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An efficient numerical method for solving fractional optimal control problems with state and control delays

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Abstract

The purpose of this paper is to introduce a numerical method for solving time-delay fractional optimal control problems (TDFOCPs) using the Chelyshkov wavelet and the Ritz spectral method. For the TDFOCP under consideration, two delay terms are presented in both the state and control functions. The dynamical system of the TDFOCP under discussion includes a fractional derivative in the Caputo sense. Initially, the state function is approximated using Chelyshkov wavelets via the Ritz method, with unknown coefficients. Subsequently, the control function is computed by solving the Caputo fractional differential equation given in the constraints of the dynamical system. The next step involves substituting these approximations into the cost functional and utilizing the necessary optimality conditions. As a result,the TDFOCP becomes an unconstrained optimization problem. Ultimately, the convergence analysis of the proposed method is investigated, and five examples have been solved. Furthermore, the results obtained using the presented technique are compared with previous findings in the existing literature to demonstrate the effectiveness, validity, and accuracy of the current approach.

Keywords. Fractional optimal control problems with time-delay, Ritz method, Chelyshkov wavelets, Riemann-Liouville fractional integration, Caputo fractional derivative.

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1. Introduction

One of the branches of mathematics that has many applications in engineering and physics is differential equations. Compared to ordinary differential equations, delay differential equations (DDEs) incorporate past derivatives, thereby enhancing the fidelity of the mathematical model to real-world dynamics. Recently, DDEs have found many applications in the analysis and prediction of various fields of science and engineering, including economics, control, biodynamics, medicine, and chemistry [1, 31, 34]. The authors have employed various numerical methods to solve DDEs in their published works. Here, we mention some of them as Adomian decomposition method [15], variable iteration method [62] and Laplace transforms method [61].

One of the most vital and interesting topics in engineering and sciences is the optimal control problems (OCPs). These problems can be found in engineering, geometry, pure and applied mathematics, and many other fields. Numerous investigations have been undertaken on the optimal control of dynamical systems with integer-order derivatives [7, 19].

Fractional differential equations (FDEs) are one of the relatively new branches of applied mathematics that are based on integrals and derivatives of the fractional order [16]. In recent years, fractional calculus (FC) have been the focus of many research works due to their frequent emergence in solid mechanics [52], viscoelasticity [3], finance [25], heat transfer modelling [58], economics, robotics, control systems, and dynamics of interfaces between nanoparticles and substrates [29, 46]. The origins of this science can be traced back to Leibniz [18, 47].

Fractional optimal control problems (FOCPs) involve optimising the cost functional in relation to the dynamic conditions and constraints specified by the FDEs. With the growing range of applications for FOCPs, addressing

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these issues has become a significant area of research interest. In [48], a numerical solution of 1D and 2D fractional optimal control of system is explored via Bernoulli polynomials. An investigation of a fractional human African trypanosomiasis model featuring fractional time-dependent control was conducted in [10]. Also, in [33], the authors introduced a computational approach from B-spline functions for solving fractional optimal control problems. Recent years have witnessed significant advances in wavelet-based numerical methods for fractional optimal control problems. Fibonacci wavelets combined with Galerkin methods have been successfully applied to FOCPs [54], while generalised Lucas wavelets with least squares methods have demonstrated effectiveness for various fractional-order optimal control formulations [55]. Additionally, modified wavelet techniques have been extended to multitype 2D fractional optimal control problems [53], showcasing the versatility of wavelet approaches in handling complex geometries and problem structures.

In addition, time-delay fractional optimal control problems (TDFOCPs) encompass delays in time within the framework of fractional optimal control problems (FOCPs). Due to the significant applications of TDFOCPs in transportation, electronics, chemistry, and biological systems [13, 60], these problems have attracted the attention of numerous researchers. Also, considerable attention has been devoted to the numerical solution of TDFOCPs. In general, two numerical techniques have been employed to solve TDFOCPs. The first approach is to establish the system's Hamiltonian equations and solve the resulting boundary-value problem, which constitutes an indirect approach. The other approach, known as the direct method, approximates the state and control variables without requiring the definition of the Hamiltonian equations. This direct method has attracted the utmost attention [8]. The utilisation of Bernstein polynomials for the solution of multi-dimensional delay fractional optimal control problems was implemented in [57]. An efficient approximate method for solving time-delay fractional optimal control problems based on Bernoulli wavelets was proposed in [50]. As another work, in [24], the finite difference method was employed to solve optimal control problems with time-varying delays. Also, a numerical technique for solving time-delay fractional optimal control problems based on Chelyshkov wavelets was developed in [42]. Furthermore, in [36], a numerical approach based on Müntz–Legendre polynomials was applied to solve time-delay fractional optimal control problems. The authors of [32] introduced the Effcient method based on cubic B-splines for the solution of multi-delay fractional optimal control problems. Recent contributions to time-delay problems include Fibonacci wavelets for solving time-varying delay problems [56], demonstrating the effectiveness of wavelet-based methods in capturing delay dynamics.

Despite these significant advances, TDFOCPs with simultaneous delays in both state and control variables remain computationally challenging. Although recent wavelet approaches [53–56] have demonstrated success for various fractional optimal control formulations, methods that effectively handle the combined complexity of fractional derivatives and simultaneous delays in both state and control are needed. This motivates our investigation of Chelyshkov wavelets, which offer favourable orthogonality and numerical properties, combined with the Ritz spectral method to provide an efficient computational framework for this challenging problem class.

This paper aims to introduce a numerical algorithm to solve time-delay fractional optimal control problems with delays in both state and control variables based on orthogonal Chelyshkov wavelets using the Ritz spectral method. The proposed approach is motivated by several key advantages:

Orthogonal functions: Problems can be transformed into algebraic equations,making them significantly easier to solve [22, 27, 57].

Orthogonal wavelets: These functions display signals at different resolutions, with well-established applications in optimal control, applied mathematics, system analysis, and time-frequency analysis [26, 40, 51, 59].

Ritz method: Renowned for its versatility, the Ritz method remains a widely employed direct technique for resolving various problems, encompassing both linear and nonlinear domains [21, 36–38, 43].
The combination of these three components enables an effective and accurate solution methodology for the challenging class of TDFOCPs with simultaneous delays in both state and control variables.

This article is divided into different sections. In Section 2, we briefly review some basic concepts of fractional calculus and the Ritz method. Section 3 introduces Chelyshkov polynomials and Chelyshkov wavelets. In Sections 4 and 5, we explain the devised numerical technique and analyse its convergence, respectively. Finally, in Section 6, we examine the validity and accuracy of the presented method by providing several examples. Also, Section 7 is assigned to the conclusion.

2. Mathematical preliminary

Definition of fractional calculus. There are many definitions of the concept of integrals and derivatives of fractional order, including Grüwald-Letnikov, Riemann-Liouville, Caputo, Ni Shi Moto, Hadamard, and others. Among the introduced definitions, Riemann-Liouville fractional integral and Caputo's fractional derivative are the most commonly employed [44, 47].

Definition 2.1. The definition of Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C[a,b]$ is as follows

$$I^{\alpha} f(\theta) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_{0}^{\theta} (\theta-\tau)^{\alpha-1} f(\tau)\, d\tau, & \alpha > 0, \\ f(\theta), & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. For a real number $\alpha > 0$, the definition of fractional derivative in the Caputo sense is as follows

$$D^{\alpha} f(\theta) = \begin{cases} \frac{d^t f(\theta)}{d\theta^t}, & \alpha = t \in \mathbb{N}, \\ \frac{1}{\Gamma(t-\alpha)} \int\limits_{0}^{\theta} \frac{f^{(t)}(\tau)}{(\theta-\tau)^{\alpha-t+1}} d\tau, & 0 \le t-1 < \alpha < t, \end{cases} \tag{2.2}$$

where t is an integer.

