Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-16
DOI:10.22034/cmde.2026.68559.3324

Periodic conditions to the problem contain $\Psi-$Hilfer nonlinear integro-fractional differential equation

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Abstract

In this paper, we discuss the ψ-Hilfer fractional differential equation of type $\mathfrak{B}$ belonging to $[0,1]$ and order $1<\alpha<2$ involving the $\Psi-$Riemann-Liouville fractional integral of order $0<\mathfrak{F}<1$ with periodic boundary conditions. The existence and uniqueness of the solution are proven using Leray-Schauder's and Banach fixed-point theorems. We also analyze the stability of our solutions; Ulam-Hyers and Ulam-Hyers-Rassias theorems are present. The problem with periodic integral boundary conditions is crucial because they provide rich mathematical challenges and help us better understand and optimize systems that exhibit periodic behavior with global constraints. For instance, in optimization, integral constraints represent total resources, such as energy or cost, while periodic integral boundary conditions help optimize cyclical systems, like minimizing energy over repeating cycles. In control theory, they model systems where output must meet total requirements (e.g., fuel consumption) over a period, with periodic conditions capturing the system's repetitive nature. Finally, an illustrative example is provided to elucidate the theorems presented in this article.

Keywords. $\Psi-$Hilfer fractional derivative, Existence, Uniqueness, Stability, Nonlinear theorem.

2010 Mathematics Subject Classification. 34K37, 26A33, 34A12, 45M10.

1. Introduction

Fractional differential equations (FDEs) are useful tools for modeling and studying natural phenomena. It can be used to describe ion channel relaxation, cellular behavior, and the dynamics of biological networks using fractional relaxation equations in biophysics [4]. In general, fractional derivatives can be defined as fractional integral operators whose kernel functions are different. There are many fractional derivatives, for example, Riemann-Liouville, Caputo, Hadamard, Katugampola, and Hilfer. As well as along with the Caputo and Riemann-Liouville fractional derivative (RLFD) operators, the Hilfer fractional derivative (HFD) operator is comprised of [15]. HFD equations are generalizations of classical FDEs, which define fractional derivatives with HFD operators. Compared to RLFD, an improved kernel is used in this operator. The concept of HFD equations was developed from exploring and investigating chaos in the physical world and new physical phenomena [20].

Kilbas et al. [13] introduced fractional operators of functions relating to each other. An additional fractional operator, the Caputo operator, was introduced by Almeida [2]. There are several applications of the operator in the papers [1, 23]. A fractional differentiable operator could be proposed that will unify the above operators and will be able to overcome the wide number of definitions while proving that solutions exist and are unique. Thus, the solution consists of working with general fractional operators. For instance, $\Psi-$Hilfer fractional operators and RLF operators concerning another function (integration) [11, 19, 21, 22]. Consequently, the fractional differential operator, or Hilfer derivative, of a function on another function was defined by Vanterler and Capelas de Oliveira [20]:

$$ {}^{H}D_{a+}^{\alpha,\mathfrak{B};\psi} f(x) = I_{a+}^{\mathfrak{B}(\kappa-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{\kappa} I_{a+}^{(1-\mathfrak{B})(\kappa-\alpha);\psi} f(x), \tag{1.1}$$

Received: 09 August 2025; Accepted: 09 August 2026.
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where $\kappa - 1 < \alpha < \kappa, \kappa = [\alpha] + 1$for $\alpha \notin N$ and $\kappa = \alpha$, for $\alpha \in N$.

Furthermore, recent research has been conducted on FDEs to determine whether Ulam-Hyers (UH) stable solutions exist. Guo et al. [10] examined the infinite delay between one and two for the existence and stability under UH conditions of the problem consisting of RLF neutral functional stochastic differential equations. Wang et al. in [24] presented the non-instantaneous differential inclusions containing Caputo derivatives to study the generalized UH stability. More recently, Vanterler et al. [18] investigated UH and Ulam-Hyers-Rassias (UHR) stabilities of the solution of the delay FDEs containing Hilfer derivatives, then developed by Benchohra et al. [3], established that implicit FDEs with RLFDs can have solutions in finite dimensions and are stable, and Kumar et al. [14] studied. An integral impulse with a noninstantaneous fractional differential equation is stable and exists. Very recently, Elsayed et al. [7] determined whether the differential equations with Hilfer-Katugampola fractional derivatives have boundary value concerns and if they are stable. For more types of stabilized boundary value problems with HFD, see [3, 12, 23].

In [5], it was Bouriah et al.'s approach to such nonlinear pantograph fractional equations with $\Psi-$CFDs:

$$\begin{cases} {}^{c}D^{\alpha;\psi}\varphi(v) = \psi(\vartheta, \varphi(\vartheta), \varphi(\varepsilon\vartheta)), \vartheta \in E := [0, \theta], \\ \varphi(0) = \varphi(\theta), \end{cases} \tag{1.2}$$

where ${}^{c}D^{\alpha;\psi}$ denotes the $\psi-$CFDs with order $0 < \alpha < 1, \varepsilon \in (0, 1)$, and continuous function $\psi : E \times R \times R \to R$.

According to the $\psi-$RLFI, the authors of [8] examined some recent findings for the $\psi-$Hilfer fractional pantograph-type differential equation.

In [6], Chohri et al. investigated some existing results for exponential fractional derivative solutions to nonlinear periodic problems

$$\begin{cases} {}^{c}D^{\mathfrak{F}}_{k_1+}\varphi(\vartheta) = g\left(\vartheta, \varphi(\vartheta), {}^{c}D^{\mu}_{k_1+}\varphi(\vartheta)\right), \ \vartheta \in E := [k_1, k_2], \\ \varphi(k_1) = \varphi(k_2) \ \ and \ {}^{c}D^{1}_{k_1+}\varphi(k_1) = {}^{c}D^{1}_{k_1+}\varphi(k_2), \end{cases} \tag{1.3}$$

where $0 < \mu \leq 1$, $g : E \times R \times R \to R$ is a continuous function, and ${}^{c}D^{\mathfrak{F}}_{k_1+} and {}^{c}D^{\mu}_{k_1+}$are the Caputo exponential fractional derivatives of order $1 < \mathfrak{F} \leq 2$.

In [17], Vanterler da et al. initiated the study on UHRS using the ψ-Hilfer operator as the follows problem:

$$\begin{cases} {}^{H}D^{\alpha,\mathfrak{B};\psi}_{a+}y(v) = f\left(v, y(v), {}^{H}D^{\alpha,\mathfrak{B};\psi}_{k_1+}y(v)\right), \\ \qquad\qquad I^{1-\gamma;\psi}_{a+}y(a) = y_a, \end{cases} \tag{1.4}$$

where ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{a+}(.)$ is the ψ-HFD [20] and $I^{1-\gamma;\psi}_{a+}(.)$ is the RLFI of order $1-\gamma, \gamma = \alpha + \mathfrak{B}(1-\alpha)$, concerning function $f\left(v, y(v), {}^{H}D^{\alpha,\mathfrak{B};\psi}_{k_1+}y(v)\right)$.

In this paper, we address the Ulam stability and existence of a solution for the $\Psi-$Hilfer fractional differential equations problem, which is inspired by the previously mentioned papers.

$$\begin{cases} {}^{H}D^{\alpha,\mathfrak{B};\psi}_{0+}x(v) = P\left(v, x(v), I^{\mathfrak{F};\psi}_{0+}x(v)\right), & v \in J = [0, \mathfrak{T}], \\ x(0+) = x(\mathfrak{T}), \ \text{and} \ I^{2-\eta;\psi}_{0+}x(0+) = I^{2-\eta;\psi}_{0+}x(\mathfrak{T}), \end{cases} \tag{1.5}$$

where ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{a+}(.)$ is the ψ- HFD [17] of order $1 < \alpha < 2$, $0 < \mathfrak{F} < 1$ and type $0 \leq \mathfrak{B} \leq 1$, and $P\left(., x(.), I^{\mathfrak{F};\psi}x(.)\right) \in C(J \times R \times R)$ and $I^{2-\eta;\psi}_{a+}(.)$ is the ψ-RLFI of order $2-\eta$, $\eta = \alpha + \mathfrak{B}(2-\alpha)$. The novelty of this work lies in the analysis of the problem by Eq. (1.5) under periodic integral boundary conditions, which are not only mathematically challenging but also highly relevant to real-world applications. Such conditions naturally arise in various fields, including physics, engineering, biology, and control theory, where systems often exhibit cyclical or repetitive behavior subject to global constraints. Unlike classical boundary conditions, periodic integral constraints capture total quantities such as energy, cost, or fuel consumption over a full cycle, making them particularly suitable for modeling and optimizing systems with repeating dynamics. For example, in optimization problems, they represent global resource limits, while in control theory, they enforce periodic performance requirements across cycles. By addressing such boundary conditions, this study contributes to a deeper understanding of the interplay between periodicity and integral constraints, which has been relatively unexplored in existing literature [16].

