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Computational approximation of fractional delay differential equations with integral boundary condition

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Abstract

In this paper, we investigate a class of fractional delay differential equations with integral boundary conditions. To solve these problems, a numerical method combining the fractional finite difference L_2 approximation and the upwind difference scheme for the convection term is employed on a uniform mesh. Error estimate is derived to assess the accuracy of the proposed method, and numerical examples are provided to validate its effectiveness.

Keywords. Fractional delay differential equation, Caputo fractional derivative, Integral boundary condition, Finite difference scheme.

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1. Introduction

Systems with memory effects and hereditary features can be described using Fractional Differential Equations (FDEs), which have become versatile tools for modeling phenomena across various scientific and engineering fields. Given the wide ranging applications of FDEs, significant research in recent years has focused on developing effective numerical and analytical techniques for their solution. The adoption of fractional order models in practical applications has gained substantial interest. FDEs are widely used in various fields, including biological and medical modeling, physics, engineering sciences, control systems, and viscoelastic materials [3, 22]. As an example, researchers [10] analyzed a two-dimensional fractional order brain tumor model using the Caputo fractional derivative and the operational matrix of conventional orthonormal Bernoulli polynomials.

Among the most frequently used fractional derivatives, the Caputo derivative is preferred due to its intuitive formulation of initial conditions, whereas the Riemann-Liouville derivative is often utilized in theoretical studies involving fractional systems. The Caputo fractional derivative is defined for $m-1<\beta\leq m$, $m\in N, x>0$ as follows (see [11])

$$^{c}D^{\beta}y(x)=\frac{1}{\Gamma(m-\beta)}\int_0^x (x-s)^{m-1-\beta}y^{(m)}(s)ds,$$

Riemann-Liouville fractional integral is defined as

$$I^{\beta}y(x)=\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}y(s)ds, \quad \beta>0.$$

Applications of integral boundary conditions are important in chemical reactions, population dynamics, hemodynamics, deep-layer water flow, and thermoelasticity. Specifically, while studying fluid flow problems, it is not always justified to assume a "circular cross-section" across the vessels. Integral boundary conditions offer a more realistic

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(workable) solution in this case. The problem of differential equations with integral boundary conditions is well presented, as demonstrated by the authors of [1]. A FDE with nonlocal and generalized Riemann-Liouville fractional integral boundary conditions has been described in [4]. The author of [16] investigates fractional differential equations with two-point boundary conditions. The fractional integral boundary value problem is resolved in [19] by employing a finite difference technique. Recently, Mary et al. [20] proposed an efficient numerical method for the variable coefficient fractional Bagley–Torvik equation with Robin boundary conditions.

Processes whose output is dependent on previous states are modeled using delay. In disciplines including biology, engineering, and economics, delay is an essential component. In [23], the study explores the existence and uniqueness of solutions to Fractional Delay Differential Equations (FDDEs) with non local and integral boundary conditions. FDDEs have been solved by using the operational matrix approach in [13], Chebyshev Pseudospectral Method (CPM) in [21], $L1$-Predictor–Corrector Method ($L1$-PCM) in [2], and the Finite Difference Method (FDM) in [5]. Recent studies have also focused on efficient numerical techniques for fractional delay differential equations and fractional optimal control problems. In [14, 15], stable and accurate computational approaches were developed for fractional systems involving delay effects. Moreover, spline-based and finite difference techniques for fractional models were investigated in [17, 18], where theoretical analysis and numerical experiments demonstrated the effectiveness of the proposed methods. Recently, Elango et al. [8] proposed efficient numerical methods for singularly perturbed delay differential equations exhibiting interior layer behavior and discontinuities due to delay terms.

In engineering applications, a double-pendulum crane system requires a PID controller for swing suppression, accurate positioning, speed control, disturbance rejection, and safety improvement. Fractional order PID controllers are employed for even higher performance. Semiconductor lasers are widely used in optics but are highly sensitive to optical feedback, where even small reflections can cause instabilities like mode hopping or chaos. Pieroux [6] investigates the Hopf bifurcation in a laser system with a large delay, focusing on how optical feedback from a reflected laser beam influences the laser's stability. By building on the work of [6], the author of [5] analyzes the effect of noise on laser light reflected by a mirror using a FDDE,

$$\begin{aligned} D^{\alpha}x(t) =& -\epsilon^{-1}x(t) + \epsilon^{-1}x(t)x(t-1), \quad 0<\alpha\le 1,\ t>0, \\ x(t) =& 0.9, \quad \forall\, t\in[-1,0]. \end{aligned}$$

Inspired by previous studies, our research examines a nonlocal boundary condition, aiming to elucidate its influence on system dynamics and its impact on the solution.

The current research is structured as follows: The problem statement appears in Section 2. The maximum principle, along with the existence and uniqueness results, is discussed in Section 3. Section 4 provides specifics on the numerical method. The convergence results for the numerical solution are shown in Section 5. The numerical results for two examples are finally given in Section 6 and are displayed using tables and graphs.

Notations:

- $C^{n,\mu}(0,2]$ denotes the space of functions $y\in C[0,2]\cap C^n(0,2]$ such that

  $|y^{(r)}(\eta)| \le C(1+\eta^{(1-\mu-r)})$ for $r=1,2\ldots n$ and $\eta\in(0,1]$,

  for all positive integer n and $\mu\in(-\infty,1)$.
- Define the supremum norm by

  $\|\lambda(\zeta)\|_\infty = \sup\limits_{\zeta\in(0,2)} \lambda(\varsigma).$
- A general positive constant, C is unaffected by the parameter N. However, C may vary in different contexts.

