Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-16
DOI:10.22034/cmde.2026.67780.3243

On the existence of positive solutions for iterative system of the 2^{nd}-Order ℓ-point impulsive boundary value problem

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Abstract

This study examines the existence of positive solutions for iterative system of the 2^{nd}-order equation, including ℓ-point impulsive boundary value problem. The four-functionals fixed-point theorem is utilized to investigate this solution. Additionally, an example is presented to demonstrate the practical application of the primary findings.

Keywords. Impulsive boundary value problems, Positive solutions, ℓ-point, Fixed point theorems, Iterative systems.

2010 Mathematics Subject Classification. 34B37, 34B18, 34B15.

1. Introduction

Impulsive differential equations are a type of differential equation that involve abrupt changes or "impulses" at certain points in time. Impulsive differential equations are used to model systems that experience sudden changes, discontinuities, or interactions at specific moments in time. Such systems can be found in various fields, including physics, engineering, economics, biology, and more. In particular, recent theoretical and applied studies have focused on impulsive models in complex dynamical systems [27] and optimal control problems [8, 26]. Solving impulsive differential equations can be more challenging than solving regular differential equations due to the non-smooth behavior introduced by the impulses. Various techniques, such as the theory of distributions, measure theory, and numerical methods, are employed to handle impulsive differential equations and study the behavior of solutions. It's important to note that impulsive differential equations should be handled with care, as some theoretical challenges arise due to the non-differentiability of the solutions at the points of impulse. In practice, modeling and analyzing impulsive systems require a deep understanding of the specific problem and the nature of the impulses involved.

Impulsive differential equations have become a crucial area of research due to their unique and intricate characteristics compared to traditional differential equations. Consequently, numerous researchers have dedicated their efforts to exploring the diverse applications of impulsive differential equations, resulting in profound insights and discoveries. Pertinent literature in this domain includes references such as [2–4, 15, 24, 25] and other notable sources.

Within the literature, significant attention has been devoted to the investigation of 2^{nd}-order impulsive boundary value problems. To explore this topic further, readers are encouraged to refer to works such as [5–7, 9–14, 16–22]. Additionally, recent works focusing on 2^{nd}-order m-point impulsive boundary value problems can be found in references such as [5, 13, 14, 20]. Furthermore, systems of 2^{nd}-order impulsive boundary value problems have attracted the interest of numerous researchers. For comprehensive insights into this area, we recommend referring to publications such as [7, 9, 11, 18, 21].

In this study, we examine the following iterative system of the 2^{nd}-order with ℓ-point impulsive boundary value problem (2^{nd}-order ℓ-point IBVP). This problem is motivated by previous findings and serves as the basis for our

Received: 11 June 2025; Accepted: 28 July 2026.
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investigation:

$$\begin{cases} \omega_i''(t) + d_i(t)\omega_i'(t) + e_i(t)\omega_i(t) + h_i(t, \omega_{i+1}(t)) = 0, & t \in J = [0,1],\ t \neq t_p,\ i = 1,2,...,n, \\ \omega_{n+1}(t) = \omega_1(t), & \\ \triangle\omega_i|_{t=t_p} = I_{ip}(\omega_i(t_p)), & p = 1,2,...,k, \\ \triangle\omega_i'|_{t=t_p} = -\overline{I}_{ip}(\omega_i(t_p)), & p = 1,2,...,k, \\ \tau_1\omega_i(0) - \tau_2\omega_i'(0) = \sum_{j=1}^{\ell-2} \kappa_j \omega_i(\chi_j), & \\ \tau_3\omega_i(1) + \tau_4\omega_i'(1) = \sum_{j=1}^{\ell-2} \varrho_j \omega_i(\chi_j). & \end{cases} \tag{1.1}$$

