.6$ and $1 + 2\beta_1 - \alpha = 1.8$ are obtained by using classical Bernstein polynomials. Additionally, we show the obtained results for the case $\alpha = \beta_1 = 1.1$, $\beta_2 = 1.7$ in Table 2. Also, the results of this table show that our expected convergences of order $r = 1 + 2\beta_2 = 4.4$ and $R = 1 + 2\beta_2 - \alpha = 3.3$ are completely compatible with the numerical results and these values become better with the increase of the value of m. In this case, using classical Bernstein polynomials, convergence rates $1 + 2\beta_1 = 3.2$ and $1 + 2\beta_1 - \alpha = 2.2$ are obtained. In addition, in Figure 5, the numerical convergence rates of R_m for $\alpha = \beta_1 = 0.3$ and different values of β_2 under different values of m are depicted. In order to reconfirm the order of theoretical convergence, in Figure 6, the L_2-error and m^{-R} for $\alpha = \beta_1 = 1.1$ and $\beta_2 = 1.4, 1.8$ are displayed. The results of this example and the following examples show that the theoretical convergence order discussed in the previous section is valid.

Example 7.2. Consider the fractional Riccati differential equation of order $\alpha > 0$ [33]

$${}_0^c D_t^{\alpha} u(t) + u^2(t) = f(t), \quad t \in (0, 1],$$

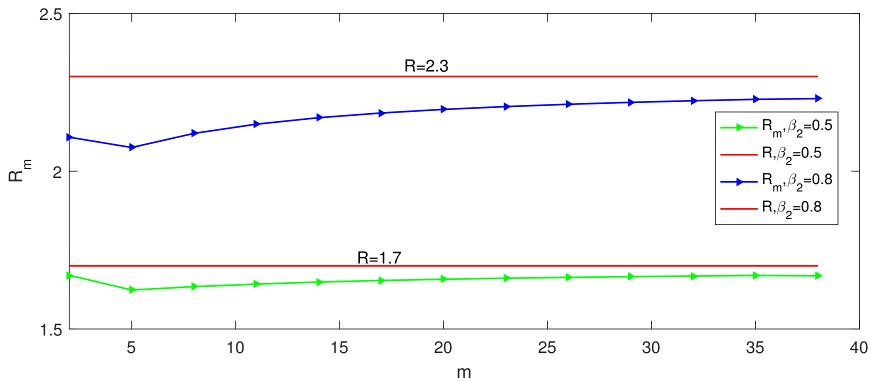

FIGURE 5. The curves of numerical rates R_m under m and our expected rates.

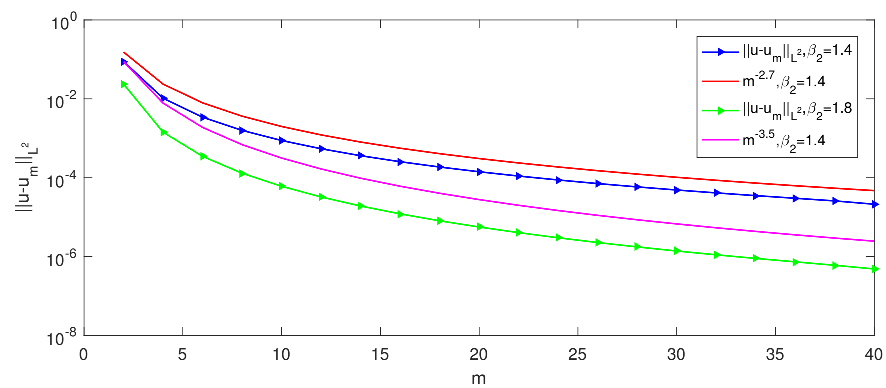

FIGURE 6. The L_2-errors and m^{-R} as a function of m.