2. Statement of the problem

We proceed by examining the following problem

$$\begin{cases} \mathcal{G}v(\eta) = -{}^cD^{\beta}v(\eta) + p(\eta)v'(\eta) + q(\eta)v(\eta) + r(\eta)v(\eta-1) = \tau(\eta), & \eta\in(0,2), 1<\beta\le 2,\\ v(\eta) = \theta(\eta), & \eta\in[-1,0],\\ \mathcal{J}v(2) = v(2) - \int_0^2 \zeta(\eta)v(\eta)\,d\eta = \mathcal{B}. \end{cases} \tag{2.1}$$

where $p(\eta), q(\eta), r(\eta)$, and $\tau(\eta)$ are smooth functions satisfying: $p(\eta) > 0, q(\eta) \geq 0,\ r(\eta) < 0,\ p(\eta)+q(\eta)+r(\eta) > 0,\ q(\eta)+r(\eta) \geq 0,\ \theta(\eta)$ is continuous function on $[-1,0]$, $\zeta(\eta) \geq 0$ is monotonic with $\int_0^2 \zeta(\eta)d\eta < 1$.
The preceding conditions imply that. $v \in X = C^0[0,2] \cap C^1(0,2) \cap C^2((0,1)\cup(1,2))$.

We employ an integral equation approach to reframe the problem (2.1). Let

$$u(\eta) = {}^cD^\beta v(\eta) \tag{2.2}$$

where u is an arbitrary function in $C[0,2]$. The solution of Equation (2.2) has the following form:

$$v(\eta) = c_1 + \eta c_2 + I^\beta u(\eta),$$

where c_1 and c_2 are constants. Applying the boundary conditions (2.1) we get:

$$c_1 = \theta(0),$$

and

$$c_2 = \frac{1}{\sigma}\left\{\mathcal{B} - I^\beta u(2) + \int_0^2 \zeta(\eta) I^\beta u(\eta)\, d\eta - \zeta(0)\left(1 - \int_0^2 \zeta(\eta)\, d\eta\right)\right\},$$

where $\sigma = 2 - \int_0^2 \eta\zeta(\eta)d\eta$.

Thus, the solution of Equation (2.2) is:

$$v(\eta) = \theta(0) + \frac{\eta}{\sigma}\left\{\mathcal{B} - I^\beta u(2) + \int_0^2 \zeta(\eta) I^\beta u(\eta)\, d\eta - \zeta(0)\left(1 - \int_0^2 \zeta(\eta)\, d\eta\right)\right\} + I^\beta u(\eta). \tag{2.3}$$

Suppose that (2.3) is the solution of the problem (2.1). Then, we get:

$$u(\eta) = \mathcal{K}u(\eta) + l(\eta) \tag{2.4}$$

where

$$\mathcal{K}u(\eta) = p(\eta)I^{\beta-1}u(\eta) + q(\eta)I^\beta u(\eta) + r(\eta)I^\beta u(\eta-1) + \frac{1}{\sigma}[p(\eta)+\eta q(\eta)+r(\eta)(\eta-1)]\left(\int_0^2 \zeta(\eta)I^\beta u(\eta)\, d\eta - I^\beta u(2)\right),$$

and

$$l(\eta) = \frac{1}{\sigma}[p(\eta)+\eta q(\eta)+r(\eta)(\eta-1)]\left[\mathcal{B} - \theta(0)\left(1 - \int_0^2 \zeta(\eta)\, d\eta\right)\right] - \tau(\eta) + q(\eta)\theta(0) + r(\eta)\theta(0).$$

Conversely, it is easy to show that if $u(\eta) \in C[0,2]$ is a solution of Equation (2.4), then $v(\eta)$ defined by (2.3) is a solution of the problem (2.1).

Therefore, Equation (2.4) is equivalent to the boundary value problem (2.1).

3. Stability result

Theorem 3.1. (Existence Theorem) *Let us assume that $p(\eta), q(\eta), r(\eta)$, and $\tau(\eta) \in C^{n,\mu}(0,2]$ for some $n \in \mathbb{N}$ and $\mu \in (-\infty, 1)$. Suppose that $v = 0$ is the only one degree polynomial that meets the boundary condition (2.1) with $\mathcal{B} = 0$. Additionally, assume that the BVP (2.1) has only the trivial solution $v = 0$ when $\tau = 0, \mathcal{B} = 0$. Then the problem (2.1) has a unique solution $v(\eta) \in C^1[0,2]$ such that ${}^cD^\beta v \in C^{n,k}(0,2]$, where $k = \max(\mu, 2-\beta)$.*

Proof. Since Equation (2.4) is the equivalent integral form of the problem (2.1), it is enough to prove it for Equation (2.4). Since $p(\eta), q(\eta), r(\eta)$, and $\tau(\eta) \in C^{n,\mu}(0,2]$, we have:

$$l(\eta) \in C^{n,\mu}(0,2] \quad \text{by Brunner and Pedas [9].}$$

From [9], $I^{\beta-i} : C^{n,\mu}(0,2] \to C^{n,\mu}(0,2]$ (where $i = 0,1$) is a linear compact operator. Therefore, $\mathcal{K}u(\eta) \in C^{n,\mu}(0,2]$. Under the Fredholm alternative theorem and the stated assumption, Equation (2.4) possesses a exclusive solution $u(\eta) \in C^{n,\mu}(0,2]$.
According to [7], an equivalent formulation of the problem (2.1) is $\mathcal{G}v(\eta) = \mathcal{T}(\eta)$, where

$$\mathcal{G}v(\eta) = \begin{cases} \mathcal{G}_1 v(\eta) = -{}^cD^\beta v(\eta) + p(\eta)v'(\eta) + q(\eta)v(\eta), & \eta \in (0,1), \\ \mathcal{G}_2 v(\eta) = -{}^cD^\beta v(\eta) + p(\eta)v'(\eta) + q(\eta)v(\eta) + r(\eta)v(\eta-1), & \eta \in (1,2). \end{cases} \tag{3.1}$$

$$\mathcal{T}(\eta) = \begin{cases} \tau(\eta) - r(\eta)\theta(\eta-1), & \eta \in (0,1), \\ \tau(\eta), & \eta \in (1,2). \end{cases} \tag{3.2}$$

with

$$\begin{cases} v(\eta) = \theta(\eta), & \eta \in [-1,0], \\ v(1^-) = v(1^+), & v'(1^-) = v'(1^+), \\ \mathcal{J}v(2) = v(2) - \int_0^2 \zeta(\eta)v(\eta)d\eta = \mathcal{B}. \end{cases} \tag{3.3}$$

