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# Numerical analysis of a nonlinear fractional Cable model using the Galerkin spectral element method

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**Abstract**

In this paper, we propose an efficient numerical method for solving the nonlinear fractional cable model, which includes multi-term time-fractional derivatives in the Riemann–Liouville sense. The model describes memory effects and anomalous diffusion through distinct fractional orders in both the diffusion and reaction terms. To discretize the problem, we employ a high-order finite difference scheme in time and a Legendre Galerkin spectral element method in space. We prove the stability and convergence of the time-discrete scheme and provide an error estimate for the fully discrete scheme. Numerical experiments are conducted to rigorously investigate the accuracy and efficiency of the proposed approach, demonstrating its effectiveness across various parameter settings.

**Keywords.** Nonlinear fractional Cable model, Galerkin spectral element method, Legendre polynomials.

**2010 Mathematics Subject Classification.** 65L05, 34K06, 34K28.



## 1. Introduction

Fractional partial differential equations (PDEs) are mathematical models that include derivatives of non-integer order and provide a more accurate representation of complex physical phenomena compared to integer-order models. These equations have attracted considerable attention in recent years due to the development of efficient numerical methods for solving fractional models arising in various scientific and engineering applications [14–16, 37, 38]. Various numerical approaches have been developed for solving fractional differential equations, including high-order and efficient computational schemes [1, 18, 23, 29, 33, 36]. These equations have gained significant attention due to their ability to model real-world phenomena in various fields [5, 12, 17, 20, 21]. The nonlinear fractional cable equation is one of the fundamental fractional mathematical models and represents an extended form of the classical cable equation, arising from the incorporation of fractional derivatives [19, 22]. Unlike traditional models that assume normal diffusion, the fractional approach captures long-range interactions and hereditary properties. These factors play a significant role in synaptic transmission and signal propagation [39]. A wide range of numerical methods such as finite difference methods, spectral collocation techniques, meshless approaches, and finite element methods have been introduced to efficiently approximate solutions to the fractional cable equation [7, 11, 24, 25, 30]. In recent years, progress in numerical analysis has paved the way for hybrid numerical schemes that offer better accuracy and computational efficiency when dealing with the nonlinear fractional cable equation. For instance, combining finite difference methods with spectral techniques has helped improve stability while lowering computational costs [26].

Spectral methods for differential equations belong to a family of weighted residual methods. Weighted residual methods are a special group of approximation techniques in which the residual is minimized in a specific manner and then continued using one of the Galerkin, Petrov-Galerkin, collocation, or Tau methods [31, 34, 40]. In [43], a fast spectral Galerkin method is investigated to solve the complex non-linear Ginzburg-Landau equation involving a fractional Laplacian. A fast algorithm for solving high-order fractional diffusion problems using a spectral fractional Laplacian method is studied in [44]. The Legendre-Tau spectral method based on Legendre polynomials is used for

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the numerical solution of the two-sided space-time Caputo fractional diffusion-wave equation in [8]. The spectral collocation method is applied to solving the fractional nonlinear Schrödinger equation with appropriate initial and boundary conditions in [45]. In [28], a hybrid method for solving the fractional Black-Scholes problem is introduced, where the discretization of time is based on a formula of second order finite difference, and the discretization of space is performed using a spectral method. The spectral element method (SEM) is a combination of the finite element method and spectral methods. This method has a higher order of accuracy than spectral methods and retains the geometric flexibility of the finite element method. Essentially, in this method, the finite element technique is applied within each element with unknown values at specific points. In [2], a Legendre spectral element method (LSEM) is developed to solve nonlinear stochastic advection-reaction-diffusion models using Legendre basis functions. In [35], a numerical method for solving the two- and three-dimensional nonlinear fractional Klein-Gordon equation is examined using the LESM. The LESM for solving the one-dimensional Sine-Gordon equation is studied in [27], where a second-order Leapfrog method is used for time discretization. In [13], a numerical approach for solving the modified nonlinear Benjamin-Bona-Mahony-Burgers equation is proposed, combining a finite difference formula and the LESM.

The study of fractional-order models has gained significant attention due to their ability to accurately describe various complex physical and biological processes. In particular, the nonlinear fractional cable model provides an effective mathematical framework for modeling electrical signal propagation in neurons and other biological tissues. However, solving such models numerically presents challenges due to the presence of multi-term time-fractional derivatives and the inherent nonlocality of fractional operators. In this paper, we propose a highly accurate and efficient numerical method specifically designed for the nonlinear fractional cable model. Our approach addresses the computational complexities associated with multi-term time-fractional derivatives while ensuring both stability and convergence. The key contributions of this work can be summarized as follows:

- We introduce an efficient numerical method for solving the nonlinear fractional cable model, incorporating multi-term time-fractional derivatives in the Riemann-Liouville sense.
- The proposed method employs a high-order finite difference scheme for temporal discretization and the LESM for spatial discretization, resulting in both semi-discrete and fully discrete schemes.
- We establish the unconditional stability and convergence of the time-discrete scheme and derive an error estimate for the fully discrete system.

The structure of the remainder of this paper is as follows. In Section 2, we introduce some preliminary concepts necessary for the subsequent analysis. In Section 3, we present a finite difference scheme for temporal discretization and analyze its unconditional stability. We also apply the spectral element method (SEM) for spatial discretization and provide an error estimate for the fully discrete scheme. In Section 4, we present numerical experiments to demonstrate the accuracy of the proposed method. Finally, Section 5 concludes the paper with a summary of the main findings.



## 2. Notation and Preliminary Concepts

In this section, we introduce the notation and fundamental concepts required for the SEM. We define the partitioning of the computational domain, the transformation between the physical and reference elements, and the basis functions used for interpolation. Additionally, we present the formulation of mass, stiffness, and advection matrices, which play a critical role in the discretization of differential equations. All the prerequisite concepts for this section are provided in [2, 13, 27, 35].

Suppose $\mathcal{P}_E$ is a partition of $\mathcal{I} = (a, b)$ with $\mathbb{N}_e$ elements:

$$\mathcal{P}_E : a = x^0 < x^1 < ... < x^{\mathbb{N}_e - 1} < x^{\mathbb{N}_e} = b.$$

Let the computational domain be partitioned by ordered nodal points $\{x^{(i)}\}_{i=0}^{\mathbb{N}_e}$. An individual element is formed by two successive nodes, and the element associated with index $e$ is given by the interval

$$\mathcal{I}_e := \left\{ \widehat{x} \in \mathbb{R} \;\middle|\; x^{(e-1)} \le \widehat{x} \le x^{(e)} \right\}, \qquad e = 1, \dots, \mathbb{N}_e.$$

For simplicity in computations, the element $\mathcal{I}_e$ is mapped to the reference element $\overline{\mathcal{I}} = \{\xi | -1 \leq \xi \leq 1\}$ using the function

$$\xi(\widehat{x}) = 2\frac{\widehat{x} - x^{(e-1)}}{h_e} - 1,$$

where $h_e$ represents the element size and $h_e = x^{(e)} - x^{(e-1)}$.
The inverse of the above function is given by

$$\widehat{x}(\xi) = \frac{1}{2}[(h_e)\xi + (x^{(e)} + x^{(e-1)})].$$

The Jacobian of the mapping is given by

$$\mathcal{J}_e = \frac{\partial x}{\partial \xi} = \frac{x^{(e)} - x^{(e-1)}}{2} = \frac{h_e}{2}.$$