with initial condition $u(0) = 0$ and

$$f(t) = \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)^2 + t.$$

The exact solution of this equation is given by $u(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$.
This problem is solved exactly using the presented technique for $m = 1$.
From definition of GBOs, we have

$${}^1\mathcal{B}^{\alpha}_{0,1}(t) = t^{\alpha} - t^{\alpha+1}, \quad {}^1\mathcal{B}^{\alpha}_{1,1}(t) = t^{\alpha+1}.$$

We seek the approximate solution as $u_1(t) = c_0^1\mathcal{B}^{\alpha}_{0,1}(t) + c_1^1\mathcal{B}^{\alpha}_{1,1}(t)$. The Caputo fractional derivatives of ${}^1\mathcal{B}^{\alpha}_{0,1}$ and ${}^1\mathcal{B}^{\alpha}_{1,1}$ can be computed at the collocation points $t = 0, 1$ and then

$$\begin{bmatrix} \Gamma(\alpha+1) & 0 \\ \Gamma(\alpha+1) - \Gamma(\alpha+2) & \Gamma(\alpha+2) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\Gamma^2(\alpha+2)} + 1 \end{bmatrix}.$$

By solving above nonlinear system, we can get $c_0 = 0$, $c_1 = \frac{1}{\Gamma(\alpha+2)}$ and the exact solution is obtained.

Table 3. The CPU times and L_2-errors of the present method and methods in [30, 41].

m	Present method	CPU time (s)	Method in [30]	CPU time (s)	Method in [41]
2	8.8775e-04	0.01242	1.3426e-02	0.344	1.7511e-02
4	3.5862e-05	0.02420	5.1379e-04	0.406	2.4436e-04
6	4.4815e-06	0.03829	1.5578e-04	0.484	3.0232e-05
8	9.4623e-07	0.05565	6.4612e-05	0.531	9.4123e-05

Table 4. The numerical convergence of order and L_2-errors of the present method and method in [51].

m	Present method	R_m	Method in [51]	R_m
5	1.1599e-05	7.3222	2.0838e-02	–
10	2.7156e-07	6.8232	5.1782e-03	2.0087
20	4.6025e-09	6.6782	1.2844e-03	2.0113
40	6.2273e-11	6.6508	3.1434e-04	2.0307

Table 5. Comparison of the absolute errors from present method and [7].

	Present method		Method in [7]	
x	$m=6$	$m=8$	$m=6$	$m=8$
0.1	5.10e-06	4.25e-06	1.58e-03	1.51e-04
0.3	7.07e-07	2.19e-08	1.10e-03	4.87e-04
0.5	7.93e-06	1.71e-06	1.16e-03	3.68e-04
0.7	1.31e-05	5.02e-07	8.54e-05	4.83e-04
0.9	3.03e-06	1.65e-07	5.10e-04	2.23e-04

Example 7.3. Consider the following fractional order integro-differential equation of order $\alpha = \frac{1}{3}$ [7]

$$ {}_0^cD_t^{\frac{1}{3}} u(t) = f(t) - \frac{32}{35}\sqrt{t}u(t) + \int_0^t (t-s)^{-\frac{1}{2}} u(s)ds, \quad u(0)=0, $$

where

$$ f(t) = \frac{6}{\Gamma(\frac{11}{3})} t^{\frac{8}{3}} + \left(\frac{32}{35} - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{17}{6})} \right) t^{\frac{11}{6}} + \Gamma(\frac{7}{3})t, \quad t \in [0,1]. $$

The exact solution is $u(t) = t^{\frac{4}{3}} + t^3$. In our meticulous analysis, we unequivocally identify a pronounced singularity at the point $t = 0$, rendering polynomial-based strategies woefully inadequate. Utilizing an advanced method we introduce in Section 5, our numerical approach not only addresses the core issue but surpasses the benchmarks set in previous studies. Table 3 compellingly demonstrates our superior performance in terms of L_2-error and computational efficiency (measured in CPU time) across a spectrum of m values, outclassing the results in referenced literature [30, 41]. Table 4 further cements the efficacy of our methodology Notably, we not only achieve the anticipated convergence order of 6.6, but as m escalates, our convergence trajectory consistently approaches its theoretical pinnacle. In addition, the error of the present method is compared with the method presented in [7] in Table 5. Figure 7 provides a clear graphical depiction of our findings, illustrating that the numerical convergence order R_m steadily tends toward the expected theoretical order. Furthermore, Figure 8 presents a decisive comparison of the L_2-error with $R = N^{-6.6}$. The congruence in their patterns robustly validates our proposed theoretical convergence rate.