□

Theorem 3.2. (Maximum Principle) *Let $\psi(\eta)$ in X satisfies $\psi(0) \geq 0$, $\mathcal{J}\psi(2) \geq 0$, $\mathcal{G}_1\psi(\eta) \geq 0, \forall \eta \in (0,1)$, $\mathcal{G}_2\psi(\eta) \geq 0, \forall \eta \in (1,2)$, and $\psi(1) = 0, \psi'(1) \leq 0$ then $\psi(\eta) \geq 0, \forall \eta \in [0,2]$.*

Proof. Let

$$\psi(\eta^*) = \min_{\eta \in [0,2]} \psi(\eta)$$

We have to prove that $\psi(\eta^*) \geq 0$. Suppose $\psi(\eta^*) < 0$.
Case (i). If $\eta^* = 0$ then $\psi(0) < 0$, which contradicts our presumption.
Case (ii). If $\eta^* = 2$, then

$$\begin{aligned} \mathcal{J}\psi(2) &= \psi(2) - \int_0^2 \zeta(\eta)\psi(\eta)d\eta, \\ &= \psi(2) - \psi(a)\int_0^2 \zeta(\eta)d\eta, \\ &\leq \psi(2)\left(1 - \int_0^2 \zeta(\eta)d\eta\right) < 0, \end{aligned}$$

which is a contradiction to $\mathcal{J}\psi(2) \geq 0$.
Case (iii). If $\eta^* \in (0,1)$, then

$$\mathcal{G}_1\psi(\eta^*) = -{}^cD^\beta\psi(\eta^*) + p(\eta)\psi'(\eta^*) + q(\eta)\psi(\eta^*).$$

By the Corollary 3.2 of [12], ${}^cD^\beta\psi(\eta^*) \geq 0$. Therefore $\mathcal{G}_1\psi(\eta^*) < 0$, which contradicts our presumption.
Case(iv) If $\eta^* \in (1,2)$, then

$$\mathcal{G}_2 v(\eta) = -{}^cD^\beta\psi(\eta^*) + p(\eta)\psi'(\eta^*) + q(\eta)\psi(\eta^*) + r(\eta)\psi(\eta^* - 1) < 0,$$

which is a contradiction.
Therefore $\eta^* \notin [0,1) \cup (1,2]$
Case (v). If $\eta^* = 1, \psi(1) < 0$.
Subcase (i). If $\psi'(1)$ does not exist, then $\psi'(1) \neq 0$ this implies that the one sided derivatives, $\psi'(1^-)$ and $\psi'(1^+)$ are not equal. Since $\psi'(1^-) \leq 0$ and $\psi'(1^+) > 0$, we have $\psi'(1) > 0$, contradicting the assumption.
Subcase (ii). $\psi(1)$ is differentiable.
Since $\psi(1)$ is minimum, $\psi'(1^-) \leq 0$ and $\psi'(1^+) \geq 0$ therefore $\psi'(1^-) = \psi'(1^+) = 0$ then $\psi''(1) \geq 0$. By the corollary 3.2 of [12], $D^\beta\psi(1) \geq 0$.
Since $\psi(1) < 0$ and $\eta = 1$ is a global minimum, the downward concavity of $\psi(\eta)$ would force $D^\beta\psi(1)$ to be negative, which contradicts the assumption that $D^\beta\psi(1) \geq 0$. Therefore, $\eta = 1$ cannot be a strict minimum, and this contradiction completes the proof. □

By Theorem 3.1, the existence of the solution to the BVP (2.1) is established, and the uniqueness of the solution is proved using the maximum principle.

Theorem 3.3. *Let $v(\eta)$ be the solution of (2.1). Then for all $\eta \in [0,2]$,*

$$|v^j(\eta)| \leq \begin{cases} C, & \text{if } j = 0,1, \\ C\eta^{\beta-j}, & \text{if } j = 2,3,\dots,n+1. \end{cases}$$

Proof. Since $v(\eta)$ is the solution of (2.1) by Theorem 3.1, $v(\eta) \in C^1[0,2]$ such that ${}^cD^\beta y \in C^{n,k}(0,2]$, where $k = \max(\mu, 2-\beta)$. Using theorem 3.4 of [16] we get $v(\eta) \in C^{n+1}(0,2]$ such that

$$|v^j(\eta)| \leq \begin{cases} C, & \text{if } j = 0,1, \\ C\eta^{\beta-j}, & \text{if } j = 2,3,\dots,n+1. \end{cases}$$

□

4. Numerical scheme

4.1. **Mesh partitioning.** Let $\{\eta_j\}_{j=0}^{2M}$ denote a uniform discretization of $[0,2]$, with mesh size $h = \frac{1}{M}$, such that $\eta_j = jh$ and $\eta_M = 1$.