When high-degree Lagrange polynomials are employed for element-wise function approximation, spurious oscillations associated with the Runge phenomenon can arise. To mitigate this issue, the interpolation is constructed using Lagrange basis functions defined at Gauss–Lobatto–Legendre (GLL) nodes.
The Gauss-Lobatto-Legendre points $\xi_j$, $(j = 0, ..., m_e)$ on $\overline{\mathcal{I}}$ are the roots of the polynomial

$$\overline{Lo}_{m_e-1}(\xi) = (1-\xi^2)Lo_{m_e-1}(\xi) \text{ or } (1-\xi^2)L'_{m_e}(\xi),$$

where $Lo_{m_e-1}(\xi)$ is the Legendre-Lobatto polynomial of degree $m_e - 1$. where $L_{m_e}(\xi)$ stands for Legendre polynomial with degree $m_e$.
Let $\{\xi_i\}_{i=0}^{m_e}$ denote the Gauss–Lobatto–Legendre (GLL) nodes. The corresponding Lagrange basis functions of degree $m_e$ are expressed as

$$\ell_i(\xi) = \frac{(\xi^2 - 1)\, L'_{m_e}(\xi)}{m_e(m_e+1)\, L_{m_e}(\xi_i)\, (\xi - \xi_i)}, \qquad i = 0, \ldots, m_e.$$

The function $u$ in the $e$-th element is approximated using a spectral element of degree $m_e$ as follows:

$$u^{\{e\}}_{m_e}(\widehat{x}) = \sum_{i=0}^{m_e} u^{\{e\}}_{m_e}(\widehat{x}_i)\ell_i(\widehat{x}).$$

The fundamental matrices required for discretizing differential operators are computed using integration over the reference element.

- **Mass Matrix**: The mass matrix is a symmetric matrix M that expresses the connection between the time derivative of the generalized coordinate vector of a system and the kinetic energy. The mass matrix is computed as:

$$(\mathbb{M}^{\{e\}})_{ij} = \int_{x^{(e-1)}}^{x^{(e)}} \ell_i(\widehat{x})\ell_j(\widehat{x})\, d\widehat{x} = \mathcal{J}_e \int_{-1}^{1} \ell_i(\xi)\ell_j(\xi)\, d\xi, \quad 0 \leq i, j \leq m_e. \tag{2.1}$$

- **Diffusion (Stiffness) Matrix**: For the numerical solution of partial differential equations, the stiffness matrix is a matrix that represents the system of linear equations that must be solved inorder to ascertain an approximate solution to the differential equation. The diffusion (stiffness) matrix is given by:

$$(\mathbb{D}^{\{e\}})_{ij} = \int_{x^{(e-1)}}^{x^{(e)}} \frac{d\ell_i}{d\widehat{x}}(\widehat{x})\frac{d\ell_j}{d\widehat{x}}(\widehat{x})\, d\widehat{x} = \frac{1}{\mathcal{J}_e}\int_{-1}^{1} \frac{d\ell_i}{d\xi}(\xi)\frac{d\ell_j}{d\xi}(\xi)\, d\xi. \tag{2.2}$$

The Gaussian quadrature formula associated with the Gauss-Lobatto-Legendre (GLL) points $\{\xi_j\}_{j=0}^{m_e}$ is utilized to compute the integral terms in (2.1) and (2.2). The local mass and stiffness matrices are evaluated by applying the Gauss–Lobatto–Legendre quadrature rule. Specifically, the entries of the element matrices for element $e$ take the form

$$(\mathbb{M}^{(e)})_{ij} = \mathcal{J}_e\, w_i\, \delta_{ij}, \qquad (\mathbb{D}^{(e)})_{ij} = \frac{1}{\mathcal{J}_e}\sum_{s=0}^{m_e} c^{m_e}_{is}\, c^{m_e}_{js}\, w_s, \quad i, j = 0, \ldots, m_e.$$

The quadrature weights corresponding to the GLL nodes $\{\xi_s\}_{s=0}^{m_e}$ are given by

$$w_s = \frac{2}{m_e(m_e+1)\,[L_{m_e}(\xi_s)]^2}, \qquad s = 0, \dots, m_e,$$

where $L_{m_e}(\xi)$ denotes the Legendre polynomial of degree $m_e$.

In the spectral collocation method, the differentiation matrix is denoted by $\mathbb{D}^{\{m_e\}}$. The matrix $\mathbb{D}^{\{e\}}$ coincides with the transpose of $\mathbb{D}^{\{m_e\}}$, whose entries are given by:

$$(\mathbb{D}^{\{m_e\}})_{ij} = \frac{d\ell_j}{d\xi}\bigg|_{\xi=\xi_i} = \begin{cases} \frac{L_{m_e}(\xi_i)}{L_{m_e}(\xi_j)} \times \frac{1}{\xi_i-\xi_j}, & i \neq j, \\ -\frac{(m_e+1)m_e}{4}, & i = j = 0, \\ \frac{(m_e+1)m_e}{4}, & i = j = m_e, \\ 0, & \text{Otherwise.} \end{cases}$$

Owing to the cardinal (interpolation) property of the basis functions, the resulting element mass matrix is diagonal. This diagonal structure leads to a substantial reduction in computational effort when solving time-dependent problems with the local spectral element method (LSEM).

## 3. A Galerkin Spectral Element Formulation for the Fractional Cable Model

In this section, we develop a Galerkin spectral element formulation for the fractional cable model. We consider the following fractional cable model:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m_1} \kappa_{1,i}\, {}^{\mathbb{RL}}\mathbb{D}_t^{1-\alpha_i} \frac{\partial^2 u}{\partial x^2} - \sum_{j=1}^{m_2} \kappa_{2,j}\, {}^{\mathbb{RL}}\mathbb{D}_t^{1-\beta_j} u + f(u(t,x),t,x), \qquad (t,x) \in J \times \mathcal{I}, \tag{3.1}$$

$$u(t,x) = 0, \qquad t \in J,\ x \in \partial\mathcal{I}, \tag{3.2}$$

$$u(0,x) = g(x), \qquad x \in \overline{\mathcal{I}}, \tag{3.3}$$

where

$$0 < \alpha_{m_1} < \alpha_{m_1-1} < \cdots < \alpha_2 < \alpha_1 < 1, \quad 0 < \beta_{m_2} < \beta_{m_2-1} < \cdots < \beta_2 < \beta_1 < 1,$$

$\kappa_{1,i}$, $i = 1, 2, \dots, m_1$, and $\kappa_{2,j}$, $j = 1, 2, \dots, m_2$, are positive constants, $J = (0, T]$ is the time interval with $0 < T < +\infty$, and the functions $f$ and $g$ are assumed to be sufficiently smooth.
The operator ${}^{\mathbb{RL}}\mathbb{D}_t^{1-\alpha} u$ denotes the Riemann–Liouville fractional derivative of order $1-\alpha$, which is defined as

$${}^{\mathbb{RL}}\mathbb{D}_t^{1-\alpha} u = \frac{\partial}{\partial t} \int_0^t \mathrm{K}_\alpha(t-s) u(s) ds,$$

where $\mathrm{K}_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$.