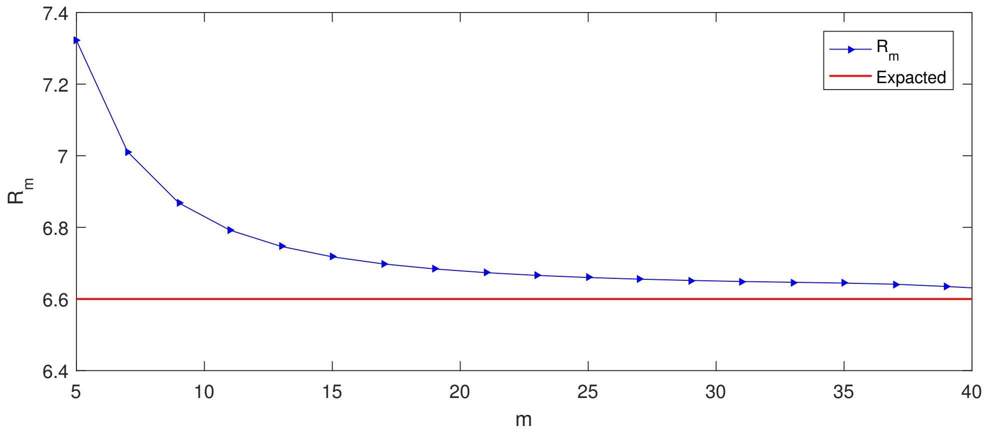

FIGURE 7. The graph of numerical rates R_m under m and our expected rate ($R = 6.6$).

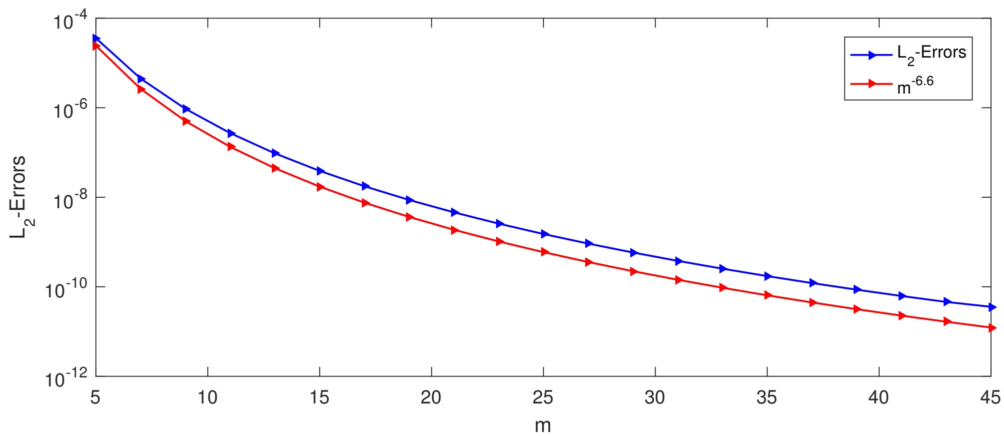

FIGURE 8. The L_2-errors and $m^{-6.6}$ as a function of m.

Example 7.4. Consider the following Volterra fractional integro-differential equations [15]

$$ {}_0^cD_t^\alpha u(t) = f(t) + \int_0^t (t-s)^{-\frac{1}{2}} u(s)ds, $$

with initial condition $u(0) = 0$ and

$$ f(t) = \frac{\Gamma(4.5)}{6} t^3 - \frac{\Gamma(4.5)\Gamma(0.5)}{24} t^4. $$

The exact solution for this problem is $u(t) = t^{3.5}$. The approximate solution is obtained by using proposed collocation method. Also, in [15], a shifted Jacobi-Gauss-collocation algorithm is applied for solving this problem. In Table 6, we compare the maximum absolute errors and computational times obtained by our method and method in [15]. Additionally, Figure 9 depicts the errors $\mid u(t) - u_m(t) \mid$ versus t for $m = 5$. It is evident that the present methods enjoys the exponential order of convergence within a few number of GBFs.

