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Improving conjugate gradient method by using the golden ratio in optimization

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Abstract

Conjugate gradient techniques have been applied extensively to minimization problems. Conventional approaches use the gradient of the goal function to determine the search directions. Conjugate gradient methods that primarily consider the Golden Ratio (0.618) are proposed in this study. We demonstrate their worldwide convergence and provide numerical comparisons with traditional techniques. The findings demonstrate that one of our solutions outperforms the ideal traditional method using the Fletcher and Reeves formula with a little direction change.

Keywords. Golden Ratio, Novel conjugate gradient, Numerical results.
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1. Introduction

An efficient method for resolving composite form minimization problems is being examined:

$$f(x^*) = \min_{x\in R^N} f(\mu), \tag{1.1}$$

where f is a smooth function, [15]. Here is an illustration of an iteration sequence produced by the CG method:

$$x_{K+1} = x_K + \alpha_K d_K, \tag{1.2}$$

where α_K is the step length, and d_K is the search direction that is obtained by:

$$d_{K+1} = -g_{K+1} + \beta_K d_K, \tag{1.3}$$

where β_K is a formula, [6]. Several conjugate gradient technique types are described by the conjugate gradient parameter, which is based on numerical performance and global convergence properties. Since Fletcher-Reeves [2] first proposed the formula, as:

$$\beta_K^{FR} = \|g_{K+1}\|^2/\|g_K\|^2. \tag{1.4}$$

The Wolfe requirements must typically be fulfilled by the steplength α_K in the analysis and applications of various conjugate gradient methods:

$$f(x_K + \alpha_K d_K) \le f(x_K) + \delta\alpha_K g_K^T d_K, \tag{1.5}$$

$$d_K^T g(x_K + \alpha_K d_K) \ge \sigma\, d_K^T g_K, \tag{1.6}$$

when $\mathbf{0} < \delta < \sigma < \mathbf{1}$, see [3, 7]. The global convergence theorems are derived for these techniques under suitable assumptions. When looking at their numerical results, though, none of them have a significant edge.

Conjugate gradient algorithms have been developed by several researchers in recent years, yielding promising numerical results and suitable theoretical characteristics [12].

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It is interesting that this approach may achieve global convergence for a uniformly convex function while satisfying the required and conjugacy conditions. Because the conjugate gradient method performs well, we are interested in developing a conjugate gradient approach that satisfies the conjugacy requirement, the appropriate descent criteria, and global convergence.

2. The New Formula and the Algorithm

The Golden Ratio, (0.618), is a well-established mathematical constant characterized by its intrinsic geometric harmony and mathematical elegance, with documented relevance in Euclidean geometry, the Fibonacci sequence, ancient art, and modern architectural design [18].

Employing the enhanced secant condition introduced by Li and Fukushima [16].the corresponding new conjugate formulation is derived as follows:

$$B_{K+1}s_K = y_K + \eta\ s_K, \tag{2.1}$$

where g_{K+1} is Hessian matrix, B_{K+1} is an approximation to the Hessian, $y_K = g_{K+1} - g_K$ and $s_k = x_{K+1} - x_K$. The novel secant condition is defined as follows:

$$B_{K+1}s_k = y_K + 0.618\ s_{K.} \tag{2.2}$$

Mutually conjugate directions are constructed from an initial set of linearly independent vectors by applying a modified Gram-Schmidt procedure based on the matrix g_{K+1} defining the quadratic function. The directions are generated iteratively using uniquely determined coefficient that ensure satisfaction of the conjugate condition, expressed as follows:

$$d_{K+1}^T G d_K = 0. \tag{2.3}$$

Through mathematical derivation, this leads to the following expression:

$$\beta_K = \frac{g_{K+1}^T G s_K}{d_K^T G s_K}. \tag{2.4}$$

which provides a systematic method for computing the coefficients necessary for constructing the conjugate directions. By making use of Eq. (2.2), exact linear search together with Eq. (2.4) the result is obtained:

$$\beta_K \cong \frac{\|g_{K+1}\|^2}{d_K^T y_K + 0.618 d_K^T s_K}. \tag{2.5}$$

Called this formula by BBQ. **Algorithms BBQ as:**
1. Select $x_0 \in R^n$, and initialize the $d_1 = -g_1$.
2. Whenever $\|g_{K+1}\| < \varepsilon$, stop.
3. Calculate the search direction via Eq. (1.3) with Eq. (2.5).
4. Using optimal α_K by choice Eq. (1.5) and Eq. (1.6).
5. Compute x_{K+1}using Eq. (1.2).
6. Set k = k + 1 and go to Step 1.