The paper is organized as follows: Section 2 introduces the necessary elements. Section 3 presents existence and uniqueness results for the problem by Eq. (1.5), based on Leray-Schauder's and Banach's fixed point theorems, and Section 4 studies the Ulam-Hyers and Ulam-Rassias stability for the problem by Eq. (1.5). Finally, Section 5 offers some illustrative examples that demonstrate the effectiveness of the major results. At the end, we will discuss the conclusion in section 6.

2. Preliminaries

Let $J = [0, \mathcal{T}]$. We consider the space $C(J)$ of all continuous functions on J, where $\psi \in C^2(J)$ is an increasing function satisfying $\psi'(v) \neq 0$ for all $v \in J$. This space is equipped with the norm $\|x\|_C := \max_{v\in J} |x(v)|$. Therefore, we define the weighted function $\tilde{\psi}_{\mathcal{T}}(v) = \psi(\mathcal{T}) - \psi(v), \quad v \in J$. For $\eta = \alpha + \mathcal{B}(2-\alpha) \in (1, 2)$, the weighted space $C_{2-\eta;\psi}(J)$ consists of all functions $x : [0, \mathcal{T}] \to \mathbb{R}$ such that $\left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta} x(v) \in C(J)$, equipped with the norm

$$\|x\|_{C_{2-\eta;\psi}} := \left\| \left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta} x(v) \right\|_C = \max_{v\in J} \left| \left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta} x(v) \right|. \tag{2.1}$$

The space $(C_{2-\eta;\psi}(J), \|\cdot\|_{C_{2-\eta;\psi}})$ is a Banach space. Note that the weight $\left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta}$ cancels the singularity at $v = \mathcal{T}$ arising from the left-sided boundary condition. Also given that $1 < \alpha < 2$ and $\eta = \alpha + \mathcal{B}(2-\alpha) \in (1, 2)$, we set:

$$\mathcal{X} = \left\{ x \in C_{2-\eta;\psi}(J) : {}^H D_{0+}^{\alpha,\mathcal{B};\psi} x \in C(J),\ I_{0+}^{F;\psi} x \in C(J) \right\}, \tag{2.2}$$

furnished with the norm $\|x\|_{\mathcal{X}} = \|x\|_{C_{2-\eta;\psi}} + \|{}^H D_{0+}^{\alpha,\mathcal{B};\psi} x\|_C + \|I_{0+}^{F;\psi} x\|_C$. Explicitly, this norm is given by:

$$\|x\|_{\mathcal{X}} = \max_{v\in J} \left| \left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta} x(v) \right| + \max_{v\in J} \left| {}^H D_{0+}^{\alpha,\mathcal{B};\psi} x(v) \right| + \max_{v\in J} \left| I_{0+}^{F;\psi} x(v) \right|.$$

A few definitions are reviewed from Vanterler et al. [20] and Kilbas et al. [13]. Now, let $x \in L^1(J)$. Afterwards, the function x is $\psi - RL$ fractional integral of order $\alpha > 0\,(\alpha \in R)$ defined by

$$I_{\sigma+}^{\alpha;\psi} x(v) = \int_\sigma^v \frac{\psi'(\rho)}{\Gamma(\alpha)} \left(\tilde{\psi}_\rho(v)\right)^{\alpha-1} x(\rho)\ d\rho. \tag{2.3}$$

Let $x \in C^\kappa(J)$. The order of the function x is $\kappa - 1 < \alpha < \kappa$, here $\kappa = 2$, type $0 \le \mathfrak{B} \le 1$, the $\psi-$HFD gives.

$${}^H D_{a+}^{\alpha,\mathfrak{B};\psi} f(v) = I_{\sigma+}^{\mathfrak{B}(1-\alpha);\psi} \left(\frac{1}{\psi'}\frac{d}{dv}\right)^\kappa I_{\sigma+}^{(1-\mathfrak{B})(1-\alpha);\psi} f(v). \tag{2.4}$$

An equivalent form is derived for the right-sided $\psi-$HFD [17]. The above-defined $\psi-$ HFD can be expressed as follows:

$${}^H D_{a+}^{\alpha,\mathfrak{B};\psi} f(v) = I_{a+}^{\gamma-\alpha;\psi} D_{a+}^{\gamma;\psi} f(v). \tag{2.5}$$

Using $\gamma = \alpha + \mathfrak{B}(\kappa - \alpha)$, and where $I_{a+}^{\gamma-\alpha;\psi}(.)$ is the $\psi-$RLFI, and the $\psi-$RLFD is $D_{a+}^{\gamma;\psi}(.)$ [12]. In this case, we are looking at the $\psi-$ HFD for $\kappa = 1$.

Lemma 2.1. *[13] Assume that* $\varepsilon, \mathfrak{F} > 0$ *and* $\rho > \kappa$. *Subsequently, (a)* $I_{\sigma+}^{\alpha;\psi} I_{\sigma+}^{\varepsilon;\psi} \Upsilon(v) = I_{\rho+}^{\alpha+\varepsilon;\psi} \Upsilon(v)$; *(b)* $I_{\sigma+}^{\alpha;\psi} \left(\tilde{\psi}(v)\right)^{\mathfrak{F}-1} = \frac{\Gamma(\mathfrak{F})}{\Gamma(\alpha+\mathfrak{F})} \left(\tilde{\psi}(v)\right)^{\alpha+\mathfrak{F}-1}$;

(c) ${}_{H}D^{\alpha,\mathfrak{B};\psi}_{\sigma+}\left(\tilde{\psi}_{\sigma}(v)\right)^{\kappa-1}=0$; (d) ${}_{H}D^{\alpha,\mathfrak{B};\psi}_{\sigma+}\left(\tilde{\psi}_{\sigma}(v)\right)^{\rho-1}=\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\left(\tilde{\psi}_{\sigma}(v)\right)^{\rho-\alpha-1}$.

Lemma 2.2. *[20] If $x\in C^{\kappa}(J)$, $\kappa-1<\alpha<\kappa$ and $0\leq\mathfrak{B}\leq 1$, then*

(i) $I^{\alpha;\psi}_{\sigma+}\,{}_{H}D^{\alpha,\mathfrak{B};\psi}_{\sigma+}x(v)=x(v)-\sum\limits_{j=1}^{\kappa}\frac{\left(\tilde{\psi}(v)\right)^{\eta-j}}{\Gamma(\eta-j+1)}x^{[\kappa-j]}_{\psi}I^{(1-\mathfrak{B})(\kappa-\alpha)}_{\sigma+}x(\sigma)$.

(ii) ${}_{H}D^{\alpha,\mathfrak{B};\psi}_{\sigma+}I^{\alpha;\psi}_{\sigma+}x(v)=x(v)$.

Definition 2.3. [3] If $c_f>0$ is a real number, then the equation is **Ulam-Hyers stable** for all $\varepsilon>0$ and all solutions $x\in C_{2-\eta;\psi}(J)$ (with ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{0+}x\in C(J)$) of the inequality

$$\left|{}^{H}D^{\alpha,\mathfrak{B};\psi}x(v)-P\left(v,\,x(v),I^{\mathfrak{F};\psi}x(v)\right)\right|\leq\varepsilon,\;\; v\in J. \tag{2.6}$$

There exists a solution $z\in C_{2-\eta;\psi}(J)$ (with ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{0+}z\in C(J)$) of Eq. (1.5) such that

$$|x(v)-z(v)|\leq c_f\varepsilon,\quad v\in J. \tag{2.7}$$

Definition 2.4. [3] In relation to $\varphi\in C(J,R_{+})$, the equation is **Ulam-Hyers-Rassias stable**. If $c_f>0$ is a real number, then for every $\varepsilon>0$ and every solution $x\in C_{2-\eta;\psi}(J)$ (with ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{0+}x\in C(J)$) of the inequality

$$\left|{}^{H}D^{\alpha,\mathfrak{B};\psi}x(v)-P\left(v,\,x(v),I^{\mathfrak{F};\psi}x(v)\right)\right|\leq\varepsilon\,\varphi(v),\;\; v\in J, \tag{2.8}$$

there exists a solution $z\in C_{2-\eta;\psi}(J)$ (with ${}^{H}D^{\alpha,\mathfrak{B};\psi}_{0+}z\in C(J)$) of Eq. (1.5) such that

$$|x(v)-z(v)|\leq c_f\varepsilon\,\varphi(v),\quad v\in J. \tag{2.9}$$

Theorem 2.5. *[9] (Banach Fixed Point Theorem(BFPT)). Assume that E is the Banach space and D is a non-empty closed subset. Consequently, there is a unique fixed point in any contraction relating T to D.*

Theorem 2.6. *[9] (Leray–Schauder(LS) alternative). In a Banach space χ, there exists a closed and bounded convex set in O and in O within Ω, there exists an open set with the property O is an element of O. $T:\bar{O}\to\Omega$ which is a function that is continuous as well as a compact mapping. Consequently, one of the following circumstances holds true.*

(i) T is the fixed point in $\bar{\text{O}}$ exists.