4.2. **Finite difference scheme.** The continuous problem (2.1) is discretized as follows:

$$\begin{cases} \mathcal{G}_1^M V_i = -{}^cD_{L2}^\beta V_i + p_i D^- V_i + q_i V_i = \tau_i - r_i\theta_{i-M}, & \text{for } i = 1,2,3,\dots,M-1, \\ \mathcal{G}_2^M V_i = -{}^cD_{L2}^\beta V_i + p_i D^- V_i + q_i V_i + r_i V_{i-M} = \tau_i, & \text{for } i = M+1, M+2,\dots,2M-1, \end{cases} \tag{4.1}$$

with

$$\begin{cases} V_i = \theta_i, & i = -M, -M+1,\dots,0, \\ \mathcal{J}^M V_{2M} = V_{2M} - \sum_{i=1}^{2M} \frac{\zeta_{i-1}V_{i-1} + \zeta_i V_i}{2} h_i, \\ D^- V_M = D^+ V_M, \end{cases} \tag{4.2}$$

where

$${}^cD_{L2}^\beta V(\eta_i) = \frac{h^{-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{i-1} w_{i-j}\left(V_{j+2} - 2V_{j+1} + V_j\right), \quad 1 < \beta \leq 2$$

$$w_{i-j} = (i-j)^{(2-\beta)} - (i-j-1)^{(2-\beta)} \quad \text{and}$$
$$D^+ V_i = \frac{V_{i+1} - V_i}{h}, D^- V_i = \frac{V_i - V_i - 1}{h}.$$

On simplification the problem (4.1)-(4.2) reduces to a system of $2M$ linear difference equations given by $AY = B$, the entries of the matrix $A = [a_{i,p}]$, $B = [b_{i,1}]$ and $Y = [V_i]$ are given by,
for $i \in \{1, 2 \dots 2M-1\} - \{M\}$.

$$a_{i,i} = \frac{p_i}{h} + q_i + S(\delta_{i1}w_2 - 2w_1);\ a_{i,i+1} = Sw_1;\ a_{i,i-1} = S(w_1 - 2w_2 + \delta_{i2}w_3) - \frac{p_i}{h}, \tag{4.3}$$

for $i \in \{3, 4 \dots 2M-1\} - \{M\}$,

$$a_{i,1} = S(w_{i-1} - 2w_i) + \varepsilon_{i,1} r_i, \tag{4.4}$$

for $i \in \{4, 5 \dots 2M-1\} - \{M\}$, $p = 2,3,\dots,i-2$,

$$a_{i,p} = S(w_{i-p} - 2w_{i-p+1} + w_{i-p+2}) + \varepsilon_{i,p} r_i, \tag{4.5}$$

where $S = \frac{-h^{-\beta}}{\Gamma(3-\beta)}$,

$$\delta_{ik} = \begin{cases} 1, & \text{if } i \neq k, \\ 0, & \text{if } i = k. \end{cases} \qquad \varepsilon_{ik} = \begin{cases} 1, & \text{if } i - k = M, \\ 0, & \text{if } i - k \neq M, \end{cases} \tag{4.6}$$

for $i = 1, 2, \dots, 2M-1$, $a_{2M,i} = -h\zeta_i$, $a_{2M,2M} = 1 - \frac{h\zeta_{2M}}{2}$.

$$\begin{aligned}
b_{1,1} &= \tau_1 - r_1\theta_{1-M} - (Sw_1 - \frac{p_1}{h})v_0, \\
b_{i,1} &= \tau_i - r_i\theta_{i-M} - Sw_i v_0, && \text{for } i = 2,3,\dots,M-1, \\
b_{i,1} &= \tau_i - Sw_i v_0, && \text{for } i = M+1, M+2, \dots, 2M-1,
\end{aligned}$$

Using [16] and our assumption, we get from (4.3), $a_{i,i} > 0$, $a_{i,i+1} < 0$ for $i \in \{1, 2 \dots 2M-1\}$ and
$a_{i,i-1} < 0$ for $i \in \{3, 4, 5 \dots, 2M-1\}$.
$a_{i,1} > 0$ for $i \in \{3, 4, 5 \dots, 2M-1\} - \{M+1\}$, from (4.4),
$a_{i,p} < 0$ for $i \in \{4, 5 \dots, 2M-1\} - \{M\}$, $\quad p = 2, 3 \dots, i-2$, from (4.5),
$a_{2M,i} < 0$ for $i \in \{1, 2, \dots, 2M-1\}$ and $a_{2M,2M} < 1$, from (4.6).

4.3. **Discrete maximum principle.**

Theorem 4.1. *Assume that $\sum_{i=1}^{2M} \frac{\zeta_{i-1}+\zeta_i}{2} h_i < 1$ and mesh function ω_i with $\omega_0 \geq 0$ and $\mathcal{J}^M \omega_{2M} \geq 0$, $\mathcal{G}_1^M \omega_i \geq 0$ for all $i = 1, 2, \dots, M-1$, $\mathcal{G}_2^M \omega_i \geq 0$ for all $i = M+1, M+2, \dots, 2M-1$ and $D^+(\omega_M) - D^-(\omega_M) \leq 0$ then $\omega_i \geq 0$ for all $i = 0, 1, 2, \dots, 2M-1, 2M$.*

Proof. Let

$$\omega(\eta_{j^*}) = \min_{0 \leq j \leq 2M} \omega(\eta_j).$$

Assume, for contradiction, that $\omega(\eta_{j^*}) < 0$. Since $\omega(\eta_0) \geq 0$ and $\mathcal{J}^M \omega(\eta_{2M}) \geq 0$, the minimum cannot occur at the boundary. Hence, $j^* \in \{1, 2, \dots, 2M-1\}$. At the minimum point,

$$\omega(\eta_{j^*}) - \omega(\eta_{j^*-1}) \leq 0, \qquad \omega(\eta_{j^*+1}) - \omega(\eta_{j^*}) \geq 0.$$

Further, the $L2$ approximation of the Caputo derivative is a positive-weight convolution operator. Therefore, by the discrete extremum property of the $L2$ scheme,

$${}^cD_{L2}^{\beta}\omega(\eta_{j^*}) \geq 0, \implies -{}^cD_{L2}^{\beta}\omega(\eta_{j^*}) \leq 0.$$

Also,

$$D^-\omega(\eta_{j^*}) \leq 0.$$

Hence,

$$\mathcal{G}_1^M \omega(\eta_{j^*}) \leq 0, \qquad \mathcal{G}_2^M \omega(\eta_{j^*}) \leq 0.$$

This contradicts the assumptions of the theorem. Therefore,

$$\omega(\eta_j) \geq 0, \qquad j = 0, 1, \dots, 2M.$$

□

5. Error analysis

The nodal error corresponding to indices $i = 1, 2, \dots, 2M$ is given by $e(\eta_i) = v(\eta_i) - V(\eta_i)$ where $V(\eta_i)$ be a numerical solution of (3.1)-(3.3) defined by (4.1)-(4.2).