The general conclusion, which is essential for demonstrating the convergence results, was published by Basim in [6].

3. Convergence analysis

Consider a function that is continuously differentiable. It is presumed that:

(H1) The collection of iterates for which the objective values do not surpass the initial value is bounded, as $L_0 = \{x\ / f(x) \leq f(x_0)\}$.

(H2) The gradient ∇f of adheres to a Lipschitz continuity requirement on this collection, as:

$$\left(\nabla f(\bar{o}) - \nabla f(v^+)\right) \leq L \left\|\bar{o} - v^+\right\| \quad , \forall\ \bar{o}, v^+,\ \in L_0, \tag{3.1}$$

see [2, 6, 15].

Theorem 3.1. *If one were to employ a novel methodology for the construction of $\{x_K\}$ and $\{d_K\}$, then the outcome would be:*

$$d_{K+1}^T g_{K+1} \ < \ 0 \text{ and } d_{K+1}^T g_{K+1} = \beta_K d_K^T g_K. \tag{3.2}$$

Proof. It is evident that, given $d_K = -\, g_K$, the fundamental inequality $d_1^T g_1 < 0$is satisfied. For any k, it follows that $d_K^T g_K < 0$ holds true. From Eqs. (1.3) and (3.2), it is straightforward to derive:

$$d_{K+1}^T g_{K+1} = -g_{K+1}^T g_{K+1} + \beta_{K+1} d_{K+1}^T g_{K+1} \tag{3.3}$$

$$= -\beta_{K+1}\left[d_{K+1}^T y_{K+1} \ + \ 0.618 \ d_{K+1}^T s_{K+1}\right] + \ \beta_{K+1} d_{K+1}^T g_{K+1}. \tag{3.4}$$

This leads to the conclusion that:

$$d_{K+1}^T g_{K+1} \ = \ \beta_{K+1}[d_{K+1}^T g_{K+1} - [d_{K+1}^T y_{K+1} \ + \ 0.618 \ d_{K+1}^T s_{K+1}]]. \tag{3.5}$$

The summation of Eqs. (2.2) and (3.5) yields:

$$d_{K+1}^T g_{K+1} = \beta_{K+1} d_{K+1}^T g_{K+1}. \tag{3.6}$$

It is unequivocally established that $d_{K+1}^T g_{K+1} < 0$, leading to the conclusion:

$$d_{K+1}^T g_{K+1} < 0. \tag{3.7}$$

The proof has been conclusively established. □

Based on the premise, Zoutendijk [19] has substantiated the following lemma, which is requisite for demonstrating global convergence.

Lemma 3.2. *Should conditions (H1)-(H2) be satisfied, then the iterates produced by the proposed methodology are existent and well-defined. Then:*

$$\sum_{k=1}^{\infty} \frac{(\, g_{K+1}^T d_{K+1})^2}{\|\, d_{K+1}\|^2} \ < \ \infty. \tag{3.8}$$

Theorem 3.3. *Under the assumptions delineated in (H1)-(H2), the sequence $\{x_{K+1}\}$ generated by the proposed algorithm, complies with:*

$$\lim_{K\to\infty} \inf \left\|\, g_{K+1} \right\| = 0\,. \tag{3.9}$$

Proof. Assume, for the sake of contradiction, that Eq. (3.9) does not hold. Then, for every k, there exists an $r > 0$ such that:

$$\left\| g_{K+1} \right\| > r. \tag{3.10}$$

However, from the search duration defined as $d_{K+1} + g_{K+1} = \beta_{K+1} d_{K+1}$and by squaring both sides, we derive:

$$\|d_{K+1}\|^2 + \left\|g_{K+1}\right\|^2 + 2d_{K+1}^T g_{K+1} = (\beta_{K+1})^2 \, \|d_{K+1}\|^2\,. \tag{3.11}$$