(ii) To have $x=N\,Tx$, there exist $x\in\partial O$ and $N\in(0,1)$.

3. Existence results

Here, using LS alternative, we demonstrate that HFDE has at least one unique solution. The problem and integral equation are equivalent according to the subsequent lemma.

Lemma 3.1. *Let $0<F<1$ and let $I^{F;\psi}_{0+}$ be the ψ-Riemann-Liouville fractional integral of order F. For any functions $x,\bar{x}\in C_{2-\eta;\psi}(J)$ (the weighted space for Problem (1.5) defined with $\tilde{\psi}_{\mathcal{T}}(v)=\psi(\mathcal{T})-\psi(v)$), the following estimates hold:*

(i) *For all $v\in J=[0,\mathcal{T}]$,*

$$\left|I^{F;\psi}_{0+}x(v)\right|=\left|\frac{1}{\Gamma(F)}\int_0^v\psi'(s)\big(\psi(v)-\psi(s)\big)^{F-1}x(s)\,ds\right|\leq\frac{\big(\psi(v)-\psi(0)\big)^F}{\Gamma(F+1)}\max_{s\in[0,v]}|x(s)|.$$

Consequently, in terms of the infinity norm, $\left\|I^{F;\psi}_{0+}x\right\|_C\leq\frac{\left(\psi(\mathcal{T})-\psi(0)\right)^F}{\Gamma(F+1)}\|x\|_C$, where $\|\cdot\|_C=\max_{v\in J}|\cdot|$.

(ii) *For all* $v \in J$,

$$\left|I_{0+}^{F;\psi}x(v) - I_{0+}^{F;\psi}\bar{x}(v)\right| \leq \frac{\left(\psi(v)-\psi(0)\right)^{F}}{\Gamma(F+1)} \max_{s\in[0,v]} |x(s)-\bar{x}(s)|.$$

Hence,

$$\left\|I_{0+}^{F;\psi}x - I_{0+}^{F;\psi}\bar{x}\right\|_{C} \leq \frac{\left(\psi(\mathcal{T})-\psi(0)\right)^{F}}{\Gamma(F+1)} \|x-\bar{x}\|_{C}.$$

(iii) *In the weighted space* $C_{2-\eta;\psi}(J)$ *with the weight* $\tilde{\psi}_{\mathcal{T}}(v) = \psi(\mathcal{T}) - \psi(v)$, *the following estimate holds:*

$$\left\|I_{0+}^{F;\psi}x\right\|_{C_{2-\eta;\psi}} \leq \frac{\left(\psi(\mathcal{T})-\psi(0)\right)^{F}}{\Gamma(F+1)} \|x\|_{C_{2-\eta;\psi}},$$

where $\|x\|_{C_{2-\eta;\psi}} = \max_{v\in J}\left|\left(\tilde{\psi}_{\mathcal{T}}(v)\right)^{2-\eta}x(v)\right|$.

Proof. The proof is directly. □

Lemma 3.2. *An equivalent way to represent the problem is through the integral equation*

$$\begin{aligned}
x(v) &= \frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)(\psi(\mathfrak{T})-\psi(0))}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv \\
&- \frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv \qquad (3.1)\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}P\left(s,\,x(s)\,,\,I_{0+}^{\mathfrak{F};\psi}x(s)\right)ds, \qquad v\in[0,\,\mathfrak{T}].
\end{aligned}$$

Proof. Since ${}^{H}D_{0+}^{\alpha,\mathfrak{B};\psi}x(v) = P\left(v, x(v)\,, I_{0+}^{\mathfrak{F};\psi}x(v)\right)$, applying to both sides of the Eq. (1.5) the fractional integral $I_{0+}^{\alpha;\,\psi}$, and by applying Lemma 2.2–(i), we get

$$x(v) = \sum_{j=1}^{2}\frac{\left(\tilde{\psi}(v)\right)^{\eta-j}}{\Gamma(\eta-j+1)}x_{\psi}^{[\kappa-j]}I_{\sigma+}^{(1-\mathfrak{B})(\kappa-\alpha)}x(\sigma) + I_{0+}^{\alpha;\,\psi}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\,\psi}x(v)\right). \qquad (3.2)$$

Because $(1-\mathfrak{B})(2-\alpha) = 2-\eta$, this gives

$$\begin{aligned}
x(v) &= \frac{1}{\Gamma(\eta)}(\psi(v)-\psi(0))^{\eta-1}\left(\frac{1}{\psi'(v)}\frac{d}{dv}\right)^{2}I_{0+}^{2-\eta;\,\psi}x(v)|_{v=0} \\
&+ \frac{1}{\Gamma(\eta-1)}(\psi(v)-\psi(0))^{\eta-2}\left(\frac{1}{\psi'(v)}\frac{d}{dv}\right)I_{0+}^{2-\eta;\,\psi}x(v)|_{v=0} + I_{0+}^{\alpha;\,\psi}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\,\psi}x(v)\right) \\
&= \frac{1}{\Gamma(\eta)}(\psi(v)-\psi(0))^{\eta-1}\,{}_{H}D_{0+}^{\eta-1,\,\mathfrak{B};\,\psi}x(v)|_{v=0} \\
&+ \frac{1}{\Gamma(\eta-1)}(\psi(v)-\psi(0))^{\eta-2}\,{}_{H}D_{0+}^{\eta-2,\,\mathfrak{B};\,\psi}x(v)|_{v=0} + I_{0+}^{\alpha;\,\psi}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\,\psi}x(v)\right).
\end{aligned}$$

Assume that

$$C_0 = {}_{H}D_{0+}^{\eta-1,\,\mathfrak{B};\,\psi}x(v)|_{v=0}, \quad C_1 = {}_{H}D_{0+}^{\eta-2,\,\mathfrak{B};\,\psi}x(v)|_{v=0}.$$

We arrive at the relation

$$x(v) = \sum_{i=0}^{1} \frac{C_i}{\Gamma(\eta - i)} (\psi(v) - \psi(0))^{\eta-(i+1)} + I_{0+}^{\alpha;\psi} P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi} x(v)\right). \tag{3.3}$$

When considering the boundary condition $x(0) = x(\mathfrak{T})$, in this case, Eq. (3.1) becomes

$$\frac{C_0}{\Gamma(\eta)} (\psi(\mathfrak{T}) - \psi(0))^{\eta-1} + \frac{C_1}{\Gamma(\eta-1)} (\psi(\mathfrak{T}) - \psi(0))^{\eta-2} + I_{0+}^{\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right) = 0. \tag{3.4}$$

And from another boundary condition $I_{0+}^{2-\eta;\psi} x(0+) = I_{0+}^{2-\eta;\psi} x(\mathfrak{T})$, we get

$$I_{0+}^{2-\eta;\psi} x(v) = \frac{C_0}{\Gamma(2)} (\psi(v) - \psi(0)) + C_1 + I_{0+}^{2-\eta+\alpha;\psi} P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi} x(v)\right).$$

$$I_{0+}^{2-\eta;\psi} x(0+) = C_1,$$

$$I_{0+}^{2-\eta;\psi} x(\mathfrak{T}) = \frac{C_0}{\Gamma(2)} (\psi(\mathfrak{T}) - \psi(0)) + C_1 + I_{0+}^{2-\eta+\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right).$$

It follows from the previous expressions that

$$C_0 = \frac{-\Gamma(2)}{(\psi(\mathfrak{T}) - \psi(0))} I_{0+}^{2-\eta+\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right). \tag{3.5}$$

To determine C_1, Eq. (3.5) in Eq. (3.4) , we obtain

$$C_1 = \frac{\Gamma(2)}{(\eta-1)} I_{0+}^{2-\eta+\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right) - \frac{\Gamma(\eta-1)}{(\psi(\mathfrak{T}) - \psi(0))^{\eta-2}} I_{0+}^{\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right). \tag{3.6}$$

Substitute equations (3.5) and (3.6) in equation (3.3), gives the following:

$$\begin{aligned} x(v) &= \frac{(\psi(\mathfrak{T}) - \psi(v)) (\psi(v) - \psi(0))^{\eta-2}}{\Gamma(\eta) (\psi(\mathfrak{T}) - \psi(0))} I_{0+}^{2-\eta+\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right) \\ &- \frac{(\psi(v) - \psi(0))^{\eta-2}}{(\psi(\mathfrak{T}) - \psi(0))^{\eta-2}} I_{0+}^{\alpha;\psi} P\left(\mathfrak{T}, x(\mathfrak{T}), I_{0+}^{\mathfrak{F};\psi} x(\mathfrak{T})\right) + I_{0+}^{\alpha;\psi} P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi} x(v)\right), \qquad v \in [0, \mathfrak{T}]. \end{aligned} \tag{3.7}$$

The proof is completed.