Example 7.5. Consider the following linear system of FDEs [2, 40]

$$ {}_0^cD_t^\alpha u(t) = -u(t) + v(t) - 1, $$

Table 6. The CPU times and maximum absolute errors of the present method and methods in [15].

m	Present method	CPU time (s)	N	Method in [15]	CPU time (s)
2	0.1160	0.0087	4	1.3181e-03	4.877
3	4.7740e-15	0.0130	8	6.1741e-06	9.89
4	4.2188e-15	0.0207	12	3.4376e-07	21.25
5	4.1078e-15	0.0233	16	5.0271e-08	43.889

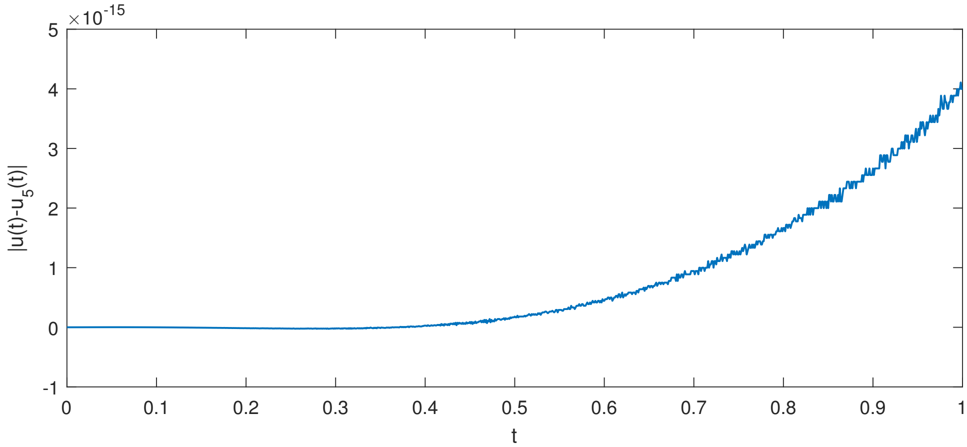

Figure 9. The behaviour of $\mid u(t) - u_5(t) \mid$ versus t.

$$ {}_{t}^{c}D_{1}^{\alpha} v(t) = -u(t) - v(t) - 1, $$

with initial conditions $u(0) = 0$ and $v(1) = 0$ and $\alpha \in (0, 1]$.

For $\alpha = 1$, the exact solution is given by

$$ u(t) = \cosh(\sqrt{2}t) + \rho \sinh(\sqrt{2}t) - 1, $$
$$ v(t) = (1 + \sqrt{2}\rho)\cosh(\sqrt{2}t) + (\sqrt{2} + \rho)\sinh(\sqrt{2}t), $$

where

$$ \rho = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}. $$

It should be mentioned that this system of fractional differential equations is derived from the necessary optimality conditions of a fractional optimal control problem (see [2]).The example is solved using the proposed method. we obtain the numerical solutions $u_{10}(t)$ and $v_{10}(t)$ for several values of α and $n = 10$, the results are depicted in Figures (10)-(11).

Conclusion

In this study, we have introduced and analyzed a novel family of Generalized Bernstein-like basis functions (GBFs) designed to provide an efficient and accurate framework for solving fractional differential equations. By constructing two distinct subclasses of GBFs and establishing their analytical properties including recurrence relations, derivative formulas, and partition of unity generalizations, we have developed a robust mathematical foundation for fractional approximation schemes.