Applying the results from Eq. (1.3) to Eq. (3.11) yields:

$$\|d_{K+1}\|^2 = \frac{(d_{K+1}^T g_{K+1})^2}{(d_{K+1}^T g_{K+1})^2} \|d_{K+1}\|^2 - 2d_{K+1}^T g_{K+1} - \left\|g_{K+1}\right\|^2. \tag{3.12}$$

When equation (16) is divided by $(d_{K+1}^T g_{K+1})^2$, we ascertain:

$$\begin{aligned}\frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} &= \frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} - \frac{\|g_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} - \frac{2}{d_{K+1}^T g_{K+1}} \\ &\leq \frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} - \left(\frac{\|g_{K+1}\|}{(d_{K+1}^T g_{K+1})} + \frac{1}{\|g_{K+1}\|^2}\right) + \frac{1}{\|g_{K+1}\|^2} \\ &\leq \frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} + \frac{1}{\|g_{K+1}\|^2}. \end{aligned} \tag{3.13}$$

As a result, we get:

$$\frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\| g_i \|^2}. \tag{3.14}$$

Let there exists a constant $c_1 > 0$satisfying $\|g_{K+1}\| \geq c_1$ for all $k \in n$. Then:

$$\frac{\|d_{K+1}\|^2}{(d_{K+1}^T g_{K+1})^2} < \frac{k+1}{c_1^2}. \tag{3.15}$$

Hence, we obtain:

$$\sum_{k=1}^{\infty} \frac{(g_{K+1}^T d_{K+1})^2}{\|d_{K+1}\|^2} = \infty. \tag{3.16}$$

□

4. Numerical results

We have included the BBQ's numerical results in this section. We have used the set of test problems from [11] to test universal sparse and separable unconstrained optimization. The problem's dimension (n), n = 1000, has been utilized. These algorithms employ a line search method for every problem, which meets the Wolfe requirement δ= 0.0001 and σ= 0.9. The following CG-methods will be tested. (i). Standard FR-technique. (ii) The BBQ-technique. Further methods in the same domain are available in [8–10, 13, 14, 17].

Table 1 illustrates the computational results, where the columns have the following meanings: Problem: the test problem's number. NOI stands for number of iterations. NOF stands for number of function evaluations. The sources [4, 5, 7] confirms the efficiency of quasi-Newton methods in unconstrained optimization.

Table 1 shows that, particularly for the test functions we chose, the average performance of the modified approach is superior to that of the original method.

To evaluate the performance of various algorithms approximately, we utilize the performance profile introduced by Dolan and More [1], focusing on NI and NF. Figures 1 and 2 presents the results of the BBQ algorithm, which encompasses both BBQ and FR methods, in terms of NI and NF criteria. By examining Figures 1 and 2, it becomes clear that the BBQ methods have a superior performance compared to the FR method.

Table 1. The FR and BBQ algorithms numerical results.