Motivated by applying Lemma (3.2), we present the functional A_1, that is described as follows:

$$\begin{aligned} (A_1 x)(v) &= \frac{(\psi(\mathfrak{T}) - \psi(v)) (\psi(v) - \psi(0))^{\eta-2}}{\Gamma(\eta) \Gamma(2-\eta+\alpha) (\psi(\mathfrak{T}) - \psi(0))} \int_0^{\mathfrak{T}} \psi'(v) \, (\psi(\mathfrak{T}) - \psi(v))^{1-\eta+\alpha} P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi} x(v)\right) dv \\ &- \frac{1}{\Gamma(\alpha)} \frac{(\psi(v) - \psi(0))^{\eta-2}}{(\psi(\mathfrak{T}) - \psi(0))^{\eta-2}} \int_0^{\mathfrak{T}} \psi'(v) \, (\psi(\mathfrak{T}) - \psi(v))^{\alpha-1} P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi} x(v)\right) dv \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{v} \psi'(s) \, (\psi(v) - \psi(s))^{\alpha-1} P\left(s, x(s), I_{0+}^{\mathfrak{F};\psi} x(s)\right) ds, \qquad v \in [0, \mathfrak{T}]. \end{aligned} \tag{3.8}$$

□

Theorem 3.3. *Assume that the functions are continuous and satisfy the conditions:*

$(K1)$ *The function $P : J \times R \times R \to R$ be such that*

$P(., x(.), w(.)) \in C_{2-\eta;\psi}(J)$ *for any* $x, w \in C_{2-\eta;\psi}(J)$.

(K2) *There exist constants* $L > 0$ *and* $M < 1$ *such that*

$$|P(v, x, w) - P(v, \bar{x}, \bar{w})| \leq L|x - \bar{x}| + M|w - \bar{w}|. \tag{3.9}$$

(K3) *There are functions* $p_1, p_2, p_3 \in C([0, \mathfrak{T}], R_+)$,

$p_1^* = \sup_{v\in[0,\mathfrak{T}]} p_1(v)$, $p_2^* = \sup_{v\in[0,\mathfrak{T}]} p_2(v)$ *and* $p_3^* = \sup_{v\in[0,\mathfrak{T}]} p_3(v) < 1$,

such that

$|P(v, x(v), w(v))| \leq p_1(v) + p_2(v)|x(v)| + p_3(v)|w(v)|$, *for all* $x, w \in R$.

For any $x, w, \bar{x}, \bar{w} \in R$ *and* $v \in J$, *where* $w(.) = I_{0+}^{\mathfrak{F};\psi} x(.)$ *and* $\bar{w}(.) = I_{0+}^{\mathfrak{F};\psi} \bar{x}(.)$ *are* $\psi-$*Riemann-Liouville fractional integral of order* $0 < \mathfrak{F} < 1$, *from* $\Gamma(\eta)\Gamma(3-\eta+\alpha) \geq \Gamma(\alpha+1)$, *gives*

$$\begin{aligned}&\max_{v\in[0,\mathfrak{T}]}\left\{\left[\left(\frac{(\psi(\mathfrak{T})-\psi(0))}{\Gamma(\eta)\,\Gamma(3-\eta+\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)(\psi(v)-\psi(0))^{\eta-2}(\psi(v)-\psi(0))^{2-\eta+\alpha}\right.\right.\\&\left.\left.+\frac{1}{\Gamma(\alpha+1)}(\psi(v)-\psi(s))^{\alpha}\right](K)\right\}\leq\Omega K<1\ ,\end{aligned} \tag{3.10}$$

where $\Omega = \frac{3}{\Gamma(\alpha+1)}(\psi(\mathfrak{T}))^{\alpha+1}$, *and* $K = \left(L + \frac{M(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}\right)$. *Then, Eq. (1.5) for the* $\psi-$*HFDEs has a unique solution.*

Proof. Let's set $r \geq \frac{\Omega\, P_1^*}{1-\Omega\, N}$, where $N = P_2^* + \frac{P_3^*(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}$, and show that $A_1 G_r \subseteq G_r$, when

$$G_r = \left\{x(v) \in C_{2-\eta;\psi}([0, \mathfrak{T}]),\ {}^H D_{0+}^{\alpha,\,\mathfrak{B};\,\psi} x(v) \in C([0, \mathfrak{T}]) \text{ exists} : \|x\| \leq r\right\},$$

$$\begin{aligned}\|A_1 x\|_{C_{2-\eta;\psi}} \leq \max_{v\in[0,\mathfrak{T}]}&\left\{\frac{\Lambda(\psi(v)-\psi(0))^{\eta-1}}{\Gamma\eta\,\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}\,|\,P(v, x(v), w(v))|\,dv\right.\\&+\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}\,|\,P(v, x(v), w(v))|\,dv\\&\left.+\frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}\left|\,P\left(s, x(s), I_{0+}^{\mathfrak{F};\psi}x(s)\right)\right|ds\right\}\\&\leq\Omega\left(P_1^*+\|x\|_{C_{2-\eta;\psi}}\left(P_2^*+\frac{P_3^*(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}\right)\right)\\&\leq\Omega(P_1^*+rN)\leq r.\end{aligned}$$

It is sufficient to demonstrate that $A_1 : G_r \to G_r$ is a contraction. Let start with A_1 which is defined in Eq. (3.8), that is $A_1 G_r \subseteq G_r$. Given $x(v) \in G_r$, we have the map $v \to A_1 x(v)$ in class C^2 and

$${}^H D_{0+}^{\alpha,\,\mathfrak{B};\,\psi} A_1 x(v) = P\left(v, x(v), I_{0+}^{\mathfrak{F};\psi}\right)$$

is continuous. Now, let $x(v), \bar{x}(v) \in G_r$ be arbitrary. Then

$$\|A_1 x - A_1 \bar{x}\|_{2-\eta;\psi} \leq \max_{v\in[0,\mathfrak{T}]} \left\{ \frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)(\psi(\mathfrak{T})-\psi(0))} \int_0^{\mathfrak{T}} \psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha} |P(v, x(v), w(v)) \right.$$

$$- P(v, \bar{x}(v), \bar{w}(v))|\, dv + \frac{1}{\Gamma(\alpha)} \frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}$$

$$\int_0^T \psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1} |P(v, x(v), w(v)) - P(v, \bar{x}(v), \bar{w}(v))|\, dv$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_0^v \psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1} |P(s, x(s), w(s)) - P(s, \bar{x}(s), \bar{w}(s))|\, ds \right\}$$

$$\leq \max_{v\in[0,\mathfrak{T}]} \left\{ \frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)(\psi(\mathfrak{T})-\psi(0))} \int_0^{\mathfrak{T}} \psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha} \right.$$

$$[L|x(v)-\bar{x}(v)| + M|w(v)-\bar{w}(v)|]\, dv + \frac{1}{\Gamma(\alpha)} \frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}$$

$$\int_0^{\mathfrak{T}} \psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1} [L|x(v)-\bar{x}(v)| + M|w(v)-\bar{w}(v)|]\, dv$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_0^v \psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1} [L|x(s)-\bar{x}(s)| + M|w(s)-\bar{w}(s)|]\, ds \right\}.$$