Theorem 5.1. *If $v(\eta_i)$ and $V(\eta_i)$ are the solution of the problem (3.1)-(3.3) and (4.1)-(4.2) then*

$$|v(\eta_i) - V(\eta_i)| \leq \begin{cases} C, & \text{if } i = 1, \\ C(i-1)^{-(\beta-1)}, & \text{if } i \in \{2, 3 \dots, M-1\}, \\ CM^{-1}, & \text{if } i = M, \\ CM^{-(\beta-1)}, & \text{if } i \in \{M+1, M+2, \dots, 2M\}. \end{cases} \tag{5.1}$$

Proof. For $i = 2M$,

$$\begin{aligned}
|e_{2M}| &= |\mathcal{J}^M v(\eta_{2M}) - \mathcal{J}^M V(\eta_{2M})|,\\
&= \left| v(\eta_{2M}) - \sum_{i=1}^{2M} \frac{\zeta_{i-1}v_{i-1} + \zeta_i v_i}{2} h_i - \int_{\eta_0}^{\eta_{2M}} \zeta(\eta)v(\eta)d\eta \right|,\\
&\leq Ch^3 \sum_{i=1}^{2M} |v''(\eta_i)|,\\
&\leq Ch^3 M,\\
&\leq CM^{-2},\\
&\leq CM^{-(\beta-1)}.
\end{aligned}$$

For $1 \leq i < M-1$, from [16], we get

$$|e_i| = \begin{cases} C, & \text{for} \quad i = 1,\\ C(i-1)^{-(\beta-1)}, & \text{for} \quad i = 2, 3, \dots, M-1. \end{cases}$$

For $i = M$, Using the regularity assumption on the exact solution $|v''(\eta)| \leq C\eta^{\beta-2}$,

$$\begin{aligned}
|e_M| &= |(D^+ - D^-)v(\eta_M)|,\\
&= h|v''(\eta_M)|,\\
&\leq hC(\eta_M)^{\beta-2},\\
&\leq CM^{-1}.
\end{aligned}$$

For $M+1 \leq i \leq 2M-1$,

$$e_i = \sum_{j=0}^{i-1} \Upsilon_{i,j} + R_i + d_i,$$

where

$$\begin{aligned}
\Upsilon_{i,j} =& \frac{1}{\Gamma(2-\beta)} \int_{\eta_j}^{\eta_{j+1}} (\eta_i - s)^{1-\beta} v''(s)ds - \frac{h^{-\beta}}{\Gamma(3-\beta)} w_{i-j} \left(v_{j+2} - 2v_{j+1} + v_j\right),\\
R_i =& p_i \frac{v(\eta_i) - v(\eta_{i-1})}{h} - p(\eta_i)v'(\eta_i),\\
d_i =& r_i v_{i-M} - r(\eta_i)v(\eta_{i-M}).
\end{aligned}$$

Using Taylor expansion,

$$\begin{aligned}
|R_i| &\leq |p_i| \frac{h}{2} |v''(\eta_i)|,\\
&\leq Ch(\eta_i)^{\beta-2},\\
&= Ch(ih)^{\beta-2}.\\
|R_i| &\leq CM^{-(\beta-1)} i^{\beta-2}.
\end{aligned}$$

As $1 < \beta < 2$, we have $\beta - 2 < 0$, and hence the function $x^{\beta-2}$ is decreasing for $x > 0$. Therefore,

$$i^{\beta-2} \leq (i-1)^{\beta-2}.$$

Consequently,

$$|R_i| \leq CM^{-(\beta-1)}(i-1)^{\beta-2}.$$

By using Taylor series expansion for $v(s)$ around ξ_j, one can write

$$v''(s) = v''(\xi_j), \qquad \xi_j \in (\eta_j, \eta_{j+1}),$$

and hence

$$\begin{aligned}
\Upsilon_{i,j} =& \frac{v''(\xi_j)}{\Gamma(2-\beta)} \int_{\eta_j}^{\eta_{j+1}} (\eta_i - s)^{1-\beta} ds - \frac{h^{-\beta}}{\Gamma(3-\beta)} w_{i-j} \left(v_{j+2} - 2v_{j+1} + v_j\right), \\
=& \frac{v''(\xi_j)}{\Gamma(3-\beta)} h^{2-\beta} \left[(i-j)^{2-\beta} - (i-j-1)^{2-\beta}\right] - \frac{h^{-\beta}}{\Gamma(3-\beta)} w_{i-j} \left(v_{j+2} - 2v_{j+1} + v_j\right), \\
=& \frac{h^{2-\beta}}{\Gamma(3-\beta)} w_{i-j} \left[v''(\xi_j) - v''(\xi_{j+1})\right] \text{ for some } \xi_{j+1} \in (\eta_j, \eta_{j+2}). \\
\leq& \frac{h^{3-\beta}}{\Gamma(3-\beta)} w_{i-j} (jh)^{\beta-3}
\end{aligned}$$