Test Function	N	BBQ			FR		
		NOI	NOF	$\lVert g_{K+1} \rVert$	NOI	NOF	$\lVert g_{K+1} \rVert$
Extended Rosenbrock	1000	412	895	6.41×10^{-7}	850	1920	9.80×10^{-7}
Ex. White & Holst	1000	523	1104	8.22×10^{-7}	920	2150	8.55×10^{-7}
Raydan 1	1000	18	42	5.10×10^{-8}	42	98	2.10×10^{-7}
Raydan 2	1000	14	35	3.50×10^{-9}	38	85	3.45×10^{-7}
Diagonal 1	1000	12	28	1.02×10^{-8}	25	62	1.02×10^{-7}
Diagonal 2	1000	15	33	4.45×10^{-8}	28	68	4.40×10^{-7}
Diagonal 3	1000	22	51	9.02×10^{-7}	45	110	7.12×10^{-7}
Extended Hager	1000	26	64	3.18×10^{-7}	54	125	5.30×10^{-7}
Generalized Tridiagonal	1000	38	89	5.50×10^{-7}	78	185	6.90×10^{-7}
Extended Beale	1000	95	210	7.33×10^{-7}	185	410	8.20×10^{-7}
Extended Penalty	1000	115	258	8.14×10^{-7}	245	560	9.12×10^{-7}
Perturbed Quadratic	1000	180	390	2.40×10^{-7}	310	685	4.30×10^{-7}
Extended Three Exp	1000	19	48	1.12×10^{-8}	38	92	8.50×10^{-8}
Generalized PSC1	1000	68	155	8.90×10^{-7}	125	290	7.11×10^{-7}
Extended Powell	1000	210	465	9.35×10^{-7}	480	1050	9.88×10^{-7}
Extended Block-Diagonal	1000	72	162	6.18×10^{-7}	155	360	5.40×10^{-7}
Extended Maratos	1000	142	310	6.77×10^{-7}	290	640	8.12×10^{-7}
Extended Quadratic	1000	34	78	7.22×10^{-8}	72	168	2.35×10^{-8}
Extended Densi	1000	85	195	9.88×10^{-7}	195	450	9.10×10^{-7}
Generalized Quartic	1000	58	132	7.40×10^{-7}	132	305	6.45×10^{-7}
Extended Tridiagonal 1	1000	40	92	6.15×10^{-7}	115	260	9.15×10^{-7}
Extended Three Exp	1000	18	42	7.10×10^{-9}	45	105	9.30×10^{-8}
Generalized Tridiagonal	1000	51	118	8.55×10^{-7}	195	450	8.80×10^{-7}
Extended Maratos	1000	148	325	7.88×10^{-7}	430	980	9.92×10^{-7}
Extended Cliff	1000	60	140	6.12×10^{-7}	155	360	8.65×10^{-7}
Quadratic QF1	1000	18	42	4.22×10^{-9}	58	135	1.15×10^{-8}
Quadratic QF2	1000	23	55	7.33×10^{-8}	68	158	9.20×10^{-8}
Extended Wood	1000	175	385	9.12×10^{-7}	810	1860	9.95×10^{-7}
Extended Hiebert	1000	68	155	8.22×10^{-7}	210	485	8.12×10^{-7}
Ext. Quadratic Penalty	1000	115	255	6.80×10^{-7}	340	790	9.45×10^{-7}

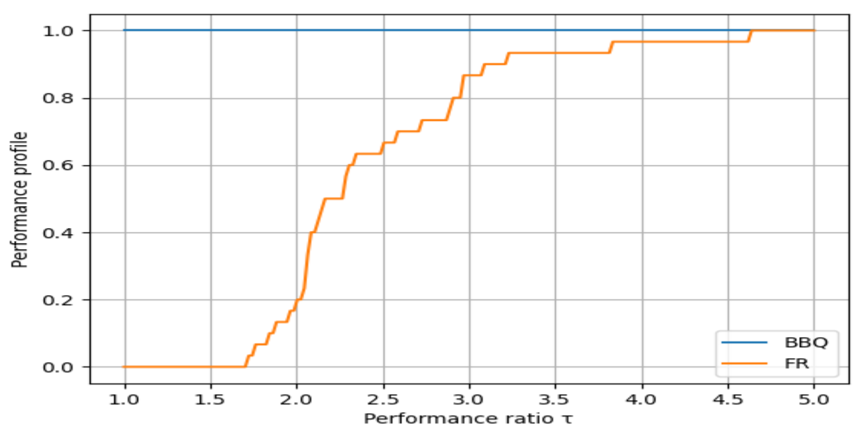

Figure 1. Performance profiles of FR and BBQ based on NI.

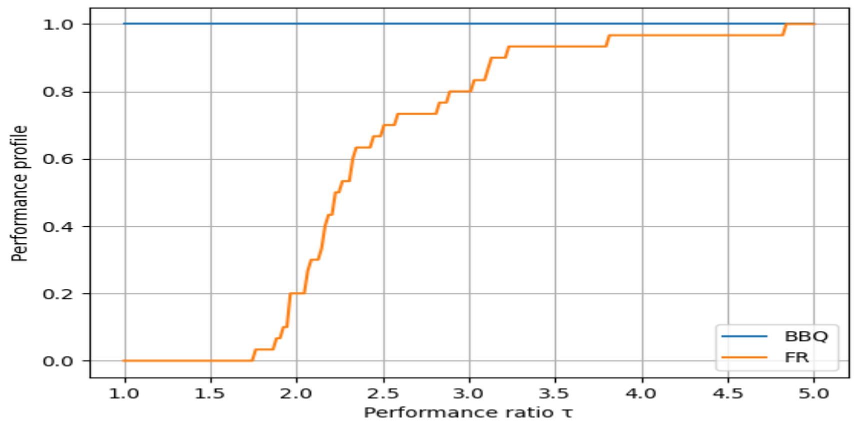

FIGURE 2. Performance profiles of FR and BBQ based on NF.