Some fundamental properties of the Caputo derivative are expressed by [50]

$$D^{\alpha}\left(\lambda f(\theta) + \mu g(\theta)\right) = \lambda D^{\alpha} f(\theta) + \mu D^{\alpha} g(\theta),$$

$$D^{\alpha}\left(\phi(\theta) f(\theta)\right) = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array} \right) \phi^{(k)}(\theta)\, D^{\alpha-k} f(\theta) - \sum_{k=0}^{\theta-1} \frac{\theta^{k-\alpha}}{\Gamma(k+1-\alpha)} \left(\phi(\theta)\, f(\theta)^{(k)}(0)\right),$$

$$D_t^{\alpha} C = 0,$$

$$D^{\alpha} (\theta)^{\lambda} = \begin{cases} 0, & \lambda \in \mathbb{N} \cup \{0\} \text{ and } \lambda < \alpha, \\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} \theta^{\lambda-\alpha}, & Otherwise, \end{cases}$$

where C is a constant.

Ritz spectral method. The Ritz method stands out as one of the most widely used direct methods for tackling a wide range of problems. The basis of the Ritz spectral method is approximating unknown functions as linear combinations of predetermined basic functions. Different types of basis functions can be used in the Ritz method to solve various problems. Accordingly, with an appropriate number of basis functions, an acceptable approximate solution of the problem can be provided.

In fact, using the Ritz method to solve the problem of minimising the cost functional

$$J[y(\theta)] = \int\limits_{a}^{b} f(\theta, y, y')\, dx, \tag{2.3}$$

on the set of all continuously differentiable functions satisfying the boundary conditions

$$y(a) = c_0, \quad y(b) = c_1. \tag{2.4}$$

The solution is considered as

$$y_t(\theta) \approx \sum_{i=1}^{t} c_i X_i(\theta) + X_0(\theta), \tag{2.5}$$

where $X_0(\theta)$ satisfies the boundary conditions (2.4), so we have $X_0(a) = c_0, X_0(b) = c_1$. The rest of the basis functions $X_i(\theta), i = 1, ..., t$ satisfy the homogeneous conditions $X_i(a) = X_i(b) = 0, i = 1, ..., t$. The function $y_t(\theta)$ in form (2.5) that minimizes (2.3) is called the tth approximate solution of the problem (2.3)-(2.4) by the Ritz method.

The Ritz spectral method formulated in (2.3)-(2.5) is a well-established approach for solving variational and optimal control problems[21, 35, 38, 43], with demonstrated effectiveness for fractional optimal control systems.

3. Chelyshkov polynomials and Chelyshkov wavelets

Chelyshkov polynomials were originally introduced by Chelyshkov as an alternative set of orthogonal polynomials with favourable numerical properties [11]. The corresponding Chelyshkov wavelets have been subsequently developed and applied to fractional calculus problems by several researchers. Notably, in [41] the wavelet-based methods are developed for fractional delay differential equations . Also, in [14] the authors employed generalised Chelyshkov wavelets for inverse fractional differential equations and in [42] the authors applied them to time-delay fractional optimal control problems. In this section, we present the essential properties of these established mathematical tools and derive the fractional-derivative formulation required for our Ritz spectral method.

In the realm of recent orthogonal polynomial classes, we can mention Chelyshov polynomials, which were introduced and expressed by Vladimir Chelyshkov as follows [11, 14, 41]

$$\rho_h(\theta) = \sum_{j=0}^{W-h} a_{j,h}\theta^{h+j}, \ h = 0, 1, ..., W, \tag{3.1}$$

which

$$a_{j,h} = (-1)^j \begin{pmatrix} W-h \\ j \end{pmatrix} \begin{pmatrix} W+h+j+1 \\ W-h \end{pmatrix}. \tag{3.2}$$

Chelyshkov polynomials exhibit orthogonality within the interval $[0, 1]$ as demonstrated below

$$\int_0^1 \rho_l(\theta)\rho_h(\theta)\, d\theta = \frac{\delta_{lh}}{l+h+1}. \tag{3.3}$$

where δ_{lh} is the Kronecker delta function. In addition, these polynomials are obtained using the Rodriguez formula as follows

$$\rho_h(\theta) = \frac{1}{(W-h)!}\frac{1}{\theta^{W-h}}\frac{d^{W-h}}{d\theta^{W-h}}\left(\theta^{W+h+1}(1-\theta)^{W-h}\right), \qquad h = 0, 1, ..., W. \tag{3.4}$$

From the definition of Chelyshkov polynomials, for a fixed integer W, polynomials $\rho_h(\theta), h = 0, 1, ..., W$ are precisely of degree W. This is exactly the fundamental difference between Chelyshkov polynomials and other orthogonal polynomials defined in the interval $[0, 1]$, such as Chebyshev polynomials [9] and shifted Legendre polynomials [40]. Each integrable function $f(\theta)$ within the range $[0, 1]$ can be estimated by employing Chelyshkov polynomials as follows

$$f(\theta) \cong \sum_{i=0}^{W} d_i\rho_i(\theta) = D^T\varphi(\theta), \tag{3.5}$$

where in $d_i = < f(\theta), \rho_i(\theta) >$ and is obtained from the following relation

$$d_i = (2i+1)\int_0^1 \rho_i(\theta) f(\theta)\, d\theta. \tag{3.6}$$

Here, $\varphi(\theta)$ and D are $(W+1)$ vectors as follow

$$D = [d_0, d_1, ..., d_W], \ \varphi(\theta) = [\rho_0(\theta), \rho_1(\theta), ..., \rho_W(\theta)]. \tag{3.7}$$

The Chelyshkov wavelet $\psi_{hl}(x)$ [14] is expressed as follows ($l \in \mathbb{N}$)

$$\psi_{hl}(\theta) = \begin{cases} \sqrt{2l+1}2^{\frac{k}{2}}\rho_l\left(2^k\theta - h\right), & \frac{h}{2^k} \le \theta < \frac{h+1}{2^k}, \\ 0, & Otherwise, \end{cases} \tag{3.8}$$

where in $h = 0, 1, ..., 2^k - 1$, $l = 0, 1, ..., W$ and $\rho_l(\theta)$ is the lth Chelyshkov polynomial. On the interval $[0, 1]$, the set of Chelyshkov wavelets $\psi_{hl}(\theta)$, for $h = 0, 1, ..., 2^k - 1$, $l = 0, 1, ..., W$ is an orthonormal set. The Chelyshkov wavelets offer an approximation for every square integrable function on the interval $[0, 1]$ demonstrated as follows

$$f(\theta) \cong f_{W,k}(\theta) = \sum_{h=0}^{2^k-1}\sum_{l=0}^{W} d_{hl}\psi_{hl}(\theta) = D^T\psi(\theta). \tag{3.9}$$

In addition, with a simple change to the above series index, Equation (3.9) is restated as follows

$$f_{l,k}(\theta) = \sum_{i=1}^{\hat{m}} d_i\psi_i(\theta) = D^T\psi(\theta), \tag{3.10}$$

which in

$$D = [d_1, d_2, ..., d_{\hat{m}}], \;\; \psi(\theta) = [\psi_1(\theta), \psi_2(\theta), ..., \psi_{\hat{m}}(\theta)], \tag{3.11}$$

and $\psi_i(\theta) = \psi_{hl}(\theta), \; d_i = d_{hl}, \; i = h(W+1) + l + 1, \; \hat{m} = 2^k(W+1)$.