Hence,

$$|\Upsilon_{i,j}| \leq C w_{i-j} j^{\beta-3}.$$

By the mean value theorem and Lemma 4.5 of [16],

$$\left|\sum_{j=1}^{i-1} \Upsilon_{i,j}\right| \leq C i^{1-\beta}.$$

Also, from [16],

$$|\Upsilon_{i,0}| \leq C(i-1)^{1-\beta}.$$

Since $\eta_{j-1} \leq \eta_{i-M} < \eta_j, \quad i = M+1, \ldots, 2M-1$, using Taylor expansion,

$$\begin{aligned}
|d_i| =& |r_i||v(\eta_{i-M}) - v(\eta_j)|, \\
\leq& C(\eta_j - \eta_{i-M})^2 |v''(\xi_j)|, \\
\leq& Ch^2 |v''(\eta_j)|, \\
\leq& Ch^2 (\eta_j)^{\beta-2}, \\
\leq& Ch^2 (hj)^{\beta-2}, \quad \text{Since } h = M^{-1}, \\
\leq& CM^{-\beta} j^{\beta-2}.
\end{aligned}$$

Again, since $\beta - 2 < 0$, $j^{\beta-2} \leq (j-1)^{\beta-2}$. Therefore,

$$|d_i| \leq CM^{-\beta}(j-1)^{\beta-2}.$$

Combining all the above estimates,

$$|e_i| \leq Ci^{1-\beta} + C(i-1)^{1-\beta} + CM^{-(\beta-1)}(i-1)^{\beta-2} + CM^{-\beta}(i-1)^{\beta-2}.$$

Since $i \geq M+1$, and $-1 \leq 1-\beta < 0$, we obtain $i^{1-\beta} \leq (i-1)^{1-\beta} \leq M^{1-\beta}$.

$$|e_i| \leq CM^{1-\beta} + CM^{1-\beta} + CM^{-(\beta-1)}(i-1)^{\beta-2} + CM^{-\beta}(i-1)^{\beta-2}.$$

Since $M^{-\beta} \leq M^{-(\beta-1)}$ for $M \geq 1$, and $-1 < \beta - 2 \leq 0$, $[(i-1)^{\beta-2} \leq 1]$ it follows that

$$|e_i| \leq CM^{-(\beta-1)}.$$

□

Remark 5.2. The estimates obtained in Theorem 5.1 show that the weak singular behaviour of the exact solution near the initial point $\eta = 0$ prevents uniform decay of the error in the region $1 \le i \le M-1$. In particular, for $2 \le i \le M-1$, the estimate

$$|e_i| \le C(i-1)^{1-\beta},$$

reflects the local singular structure of the solution and does not yield convergence with respect to the mesh parameter M near the initial point. However, away from the singular region, namely for $M+1 \le i \le 2M$, the proposed numerical method satisfies

$$|e_i| \le CM^{-(\beta-1)}.$$

Therefore, the scheme achieves the expected fractional-order accuracy outside the boundary-singular region.

6. Numerical experiments

The precise solution of the test cases remains elusive. Thus, the double mesh approach is employed to gauge the deviation and derive the empirical rate of convergence of the obtained solution. In this regard, the discrepancy is characterized as:

$$D_\beta^M = \max_{0\le i\le 2M} |V_i^M - V_i^{2M}|.$$

Here, V_i^M and V_i^{2M} denote the i^{th} elements of the numerical approximations on grids with M and $2M$ divisions, respectively. The consistent deviation and the observed convergence rate are determined as:

$$P_\beta^M = log_2\left(\frac{D_\beta^M}{D_\beta^{2M}}\right).$$

Example 6.1.

$$-{}^cD^\beta v(\eta) + \eta v'(\eta) + (1+\eta)\,v(\eta) - v(\eta-1) = \tau(\eta), \quad \eta \in (0,1)\cup(1,2),$$
$$v(\eta) = 0, \quad \eta \in [-1,0], \quad v(2) - \int_0^2 \frac{\eta}{3} v(\eta)\,d\eta = \mathcal{B}.$$

If the function $\tau(\eta)$ and the constant $\mathcal{B}$ are selected so that $v_e(\eta) = \eta^6 - \eta^{\beta+5}$ is the exact solution. The maximum pointwise error between the numerical solution $V(\eta_i)$ and the actual solution is measured as

$$D_\beta^M = \max_{0\le i\le 2M} |v_e(\eta_i) - V_M(\eta_i)|.$$

Example 6.2.

$$-{}^cD^\beta v(\eta) + \left(\frac{\eta^2}{2}\right) v'(\eta) + (2+\eta)\,v(\eta) - v(\eta-1) = \frac{\eta^2}{2}, \quad \eta \in (0,1)\cup(1,2),$$
$$v(\eta) = 1, \quad \eta \in [-1,0], \quad v(2) - \int_0^2 \frac{\eta}{3} v(\eta)\,d\eta = 2.$$

For Examples 6.1 and 6.2 it is readily verified that the associated homogeneous problems corresponding to $\tau(\eta) = 0$ and $\mathcal{B} = 0$ admit only the trivial solution. Hence, the assumptions of Theorem 3.1 are satisfied for both test problems.

Remark 6.3. The theoretical convergence estimate established in Section 5 provides a conservative lower-bound rate under weak regularity assumptions on the exact solution. In the numerical experiments considered here, the exact solutions possess higher effective smoothness than the minimal regularity required in the analysis. Consequently, the observed convergence rates are noticeably better than the theoretical prediction, particularly for values of β close to 1.

TABLE 1. Maximum pointwise error D_β^M and their order of convergence P_β^M of Example 6.1.