where $J = [0,1],\ t \neq t_p,\ \ p = 1,2,...,k$ with $0 < t_1 < t_2 < ... < t_k < 1$. For $i = 1,2,...n,\ \ \triangle\omega_i|_{t=t_p}$ and $\triangle\omega_i'|_{t=t_p}$ indicate the jump of $\omega_i(t)$ and $\omega_i'(t)$ at $t = t_p$, i.e.,

$$\triangle\omega_i|_{t=t_p} = \omega_i(t_p^+) - \omega_i(t_p^-), \quad \triangle\omega_i'|_{t=t_p} = \omega_i'(t_p^+) - \omega_i'(t_p^-),$$

where $\omega_i'(t_p^-),\ \omega_i(t_p^-)$ and $\omega_i'(t_p^+),\ \omega_i(t_p^+)$ represent the left-hand limit and right-hand limit of $\omega_i'(t)$ and $\omega_i(t)$ at $t = t_p$, $p = 1,2,...,k$, respectively.
We assume that the following conditions are provided throughout this paper.

- $(H1)$ For $j \in \{1,...,\ell-2\}$, $\kappa_j, \varrho_j \in [0,\infty)$, $\chi_j \in (0,1)$ and $\tau_1,\tau_2,\tau_3,\tau_4 \in [0,\infty)$,
- $(H2)$ For $i = 1,2,\ldots,n$, the function $h_i \in C(J \times \mathbb{R}, \mathbb{R}^+)$ where $\mathbb{R}^+ = (0,+\infty)$,
- $(H3)$ $d_i(t) \in C(J, \mathbb{R}^+)$, $e_i(t) \in C(J, (-\infty,0))$ such that $d_i(t) + e_i(t) \geq 0$ for $t \in J$, $i = 1,2,...n$,
- $(H4)$ The impulsive functions $I_{ip}, \overline{I}_{ip} : \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded for all $p = 1,2,\ldots,k$ and $i = 1,2,\ldots,n$,
- $(H5)$ For $i = 1,2,\ldots,n$, $\max\{X_{i1}, X_{i2}, \ldots, X_{ik}\} \leq \frac{Z}{W_i}$ where

  $$X_{ip} = \sup_{u \geq 0}\{I_{ip}(u), \overline{I}_{ip}(u)\}, \quad (p = 1,2,\ldots,k),$$

  with constants Z and W_i given by Theorem 3.2.
- $(H6)$ The constant ∇_i defined in (2.17) satisfies $\nabla_i < 0$. This is guaranteed if the parameters satisfy the condition:

  $$\left(\sum_{j=1}^{l-2}\kappa_j K_{ij}(\chi_j)\right)\left(\sum_{j=1}^{l-2}\rho_j z_i(\chi_j)\right) - \left(\rho_i - \sum_{j=1}^{l-2}\kappa_j z_i(\chi_j)\right)\left(\frac{\rho_i}{v_i(1)} - \sum_{j=1}^{l-2}\rho_j y_i(\chi_j)\right) < 0,$$
- $(H7)$ For $p = 1,2,\ldots,k$ and $i = 1,2,\ldots,n$, the following inequalities hold:

  $$\frac{\rho_i}{v_i(1)} - \sum_{j=1}^{l-2}\rho_j y_i(\chi_j) > 0, \quad \rho_i - \sum_{j=1}^{l-2}\kappa_j z_i(\chi_j) > 0.$$

  Additionally, we assume that for any $u \geq 0$:

  $$z_i(t_p)\overline{I}_{ip}(u) + z_i'(t_p)I_{ip}(u) > 0,$$
- $(H8)$ $\sum_{j=1}^{l-2}(\tau_1 e_j - \tau_3\kappa_j) \geq 0$ where parameters are given by condition (H1).

Remark 1.1. It is important to emphasize that the functions $y_i(t), z_i(t)$ and the quantities $\rho_i, \nabla_i, \Lambda_i, W_i$ are derived solely from the linear part of the operator and are independent of the solution ω. Furthermore, the conditions in (H5) and (H7) involving the impulsive functions I_{ip} are imposed on the *structure* of these functions for all non-negative arguments $u \geq 0$, rather than depending on the specific unknown solution $\omega(t_p)$.