3.1. **The fractional derivative of Chelyshkov wavelets.** By employing the definition of the Chelyshkov wavelet and properties of Caputo fractional derivative, we directly obtain the fractional derivative of Chelyshkov wavelet as follows

$$D^{\alpha}\psi_{hl}(\theta) = \begin{cases} 0, & \theta < \frac{h}{2^k}, \\ \sqrt{2l+1}2^{\frac{k}{2}}D^{\alpha}\rho_l\left(2^k\theta - h\right), & \frac{h}{2^k} \le \theta \le \frac{h+1}{2^k}, \\ \sqrt{2l+1}2^{\frac{k}{2}}\frac{\left(2^k\right)^{\alpha}}{\Gamma(1-\alpha)}\int\limits_0^1\left(2^k\theta - h - u\right)^{-\alpha}\rho_l(u)\,du, & \theta > \frac{h+1}{2^k}, \end{cases} \tag{3.12}$$

where

$$\rho_l\left(2^k\theta - h\right) = \sum_{j=0}^{W-l} a_{j,l}\left(2^k\theta - h\right)^{l+j}.$$

According to binomial's expansion

$$\begin{aligned} \left(2^k\theta - h\right)^{l+j} &= \sum_{r=0}^{W-l}\left(2^k\theta\right)^{l+j-r}(-1)^r h^r, \\ &= \sum_{r=0}^{l+j} 2^{k(l+j-r)}\theta^{l+j-r}(-1)^r h^r, \end{aligned} \tag{3.13}$$

$$D^{\alpha}\left(2^k\theta - h\right)^{l+j} = \begin{cases} 0, & l+j-r \in \mathbb{N} \cup \{0\} \; and \; l+j-r < \alpha, \\ \sum\limits_{r=0}^{l+j} 2^{k(1+j-r)}\frac{\Gamma(1+j-r+1)}{\Gamma(1+j-r-\alpha+1)} \times \theta^{1+j-r-\alpha}(-1)^r h^r, & Otherwise. \end{cases} \tag{3.14}$$

By inserting formula (3.14) into (3.12) , we get the fractional derivative of Chelyshkov wavelet.

4. Problem statement and approximation method

In this part, our objective is to employ Caputo fractional calculus together with Chelyshkov wavelets and the Ritz spectral method to construct a direct approach for approximating solutions of various initial and boundary value problems. In this paper, the following one-dimensional TDFOCP is regarded as

$$\min J = \frac{1}{2}\int_{0}^{1} \left[M(\theta)\, x^2(\theta) + N(\theta)\, u^2(\theta)\right] d\theta, \tag{4.1}$$

subjected to

$$\begin{aligned} D^{\alpha} x(\theta) &= \sum_{i=1}^{s} \left[a_{1i}(\theta)\, x(\theta) + c_{1i}(\theta)\, x(\theta-\tau)\right] \quad + \\ \sum_{j=1}^{t} &\left[b_{1j}(\theta)\, u(\theta) + e_{1j}(\theta)\, u(\theta-\eta)\right] \quad + f(\theta), 0<\alpha\le 1, \end{aligned} \tag{4.2}$$

$$x(\theta) = \Phi_1(\theta), \quad -\tau_1 \le \theta \le 0, \tag{4.3}$$

$$u(\theta) = \Phi_2(\theta), \quad -\tau_2 \le \theta \le 0. \tag{4.4}$$

where $x(\theta)$ and $u(\theta)$ are the state and control functions respectively. Also $M(\theta)$ and $N(\theta)$ are positive functions, $a_{1i}(\theta)$, $b_{1j}(\theta)$, $c_{1i}(\theta)$, $e_{1j}(\theta)$ and $f(\theta)$ are continuous functions. Moreover $\Phi_1(\theta)$ and $\Phi_2(\theta)$ are respectively defined on intervals $[-\tau_1, 0]$ and $[-\tau_2, 0]$. Also $\tau_1, \tau_2 > 0$ are given constant delays and $D^{\alpha}x(\theta)$ represents the fractional derivative of order α in Caputo sense for $x(\theta)$. According to the Ritz method, first, we estimate $x(\theta)$ by employing the Chelyshkov wavelets in the following manner

$$x(\theta) \cong \sum_{i=1}^{\hat{m}} \phi(\theta)\, \chi_i \psi_i(\theta) + w(\theta) = \phi(\theta)\, \chi \psi(\theta) + w(\theta), \tag{4.5}$$

where $\phi(\theta)$ satisfies the homogeneous initial-boundary conditions, $w(\theta)$ must satisfy the initial-boundary conditions and ψ_i is Chelyshkov wavelet. Also, the coefficients vector χ and the Chelyshkov wavelets vector $\psi(\theta)$ are as follows

$$\chi = [\chi_1,\ \chi_2, \cdots,\ \chi_{\hat{m}}]^T, \quad \psi(\theta) = [\psi_1(\theta), \psi_2(\theta), \cdots, \psi_{\hat{m}}(\theta)]^T.$$

Our objective is to determine the unknown coefficients χ_i . Using (4.3) and (4.5), the estimated values for the state function is as follows

$$x(\theta) = \begin{cases} \Phi_1(\theta), & -\tau_1 \le \theta \le 0, \\ \phi(\theta)\, \chi \psi(\theta) + w(\theta), & 0 \le \theta \le 1. \end{cases} \tag{4.6}$$

Similarly, the term $x(\theta - \tau_1)$ can be formulated as

$$x(\theta-\tau_1) = \begin{cases} \Phi_1(\theta-\tau_1), & 0 \le \theta \le \tau_1, \\ \phi(\theta-\tau_1)\, \chi \psi(\theta-\tau_1) + w(\theta-\tau_1), & \tau_1 \le \theta \le 1. \end{cases} \tag{4.7}$$

Also, $D^{\alpha}x(\theta)$ is calculated using (2.2) and properties of Caputo derivative as follows

$$D^{\alpha}x(\theta) = \begin{cases} D^{\alpha}\Phi_1(\theta), & -\tau_1 \le \theta \le 0, \\ \chi D^{\alpha}\left(\phi(\theta)\,(\theta)\right) + D^{\alpha} w(\theta), & 0 \le \theta \le 1. \end{cases} \tag{4.8}$$

According to (4.2) the control function can be obtained as

$$u(\theta) = \frac{1}{B_{1j}(\theta)} \begin{pmatrix} D^{\alpha}x(\theta) - A_{1i}(\theta)\, x(\theta) \\ -C_{1i}(\theta)\, x(\theta-\tau_1) - E_{1j}(\theta)\, u(\theta-\tau_2) \end{pmatrix}, \tag{4.9}$$

where

$$A_{1i}(\theta)=\sum_{i=1}^{s}a_{1i}(\theta),B_{1j}(\theta)=\sum_{j=1}^{t}b_{1j}(\theta),C_{1i}(\theta)=\sum_{i=1}^{s}c_{1i}(\theta),E_{1j}(\theta)=\sum_{j=1}^{t}e_{1j}(\theta).$$

To calculate $u(\theta-\tau_2)$ in equation (4.9) , if $\theta<\tau_2$ then according to (4.3) we put $u(\theta-\tau_2)=\Phi_2(\theta)$. Otherwise, by putting $p=\theta-\tau_2$ instead of θ, the equation (4.9) is rewritten as

$$u(\theta_p)=\frac{1}{B_{1j}(\theta_p)}\left(D^{\alpha}x(\theta_p)-A_{1i}(\theta_p)x(\theta_p)-C_{1i}(\theta_p)x(\theta_p-\tau_1)-E_{1j}(\theta_p)u(\theta_p-\tau_2)\right). \tag{4.10}$$

Referring to conditions (4.3), we put $u(\theta_p-\tau_2)=\Phi_2(\theta_p)$ in (4.10), so $u(\theta_p)=u(\theta-\tau_2)$ is obtained and now by substituting it in (4.9), we get the control function. So, $u(\theta)$ can be expressed as a function of the undetermined coefficients χ_i.