By the same way as Lemma (3.1) (ii), we get

$$\|A_1 x - A_1 \bar{x}\|_{2-\eta;\psi} \leq \max_{v\in[0,\mathfrak{T}]} \left\{ \frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\,\Gamma(3-\eta+\alpha)} (\psi(\mathfrak{T})-\psi(0))^{2-\eta+\alpha} \left[L|x(v)-\bar{x}(v)| \right.\right.$$

$$\left. + \frac{M(\psi(v)-\psi(0))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} |x(v)-\bar{x}(v)| \right] + \frac{1}{\Gamma(\alpha+1)} \frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}$$

$$(\psi(\mathfrak{T})-\psi(0))^{\alpha} \left[L|x(v)-\bar{x}(v)| + \frac{M(\psi(\mathfrak{T})-\psi(0))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} |x(v)-\bar{x}(v)| \right]$$

$$\left. + \frac{1}{\Gamma(\alpha+1)} (\psi(v)-\psi(s))^{\alpha} \left[L|x(s)-\bar{x}(s)| + M \frac{M(\psi(s)-\psi(0))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} |x(s)-\bar{x}(s)| \right] \right\}$$

$$\leq \max_{v\in[0,\mathfrak{T}]} \left\{ \left[\left(\frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\,\Gamma(3-\eta+\alpha)} + \frac{(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\alpha+1)} \right) (\psi(\mathfrak{T})-\psi(0))^{2-\eta+\alpha} \right.\right.$$

$$\left.\left. + \frac{1}{\Gamma(\alpha+1)} (\psi(v)-\psi(0))^{\alpha} \right] \left(L + \frac{M(\psi(v)-\psi(0))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} \right) \right\} \|x-\bar{x}\|_{2-\eta;\psi} \leq \Omega K \|x-\bar{x}\|_{2-\eta;\psi}.$$

By establishing the contraction property of A_1. A unique solution has been found to the problem Eq.(1.5) based on the Banach contraction principle.

□

Theorem 3.4. *Assume that the following conditions are hold*

(K4) *The inequality is satisfied when a real number $R > 0$,*

$$\Omega\left(P_1^* + g\left(R\right)\left(P_2^* + \frac{P_3^*(\psi\left(\mathfrak{T}\right))^{\mathfrak{F}}}{\Gamma\left(\mathfrak{F}+1\right)}\right)\right) \leq R,$$

and also from $\Gamma\left(\eta\right)\Gamma\left(3-\eta+\alpha\right) \geq \Gamma\left(\alpha+1\right)$, we get

$$\max_{v\in[0,\,\mathfrak{T}]}\left\{\left(\frac{\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)}{\Gamma\left(\eta\right)\Gamma\left(3-\eta+\alpha\right)}+\frac{1}{\Gamma\left(\alpha+1\right)}\right)\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)^{2-\eta+\alpha}+\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right\} \leq \Omega,$$

where $\Omega = \frac{3}{\Gamma(\alpha+1)}\left(\psi\left(\mathfrak{T}\right)\right)^{\alpha+1}$. Then, the problem Eq. (1.5) at least one solution.

Proof. Define the functional A_1 and take consideration for the closed ball $B_R = \{x\left(v\right) \in C_{2-\eta;\psi} :\ \|x\| \leq R\}$.

Step 1. $A_1\left(B_R\right) \subseteq B_R$. Given $x\left(v\right) \in B_R$, we contain the following:

$$\begin{aligned}
\|A_1x\|_{C_{2-\eta;\psi}} \leq \max_{v\in[0,\,\mathfrak{T}]}&\left\{\frac{\left(\psi\left(\mathfrak{T}\right)-\psi\left(v\right)\right)\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\Gamma\eta\ \Gamma\left(2-\eta+\alpha\right)}\int_0^{\mathfrak{T}}\psi'\left(v\right)\ \left(\psi\left(\mathfrak{T}\right)-\psi\left(v\right)\right)^{1-\eta+\alpha}\left|P\left(v,\,x\left(v\right),\,w\left(v\right)\right)\right|dv\right.\\
&+\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)^{\eta-2}}\int_0^{\mathfrak{T}}\psi'\left(v\right)\ \left(\psi\left(\mathfrak{T}\right)-\psi\left(v\right)\right)^{\alpha-1}\left|P\left(v,\,x\left(v\right),\,w\left(v\right)\right)\right|dv\\
&\left.+\frac{1}{\Gamma\left(\alpha\right)}\times\int_0^{v}\psi'\left(s\right)\ \left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha-1}\left|P\left(s,\,x\left(s\right),\,I_{0+}^{\mathfrak{F};\psi}x\left(s\right)\right)\right|ds\right\}\\
&\leq \max_{v\in[0,\,\mathfrak{T}]}\left\{\frac{\left(\psi\left(\mathfrak{T}\right)-\psi\left(v\right)\right)\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\Gamma\eta}\frac{\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)^{2-\eta+\alpha}}{\Gamma\left(3-\eta+\alpha\right)}\left[p_1\left(v\right)+p_2\left(v\right)\left|x\left(v\right)\right|+p_3(v)\left|w(v)\right|\right]\right.\\
&+\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\Gamma\left(\alpha+1\right)}\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)^{2-\eta+\alpha}\left[p_1\left(v\right)+p_2\left(v\right)\left|x\left(v\right)\right|+p_3\left(v\right)\left|w\left(v\right)\right|\right]\\
&\left.+\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left[p_1\left(s\right)+p_2\left(s\right)\left|x\left(s\right)\right|+p_3\left(s\right)\left|w\left(s\right)\right|\right]\right\}.
\end{aligned}$$

By the same way as Lemma 3.1–(i), we get

$$\begin{aligned}
&\leq \max_{v\in[0,\,\mathfrak{T}]}\left\{\left(\frac{\left(\psi\left(\mathfrak{T}\right)-\psi\left(v\right)\right)\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\Gamma\left(\eta\right)\Gamma\left(3-\eta+\alpha\right)}+\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\eta-2}}{\Gamma\left(\alpha+1\right)}\right)\left(\psi\left(\mathfrak{T}\right)-\psi\left(0\right)\right)^{2-\eta+\alpha}\right.\\
&\left.+\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right\}\left[p_1^*+\|x\|_{C_{2-\eta;\psi}}\left(p_2^*+p_3^*\frac{\left(\psi\left(v\right)-\psi\left(0\right)\right)^{\mathfrak{F}}}{\Gamma\left(\mathfrak{F}+1\right)}\right)\right]\\
&\leq \Omega\left(P_1^*+g\left(R\right)\left(P_2^*+\frac{P_3^*(\psi\left(\mathfrak{T}\right))^{\mathfrak{F}}}{\Gamma\left(\mathfrak{F}+1\right)}\right)\right) \leq R.
\end{aligned}$$

Step 2. $A_1\left(B_R\right)$ is equicontinuous. Let $v_1,\,v_2 \in [0,\,\mathfrak{T}]$ with $v_1 > v_2$ and

$$Q = \max\left\{P\left(v,\,x\left(v\right),\,I_{0+}^{\mathfrak{F};\,\psi}\right) :\ \left(v,\,x\left(v\right),\,I_{0+}^{\mathfrak{F};\,\psi}\right) \in [0,\,\mathfrak{T}]\times B_R\times B_R\right\}.$$

Then

$$\begin{aligned}
|A_1x(v_1) - A_1x(v_2)| \leq & \left[(\psi(\mathfrak{T}) - \psi(v_1))(\psi(v_1) - \psi(0))^{\eta-2} - (\psi(\mathfrak{T}) - \psi(v_2))(\psi(v_2) - \psi(0))^{\eta-2}\right] \frac{Q(\psi(\mathfrak{T}) - \psi(0))^{2-\eta+\alpha}}{\Gamma(\eta)\Gamma(3-\eta+\alpha)} \\
& + \left[(\psi(v_2) - \psi(0))^{\eta-2} - (\psi(v_1) - \psi(0))^{\eta-2}\right] \frac{Q(\psi(\mathfrak{T}) - \psi(0))^{2-\eta+\alpha}}{\Gamma(\alpha+1)} \\
& + \left[(\psi(v_1) - \psi(0))^{\alpha} - (\psi(v_1) - \psi(v_2))^{\alpha} - (\psi(v_2) - \psi(0))^{\alpha}\right] \frac{Q}{\Gamma(\alpha+1)} \\
& + \frac{Q}{\Gamma(\alpha+1)} (\psi(v_1) - \psi(0))^{\alpha},
\end{aligned}$$

which converges to zero as $v_1 \to v_2$. $A_1(B_R)$ included in a compact set by the Arzela-Ascoli theorem. Lastly, we show that $x(v) = N\,A_1x(v)$ does not exist in ∂B_R and $N \in (0,1)$. As previously demonstrated, $R = \|x\|_{C_{2-\eta;\psi}} < \|A_1x\|_{C_{2-\eta;\psi}} \leq R$ is obtained by assume there exist $x(v)$ and N. Our argument is supported by this contradiction. Therefore, by using the LS alternative, we can make sure that A_1 has a fixed point. The proof consequently becomes complete because this fixed point represents to the problem's solution (1.5). □

4. UH and UHR stabilities

In this section, we demonstrate the UH and UHR stabilities for the nonlinear problem, Eq. (1.5), using the $\Psi-$Hilfer fractional derivative.