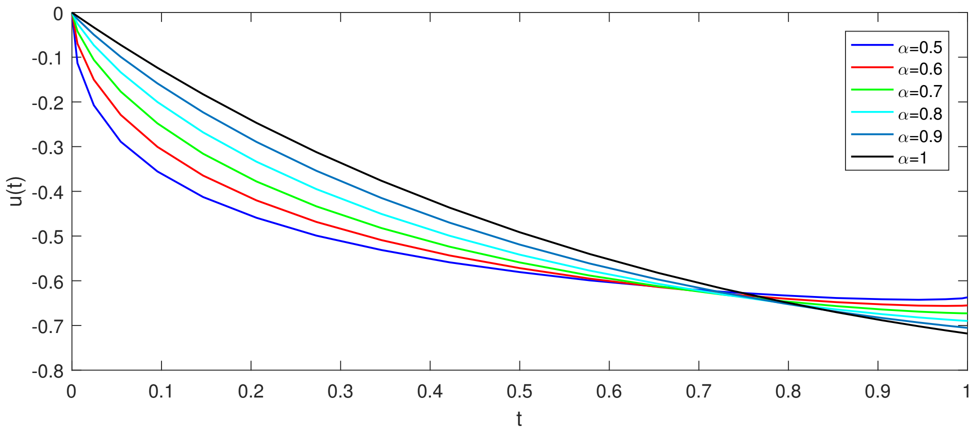

FIGURE 10. Numerical solution of $u(t)$ for some values of α.

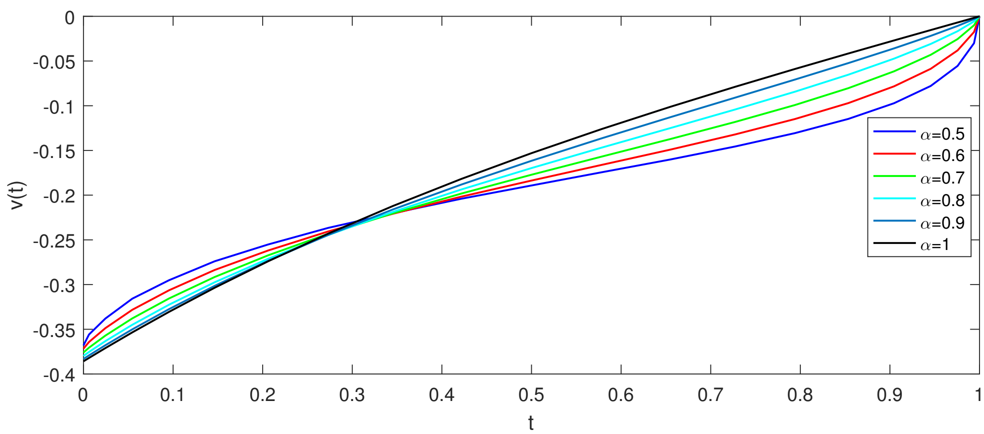

FIGURE 11. Numerical solution of $v(t)$ for some values of α.

Through rigorous analysis and implementation, the proposed GBFs have demonstrated remarkable stability and computational efficiency, particularly when applied to problems exhibiting singular behavior at $t = 0$. The numerical experiments confirm that our approach consistently yields smaller L_2-errors compared to classical polynomial-based techniques, as well as improved convergence profiles across a broad range of fractional orders.

Theoretical convergence analysis, complemented by graphical validations, demonstrates that the proposed method achieves exponential accuracy when the solution admits a representation of the form $t^\alpha P_n(t)$, where $P_n(t)$ is a polynomial of degree at most n. For more general functions, the convergence rate is algebraic. The applicability of the method has been successfully illustrated in several complex models, including multi-order initial value problems, fractional boundary value problems, and integro-differential systems of Volterra-Fredholm type.

Overall, the findings of this research highlight the potential of GBFs as a powerful and flexible basis for fractional calculus. The methodology not only bridges theoretical advancements with practical computational frameworks but also opens new avenues for future investigations in fractional optimal control, hybrid basis function design, and high-dimensional fractional modeling.

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