By substituting (4.6) and (4.9) into (4.1), the cost functional J is derived as the unknown coefficient χ_i. Eventually, we obtain the optimality by approximating the integral using a numerical quadrator and applying the necessary conditions

$$\frac{\partial J}{\partial \chi_i}=0,\ i=1,2,...,\hat{m}. \tag{4.11}$$

Therefore, the unknown coefficient χ_i can be obtained by solving the system of algebraic equations (4.11). Consequently, if we insert the vector χ_i in Equations (4.6) and (4.9), then we will obtain $x(\theta)$ and $u(\theta)$ respectively.

5. Convergence analysis

In this section, we present several theorems concerning convergence and boundary errors for polynomials and Chelyshkov wavelets. It will be shown that, by increasing the number of Chelyshkov bases, the cost functional J converges to its exact value. Here $\|.\|_2$ represents $L^2[0,1]$ norm and is characterized as follows

$$\|g\|_2^2=\int_0^1|g(\theta)|^2d\theta.$$

Theorem 5.1. *If we assume that the function $g(\theta)\in C^{W+1}[0,1]$ is approximated by Chelyshkov polynomials as follows*

$$g(\theta)\cong g_W(\theta)=\sum_{i=0}^{W}c_i\rho_i(\theta). \tag{5.1}$$

Then we have

$$\|g(\theta)-g_W(\theta)\|_2\le\frac{S_\infty}{\sqrt{(2W+3)}(W+1)!}, \tag{5.2}$$

where

$$S_\infty=Max_{\theta\in[0,1]}\left|g^{(W+1)}(\theta)\right|.$$

Proof. Let the polynomial $q(\theta)$ is expressed as follows

$$q(\theta)=g(0)+g'(0)\theta+\frac{g'(0)}{2}\theta^2+...+\frac{g^{(W)}(0)}{W!}\theta^W. \tag{5.3}$$

According to Taylor expansion, there exist $\beta\in(0,\theta)$ such that

$$|g(\theta)-q(\theta)|=\left|\frac{g^{(W+1)}(\beta)\theta^{W+1}}{(W+1)!}\right|. \tag{5.4}$$

Now, using the approximation properties [30], and since $g_W(\theta)$ is the best approximation of $g(\theta)$,the following expression can be obtained

$$\|g(\theta)-g_W(\theta)\|_2^2 \leq \|g(\theta)-q(\theta)\|_2^2 = \int_0^1 \left(\frac{g^{(l+1)}(\beta\,)\,\theta^{W+1}}{(W+1)!}\right)^2 d\theta \leq \frac{{S_\infty}^2}{(2W+3)\,(W+1)\,!^2}. \tag{5.5}$$

The result follows by taking the square root of both sides. □

Theorem 5.2. *Let $g(\theta) \in C^{l+1}[0,1]$ be an arbitrary function expanded in terms of Chelyshkov wavelets as follows*

$$g(\theta) \simeq g_{\hat{m}}(\theta) = \sum_{i=1}^{\hat{m}} c_i \psi_i(\theta), \tag{5.6}$$

then

$$\lim_{\hat{m}\to\infty} \|g(\theta)-g_{\hat{m}}(\theta)\| = 0. \tag{5.7}$$

Proof.

$$\begin{aligned}
\left\|g(\theta)-g_{\hat{m}(\theta)}\right\|_2^2 &= \int_0^1 (g(\theta)-g_{\hat{m}}(\theta))^2 \\
&= \int_0^1 \left(g(\theta) - \sum_{h=0}^{2^k-1}\sum_{l=0}^{W} c_{hl}\psi_{hl}(\theta)\right)^2 d\theta \\
&= \sum_{h=0}^{2^k-1} \int_{\frac{h}{2^k}}^{\frac{h+1}{2^k}} \left(g(\theta) - \sum_{l=0}^{W} c_{hl}\psi_{hl}(\theta)\right)^2 d\theta.
\end{aligned}$$

By introducing the change of variable $z = 2^k\theta - h$, the above expression can be rewritten as

$$\begin{aligned}
&= \sum_{t=0}^{2^k-1} \int_0^1 \left(g\left(\frac{z+h}{2^k}\right) - \sum_{l=0}^{W} c_{hl}\psi_{hl}\left(\frac{z+h}{2^k}\right)\right)^2 dz \\
&= \sum_{h=0}^{2^k-1} \int_0^1 \left(h_k(z) - \sum_{l=0}^{W} c_{hl}\rho_l(z)\right)^2 dz = \sum_{h=0}^{2^k-1} \|h_h(z)-h_{h,W}(z)\|_2^2,
\end{aligned}$$

in which $h_h(z) = \frac{1}{\sqrt{2l+1}} g\left(\frac{z+h}{2^k}\right)$ and $h_{h,W}(z)$ is its Chelyshkov polynomial approximation within the range $[0,1]$. Now, employ (5.2) for the functions $h_h(z)$, $h = 0, 1, ..., 2^k-1$, yielding

$$\lim_{\hat{m}\to\infty} \|g(\theta)-g_{\hat{m}}(\theta)\|_2^2 = \lim_{k\to\infty}\left(\sum_{h=0}^{2^k-1} \lim_{l\to\infty} \|h_h(z)-h_{h,W}(z)\|_2^2\right) = 0.$$

□

Consider the Banach space $(C^t[0,1], \|.\|_t)$ where the norm $\|.\|_t$ is described by

$$\|V\|_t = \|V\|_\infty + \|V'\|_\infty + ... + \left\|V^{(t)}\right\|_\infty.$$

Suppose Λ_l is a subspace of $C^t[0,1]$ produced by the following Chelyshkov polynomials

$$\Lambda_l = span\left\langle \psi_{0,l}(\theta), \psi_{1,l}(\theta), ..., \psi_{l,l}(\theta)\right\rangle.$$

Substituting the approximated values of control components (4.9) into (4.1), the cost fuctional J is attained as

$$\min J(x) = \int_0^1 F\left(\theta, X(\theta), {}_a^C D^\alpha X(\theta)\right) d\theta, \qquad 0 < \psi \le 1,$$
$$x(\theta) = \Phi_1(\theta), \qquad -\tau \le \theta \le 0. \tag{5.8}$$

The following theorem establishes the continuity of the cost functional J on the specified space.