3.1. **Temporal Discretization.** We first aim to discretize the time variable in the equation presented in (3.1). The time step, denoted by $\delta t$, is defined as $\delta t = \frac{T}{N}$. Specifically, the discrete time points are given by

$$t_n = n\,\delta t, \quad n = 0, 1, \dots, N.$$

**Lemma 3.1.** *([3, 41]) The Riemann-Liouville fractional derivatives in time, using the weighted shifted Grunwald derivative of second-order, can be approximated at $t = t_{n+1}$ as:*

$${}^{\mathbb{RL}}\mathbb{D}_t^{1-\alpha_i} \frac{\partial^2 u^{n+1}}{\partial x^2} \approx \delta t^{-(1-\alpha_i)} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \frac{\partial^2 u^{n+1-k}}{\partial x^2} + O(\delta t^2),\ i = 1, 2, \cdots, m_1,$$

*and*

$${}^{\mathbb{RL}}\mathbb{D}_t^{1-\beta_i} u^{n+1} \approx \delta t^{-(1-\beta_i)} \sum_{k=0}^{n+1} C_k^{(\beta_i)} u^{n+1-k} + O(\delta t^2),\ i = 1, 2, \cdots, m_2,$$

*where* $u^n = u(t_n)$, *and the coefficients* $C_k^{(\alpha_i)}$ *and* $C_k^{(\beta_i)}$ *are defined as:*

$$C_k^{(\alpha_i)} = \begin{cases} \frac{1-\alpha_i+2}{2} g_0^{(\alpha_i)}, & \textit{for } k=0, \\ \frac{1-\alpha_i+2}{2} g_k^{(\alpha_i)} - \frac{1-\alpha_i}{2} g_{k-1}^{(\alpha_i)}, & \textit{for } k>0, \end{cases} \tag{3.4}$$

*and*

$$C_k^{(\beta_i)} = \begin{cases} \frac{1-\beta_i+2}{2} g_0^{(\beta_i)}, & \textit{for } k=0, \\ \frac{1-\beta_i+2}{2} g_k^{(\beta_i)} - \frac{1-\beta_i}{2} g_{k-1}^{(\beta_i)}, & \textit{for } k>0. \end{cases} \tag{3.5}$$

*The sequences* $g_k^{(\alpha_i)}$ *and* $g_k^{(\beta_i)}$ *are recursively defined as:*

$$g_0^{(\alpha_i)} = 1, \quad g_k^{(\alpha_i)} = \left(1 - \frac{1-\alpha_i+1}{k}\right) g_{k-1}^{(\alpha_i)}, \quad k \geq 1,$$

*and*

$$g_0^{(\beta_i)} = 1, \quad g_k^{(\beta_i)} = \left(1 - \frac{1-\beta_i+1}{k}\right) g_{k-1}^{(\beta_i)}, \quad k \geq 1.$$

Using Taylor series expansion we can write [46]

$$\begin{aligned} f(u(t_{n+1},x),t_{n+1},x) &= f(u^{n+1},t_{n+1},x) = f(2u^n - u^{n-1},t_{n+1},x) + (u^{n+1} - 2u^n + u^{n-1}) f_\xi(\xi^{n+1},t_{n+1},x) \\ &= f(2u^n - u^{n-1},t_{n+1},x) + \delta t^2 \frac{\partial^2 u}{\partial t^2}(\rho^{n+1},x) f_\xi(\xi^{n+1},t_{n+1},x), \end{aligned} \tag{3.6}$$

where $\rho^{n+1} \in (t_{n-1}, t_{n+1})$ and $\xi^{n+1}$ is between $u^{n+1}$ and $2u^n - u^{n-1}$.

Thus, the semi-discrete scheme for (3.1) can be written as:

$$\begin{aligned} \frac{\overline{u}^{n+1} - \overline{u}^n}{\delta t} = &\sum_{i=1}^{m_1} k_{1,i} \left( \delta t^{-(1-\alpha_i)} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \frac{\partial^2 \overline{u}^{n+1-k}}{\partial x^2} \right) \\ &- \sum_{j=1}^{m_2} k_{2,j} \left( \delta t^{-(1-\beta_j)} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \overline{u}^{n+1-k} \right) \\ &+ f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), \quad n = 0,1,\ldots,N-1, \end{aligned} \tag{3.7}$$

where $\overline{u}^n$ is an approximation of the exact solution $u^n$.

Simplifying the above equation, we obtain the following expression for $\overline{u}^{n+1}$:

$$\begin{aligned} \overline{u}^{n+1} = &\sum_{i=1}^{m_1} k_{1,i} \left( \delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \frac{\partial^2 \overline{u}^{n+1-k}}{\partial x^2} \right) \\ &- \sum_{j=1}^{m_2} k_{2,j} \left( \delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \overline{u}^{n+1-k} \right) \\ &+ \overline{u}^n + \delta t f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), \quad n = 0,1,\ldots,N-1. \end{aligned} \tag{3.8}$$

We assume that the function $f(\xi,t,x)$ possesses continuous second-order derivatives with respect to both $\xi$ and $t$ within an $\varepsilon_0$-neighborhood of the solution, where $\varepsilon_0 > 0$ is a given constant, and define [46]

$$r_0 = \max_{(t,x)\in J\times\mathcal{I}} |u(t,x)|,$$

$$r_1 = \max_{(t,x)\in J\times\mathcal{I},\ |\varepsilon_1|\leq\varepsilon_0} |f_\xi(u(t,x)+\varepsilon_1,t,x)|, \tag{3.9}$$

$$r_2 = \max_{(t,x)\in J\times\mathcal{I}} |f(u(t,x),t,x)|. \tag{3.10}$$

Note that we can write

$$|f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x)| = |f(2u^n - u^{n-1}, t_{n+1}, x)| + r_1|2(u^n - \overline{u}^n) - (u^{n-1} - \overline{u}^{n-1})| \\ \leq r_2 + r_1|2(u^n - \overline{u}^n) - (u^{n-1} - \overline{u}^{n-1})|. \tag{3.11}$$

Assume the existence of a positive constant $C$ for which

$$\|\overline{u}^j\|^2 \leq C, \quad j = 0, \dots, n. \tag{3.12}$$

Based on the assumptions (3.12), (3.9), and (3.11), it follows that

$$|f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x)| \leq r_2 + r_1|2(u^n - \overline{u}^n) - (u^{n-1} - \overline{u}^{n-1})| \leq r_2 + 3r_1(C + r_0) = C_1. \tag{3.13}$$

3.1.1. *Stability of the Semi-Discrete Scheme.* In this subsection, we analyze the stability of the semi-discrete scheme (3.8) using the energy method. For this purpose, we consider the following functional spaces, equipped with the usual norms [6]:

$$H^1(\mathcal{I}) = \left\{ u \in L^2(\mathcal{I}) \mid \frac{\partial u}{\partial x} \in L^2(\mathcal{I}) \right\}, \tag{3.14}$$

$$H_0^1(\mathcal{I}) = \left\{ u \in H^1(\mathcal{I}) \mid u\big|_{\partial\mathcal{I}} = 0 \right\}. \tag{3.15}$$

The $L^2$-space is defined by

$$L^2(\mathcal{I}) = \left\{ u \in L^2(\mathcal{I}) \mid \int_{\mathcal{I}} |u(x)|^2 \, dx < \infty \right\}. \tag{3.16}$$

The inner product in $L^2(\mathcal{I})$ and the corresponding inner product in $H^1(\mathcal{I})$ are given by

$$\langle u, v \rangle = \int_{\mathcal{I}} u(x) v(x) \, dx, \tag{3.17}$$

$$\langle u, v \rangle_{H^1} = \langle u, v \rangle + \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle. \tag{3.18}$$

The norms associated with these spaces are expressed as

$$\|u\|_{L^2(\mathcal{I})} = \sqrt{\langle u, u \rangle}, \tag{3.19}$$

$$\|u\|_{H^1(\mathcal{I})} = \sqrt{\|u\|^2_{L^2(\mathcal{I})} + \left\| \frac{\partial u}{\partial x} \right\|^2_{L^2(\mathcal{I})}}. \tag{3.20}$$