Remark 4.1. As long as there exists a function $h \in C_{2-\eta;\psi}(J, R))$ (that depends on x), then $x \in C_{2-\eta;\psi}(J, R))$ solves inequality Eq. (2.6) such that:
i. $|h(v)| \leq \varepsilon, \ \forall v \in [0, T]$.
ii. ${}^{H}D^{\alpha,\mathfrak{B};\psi}x(v) - P\left(v, x(v), I^{\mathfrak{F};\psi}x(v)\right) = h(v), v \in [0, T]$.

Remark 4.2. It can be argued that the inequality Eq.(2.8) can be solved by a function $x \in C_{2-\eta;\psi}(J, R))$, iff $\varsigma \in C_{2-\eta;\psi}(J, R))$ (and only depends on x) is:
i. $|\varsigma(v)| \leq \varepsilon\varphi(v), \ \forall v \in [0, T]$.
ii. ${}^{H}D^{\alpha,\mathfrak{B};\psi}x(v) - P\left(v, x(v), I^{\mathfrak{F};\psi}x(v)\right) = \varsigma(v), v \in [0, T]$.
First, there is the UH stability theorem.

Theorem 4.3. *Assume $\Omega K < 1$ and the validity of (K1), (K2), and Eq. (3.10). Consequently, Eq. (1.5) is UH stable.*

Proof. Let $\varepsilon > 0$, and each solution $x \in C_{2-\eta;\psi}(J, R)$. By Remark 4.1, the result is ${}^{H}D^{\alpha,\mathfrak{B};\psi}x(v) - P\left(v, x(v), I^{\mathfrak{F};\psi}x(v)\right) = h(v), v \in [0, T]$. which implies that

$$\left|{}^{H}D^{\alpha,\mathfrak{B};\psi}x(v) - P\left(v, x(v), I^{\mathfrak{F};\psi}x(v)\right)\right| \leq \varepsilon, \quad v \in J. \tag{4.1}$$

Let us indicate the unique solution to the problem (1.5) by $z \in C_{2-\eta}(J, R)$ as follows, then $z(v)$, is given by

$${}^{H}D^{\alpha,\mathfrak{B};\psi}z(v) - P\left(v, z(v), I^{\mathfrak{F};\psi}z(v)\right) = h(v), v \in [0, T].$$

and

$$z(v)=\frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,z(v)\,,\,I_{0+}^{\mathfrak{F};\psi}z(v)\right)dv$$
$$-\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,z(v)\,,\,I_{0+}^{\mathfrak{F};\psi}z(v)\right)dv$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}P\left(s,\,z(s)\,,\,I_{0+}^{\mathfrak{F};\psi}z(s)\right)ds.$$

Then, we have

$$\|x-z\|_{C_{2-\eta;\psi}}\leq\max_{v\in[0,\,\mathfrak{T}]}\left\{\left|x(v)-\frac{\Lambda(\psi(v)-\psi(0))^{\eta-1}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv\right.\right.$$
$$+\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv$$
$$\left.-\frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}P\left(s,\,x(s)\,,\,I_{0+}^{\mathfrak{F};\psi}x(s)\right)ds\right|$$
$$+\left|\frac{\Lambda(\psi(v)-\psi(0))^{\eta-1}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv\right.$$
$$-\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}P\left(s,\,x(s)\,,\,I_{0+}^{\mathfrak{F};\psi}x(s)\right)ds$$
$$-\frac{\Lambda(\psi(v)-\psi(0))^{\eta-1}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,z(v)\,,\,I_{0+}^{\mathfrak{F};\psi}z(v)\right)dv$$
$$+\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,z(v)\,,\,I_{0+}^{\mathfrak{F};\psi}z(v)\right)dv$$
$$\left.\left.-\frac{1}{\Gamma(\alpha)}\int_0^{v}\psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1}P\left(s,\,z(s)\,,\,I_{0+}^{\mathfrak{F};\psi}z(s)\right)ds\right|\right\},$$

for each $v\in J$, $1<\alpha<2,\ 0<\mathfrak{F}<1$ and $0\leq\mathfrak{B}\leq1$; $I_{0+}^{(1-\mathfrak{B})(2-\alpha);\psi}x(0+)=I_{0+}^{(1-\mathfrak{B})(2-\alpha);\psi}z(0)$,where

$$\left\|x(v)-\frac{(\psi(\mathfrak{T})-\psi(v))(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\Gamma(2-\eta+\alpha)}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv\right.$$
$$+\frac{1}{\Gamma(\alpha)}\frac{(\psi(v)-\psi(0))^{\eta-2}}{(\psi(\mathfrak{T})-\psi(0))^{\eta-2}}\int_0^{\mathfrak{T}}\psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{\alpha-1}P\left(v,\,x(v)\,,\,I_{0+}^{\mathfrak{F};\psi}x(v)\right)dv$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_0^v \psi'(s)\,(\psi(v)-\psi(s))^{\alpha-1} P\left(s,\, x(s),\, I_{0+}^{\mathfrak{F};\psi} x(s)\right) ds \Bigg\| \\
&\leq \left(\frac{(\psi(\mathfrak{T})-\psi(v))\,(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\,\Gamma(3-\eta+\alpha)} + \frac{(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\alpha+1)} \right) (\psi(\mathfrak{T})-\psi(0))^{2-\eta+\alpha} \|h(v)\| + \|h(v)\| \frac{(\psi(v)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \left(\left(\frac{(\psi(\mathfrak{T})-\psi(v))}{\Gamma(\eta)\,\Gamma(3-\eta+\alpha)} + \frac{(1)}{\Gamma(\alpha+1)} \right) (\psi(v)-\psi(0))^{\eta-2} (\psi(\mathfrak{T})-\psi(0))^{2-\eta+\alpha} + \frac{(\psi(v)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \right) \|h(v)\| \\
&\leq \Omega\,\varepsilon\,, \qquad v \in J = [0,\, \mathfrak{T}]\,.
\end{aligned} \tag{4.2}
$$

Then, we get

$$
\begin{aligned}
\|x-z\|_{C_{2-\eta;\psi}} &\leq \Omega\,\varepsilon + \Omega \max_{v\in[0,\,\mathfrak{T}]} \left\{ \left| P\left(v,\, x(v),\, I_{0+}^{\mathfrak{F};\,\psi} x(v)\right) - P\left(v,\, z(v),\, I_{0+}^{\mathfrak{F};\,\psi} z(v)\right) \right| \right\} \\
&\leq \Omega\,\varepsilon + \Omega \left(L + \frac{M(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} \right) \|x-z\|_{C_{2-\eta;\psi}}, \;\; v \in [0, \mathfrak{T}],
\end{aligned}
$$

$$
\|x-z\|_{C_{2-\eta;\psi}} (1-\Omega K) \leq \Omega\,\varepsilon,
$$

where $K = L + \frac{M(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}$, can be rewritten in the form

$$
\|x-z\|_{C_{2-\eta;\psi}} \leq (1-\Omega K)^{-1} \Omega\,\varepsilon.
$$

Then, for $C_f = (1-\Omega K)^{-1}\Omega$ with v $\in J = [0,\, T]\,.$

Based on Eq. (4.2), we may conclude that Eq. (1.5) is stable for Ulam-Hyers.

□

Theorem 4.4. *Let us assume that $\Omega K < 1$ and that (K1), (K2), and Eq. (3.10) are appropriate. The condition and the hypothesis (K4) are satisfied, i.e., $\lambda_\varphi > 0$ exists and that function $\varphi \in C(J,\, R_+)$ increases such that for each $v \in J$, gives*

$$
I_{0+}^{\alpha;\,\psi} \varphi(v) \leq \lambda_\varphi \varphi(v)\,.
$$

For φ, the Eq. (1.5) is UHR stable.