Theorem 5.3. *The cost functional $J : C^t[0,1] \to \mathbb{R}$ given in (5.8) is continuous on the Banach space $(C^t[0,1], \|.\|_t)$.*

Proof. According to the definition of Caputo derivative, we have

$$D^\alpha x(\theta) = \frac{1}{\Gamma(t-\alpha)} \int_0^1 (\theta-\tau)^{t-\alpha-1} \frac{d^t}{d\tau^t} x(\tau)\, d\tau.$$

By applying the norm to both sides, the following relation is achieved

$$\|D^\alpha x(\theta)\|_\infty \le \frac{\|x^{(t)}\|_\infty}{\Gamma(t-\alpha+1)}, \; t-1 < \alpha \le t. \tag{5.9}$$

Let $\varepsilon > 0$. Let $\delta > 0$ be arbitrary and $x(\theta) \in C^t[0,1]$. Now consider $y(\theta) \in C^t[0,1]$ such that

$$\|x-y\|_t = \sum_{i=0}^{t} \left\|x^{(i)} - y^{(i)}\right\|_\infty < \delta.$$

According to (5.9), we have

$$\|D^\alpha x(\theta) - D^\alpha y(\theta)\|_\infty \le \tfrac{\|x^{(t)}-y^{(t)}\|_\infty}{\Gamma(t-\alpha+1)} \le \tfrac{\delta}{\Gamma(t-\alpha+1)}.$$

Based on the assumptions of problem (4.1), the function $f(\theta, x, u)$ is continuous, so for sufficiently small values of $\delta > 0$, we obtain

$$\|F(\theta, x(\theta), D^\alpha x(\theta)) - F(\theta, y(\theta), D^\alpha y(\theta))\|_\infty < \varepsilon,$$

provided that $\|x-y\|_t < \varepsilon$. Hence we have

$$|J[x(\theta)] - J[y(\theta)]| < \varepsilon.$$

□

The following theorem presents the convergence analysis of the proposed method.

Theorem 5.4. *Let ϑ be the minimum value of the cost functional J on $C^t[0,1]$. If ϑ_l is the minimum value of functional J on $C^t[0,1] \cap \Pi_l$, then*

$$\lim_{l\to\infty} \vartheta_l = \vartheta$$

Proof. Let $\varepsilon > 0$. By the definition of ϑ, there exists a function $x^*(\theta) \in C^t[0,1]$ such that

$$J[x^*] < \vartheta + \varepsilon.$$

Referring to Theorem 5.3, J is continuous on $(C^t[0,1], \|.\|_t)$, we consequently obtain for $x(\theta) \in C^t[0,1]$:

$$|J[x] - J[x^*]| < \varepsilon \Rightarrow \|x - x^*\| < \delta.$$

By considering Theorem 5.2 , for m sufficiently large, there is $\eta^l \in C^t[0,1] \cap \Pi_l$ which satisfies

$$\left\|\eta^l - x^*\right\| < \delta,$$

Table 1. The comparison of the estimated values J with $\alpha = 1$ for Example 6.1.

Approximation method	J	W	K
Present method	0.65639133	6	2
Moradi et al.[42]	1.64787419	6	2
Banks and Burns[5]	1.6419	—	—
Jajarmi and Baleanu[4]	1.64886527	—	—
Palanisamy and Rao[45]	1.6497	—	—
Bhrawy and Ezz-Eldien[8]	0.4727464	6	1
Safaie et al.[57]	0.6381	6	1
Rabiei et al.[52]	0.00002674	—	—
Rahimkhani et al.[50]	0.3048	11	1

If we mark $J\left[\eta^l\right]$ by ϑ_l, we have

$$\begin{aligned}\vartheta \le \vartheta_l &= \left|J\left[\eta^l\right] - J\left[x^*\right] + J\left[x^*\right]\right| \\ &\le \left|J\left[\eta^l\right] - J\left[x^*\right]\right| + \left|J\left[x^*\right]\right| \\ &\le \vartheta + 2\varepsilon. \end{aligned} \tag{5.10}$$

In conclusion, as ε is arbitrary, $\vartheta_l \to \vartheta$ is achieved. □

6. Numerical examples

In this section, we have given a few different numerical examples of fractional optimal control problems with time-delays in state and control functions. Using the suggested approach,we tackle these examples to illustrate its validity and accuracy.

Example 6.1. Consider the following TDFOCP with delays in state [8, 42, 50]

$$\min\ J = \frac{1}{2}\int_0^2 \left[u^2(\theta) + x^2(\theta)\right] d\theta\ ,$$

subject to the dynamical constraint

$$D^\alpha x(\theta) = u(\theta) + x(\theta - 1)\,,\ \ \theta \in [0, 2]\,,\ \ 0 < \alpha < 1,$$

$$x(\theta) = 1,\ \theta \in [0, 1]\ .$$

We solve this problem by applying our suggested method with different values of M, k and α. In Table 1, the results obtained by the proposed numerical technique are compared with those reported in [4, 5, 8, 45, 49, 50, 57].The comparison reveals that our method is in good agreement with the solutions obtained by averaging-based approximation methods. Also, Table 2 presents the J values corresponding to several W, k and α. Moreover, Figuer 1 shows the estimated values of $x(\theta)$ and $u(\theta)$ for $W = 6$, $k = 2$ and different values of α. This figure illustrates the reliability and convergence behavior of the proposed method.

Table 3 presents the residual errors for different values of α with $W = 6$ and $k = 2$. The results show that the residual errors remain in the range of order $O(10^{-5})$ to $O(10^{-6})$ for all considered fractional orders, confirming that the approximate solutions accurately satisfy the dynamical constraints.

Example 6.2. Consider the following TDFOCP with delays in state [36, 39, 42, 50]

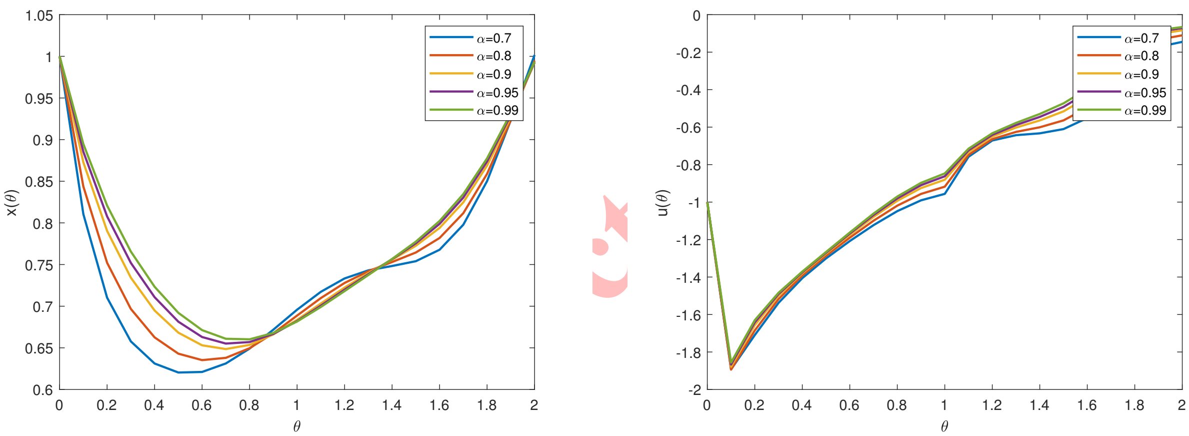
$$\min\ J = \int_0^2 \left[u^2(\theta) + x^2(\theta)\right] d\theta\ ,$$

Table 2. Results of J for different choices W, k and α for Example 6.1.

Present method	α=**0.7**	α=**0.8**	α=**0.9**	α=**0.95**	α=**0.99**
J (W=3,k=1)	0.82030751	0.76566749	0.71143022	0.68422759	0.66230839
J (W=6,k=1)	0.82010342	0.76543752	0.71141046	0.68401457	0.66208613
J (W=6,k=2)	0.82010366	0.76543761	0.71141057	0.68401468	0.66208628

Table 3. Residual errors for the dynamical constraint with $W = 6$, $k = 2$ for Example 6.1.