To the best of our knowledge, the study of iterative system of 2^{nd}-order ℓ-point ibvps has not been explored in existing literature.
Compared to previous works, in particular [20], the type of differential equation considered is the more general type of equation of 2nd order. In fact, it is considered as an iterative system in this study. In the study of [9], this equation type (without impulsive conditions) is treated as a binary system, even the system in the mentioned study is a discrete system, not iterative. However, finding the solution of the iterative system is related to finding the solutions in the previous step. In this sense, it is more comprehensive and difficult. In this study, the 2^{nd}-order impulsive boundary value problem studied by the authors in [20] is considered as an iterative system. By utilising the fixed point theorem called four functionals, the necessary criteria for the existence of at least one solution are obtained. Then, the study is completed with an example.

This paper is structured as follows: In Section 2, we present several definitions and basic lemmas that serve as fundamental tools for obtaining our main result. In Section 3, we establish and prove our main result using the four functional fixed point theorem. Finally, in Section 4, we provide an example to illustrate the practical application of our main result.

2. Preliminaries

In this part, we will start with some basic descriptions in the corresponding Banach spaces and then go on to supplementary lemmas that will be helpful later.

Let $J' = J\backslash\{t_1, t_2, ..., t_k\}$. $C(J)$ denote the Banach space of all continuous mapping $\omega(t) : J \to \mathbb{R}$ with the norm $\|\omega\| = \sup_{t\in J} |\omega(t)|$, $PC(J) = \{\omega(t) : J \to \mathbb{R} : \omega(t) \in C(J'),\ \omega(t_p^+)$ and $\omega(t_p^-)$ exist and $\omega(t_p^-) = \omega(t_p),\ p = 1, 2, ..., k\}$ is also a Banach space with norm $\|\omega\|_{PC} = \sup_{t\in J} |\omega(t)|$ and $PC^1(J) = \{\omega(t) \in PC(J) : \omega'(t) \in C(J'),\ \omega'(t_p^+)$ and $\omega'(t_p^-)$ exist and $\omega'(t_p^-) = \omega'(t_p),\ p = 1, 2, ..., k\}$ is a real Banach space with norm $\|\omega\|_{PC^1} = \max\{\|\omega\|_{PC}, \|\omega'\|_{PC}\}$ where $\|\omega\|_{PC} = \sup_{t\in J} |\omega(t)|$, $\|\omega'\|_{PC} = \sup_{t\in J} |\omega'(t)|$. Let $\mathbb{B} = PC^1(J) \cap C^2(J')$. A function $(\omega_1(t), ..., \omega_n(t)) \in \mathbb{B}$ is called a solution of the iterative system of 2^{nd}-order ℓ-point ibvp (1.1) provided that it yields the iterative system of 2^{nd}-order ℓ-point ibvp (1.1).

Let us begin by examining the case where $i = 1$ within the iterative system of 2^{nd}-order ℓ-point ibvp (1.1). This will enable us to derive the solution $\omega_1(t)$ of the 2^{nd}-order ℓ-point ibvp (2.1), thereby allowing us to compute $\omega_n(t)$ based on the known value of $\omega_1(t)$. By repeating this process, we can calculate $\omega_{n-1}(t)$, followed by $\omega_{n-2}(t)$ and so on, ultimately arriving at the solution $\omega_2(t)$. As a result, we will obtain the solution $(\omega_1(t), ..., \omega_n(t))$ for the iterative system of 2^{nd}-order ℓ-point ibvp (1.1).