**Theorem 3.1.** *([32]) Let $\phi_n$ and $\delta_k$ be non-negative sequences, and let $c_0 > 0$, where the sequence $\theta_n$ satisfies:*

$$\begin{cases} \theta_0 \leq c_0, \\ \theta_n \leq c_0 + \sum_{k=0}^{n-1} \delta_k + \sum_{k=0}^{n-1} \phi_k \theta_k, \quad n \geq 1. \end{cases} \tag{3.21}$$

*If $c_0 \geq 0$ and $\delta_0 \geq 0$, it follows*

$$\theta_n \leq \left( c_0 + \sum_{k=0}^{n-1} \delta_k \right) \exp\left( \sum_{k=0}^{n-1} \phi_k \right), \quad n \geq 1. \tag{3.22}$$

**Lemma 3.2.** *([41, 42]).Assume $\{C_k^{\alpha_i}\}_{k=1}^{\infty}$ and $\{C_k^{\beta_i}\}_{k=1}^{\infty}$ are introduced in (3.4) and (3.5), respectively. Then for any positive integer $N$ and real vector $(u^0, u^1, ..., u^N) \in R^{N+1}$ the following inequalities hold*

$$\sum_{n=0}^{N} \sum_{k=0}^{n} C_k^{\alpha_i} (u^{n-k}, u^n) \geq 0$$

*and*

$$\sum_{n=0}^{N}\sum_{k=0}^{n} C_k^{\beta_i}(u^{n-k},u^n) \geq 0.$$

**Theorem 3.2.** *Let $u^n \in H_0^1(\mathcal{I}) \cap H^2(\mathcal{I})$. Hence, the time-discrete scheme (3.8) is unconditionally stable.*

*Proof.* The variational weak form of Equation (3.8) can be written as:

$$\begin{aligned}\langle \overline{u}^{n+1}, v\rangle = &\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial^2 \overline{u}^{n+1-k}}{\partial x^2}, v\rangle \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \overline{u}^{n+1-k}, v\rangle \\ &+\langle \overline{u}^{n}, v\rangle + \delta t\langle f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), v\rangle,\end{aligned} \tag{3.23}$$

Setting $v = \overline{u}^{n+1}$, arrives at

$$\begin{aligned}\langle \overline{u}^{n+1}, \overline{u}^{n+1}\rangle = &\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial^2 \overline{u}^{n+1-k}}{\partial x^2}, \overline{u}^{n+1}\rangle \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \overline{u}^{n+1-k}, \overline{u}^{n+1}\rangle \\ &+\langle \overline{u}^{n}, \overline{u}^{n+1}\rangle + \delta t\langle f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), \overline{u}^{n+1}\rangle,\end{aligned}$$

or

$$\begin{aligned}\|\overline{u}^{n+1}\|^2_{L^2(\mathcal{I})} = &-\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \overline{u}^{n+1-k}}{\partial x}, \frac{\partial \overline{u}^{n+1}}{\partial x}\rangle \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \overline{u}^{n+1-k}, \overline{u}^{n+1}\rangle \\ &+\langle \overline{u}^{n}, \overline{u}^{n+1}\rangle + \delta t\langle f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), \overline{u}^{n+1}\rangle.\end{aligned} \tag{3.24}$$

Summing Equation (3.24) from $n = 0$ to $N$ yields:

$$\begin{aligned}\sum_{n=0}^{N}\|\overline{u}^{n+1}\|^2_{L^2(\mathcal{I})} = &-\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \overline{u}^{n+1-k}}{\partial x}, \frac{\partial \overline{u}^{n+1}}{\partial x}\rangle \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \overline{u}^{n+1-k}, \overline{u}^{n+1}\rangle \\ &+\sum_{n=0}^{N}\langle \overline{u}^{n}, \overline{u}^{n+1}\rangle + \delta t\sum_{n=0}^{N}\langle f(2\overline{u}^n - \overline{u}^{n-1}, t_{n+1}, x), \overline{u}^{n+1}\rangle.\end{aligned}$$

The Cauchy-Schwarz inequality results

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{N}\|\overline{u}^{n+1}\|^2_{L^2(\mathcal{I})} \leq &-\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \overline{u}^{n+1-k}}{\partial x}, \frac{\partial \overline{u}^{n+1}}{\partial x}\rangle \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \overline{u}^{n+1-k}, \overline{u}^{n+1}\rangle\end{aligned}$$

$$+\frac{1}{2}\sum_{n=0}^{N}\|\overline{u}^{n}\|^2_{L^2(\mathcal{I})}+\delta t\sum_{n=0}^{N}\langle f(2\overline{u}^{n}-\overline{u}^{n-1},t_{n+1},x),\overline{u}^{n+1}\rangle. \tag{3.25}$$

By using Lemma 3.2, we have

$$-\sum_{i=1}^{m_1}k_{1,i}\delta t^{\alpha_i}\sum_{n=0}^{N}\sum_{k=0}^{n+1}C_k^{(\alpha_i)}\langle\frac{\partial\overline{u}^{n+1-k}}{\partial x},\frac{\partial\overline{u}^{n+1}}{\partial x}\rangle\le 0,$$
$$-\sum_{j=1}^{m_2}k_{2,j}\delta t^{\beta_j}\sum_{n=0}^{N}\sum_{k=0}^{n+1}C_k^{(\beta_j)}\langle\overline{u}^{n+1-k},\overline{u}^{n+1}\rangle\le 0.$$

Therefore, from the inequality above, Equation (3.25) can be rewritten as:

$$\frac{1}{2}\sum_{n=0}^{N}\|\overline{u}^{n+1}\|^2_{L^2(\mathcal{I})}\le\frac{1}{2}\sum_{n=0}^{N}\|\overline{u}^{n}\|^2_{L^2(\mathcal{I})}+\delta t\sum_{n=0}^{N}\langle f(2\overline{u}^{n}-\overline{u}^{n-1},t_{n+1},x),\overline{u}^{n+1}\rangle.$$

Alternatively, the equivalent expression can be written as:

$$\|\overline{u}^{N+1}\|^2_{L^2(\mathcal{I})}\le\|\overline{u}^{0}\|^2_{L^2(\mathcal{I})}+\delta t\sum_{n=0}^{N}\|f(2\overline{u}^{n}-\overline{u}^{n-1},t_{n+1},x)\|^2_{L^2(\mathcal{I})}+\delta t\sum_{n=0}^{N}\|\overline{u}^{n+1}\|^2_{L^2(\mathcal{I})}.$$

By applying Theorem 3.1 and Equation (3.13), we derive:

$$\|\overline{u}^{N+1}\|^2_{L^2(\mathcal{I})}\le\left(\|\overline{u}^{0}\|^2_{L^2(\mathcal{I})}+\delta tNC_1^2\right)\exp(N\delta t)\le\left(\|\overline{u}^{0}\|^2_{L^2(\mathcal{I})}+TC_1^2\right)\exp(T),$$

in which the last inequality confirms that the proposed time-stepping scheme is unconditionally stable. □