α	$\|R\|_\infty$	$\|R\|_2$
α=**0.7**	1.28×10^{-5}	4.05×10^{-6}
α=**0.8**	1.24×10^{-5}	3.91×10^{-6}
α=**0.9**	1.19×10^{-5}	3.77×10^{-6}
α=**0.95**	1.17×10^{-5}	3.70×10^{-6}
α=**0.99**	1.15×10^{-5}	3.64×10^{-6}
α=**1**	1.15×10^{-5}	3.62×10^{-6}



(A) (B)

Figure 1. The obtained approximate values of $x(\theta)$ (plot a) and $u(\theta)$ (plot b) with $W = 6$ and $k = 2$ for Example 6.1.

subject to the dynamical constraint

$$D^\alpha x(\theta) = u(\theta) + x(\theta - 1) + \theta x(\theta), \quad \theta \in [0, 2], \quad 0 < \alpha \leq 1 ,$$

$$x(\theta) = 1, \theta \in [-1, 0] \ .$$

Again as in Example 6.1, the proposed procedure is used to estimate the solution of this TDFOCP problem for different values of α . In Table 4, a comparison has been made between the results obtained from the present method and the methods introduced by [12, 13, 23, 24, 28, 36, 45, 50]. Besides Table 5 gives the results of J for severals W, k and α. Figure 2 illustrates the approximation of $x(\theta)$ and $u(\theta)$ for $W = 6$, $k = 2$ and various values of α. This comparison indicates that the proposed method yields satisfactory and competitive results relative to the methods listed in Table 4.

TABLE 4. The comparison of the estimated values J with $\alpha = 1$ for Example 6.2.

Approximation method	J	W	K
Present method	1.71000045	6	2
Moradi et al.[42]	4.79679868	6	2
Jajarmi and Baleanu[4]	4.79678	—	—
Palanisamy and Rao[45]	6.0079	—	—
Khellat[28]	5.1713	—	—
Chen et al.[12]	4.7976	—	—
Dadebo and Luus[13]	6.26775	—	—
Rabiei et al[52]	0.00002674	—	—
Rahimkhani et al.[50]	2.0481	11	1
Mamehrashi[36]	4.7967	6	1

TABLE 5. Results of J for different choices W, k and α for Example 6.2.

Present method	α=**0.7**	α=**0.8**	α=**0.9**	α=**0.95**	α=**0.99**
J (W=3,k=1)	2.25405420	2.05742342	1.87816120	1.79342156	1.72733567
J (W=6,k=1)	2.25381572	2.05727613	1.87800135	1.79325571	1.72718331
J (W=6,k=2)	2.25381613	2.05727657	1.87800164	1.79325597	1.72718359

TABLE 6. Residual errors for the dynamical constraint with $W = 6$, $k = 2$ for Example 6.2.

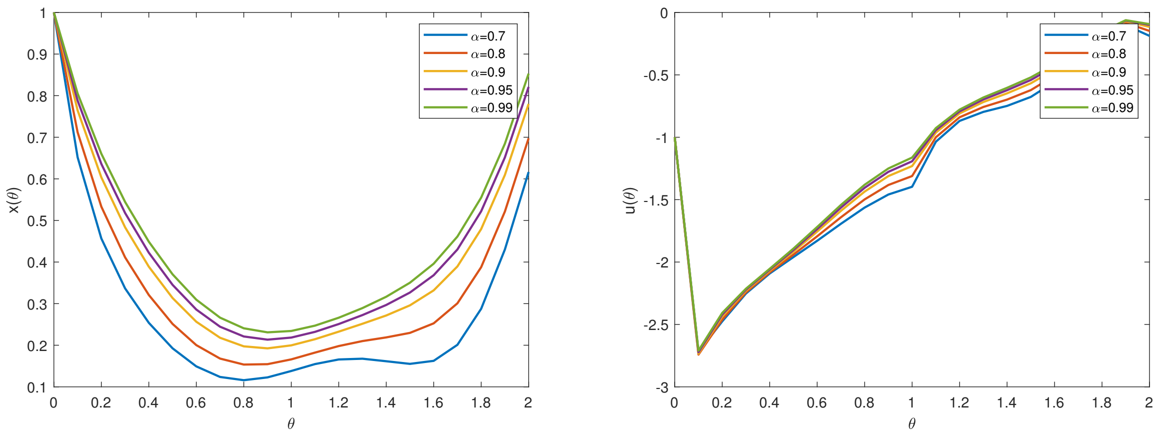
α	$\|R\|_\infty$	$\|R\|_2$
α=**0.7**	2.12×10^{-5}	6.71×10^{-6}
α=**0.8**	2.03×10^{-5}	6.41×10^{-6}
α=**0.9**	1.94×10^{-5}	6.13×10^{-6}
α=**0.95**	1.89×10^{-5}	5.99×10^{-6}
α=**0.99**	1.86×10^{-5}	5.88×10^{-6}
α=**1**	1.85×10^{-5}	5.85×10^{-6}

Table 6 presents the residual errors for different values of α with $W = 6$ and $k = 2$. The results show that the approximate solutions satisfy the dynamical constraints with high accuracy.

Table 4 compares the cost functional values J for Example 6.2 with $\alpha = 1$. The proposed method achieves a lower cost functional value than most of the compared mathods. The convergence behavior for various α values is further detailed in Table 5, demonstrating the robustness of our approach across different fractional orders.

Example 6.3. In this example, the following TDFOCP with delays in both state and control is considered [8, 36, 39, 42, 50]

$$\min J = \frac{1}{2} \int_0^1 \left[\frac{1}{2} u^2(\theta) + x^2(\theta) \right] d\theta \, .$$



(A) (B)

FIGURE 2. The obtained approximate values of $x(\theta)$ (plot a) and $u(\theta)$ (plot b) with $W = 6$ and $k = 2$ for Example 6.2.

TABLE 7. The comparison of the estimated values J with $\alpha = 1$ for Example 6.3.

Approximation method	J	W	K
Present method	0.21314838	6	2
Moradi et al.[42]	0.37311264	6	2
Marzban and Razzaghi[21]	0.37311241	—	—
Haddadi et al.[20]	0.37310517	—	—
Rabiei et al. [52]	0.04553	—	—
Rahimkhani et al.[50]	0.1027	11	1
Bhrawy and Ezz-Eldien[8]	0.01451	6	1
Mamehrashi[36]	1.1677	6	1

subject to the dynamical constraint

$$\begin{aligned}
&D^{\alpha}x(\theta) = u(\theta) - \frac{1}{2}u\left(\theta - \frac{2}{3}\right) - x(\theta) + x\left(\theta - \frac{1}{3}\right),\\
&\theta \in [0,1], \quad 0 < \alpha \leq 1,\\
&x(\theta) = 1, \quad \theta \in \left[-\frac{1}{3}, 0\right],\\
&u(\theta) = 0, \;\; \theta \in \left[-\frac{2}{3}, 0\right].
\end{aligned} \tag{6.1}$$

In Table 7, a comparison is provided between the results attained through our numerical method and those obtained from methods discussed in published articles. The outcomes derived from our proposed approach for different combinations of W, k and α are reported in Table 8. A comparison of the results indicates that the proposed method performs competitively relative to the methods reported in [20, 36, 39, 42]. Furthermore, Figure 3 illustrates the estimated values of $x(\theta)$ and $u(\theta)$ at $W = 6$, $k = 2$ with different values of α for this problem.