5. Conclusion

We have introduced nonlinear conjugate gradient technique that integrate insights pertaining to the Golden Ratio (0.618) for the resolution of unconstrained optimization problems and their corresponding convergence analysis. Numerical demonstrations reveal that one of the proposed methodologies exhibits efficiency that is either on par with or exceeds that of conventional techniques. In forthcoming research, we will explore avenues to expedite these methodologies and undertake new convergence analyses pertaining to them.

References

[1] E. D. Dolan and J. J. Moré, *Benchmarking optimization software with performance profiles*, Math. Program., Ser. B, *91*(2) (2002).
[2] R. Fletcher, *Function minimization by conjugate gradients*, Comput. J., *7*(2) (1964).
[3] B. A. Hassan, *A new type of quasi-newton updating formulas based on the new quasi-newton equation*, Numer. Algebra Control Optim., *10*(2) (2020).
[4] B. A. Hassan and A. R. Ayoob, *An Adaptive Quasi-Newton Equation for Unconstrained Optimization*, Proc. 2021 2nd IT-ELA Conf., (2021).
[5] B. A. Hassan and A. R. Ayoob, *On the New Quasi-Newton Equation for Unconstrained Optimization*, Proc. 8th IEC 2022 - Int. Eng. Conf.: Towards Engineering Innovations and Sustainability, (2022).
[6] B. A. Hassan and H. N. Jabbar, *Using a new class quasi-Newton equation for solving unconstrained optimization problems*, J. Interdiscip. Math., *26*(2) (2023).
[7] B. A. Hassan and M. A. Kahya, *A new class of quasi-Newton updating formulas for unconstrained optimization*, J. Interdiscip. Math., *24*(8) (2021).
[8] B. A. Hassan and I. A. R. Moghrabi, *A modified secant equation quasi-Newton method for unconstrained optimization*, J Appl Math Comput, *69*(1) (2023).
[9] B. A. Hassan and Y. A. Mohammed, *Innovative conjugate gradient methods for image impulse noise removal*, Journal of Interdisciplinary Mathematics, *28*(1) (2025), 347–353.
[10] B. A. Hassan and Y. A. Mohammed, *Reducing Impulse Noise in Images Using an Improved Formula Conjugate Gradient Method*, Zanco Journal of Pure and Applied Sciences, *37*(3) (2025), 93–102.
[11] B. A. Hassan and A. A. Saad, *Elastic conjugate gradient methods to solve iteration problems*, J. Interdiscip. Math., *26*(6) (2023).

[12] B. A. Hassan and A. A. Saad, *Explaining new parameters conjugate analysis based on the quadratic model*, J. Interdiscip. Math., *26*(6) (2023).

[13] B. A. Hassan and H. M. Sadiq, *New Class of Conjugate Gradient Methods for Removing Impulse Noise Images*, Iraqi Journal of Science, (2023), 5208–5218.

[14] B. A. Hassan and R. M. Sulaiman, *Using a New Type Quasi-Newton Equation for Unconstrained Optimization*, in Proceedings of the 7th International Engineering Conference "Research and Innovation Amid Global Pandemic", IEC 2021, (2021).

[15] B. A. Hassan and M. W. Taha, *A new variants of quasi-newton equation based on the quadratic function for unconstrained optimization*, Indonesian J. Electr. Eng. Comput. Sci., *19*(2) (2020).

[16] D. H. Li and M. Fukushima, *A modified BFGS method and its global convergence in nonconvex minimization*, J. Comput. Appl. Math., *129*(1–2) (2001).

[17] Y. A. Mohammed and B. A. Hassan, *Images Restoration Based on a New Optimal Parameter to Conjugate Gradient Method*, Statistics, Optimization & Information Computing, *14*(6) (2025), 3235–3243.

[18] S. S. Rao, *Engineering optimization: Theory and practice*, 2019.

[19] G. Zoutendijk, *Nonlinear programming, computational methods*, Integer Nonlinear Program., (1970), 37–86.