Proof. Suppose $z \in C_{2-\eta;\psi}(J,\, R)$ is a solution. By Remark 4.2, the outcome is

$$
{}^{H}D^{\alpha,\mathfrak{B};\psi} z(v) - P\left(v,\, z(v)\,, I^{\mathfrak{F};\psi} z(v)\right) = \varsigma(v)\,, v \in [0, T]\,.
$$

To the inequality equation:

$$
\left| {}^{H}D^{\alpha,\mathfrak{B};\psi} z(v) - P\left(v,\, z(v)\,, I^{\mathfrak{F};\psi} z(v)\right) \right| \leq \varepsilon\,\varphi(v)\,, \;\; v \in J, \qquad \text{and} \qquad \varepsilon > 0. \tag{4.3}
$$

Conversely, let us indicate the unique solution to the problem by $x \in C_{2-\eta;\psi}(J,\, R)$.

$$
\begin{aligned}
&{}^{H}D_{0+}^{\alpha,\mathfrak{B};\psi} x(v) = P\left(v, x(v)\,, I_{0+}^{\mathfrak{F};\psi} x(v)\right), \qquad v \in J,\, 1 < \alpha < 2, 0 < \mathfrak{F} < 1, 0 \leq \mathfrak{B} \leq 1, \\
&\quad I_{0+}^{\alpha;\,\psi} x(0+) = I_{0+}^{\alpha;\,\psi} z(0+)\,,
\end{aligned}
$$

and

$$
x(v) = \frac{(\psi(\mathfrak{T})-\psi(v))\,(\psi(v)-\psi(0))^{\eta-2}}{\Gamma(\eta)\,\Gamma(2-\eta+\alpha)} \int_0^{\mathfrak{T}} \psi'(v)\,(\psi(\mathfrak{T})-\psi(v))^{1-\eta+\alpha} P\left(v,\, x(v)\,,\, I_{0+}^{\mathfrak{F};\psi} x(v)\right) dv
$$

$$- \frac{1}{\Gamma(\alpha)} \frac{(\psi(v) - \psi(0))^{\eta-2}}{(\psi(\mathfrak{T}) - \psi(0))^{\eta-2}} \int_0^{\mathfrak{T}} \psi'(v) \, (\psi(\mathfrak{T}) - \psi(v))^{\alpha-1} P\left(v, \, x(v), \, I_{0+}^{\mathfrak{F};\psi} x(v)\right) dv$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{v} \psi'(s) \, (\psi(v) - \psi(s))^{\alpha-1} P\left(s, \, x(s), \, I_{0+}^{\mathfrak{F};\psi} x(s)\right) ds.$$

Applying the operator $I_{0+}^{\alpha;\psi}$ on both sides of Eq. (4.3) and using Lemma 2.2, the same steps as in Theorem 5, we get

$$\|x - z\|_{C_{2-\eta;\psi}} \leq \Omega \, \lambda_\varphi \varphi(v) + \Omega \max_{v \in [0, \, \mathfrak{T}]} \left\{ \left| P\left(v, \, x(v), \, I_{0+}^{\mathfrak{F};\psi} x(v)\right) - P\left(v, \, z(v), \, I_{0+}^{\mathfrak{F};\psi} z(v)\right) \right| \right\}.$$

This gives

$$\|x - z\|_{C_{2-\eta;\psi}} \leq \Omega \lambda_\varphi \, \varphi(v) + \Omega \left(L + \frac{M(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} \right) \|x - z\|_{C_{2-\eta;\psi}}.$$

$$\|x - z\|_{C_{2-\eta;\psi}} (1 - \Omega K) \leq \Omega \lambda_\varphi \, \varphi(v), \tag{4.4}$$

where $K = L + \frac{M(\psi(\mathfrak{T}))^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}$. By using Eq. (3.10), we can rewrite Eq. (4.4) as follows:

$$\|x - z\|_{C_{2-\eta;\psi}} \leq (1 - \Omega K)^{-1} \Omega \lambda_\varphi \, \varphi(v).$$

Then, for $C_f = (1 - \Omega K)^{-1} \lambda_\varphi$ with $v \in J = [0, \mathfrak{T}]$. Finally, from Eq. (4.4) then the problem (1.5) is UHR stable. □

5. Related numerical example

The particular example of $\psi-$Hilfer fractional derivatives is examined to numerically demonstrate the existence result. The following example describes one of the valuable applications of the problem (1.5).

Example 5.1. Consider the following nonlinear problem

$${}^{H}D_{0+}^{\alpha,\mathfrak{B};\psi} x(v) = 2v + e^{-v^3} + \frac{\sin(2v)\, x(v)}{5} + \frac{I_{0+}^{\mathfrak{F};\psi} x(v)}{20 + v^2}.$$

With periodic conditions $x(0) = x(1)$ and $I_{0+}^{2-\eta;\psi} x(0+) = I_{0+}^{2-\eta;\psi} x(1)$, when $\alpha = 1.3$, $\mathfrak{B} = 0.5$, $\mathfrak{F} = 0.6$.

Here $P(v, x, I_{0+}^{\mathfrak{F};\psi} x(v)) = 2v + e^{-v^3} + \frac{\sin(2v)x(v)}{5} + \frac{I_{0+}^{\mathfrak{F};\psi} x(v)}{20+v^2}$, we have

$$\begin{aligned} |P(v, x_1, I_{0+}^{\mathfrak{F};\psi} x_1(v)) - P(v, x_2, I_{0+}^{\mathfrak{F};\psi} x_2(v))| &= \left| \frac{\sin(2v)\, x_1(v)}{5} + \frac{I_{0+}^{\mathfrak{F};\psi} x_1(v)}{20 + v^2} - \frac{\sin(2v)\, x_2(v)}{5} + \frac{I_{0+}^{\mathfrak{F};\psi} x_2(v)}{20 + v^2} \right| \\ &\leq 0.2 \, |x_1(v) - x_2(v)| + 0.05 \left| I_{0+}^{\mathfrak{F};\psi} x_1(v) - I_{0+}^{\mathfrak{F};\psi} x_2(v) \right|, \end{aligned}$$

$$L = \max_{v \in [0,1]} \frac{|\sin 2v|}{5} = 0.2, \quad M = \max_{v \in [0,1]} \frac{1}{20 + v^2} \leq 0.05.$$

Then, we get $L = 0.2$, $M = 0.05$. Let $W = \Omega K$ when $\psi(v) = 2^{v-1}$, $\psi(v) = \sin(v)$, and $\psi(v) = 0.8(v+0.1)^2$, then calculate W:

$$\psi(v) = \begin{cases} 2^{v-1}, & W = 0.6582, \\ \sin(v), & W = 0.4330, \\ 0.8(v+0.1)^2, & W = 0.6081. \end{cases} \tag{5.1}$$

The following figure shows the different values of v with different functions of $\psi(v)$.

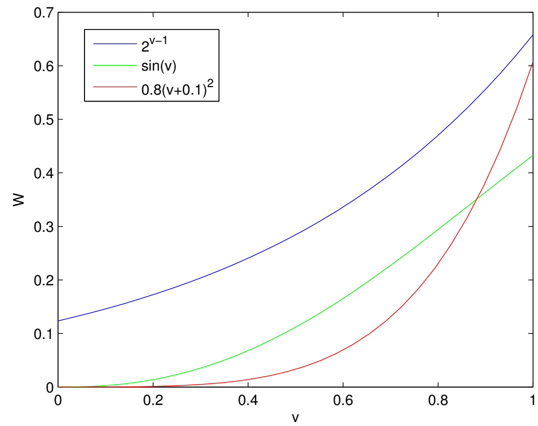
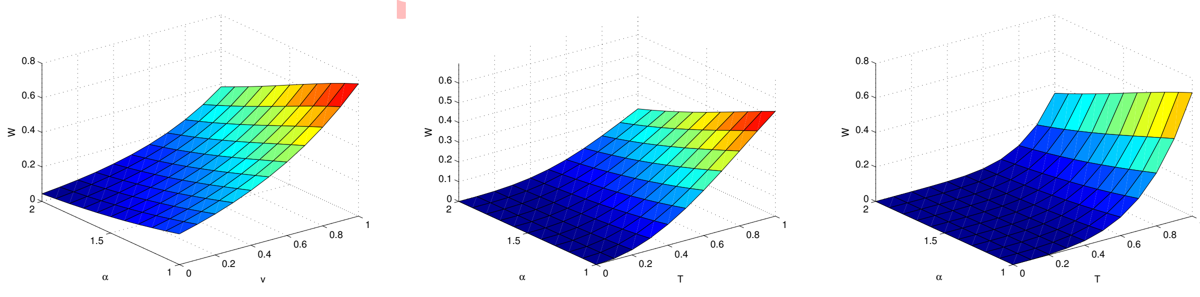


FIGURE 1. Different values of v with different functions of $\psi\,(v)$.