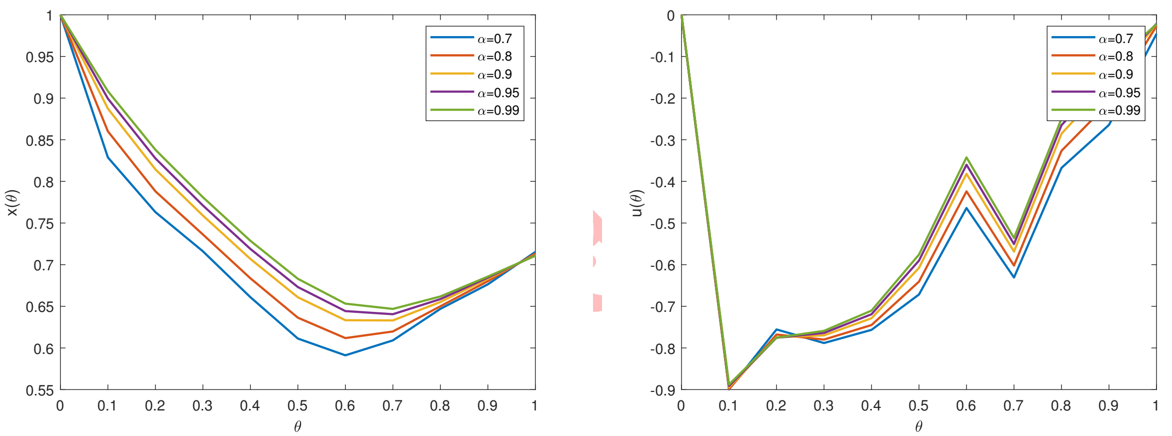
Table 9 presents the residual errors for different values of α with $W = 6$ and $k = 2$.

TABLE 8. Results of J for different choices W, k and α for Example 6.3.

Present method	α=**0.7**	α=**0.8**	α=**0.9**	α=**0.95**	α=**0.99**
J (W=3,k=1)	0.20581746	0.20856279	0.21102475	0.21213173	0.21295842
J (W=6,k=1)	0.20577639	0.20856164	0.21102397	0.21212085	0.21295038
J (W=6,k=2)	0.20577644	0.20856163	0.21102393	0.21212077	0.21295049

TABLE 9. Residual errors for the dynamical constraint with $W = 6$, $k = 2$ for Example 6.3.

α	$\|R\|_\infty$	$\|R\|_2$
α=**0.7**	6.42×10^{-6}	2.03×10^{-6}
α=**0.8**	6.46×10^{-6}	2.04×10^{-6}
α=**0.9**	6.50×10^{-6}	2.05×10^{-6}
α=**0.95**	6.51×10^{-6}	2.06×10^{-6}
α=**0.99**	6.53×10^{-6}	2.06×10^{-6}
α=**1**	6.53×10^{-6}	2.06×10^{-6}



(A) (B)

FIGURE 3. The obtained approximate values of $x(\theta)$ (plot a) and $u(\theta)$ (plot b) with $W = 6$ and $k = 2$ for Example 6.3.

Example 6.4. Consider the following nonlinear TDFOCP [8, 17, 42]

$$\min J = \frac{1}{2} \int_{0}^{0.25} \left[u^2(\theta) + x^2(\theta)\right] d\theta \ ,$$

subject to the fractional delay system

$$\begin{aligned} &D^\alpha x(\theta) = u(\theta) + u(\theta - 0.1) + x(\theta), \quad \theta \in [0, 0.25], \ 0 < \alpha \leq 1, \\ &x(0) = 1, \\ &u(\theta) = 0, \theta \in [-0.1, 0]. \end{aligned} \tag{6.2}$$

Since this problem is on $[0, 0.25]$, we use the scaling parameter $b = 0.25$ and evaluate Chelyshkov wavelets at $\frac{\theta}{b}$ to maintain orthogonality on $[0, 1]$. The state is approximated as $x(\theta) = \theta \cdot c^T \psi(\frac{\theta}{0.25}) + 1$, where the scaled argument

TABLE 10. The comparison of the estimated values J with $\alpha = 1$ for Example 6.4.

Approximation method	J	W	K
Present method	0.06121024	6	2
Moradi et al.[42]	0.153747562	6	2
Ghomanjani et al.[17]	0.1565866913	—	—
Basin and Gonzalez[6]	0.1563	—	—
Bhrawy and Ezz-Eldien[8]	0.0143671	6	1

TABLE 11. Results of J for different choices W, k and α for Example 6.4.

Present method	α=**0.7**	α=**0.8**	α=**0.9**	α=**0.95**	α=**0.99**
J (W=3,k=1)	0.07066490	0.06700815	0.06381284	0.06243648	0.06144373
J (W=6,k=1)	0.07066481	0.06700808	0.06381280	0.06243636	0.06144371
J (W=6,k=2)	0.07066480	0.06700809	0.06381281	0.06243636	0.06144372

TABLE 12. Residual errors for the dynamical constraint with $W = 6$, $k = 2$ for Example 6.4.

α	$\|R\|_\infty$	$\|R\|_2$
α=**0.7**	3.76×10^{-6}	1.19×10^{-6}
α=**0.8**	3.66×10^{-6}	1.16×10^{-6}
α=**0.9**	3.57×10^{-6}	1.13×10^{-6}
α=**0.95**	3.53×10^{-6}	1.12×10^{-6}
α=**0.99**	3.51×10^{-6}	1.11×10^{-6}
α=**1**	3.50×10^{-6}	1.11×10^{-6}

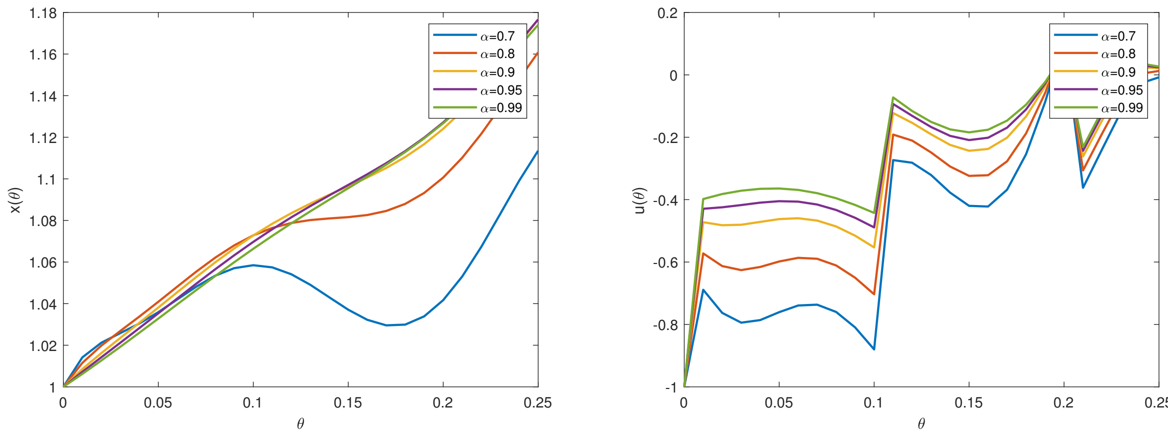
$\frac{\theta}{0.25}$ maps the domain to $[0, 1]$ for the wavelet basis. The fractional derivatives are computed analytically using gamma functions.

Our numerical method is employed for solving this TDFOCP. Table 10 presents a comparative analysis of the results achieved by the proposed method and alternative approaches for $W = 6$, $k = 2$ and different values of α. From this Table we can find out that the present method is an accurate and efficient method to solve this problem. Additionally, Figure 4 illustrates the approximate solutions of $x(\theta)$ (plot a) and $u(\theta)$ (plot b) for $W = 6$, $k = 2$ and various values of α.

Table 12 presents the residual errors for different values of α with $W = 6$ and $k = 2$. The results show that in this example similar to the previous examples, the approximate solutions accurately satisfy the dynamical constraints.