3.2. **Fully discrete scheme.** Initially, we define $\mathbb{V}^0_{m_e}=\{u\in H^1_0(\mathcal{I})\mid u|_{\mathcal{I}_e}\in\mathbb{P}_{m_e}(\mathcal{I}),\quad e=1,2,\dots,\mathbb{N}_e\}$, where $\mathbb{P}_{m_e}$ is a polynomial space less than or equal to $m_e$. The Galerkin formulation corresponding to (3.8) can then be expressed as:
The goal is to find $\widehat{u}^{n+1}\in\mathbb{V}^0_{m_e}$ such that for all $v\in\mathbb{V}^0_{m_e}$, the following relation holds:

$$\begin{aligned}\langle\widehat{u}^{n+1},v\rangle&=\sum_{i=1}^{m_1}k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1}C_k^{(\alpha_i)}\langle\frac{\partial^2\widehat{u}^{n+1-k}}{\partial x^2},v\rangle\\&-\sum_{j=1}^{m_2}k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1}C_k^{(\beta_j)}\langle\widehat{u}^{n+1-k},v\rangle\\&+\langle\widehat{u}^{n},v\rangle+\delta t\langle f(2\widehat{u}^{n}-\widehat{u}^{n-1},t_{n+1},x),v\rangle.\end{aligned}$$

Alternatively, we can express it as:

$$\begin{aligned}\langle\widehat{u}^{n+1},v\rangle&=-\sum_{i=1}^{m_1}k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1}C_k^{(\alpha_i)}\langle\frac{\partial\widehat{u}^{n+1-k}}{\partial x},\frac{\partial v}{\partial x}\rangle\\&-\sum_{j=1}^{m_2}k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1}C_k^{(\beta_j)}\langle\widehat{u}^{n+1-k},v\rangle\\&+\langle\widehat{u}^{n},v\rangle+\delta t\langle f(2\widehat{u}^{n}-\widehat{u}^{n-1},t_{n+1},x),v\rangle.\end{aligned} \tag{3.26}$$

The approximate solution using Lagrange interpolation polynomials is given by:

$$\widehat{u}^{n+1}(x)=\sum_{q=1}^{\mathbb{N}_g}\widehat{u}^{n+1}(x_q)\ell_q(x),$$

where $\mathbb{N}_g=\mathbb{N}_e(m_e+1)-\mathbb{N}_c$, and $\mathbb{N}_e$ is the number of elements, $m_e$ is the degree of the Legendre polynomial, and $\mathbb{N}_c$ is the number of points in common between elements.

Now, substituting this approximate solution into the relation obtained from the Galerkin formulation and setting $v = \ell_j$, we get:

$$\sum_{q=1}^{\mathbb{N}_g} \widehat{u}^{n+1}(x_q)\langle \ell_q, \ell_j\rangle = -\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \sum_{q=1}^{\mathbb{N}_g} \widehat{u}^{n+1-q}(x_q)\langle \frac{d\ell_q}{dx}, \frac{d\ell_j}{dx}\rangle$$
$$- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \sum_{q=1}^{\mathbb{N}_g} \widehat{u}^{n+1-q}(x_q)\langle \ell_q, \ell_j\rangle$$
$$+ \sum_{q=1}^{\mathbb{N}_q} \widehat{u}^{n}(x_q)\langle \ell_q, \ell_j\rangle + \delta t \sum_{q=1}^{\mathbb{N}_q} \langle f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x_q), l_j\rangle, \;\; j = 1, 2, \cdots, \mathbb{N}_g.$$

Once the local element matrices are computed, they are assembled to form the global system. The global mass and diffusion matrices are constructed as:

$$(\mathbb{M}^{\{g\}})_{ij} = \int_a^b \ell_j(x)\ell_i(x)dx = \sum_{e=1}^{\mathbb{N}_e} \int_{x^{(e-1)}}^{x^{(e)}} \ell_j(\widehat{x})\ell_i(\widehat{x})d\widehat{x},$$

$$(\mathbb{D}^{\{g\}})_{ij} = \int_a^b \frac{d\ell_j}{dx}(x)\frac{d\ell_i}{dx}(x)dx = \sum_{e=1}^{\mathbb{N}_e} \int_{x^{(e-1)}}^{x^{(e)}} \frac{d\ell_j}{d\widehat{x}}(\widehat{x})\frac{d\ell_i}{d\widehat{x}}(\widehat{x})d\widehat{x}.$$

Using the above relations, we obtain:

$$\mathbb{M}^{\{g\}}\widehat{\mathbf{u}}^{n+1} = -\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \mathbb{D}^{\{g\}}\widehat{\mathbf{u}}^{n+1-k}$$
$$- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \mathbb{M}^{\{g\}}\widehat{\mathbf{u}}^{n+1-k}$$
$$+ \mathbb{M}^{\{g\}}\widehat{\mathbf{u}}^{n} + \delta t \mathbb{M}^{\{g\}} \odot f\left(2\widehat{\mathbf{u}}^n - \widehat{\mathbf{u}}^{n-1}, t_{n+1}\right).$$

where $\widehat{\mathbf{u}}^n$ represents the vector of the approximate solution over the general domain $\mathcal{I}$ at time $t = t_n$. The nonlinear function $f$ is applied componentwise, and $\odot$ denotes the elementwise product.

3.3. **Convergence study for the fully discrete scheme.** This section focuses on deriving an error estimate for the fully discrete scheme (3.26). Before proceeding, we introduce some important lemmas and definitions. We denote by $H^m$ and $H_0^m$ the Sobolev spaces endowed with the norm $\|\cdot\|_m$ and the seminorm $|\cdot|_m$, respectively.

**Definition 3.1.** ([9]) The spatial projection operator is defined as

$$\widehat{\Pi}_h : H_0^1(\mathcal{I}) \to \mathbb{V}_{m_e}^0, \tag{3.27}$$

where it satisfies the following orthogonality condition:

$$\left\langle \frac{d}{dx}(u - \widehat{\Pi}_h u), \frac{d}{dx}v \right\rangle = 0, \quad \forall u \in H_0^1(\mathcal{I}), \quad \forall v \in \mathbb{V}_{m_e}^0. \tag{3.28}$$

$$\left\langle u - \widehat{\Pi}_h u, v \right\rangle = 0, \quad \forall u \in H_0^1(\mathcal{I}), \quad \forall v \in \mathbb{V}_{m_e}^0. \tag{3.29}$$

**Lemma 3.3.** *([10]) Let $u \in H^s$ with $s \geq 1$. Then, the following error estimate holds:*

$$\|u - \widehat{\Pi}_h u\| \leq C \left( \sum_{l=1}^{\mathbb{N}_e} h_l^{2(\min((m_e)_l+1,s)-1)} (m_e)_l^{2(1-s)} \|u\|_s^2 \right)^{\frac{1}{2}}. \tag{3.30}$$

*In the special case where* $(m_e)_l = m_e$ *and* $h \le h_l \le \widetilde{c}h$*, we obtain the simplified bound:*

$$\|u - \widehat{\Pi}_h u\| \le Ch^{\min(m_e+1,s)-1} m_e^{1-s} \|u\|_s. \tag{3.31}$$

**Lemma 3.4.** *([4]) we assume that*

$$\varrho^{n+1} = (u^{n+1} - u^n) - (\widehat{\Pi}_h u^{n+1} - \widehat{\Pi}_h u^n), \tag{3.32}$$

*then*

$$\|\varrho^{n+1}\| \le C_u \delta t m_e^{1-s}. \tag{3.33}$$

**Theorem 3.3.** *Let* $u^{n+1}$ *be solution of* (3.1) *and* $\widehat{u}^{n+1}$ *be solution of* (3.8)*. So*

$$\left\|e^{n+1}\right\| \le (\delta t^2 + m_e^{1-s}),$$

*in which* $e^{n+1} = u^{n+1} - \widehat{u}^{n+1}$ *and* $C$ *is a positive constant that is independent of* $\delta t$ *and* $m_e$*.*