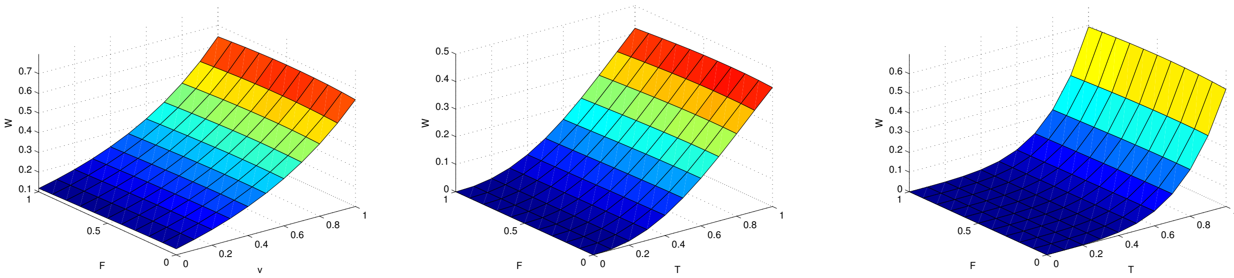
TABLE 1. Numerical values of W for different choices of $\psi(v)$ and α.

f_v	$\psi(v) = 2^{v-1}$		$\psi(v) = \sin(v)$		$\psi(v) = 0.8(v + 0.1)^2$	
	$1 < \alpha < 2$	$\alpha = 1.3$	$1 < \alpha < 2$	$\alpha = 1.3$	$1 < \alpha < 2$	$\alpha = 1.3$
0.0001	0.1777	0.1306	0.0000	0.0000	0.0000	0.0000
0.1	0.1845	0.1527	0.0031	0.0049	0.0002	0.0004
0.2	0.1930	0.1787	0.0150	0.0172	0.0014	0.0018
0.3	0.2037	0.2090	0.0372	0.0354	0.0052	0.0049
0.4	0.2168	0.2447	0.0701	0.0582	0.0146	0.0112
0.5	0.2329	0.2867	0.1133	0.0848	0.0338	0.0228
0.6	0.2525	0.3362	0.1655	0.1138	0.0692	0.0426
0.7	0.2763	0.3946	0.2249	0.1440	0.1293	0.0757
0.8	0.3053	0.4637	0.2888	0.1736	0.2256	0.1291
0.9	0.3406	0.5456	0.3545	0.2008	0.3733	0.2140
0.9999	0.3839	0.6427	0.4184	0.2238	0.5924	0.3466



(A) (B) (C)

FIGURE 2. The values of W when $0 < v < 1$, $1 < \alpha < 2$, $\mathfrak{F} = 0.6$ for (a) $\psi\,(v) = 2^{v-1}$, (b) $\psi\,(v) = \sin\,(v)$, and (c) $\psi\,(v) = 0.8(v + 0.1)^2$ of the Example 5.1.



(A) (B) (C)

Figure 3. Values of W when $0 < v < 1,\ 0 < \mathfrak{F} < 1$, $\alpha = 1.3$, for (a) $\psi\left(v\right) = 2^{v-1}$, (b) $\psi\left(v\right) = \sin\left(v\right)$, and (c) $\psi\left(v\right) = 0.8(v + 0.1)^2$ of the Example 5.1.

Based on these figures, all conditions of Theorems 3.3, 3.4, 4.3, and 4.4 are satisfied. In this case, Theorem 3.3 and Theorem 3.4 prove that the problem previously existed and had a unique solution. Additionally, Theorems 4.3, 4.4 implies that the problem is UH and UHR stable.

6. Conclusion

In conclusion, this paper provides a thorough examination of generalized boundary value problems for ψ-HFDEs with periodic integral boundary conditions, establishing the existence and uniqueness of the solutions through an equivalent fractional integral equation. The use of fractional calculus techniques, BFPT, and LS nonlinear theorem strengthens the analysis. Additionally, the exploration of UH and UHR stabilities adds further depth to the understanding of the solutions. The key findings are illustrated through multiple examples, highlighting the practical implications of the theoretical results research.

Acknowledgment

The authors sincerely thank the esteemed editorial board and reviewers for their invaluable comments and insightful suggestions, which have significantly enhanced the quality and depth of this work.

References

[1] M. S. Abdo, S. K. Panchal, and A. M. Saeed, *Fractional boundary value problem with $\psi-$ Caputo fractional derivative*, Proceedings of the Indian Academy of Sciences: Mathematical Sciences, *129*(5)(2019).

[2] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Communications in Nonlinear Science and Numerical Simulation, *44* (2017), 460–481.

[3] M. Benchohra and J. E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative*, Studia Universitatis Babes-Bolyai Mathematica, *62*(1) (2017), 27–38.

[4] Bhupeshwar, D. K. Patel, M. Gabeleh, and M. E. Samei, *On existence of solutions for $\psi-$ Hilfer type fractional BVP of generalized higher order*, Computational and Applied Mathematics, *43*(3) (2024).

[5] S. Bouriah, D. Foukrach, M. Benchohra, and J. Graef, *Existence and uniqueness of periodic solutions for some nonlinear fractional pantograph differential equations with $\psi-$ Caputo derivative*, Arabian Journal of Mathematics, *10*(3) (2021), 575–587.

[6] M. Chohri, S. Bouriah, A. Salim, and M. Benchohra, *On Nonlinear Periodic Problems with Caputo's Exponential Fractional Derivative* Advances in the Theory of Nonlinear Analysis and Its Applications, *7*(1) (2023), 103–120.

[7] E. M. Elsayed, S. Harikrishnan, and K. Kanagarajan, *On the Existence and Stability of Boundary Value Problem for Differential Equation with Hilfer-Katugampola Fractional Derivative*, Acta Mathematica Scientia, *39*(6) (2019), 1568–1578.

[8] D. Foukrach, S. Bouriah, M. Benchohra, and E. Karapinar, *Some new results for $\psi-$ Hilfer fractional pantograph-type differential equation depending on $\psi-$ Riemann–Liouville integral*, Journal of Analysis, *30*(1) (2022), 195–219.

[9] A. Granas and J. Dugundji, *Fixed point theory*, New York: Springer, (2003), 15-16.

[10] Y. Guo, X. B. Shu, Y. Li, and F. Xu, *The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order* $1 < \beta < 2$, Boundary Value Problems, *2019*(1) (2019).

[11] F. Haddouchi, M. E. Samei, and S. Rezapour, *Study of a sequential $\psi-$Hilfer fractional integro-differential equations with nonlocal BCs*, Journal of Pseudo-Differential Operators and Applications,*14*(4) (2023).

[12] J. Huang and Y. Li, *Hyers-Ulam stability of delay differential equations of first order*, Mathematische Nachrichten, *289*(1) (2016), 60–66.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, elsevier, *204* (2006).

[14] V. Kumar and M. Malik,*Existence and stability of fractional integro differential equation with non-instantaneous integrable impulses and periodic boundary condition on time scales*, Journal of King Saud University - Science, *31*(4) (2019), 1311–1317.

[15] S. K. Ntouyas, B. Ahmad, and J. Tariboon,*Nonlocal Boundary Value Problems for (k,ψ)-Hilfer Fractional Differential Equations and Inclusions*. Foundations, *2*(3) (2022), 681–696.

[16] L.S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko , *The Mathematical Theory of Optimal Processes*, Interscience Publishers John Wiley and Sons, Inc., New York, 1962.

[17] J. V. C. Sousa and E. C. de Oliveira, *On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the $\psi-$ Hilfer operator*, Journal of Fixed Point Theory and Applications, *20*(3) (2018).

[18] J. V. Da C. Sousa, E. C. de Oliveira, and F. G. Rodrigues, *Stability of fractional functional differential equations*,arXiv:1807.06145, (2018).

[19] S. T. M. Thabet, M. Vivas-Cortez, I. Kedim, M. E. Samei, and M. I. Ayari, *Solvability of a $\varrho-$ Hilfer Fractional Snap Dynamic System on Unbounded Domains*, Fractal and Fractional, *7*(8) (2023).

[20] J. Vanterler Da and E. Capelas De Oliveira, *On the $\psi-$ Hilfer fractional derivative*, Communications in Nonlinear Science and Numerical Simulation, *60*(2018), 72–91.

[21] J. Vanterler Da and E. Capelas De Oliveira, *Leibniz type rule: $\psi-$Hilfer fractional operator*. Communications in Nonlinear Science and Numerical Simulation, *77*(2019), 305–311.

[22] J. Vanterler Da, M. Benchohra, and G. M. N'Guérékata,*Attractivity for differential equations systems of fractional order*, *23*(2020),1188–1207.

[23] D. Vivek, K. Kanagarajan, and E. M. Elsayed, *Some Existence and Stability Results for Hilfer-fractional Implicit Differential Equations with Nonlocal Conditions*, Mediterranean Journal of Mathematics, *15*(1) (2018).

[24] J. Wang, A. G. Ibrahim, and D. O'Regan, *Nonemptyness and compactness of the solution set for fractional evolution inclusions with non-instantaneous impulses*, Electronic Journal of Differential Equations, *2019*(37) (2019), 1–17.