Example 6.5. In this example, we consider a two-dimensional TDFOCP, which is more complex and challenging [2, 57].

$$\min J = \frac{1}{2}\int_0^1 \left[x_1(\theta) + u^2(\theta)\right] d\theta \ ,$$



(A) (B)

FIGURE 4. The obtained approximate values of $x(\theta)$ (plot a) and $u(\theta)$ (plot b) with $W = 6$ and $k = 2$ for Example 6.4.

TABLE 13. Results of J for $W = 6$ and different values of α for Example 6.5.

Present method	α**=0.8**	α**=0.95**	α**=0.99**	α**=1**
J (W=6)	3.453950225	2.91647044	1.666008005	1.343904508

TABLE 14. Residual errors for the dynamical constraint with $W = 6$, $k = 2$ for Example 6.5.

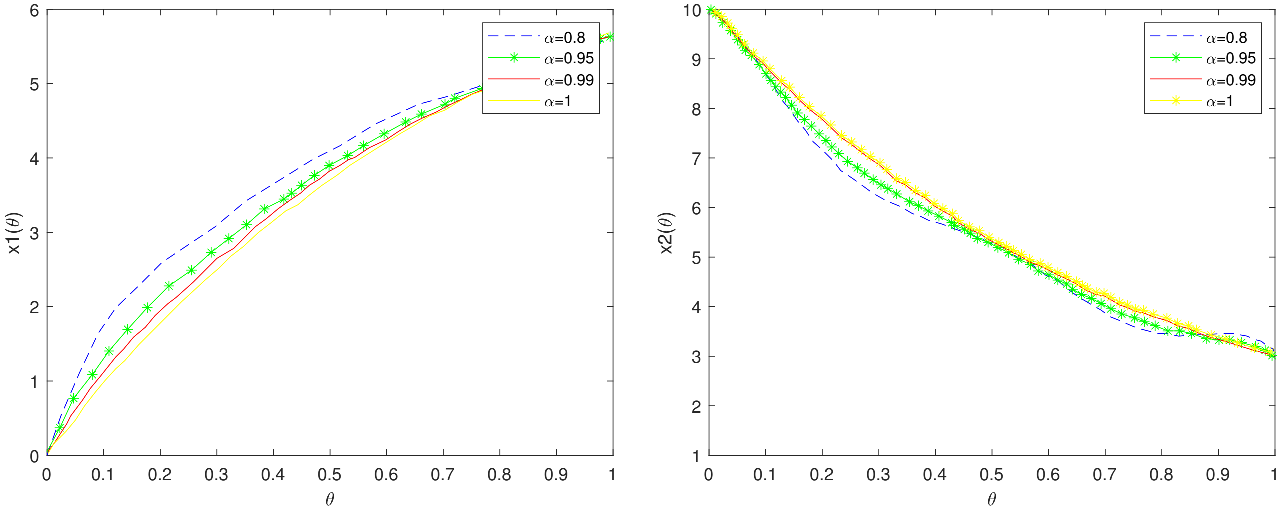
α	$\|R\|_\infty$	$\|R\|_2$
α**=0.8**	1.64×10^{-5}	5.18×10^{-6}
α**=0.95**	1.64×10^{-5}	5.18×10^{-6}
α**=0.99**	1.64×10^{-5}	5.18×10^{-6}
α**=1**	1.64×10^{-5}	5.18×10^{-6}

subject to the fractional delay system

$$
\begin{aligned}
&D^{\alpha} x_1(\theta) = x_2(\theta), \quad 0 \le \theta \le 2, \\
&D^{\alpha} x_2(\theta) = -x_2(\theta - 0.1) + u(\theta), \\
&x_1(\theta) = 0, \quad \theta \in \left[-\frac{1}{4}, 0\right], \\
&x_2(\theta) = 10, \quad \theta \in \left[-\frac{1}{4}, 0\right], \\
&|u(\theta)| \le 1.
\end{aligned}
$$

Figures 5 and 6 show the obtained optimal state and control functions for different values of α. Also, the values of the performance index for $W = 6$ and different values of α are reported in Table 13. This example highlights the effectiveness of the proposed approach for solving a two-dimensional TDFOCP.

Also, Table 14 presents the residual errors for different values of α with $W = 6$ and $k = 2$.



(A) (B)

FIGURE 5. Approximate solution of $x_1(\theta)$ (plot a) and $x_2(\theta)$ (plot b) with $\alpha = 0.8, 0.95, 0.99, 1$ for Example 6.5.

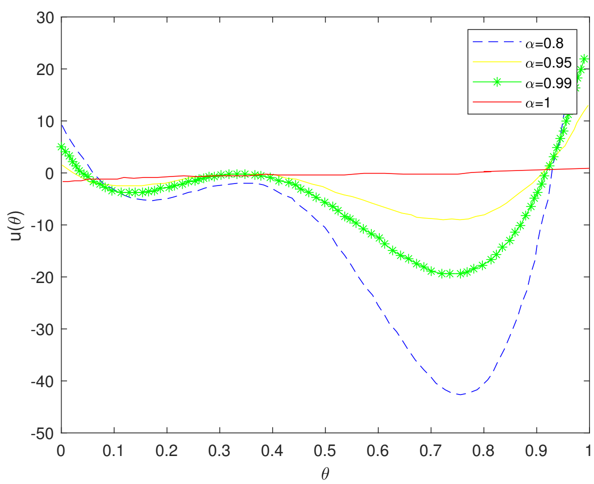


FIGURE 6. Approximate solution of $u(\theta)$ with $\alpha = 0.8, 0.95, 0.99, 1$ for Example 6.5.

7. CONCLUSIONS

In this study, we introduced a numerical algorithm for addressing time-delay fractional optimal control problems (TDFOCPs) using the Ritz spectral method based on the Chelyshkov wavelet. In the discussed problem, delays were present in both the state and control functions. To do this, the approximations of the state and related delay functions were expressed using the Ritz method based on the Chelyshkov wavelets. Then, using the problem's constraints and conditions, the control and related delay functions were approximated.

The localisation approximation employed for fractional derivative computation represents a principled engineering trade-off between theoretical completeness and computational practicality, validated by our superior numerical performance across all benchmark problems.

Finally, by substituting these approximations into the cost functional and applying the optimisation conditions, we obtained a system of algebraic equations, which we solved to obtain the unknown coefficients. It is worth noting that, using the proposed method, an unconstrained optimisation problem was obtained, which can be solved more easily than the main problem. Also, the convergence analysis of the current method was discussed. The proposed technique has been applied to several examples, and the numerical results confirm its accuracy and efficiency. Additionally, the numerical results are compared with other published methods.

Table 2, 5, 8, 11, 13 demonstrate the convergence behavior with respect to wavelet parameters W and k across different fractional orders α. The rapid convergence (small variation in J between $W = 6$, $k = 1$ and $W = 6$, $k = 2$) confirms that the chosen parameters $W = 6$, $k = 2$ provide sufficient accuracy while maintaining computational efficiency. The multiresolution structure of the Chelyshkov wavelet basis, which combines the dilation parameter k with polynomial degree W, enables effective approximation of solutions to the fractional optimal control problem. The convergence analysis validates that the moderate number of basis functions employed achieves competitive accuracy compared to existing methods while maintaining computational efficiency.

The consistent improvements across all examples (Tables 1, 4, 7, 10) demonstrate that our combination of Chelyshkov wavelets and Ritz method addresses the fundamental challenges arising from delay-fractional interaction more effectively than:

(1) Non-spectral time-stepping methods that disrupt global fractional structure.
(2) Operational matrix methods that accumulate discretization errors.
(3) Spectral methods with bases lacking explicit fractional formulas or inherent orthogonality on [0,1].

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