*Proof.* For the exact solution, the corresponding variational weak form is given by:

$$\begin{aligned}\langle u^{n+1}, v\rangle &= \sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \langle \frac{\partial^2 u^{n+1-k}}{\partial x^2}, v\rangle \\ &- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \langle u^{n+1-k}, v\rangle \\ &+ \langle u^n, v\rangle + \delta t \langle f(2u^n - u^{n-1}, t_{n+1}, x), v\rangle + \delta t \langle \mathcal{E}_t^n, v\rangle.\end{aligned} \tag{3.34}$$

where $\mathcal{E}_t^n = \mathcal{O}(\delta t^2)$.
Also

$$\begin{aligned}\langle \widehat{u}^{n+1}, v\rangle &= \sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \langle \frac{\partial^2 \widehat{u}^{n+1-k}}{\partial x^2}, v\rangle \\ &- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \langle \widehat{u}^{n+1-k}, v\rangle \\ &+ \langle \widehat{u}^n, v\rangle + \delta t \langle f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x), v\rangle.\end{aligned} \tag{3.35}$$

We define the following notations

$$\zeta^n = u^n - \widehat{\Pi}_h u^n, \quad \eta^n = \widehat{\Pi}_h u^n - \widehat{u}^n. \tag{3.36}$$

Subtracting Eq. (3.35) from Eq. (3.34), results

$$\begin{aligned}\langle u^{n+1} - \widehat{u}^{n+1}, v\rangle &= \sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \langle \frac{\partial^2 u^{n+1-k} - \partial^2 \widehat{u}^{n+1-k}}{\partial x^2}, v\rangle \\ &- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \langle u^{n+1-k} - \widehat{u}^{n+1-k}, v\rangle + \langle u^n - \widehat{u}^n, v\rangle \\ &+ \delta t \langle f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x), v\rangle + \delta t \langle \mathcal{E}_t^n, v\rangle.\end{aligned} \tag{3.37}$$

From the relation above, along with Eq. (3.28) and the definitions in (3.36), we derive:

$$\begin{aligned}\langle \eta^{n+1}, v\rangle &= \sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i} \sum_{k=0}^{n+1} C_k^{(\alpha_i)} \langle \frac{\partial^2 \eta^{n+1-k}}{\partial x^2}, v\rangle \\ &- \sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j} \sum_{k=0}^{n+1} C_k^{(\beta_j)} \langle \eta^{n+1-k}, v\rangle + \langle \eta^n, v\rangle + \delta t \langle \mathcal{E}_t^n, v\rangle\end{aligned}$$

$$-\langle \zeta^{n+1}, v\rangle + \langle \zeta^{n}, v\rangle + \delta t\langle f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x), v\rangle, \tag{3.38}$$

in which the equivalent expression for the final equality is:

$$\begin{aligned}\left\|\eta^{n+1}\right\|^2 &= -\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \eta^{n+1-k}}{\partial x}, \frac{\partial \eta^{n+1}}{\partial x}\rangle \\ &\quad -\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \eta^{n+1-k}, \eta^{n+1}\rangle + \langle \eta^n, \eta^{n+1}\rangle + \delta t\langle \mathcal{E}_t^n, \eta^{n+1}\rangle - \langle \zeta^{n+1}, \eta^{n+1}\rangle \\ &\quad + \langle \zeta^n, \eta^{n+1}\rangle + \delta t\langle f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x), \eta^{n+1}\rangle.\end{aligned} \tag{3.39}$$

By taking the sum of Eq. (3.39) over $n$ from 0 to $N$, we obtain:

$$\begin{aligned}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2 &= -\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \eta^{n+1-k}}{\partial x}, \frac{\partial \eta^{n+1}}{\partial x}\rangle \\ &\quad -\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \eta^{n+1-k}, \eta^{n+1}\rangle + \sum_{n=0}^{N}\langle \eta^n, \eta^{n+1}\rangle \\ &\quad + \delta t\sum_{n=0}^{N}\langle \mathcal{E}_t^n, \eta^{n+1}\rangle - \sum_{n=0}^{N}\langle \zeta^{n+1}, \eta^{n+1}\rangle + \sum_{n=0}^{N}\langle \zeta^n, \eta^{n+1}\rangle \\ &\quad + \delta t\sum_{n=0}^{N}\langle f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x), \eta^{n+1}\rangle.\end{aligned} \tag{3.40}$$

Lemma 3.2 results

$$\begin{aligned}&-\sum_{i=1}^{m_1} k_{1,i}\delta t^{\alpha_i}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\alpha_i)}\langle \frac{\partial \eta^{n+1-k}}{\partial x}, \frac{\partial \eta^{n+1}}{\partial x}\rangle \le 0, \\ &-\sum_{j=1}^{m_2} k_{2,j}\delta t^{\beta_j}\sum_{n=0}^{N}\sum_{k=0}^{n+1} C_k^{(\beta_j)}\langle \eta^{n+1-k}, \eta^{n+1}\rangle \le 0.\end{aligned}$$

Therefore, based on the inequality above and Eq. (3.40), by applying Young's inequality with $\epsilon > 0$, we obtain:

$$\begin{aligned}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2 &\le \frac{1}{2\epsilon_1}\sum_{n=0}^{N}\left\|\eta^n\right\|^2 + \frac{\epsilon_1}{2}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2 \\ &\quad + \frac{\delta t}{2\epsilon_2}\sum_{n=0}^{N}\left\|\mathcal{E}_t^n\right\|^2 + \frac{\epsilon_2\delta t}{2}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2 \\ &\quad + \frac{1}{2\epsilon_3}\sum_{n=0}^{N}\left\|\varrho^{n+1}\right\|^2 + \frac{\epsilon_3}{2}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2 \\ &\quad + \frac{\delta t}{2\epsilon_4}\sum_{n=0}^{N}\left\|f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x)\right\|^2 + \frac{\epsilon_4\delta t}{2}\sum_{n=0}^{N}\left\|\eta^{n+1}\right\|^2.\end{aligned} \tag{3.41}$$

Assuming that the function $f$ satisfies the Lipschitz condition with respect to the first variable with Lipschitz constant $L$, namely $\|f(u,t,x) - f(v,t,x)\| \le L\|u - v\|,$, we obtain the following upper bound:

$$\begin{aligned}\left\|f(2u^n - u^{n-1}, t_{n+1}, x) - f(2\widehat{u}^n - \widehat{u}^{n-1}, t_{n+1}, x)\right\|^2 &\le L^2\left\|(2u^n - u^{n-1}) - (2\widehat{u}^n - \widehat{u}^{n-1})\right\|^2 \\ &= L^2\left\|2(u^n - \widehat{u}^n) - (u^{n-1} - \widehat{u}^{n-1})\right\|^2 \\ &\le 2L^2\left(4\|\zeta^n + \eta^n\|^2 + \|\zeta^{n-1} + \eta^{n-1}\|^2\right)\end{aligned}$$

$$\leq 16L^2 \left(\|\zeta^n\|^2 + \|\eta^n\|^2\right) + 4L^2 \left(\|\zeta^{n-1}\|^2 + \|\eta^{n-1}\|^2\right). \tag{3.42}$$