β/M	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
1.1	1.8909e-01	1.0378e-01	5.5356e-02	2.8905e-02	1.4876e-02	7.5800e-03	3.8368e-03
	8.6554e-01	9.0671e-01	9.3741e-01	9.5837e-01	9.7269e-01	9.8231e-01	
1.2	3.6314e-01	1.9798e-01	1.0490e-01	5.4423e-02	2.7817e-02	1.4082e-02	7.0880e-03
	8.7514e-01	9.1641e-01	9.4668e-01	9.6824e-01	9.8213e-01	9.9038e-01	
1.3	5.0398e-01	2.6913e-01	1.4001e-01	7.1611e-02	3.6253e-02	1.8246e-02	9.1548e-03
	9.0509e-01	9.4273e-01	9.6729e-01	9.8211e-01	9.9048e-01	9.9500e-01	
1.4	6.1871e-01	3.2586e-01	1.6793e-01	8.5417e-02	4.3117e-02	2.1673e-02	1.0869e-02
	9.2501e-01	9.5637e-01	9.7528e-01	9.8627e-01	9.9237e-01	9.9573e-01	
1.5	7.1456e-01	3.7388e-01	1.9215e-01	9.7665e-02	4.9319e-02	2.4810e-02	1.2452e-02
	9.3449e-01	9.6035e-01	9.7630e-01	9.8571e-01	9.9123e-01	9.9450e-01	
1.6	7.9024e-01	4.1320e-01	2.1268e-01	1.0839e-01	5.4895e-02	2.7691e-02	1.3932e-02
	9.3545e-01	9.5813e-01	9.7247e-01	9.8148e-01	9.8724e-01	9.9100e-01	
1.7	8.3651e-01	4.3861e-01	2.2687e-01	1.1628e-01	5.9224e-02	3.0033e-02	1.5182e-02
	9.3144e-01	9.5106e-01	9.6430e-01	9.7332e-01	9.7963e-01	9.8417e-01	
1.8	8.3141e-01	4.3768e-01	2.2791e-01	1.1773e-01	6.0457e-02	3.0906e-02	1.5744e-02
	9.2568e-01	9.4140e-01	9.5299e-01	9.6152e-01	9.6802e-01	9.7311e-01	
1.9	7.3375e-01	3.8552e-01	2.0125e-01	1.0450e-01	5.4035e-02	2.7842e-02	1.4304e-02
	9.2849e-01	9.3784e-01	9.4546e-01	9.5155e-01	9.5660e-01	9.6086e-01	

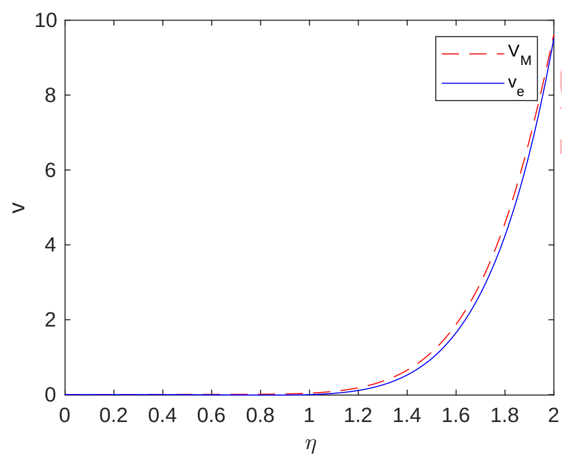

FIGURE 1. Exact and numerical solution of Example 6.1 for $\beta = 1.2$ and $M = 2^6$.

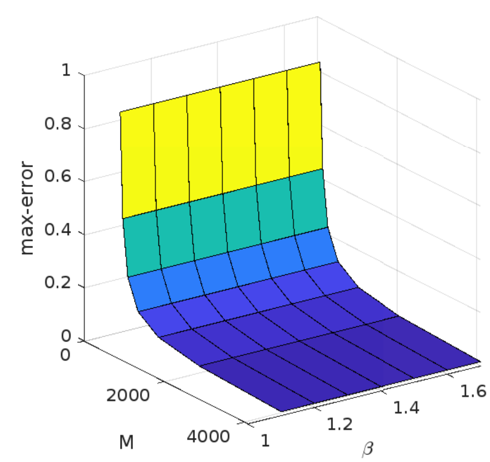

FIGURE 2. Error plot for Example 6.1.

TABLE 2. Maximum pointwise error D_β^M and their order of convergence P_β^M of Example 6.2.

β/M	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
1.1	3.2653e-01	3.1773e-01	3.0472e-01	2.8519e-01	2.6219e-01	2.3272e-01	1.9479e-01
	3.9412e-02	6.0312e-02	9.5563e-02	1.2129e-01	1.7204e-01	2.5668e-01	
1.2	1.8519e-01	1.4596e-01	1.0521e-01	6.8677e-02	4.0978e-02	2.2790e-02	1.2095e-02
	3.4344e-01	4.7229e-01	6.1541e-01	7.4498e-01	8.4647e-01	9.1394e-01	
1.3	9.8063e-02	6.3013e-02	3.7179e-02	2.0551e-02	1.0880e-02	5.6128e-03	2.8537e-03
	6.3806e-01	7.6117e-01	8.5523e-01	9.1751e-01	9.5493e-01	9.7589e-01	
1.4	5.7479e-02	3.3491e-02	1.8391e-02	9.7151e-03	5.0121e-03	2.5507e-03	1.2882e-03
	7.7926e-01	8.6474e-01	9.2072e-01	9.5483e-01	9.7450e-01	9.8555e-01	
1.5	3.7331e-02	2.0797e-02	1.1120e-02	5.7929e-03	2.9697e-03	1.5079e-03	7.6129e-04
	8.4400e-01	9.0318e-01	9.4084e-01	9.6398e-01	9.7778e-01	9.8603e-01	
1.6	2.5907e-02	1.4132e-02	7.4853e-03	3.8882e-03	1.9946e-03	1.0150e-03	5.1379e-04
	8.7442e-01	9.1679e-01	9.4497e-01	9.6303e-01	9.7462e-01	9.8219e-01	
1.7	1.8569e-02	1.0029e-02	5.3002e-03	2.7596e-03	1.4219e-03	7.2735e-04	3.7012e-04
	8.8875e-01	9.2000e-01	9.4159e-01	9.5662e-01	9.6712e-01	9.7464e-01	
1.8	1.3297e-02	7.1243e-03	3.7642e-03	1.9676e-03	1.0200e-03	5.2551e-04	2.6940e-04
	9.0023e-01	9.2041e-01	9.3594e-01	9.4778e-01	9.5686e-01	9.6395e-01	
1.9	9.0433e-03	4.7694e-03	2.5018e-03	1.3057e-03	6.7831e-04	3.5100e-04	1.8101e-04
	9.2303e-01	9.3086e-01	9.3818e-01	9.4477e-01	9.5047e-01	9.5540e-01	