Using Eqs. (3.41)–(3.42), we obtain

$$\begin{aligned}\sum_{n=0}^{N} \|\eta^{n+1}\|^2 \leq \frac{1}{2\epsilon_1} \sum_{n=0}^{N} \|\eta^n\|^2 + \frac{\epsilon_1}{2} \sum_{n=0}^{N} \|\eta^{n+1}\|^2 \\ + \frac{\delta t}{2\epsilon_2} \sum_{n=0}^{N} \|\mathcal{E}_t^n\|^2 + \frac{\epsilon_2 \delta t}{2} \sum_{n=0}^{N} \|\eta^{n+1}\|^2 \\ + \frac{1}{2\epsilon_3} \sum_{n=0}^{N} \|\varrho^{n+1}\|^2 + \frac{\epsilon_3}{2} \sum_{n=0}^{N} \|\eta^{n+1}\|^2 \\ + \frac{20L^2\delta t}{2\epsilon_4} \sum_{n=0}^{N} (\|\zeta^n\|^2 + \|\eta^n\|^2) \\ + \frac{\epsilon_4 \delta t}{2} \sum_{n=0}^{N} \|\eta^{n+1}\|^2.\end{aligned} \tag{3.43}$$

Suppose that $\epsilon_1 = \epsilon_2 = \epsilon_4 = 1$ and $\epsilon_3 = \delta t$. Applying Lemma 3.3, there exist positive constants $C_1$, $C_2$, and $C_3$ such that

$$\|\eta^{N+1}\|^2 \leq \|\eta^0\|^2 + C_1 \delta t^4 + C_2 (m_e^{1-s})^2 + C_3 \delta t \sum_{n=0}^{N} \|\eta^{n+1}\|^2. \tag{3.44}$$

From Theorem 3.1, we obtain

$$\left\|\eta^{N+1}\right\|^2 \leq C(\delta t^2 + m_e^{1-s})^2.$$

By applying the relation in (3.36) along with the triangle inequality, one can derive the following:

$$\left\|e^{n+1}\right\|^2 = \left\|u^{n+1} - \widehat{u}^{n+1}\right\|^2 = \left\|\eta^{n+1} + \zeta^{n+1}\right\|^2 \leq \left\|\eta^{n+1}\right\|^2 + \left\|\zeta^{n+1}\right\|^2 \leq \left[C(\delta t^2 + m_e^{1-s})\right]^2 + (m_e^{1-s})^2.$$

Thus

$$\left\|e^{n+1}\right\| \leq C(\delta t^2 + m_e^{1-s}),$$

which completes the proof. □



## 4. Illustrative Test Problems and Discussion

**Example 4.1.** *We consider the following fractional cable equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^{1-\beta} u}{\partial t} + u^3 - u + f(t,x), \quad (t,x) \in (0,T] \times (0,1). \tag{4.1}$$

*Here, the source term $f(t,x)$ is a given function such that the exact solution to the problem is*

$$u(t,x) = t^3 \cos(3\pi(1-x)).$$

*In this example, we investigated the accuracy and efficiency of the proposed numerical method for solving the fractional cable problem. Table 1 presents the norm infinity errors for different values of the fractional parameters $\alpha$ and $\beta$ with $T = 1$. As the number of time steps $N$ increased, the numerical error decreased consistently, demonstrating the convergence of the proposed method. The computational order of convergence (COC) confirmed that the method achieved nearly first-order accuracy in time.*

*Figure 1 illustrates the norm infinity errors for different values of $\alpha$ and $\beta$ with $T = 2$, further validating the stability and effectiveness of the approach across various fractional orders. The numerical errors remained well-behaved, reinforcing the reliability of the method. Overall, the results demonstrate that the proposed numerical scheme*

TABLE 1. The norm infinity errors for different values of $\alpha$ and $\beta$ with $T = 1$ (Example 4.1).

| $N$ | $(\alpha,\beta)=(0.7,0.7)$ Error | COC | $(\alpha,\beta)=(0.8,0.6)$ Error | COC | $(\alpha,\beta)=(0.9,0.5)$ Error | COC |
|---|---|---|---|---|---|---|
| 200 | $1.3446e-04$ | - | $1.3683e-04$ | - | $1.3962e-04$ | - |
| 400 | $6.0268e-05$ | 1.1577 | $6.2953e-05$ | 1.1200 | $6.5785e-05$ | 1.0856 |
| 800 | $2.8395e-05$ | 1.0857 | $3.0114e-05$ | 1.0638 | $3.1892e-05$ | 1.0445 |
| 1600 | $1.3762e-05$ | 1.0448 | $1.4716e-05$ | 1.0329 | $1.5696e-05$ | 1.0227 |
| 3200 | $6.7728e-06$ | 1.0229 | $7.2735e-06$ | 1.0167 | $7.7862e-06$ | 1.0114 |

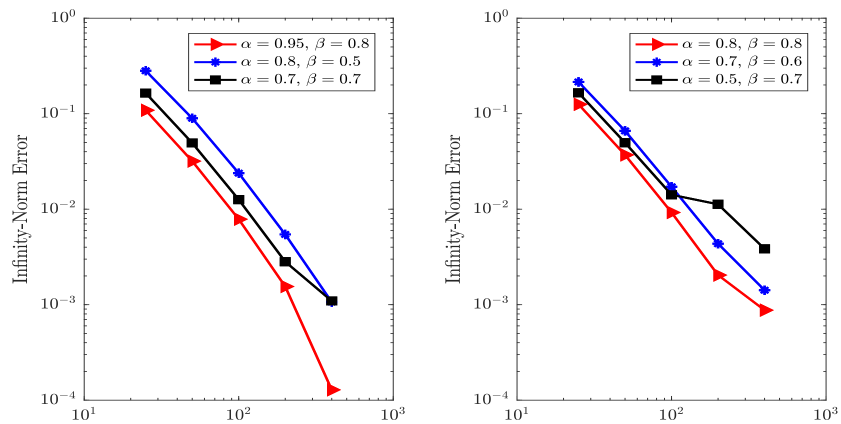


FIGURE 1. The norm infinity errors for different values of $\alpha$ and $\beta$ with $T = 2$ (Example 4.1).



TABLE 2. The norm infinity errors for different values of $\alpha_i, i = 1, 2$ and $\beta_i, i = 1, 2$ with $T = 1$ (Example 4.2).

| $N$ | $(\alpha_1,\beta_1)=(0.4,0.4)$, $(\alpha_2,\beta_2)=(0.5,0.5)$ Error | COC | $(\alpha_1,\beta_1)=(0.8,0.6)$, $(\alpha_2,\beta_2)=(0.6,0.5)$ Error | COC |
|---|---|---|---|---|
| 200 | $3.0119e-04$ | - | $1.2972e-03$ | - |
| 400 | $1.0453e-04$ | 1.5268 | $5.2523e-04$ | 1.3044 |
| 800 | $3.6281e-05$ | 1.5266 | $2.1392e-04$ | 1.2959 |
| 1600 | $1.2595e-05$ | 1.5264 | $8.7640e-05$ | 1.2874 |
| 3200 | $4.3743e-06$ | 1.5257 | $3.6108e-06$ | 1.2793 |

*provides accurate and efficient approximations for the fractional cable model, confirming its effectiveness for different fractional orders and computational settings.*

**Example 4.2.** *We consider the following fractional cable equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\alpha_1}}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^{1-\alpha_2}}{\partial t}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^{1-\beta_1} u}{\partial t} - \frac{\partial^{1-\beta_2} u}{\partial t} + f(t,x), \quad (t,x) \in (0,T] \times (0,2\pi). \tag{4.2}$$

*Here, the source term $f(t,x)$ is a given function such that the exact solution to the problem is*

$$u(t,x) = t(x - \sin(x) - 2\pi\cos(x)).$$

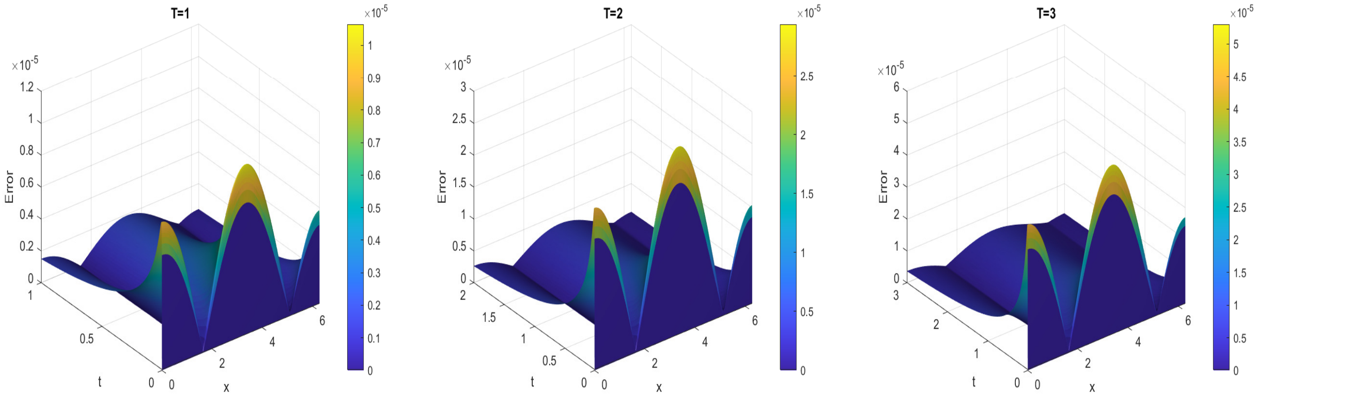


FIGURE 2. The error for $(\alpha_1, \beta_1) = (0.5, 0.3)$ and $(\alpha_2, \beta_2) = (0.2, 0.4)$ for $T = 1, 2, 3$ with $N = 3200$ (Example 4.2).

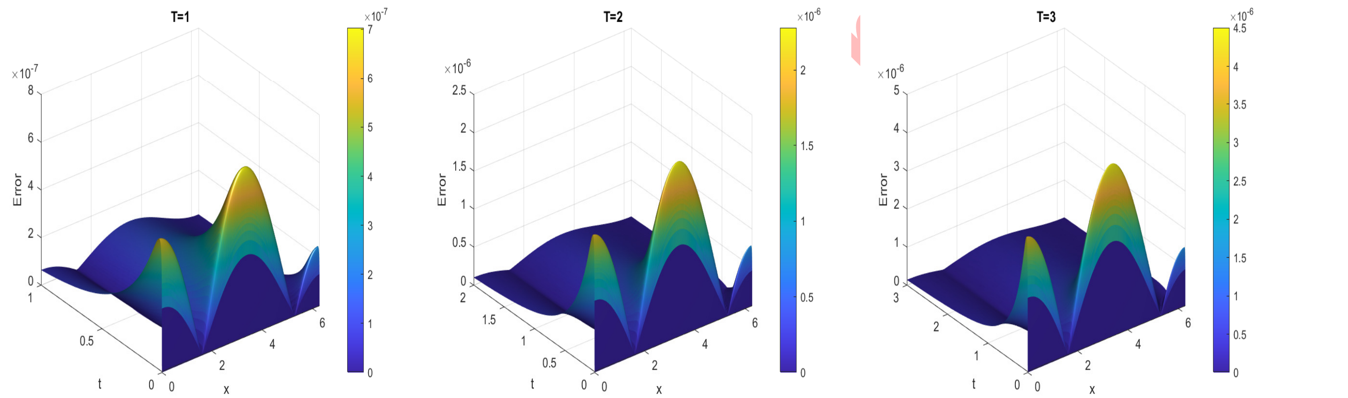


FIGURE 3. The error for $(\alpha_1, \beta_1) = (0.1, 0.1)$ and $(\alpha_2, \beta_2) = (0.2, 0.2)$ for $T = 1, 2, 3$ with $N = 3200$ (Example 4.2).

TABLE 3. The norm infinity errors for different values of $\alpha_i, i = 1, 2$ and $\beta_i, i = 1, 2$ with $T = 2$ (Example 4.2).

| | $(\alpha_1, \beta_1) = (0.4, 0.4)$ $(\alpha_2, \beta_2) = (0.5, 0.5)$ | | $(\alpha_1, \beta_1) = (0.8, 0.6)$ $(\alpha_2, \beta_2) = (0.6, 0.5)$ | |
|---|---|---|---|---|
| 200 | $6.0454e-04$ | - | $2.5204e-03$ | - |
| 400 | $2.0935e-04$ | 1.5299 | $1.0139e-03$ | 1.3044 |
| 800 | $7.2526e-05$ | 1.5293 | $4.1041e-04$ | 1.2959 |
| 1600 | $2.5137e-05$ | 1.5286 | $1.6714e-04$ | 1.2874 |
| 3200 | $8.7182e-06$ | 1.5277 | $6.8478e-05$ | 1.2793 |

*In this example, we examined the accuracy and convergence of the proposed numerical method for solving the fractional model under different fractional parameter settings. Tables 2, 3 and 4 present the norm infinity errors for different values of the fractional parameters $(\alpha, \beta)$ at $T = 1, 2, 3$. The results show a consistent decrease in the error as*

TABLE 4. The norm infinity errors for different values of $\alpha_i, i = 1, 2$ and $\beta_i, i = 1, 2$ with $T = 3$ (Example 4.2).

| | $(\alpha_1, \beta_1) = (0.4, 0.4)$ $(\alpha_2, \beta_2) = (0.5, 0.5)$ | | $(\alpha_1, \beta_1) = (0.8, 0.6)$ $(\alpha_2, \beta_2) = (0.6, 0.5)$ | |
|---|---|---|---|---|
| $N$ | *Error* | *COC* | *Error* | *COC* |
| 200 | $9.0312e-04$ | - | $3.7032e-03$ | - |
| 400 | $3.1238e-04$ | 1.5315 | $1.4841e-03$ | 1.3191 |
| 800 | $1.0810e-04$ | 1.5308 | $5.9854e-04$ | 1.3100 |
| 1600 | $3.7437e-05$ | 1.5299 | $2.4229e-04$ | 1.3047 |
| 3200 | $1.2973e-05$ | 1.5288 | $9.9180e-05$ | 1.2886 |

*the number of time steps $N$ increases, indicating the convergence of the method. The computed order of convergence (COC) remains approximately 1.5 in most cases, confirming the expected accuracy of the scheme.*

*Figures 2 and 3 depict the spatial distribution of errors for different choices of $(\alpha, \beta)$ at different times. These visualizations demonstrate the error behavior over the domain, highlighting that the proposed method effectively captures the solution with minimal numerical artifacts. The results indicate that the method is robust for varying fractional orders.*

*Overall, the numerical findings confirm that the proposed approach provides a stable and accurate solution to the fractional model, achieving the desired convergence properties for different values of $(\alpha, \beta)$ and computational settings.*

## 5. Conclusion

In this study, we proposed an efficient numerical method for solving the nonlinear fractional cable model, incorporating multi-term time-fractional derivatives in the Riemann-Liouville sense. To discretize the problem, a high-order finite difference scheme was utilized in the time direction, while the Legendre spectral element method was applied in the spatial direction. The stability and convergence of the time-discrete scheme were rigorously analyzed, and an error estimate for the fully discrete system was established. The results demonstrated that the proposed approach provided highly accurate solutions across various parameter settings, confirming its effectiveness in solving fractional cable models. The findings of this work contributed to the advancement of numerical techniques for fractional differential equations, particularly those modeling anomalous diffusion and memory effects.

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