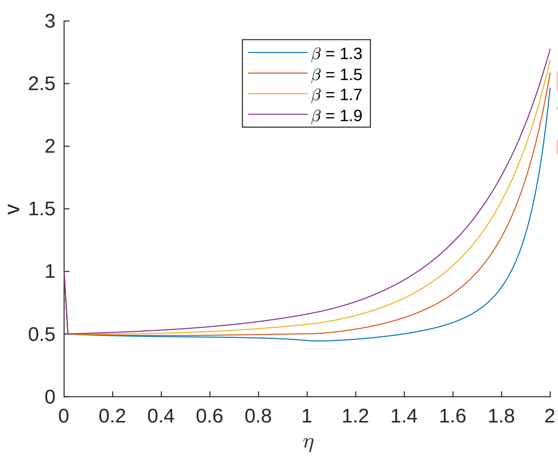

FIGURE 3. Numerical solution graph for Example 6.2 for various values of β and $M = 2^6$

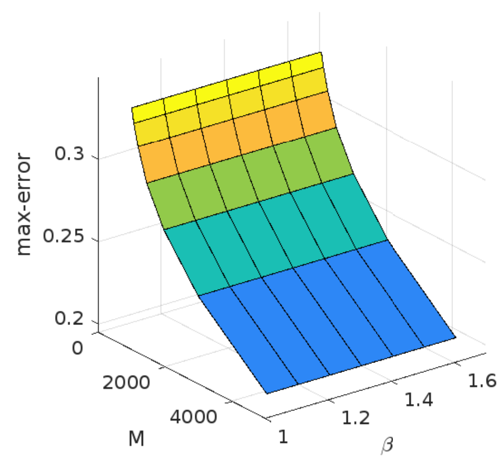

FIGURE 4. Error plot for Example 6.2.

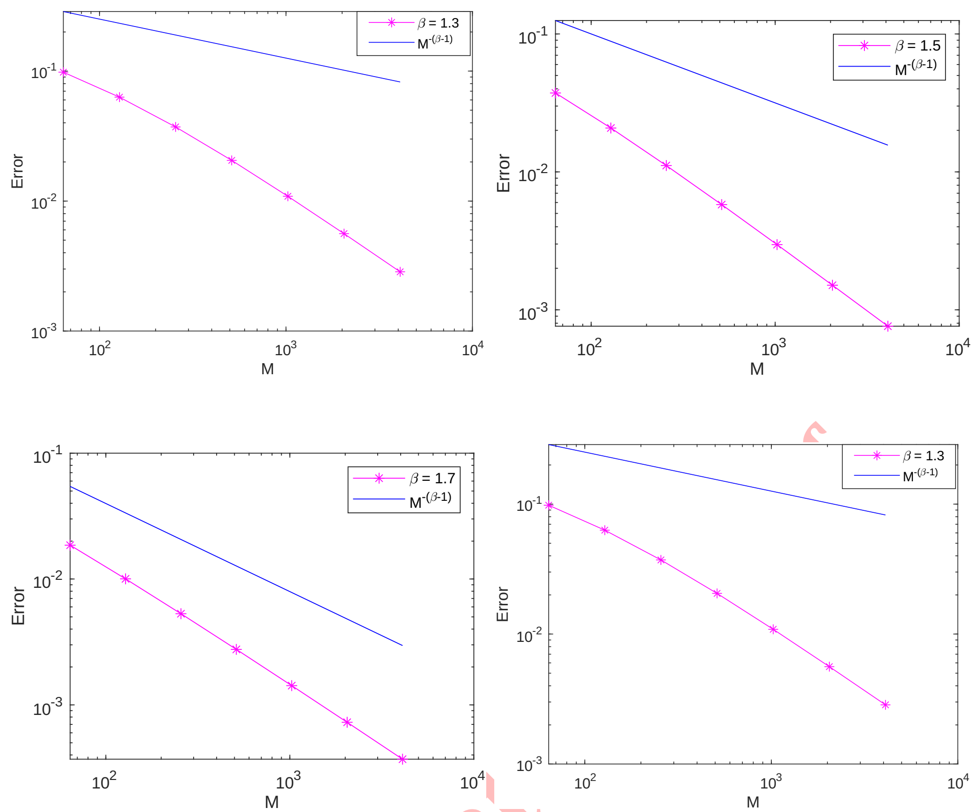

FIGURE 5. Log-log plots for Example 6.2 with varying β.

7. CONCLUSION

The fractional delay differential of order $1 < \beta \leq 2$ with integral boundary condition is examined in this work. Our numerical approach consists of a L_2 approximation on a uniform mesh and an upwind difference strategy. It is demonstrated that the method is of order $O(M^{-(\beta-1)})$. To highlight the numerical method, two examples are given. Figures 1 and 3 respectively, exhibit the numerical solutions for Examples 6.1 and 6.2. In Tables 1 and 2, the order of convergence and maximum pointwise errors are displayed. As M increases, the errors diminish, as seen in Figures 2 and 4. Figure 5 present a log-log plot comparing the theoretical outcome with the maximum pointwise error for different values of β for Example 6.2. Higher order difference methods for systems of fractional differential boundary value problems will be covered in the future.
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Declaration of competing interest

The authors declare that they have no competing interests concerning the publication of the manuscript.

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