We consider the following 2^{nd}-order ℓ-point ibvp:

Let $f \in C[0, 1]$,

$$\begin{cases} \omega_1''(t) + d_1(t)\omega_1'(t) + e_1(t)\omega_1(t) + f(t) = 0, & t \in J = [0, 1],\ t \neq t_p, \\ \triangle\omega_1|_{t=t_p} = I_{1p}(\omega_1(t_p)), & p = 1, 2, ..., k, \\ \triangle\omega_1'|_{t=t_p} = -\overline{I}_{1p}(\omega_1(t_p)), & p = 1, 2, ..., k, \\ \tau_1\omega_1(0) - \tau_2\omega_1'(0) = \displaystyle\sum_{j=1}^{\ell-2} \kappa_j\omega_1(\chi_j), & \\ \tau_3\omega_1(1) + \tau_4\omega_1'(1) = \displaystyle\sum_{j=1}^{\ell-2} \varrho_j\omega_1(\chi_j). & \end{cases} \tag{2.1}$$

Lemma 2.1. *[23] Let (H1) and (H3) be valid. Let y_1 and z_1 be the unique solution of boundary value problem:*

$$\begin{cases} y_1''(t) + d_1(t)y_1'(t) + e_1(t)y_1(t) = 0, \quad t \in J, \\ y_1(0) = \tau_2, \qquad y_1'(0) = \tau_1 \end{cases} \tag{2.2}$$

and

$$\begin{cases} z_1''(t) + d_1(t)z_1'(t) + e_1(t)z_1(t) = 0, & t \in J, \\ z_1(1) = \tau_4, \qquad z_1'(1) = -\tau_3 \end{cases} \tag{2.3}$$

respectively. Therefore,
(i) *On $(0,1]$, y_1 is strictly increasing on J and $y_1 > 0$;*
(ii) *On $[0,1)$, z_1 is strictly decreasing on J and $z_1 > 0$.*

Set

$$\rho_1 := v_1(t)\begin{vmatrix} z_1(t) & y_1(t) \\ z_1'(t) & y_1'(t) \end{vmatrix} = \tau_1 z_1(0) - \tau_2 z_1'(0) > 0 \text{ with } v_1(t) = exp\left(\int_0^t d_1(s)ds\right), \ \forall t \in J$$

and

$$\nabla_1 = \begin{vmatrix} -\sum_{j=1}^{\ell-2}\kappa_j y_1(\chi_j) & \rho_1 - \sum_{j=1}^{\ell-2}\kappa_j z_1(\chi_j) \\ \frac{\rho_1}{v_1(1)} - \sum_{j=1}^{\ell-2}\varrho_j y_1(\chi_j) & -\sum_{j=1}^{\ell-2}\varrho_j z_1(\chi_j) \end{vmatrix}. \tag{2.4}$$

Lemma 2.2. *Let (H1)-(H8) hold.*
If $\omega_1(t) \in \mathbb{B}$ is a solution of the equation

$$\omega_1(t) = \int_0^1 H_1(t,s)v_1(s)f(s)ds + \sum_{p=1}^{k} W_{1p}(t,t_p) + y_1(t)\Theta_1(f(s)) + z_1(t)\Gamma_1(f(s)), \tag{2.5}$$

where

$$H_1(t,s) = \frac{1}{\rho_1}\begin{cases} y_1(s)z_1(t), & s \le t, \\ y_1(t)z_1(s), & t \le s, \end{cases} \tag{2.6}$$

$$W_{1p}(t,t_p) = \frac{1}{\rho_1}\begin{cases} y_1(t)v_1(t_p)[z_1'(t_p)I_{1p}(\omega_1(t_p)) + z_1(t_p)\overline{I}_{1p}(\omega_1(t_p))], & t < t_p, \\ z_1(t)v_1(t_p)[y_1'(t_p)I_{1p}(\omega_1(t_p)) + y_1(t_p)\overline{I}_{1p}(\omega_1(t_p))], & t_p \le t, \end{cases} \tag{2.7}$$

$$K_{1j}(\chi_j) = \int_0^1 H_1(t,s)|_{t=\chi_j} v_1(s)f(s)ds + \sum_{p=1}^{k} W_{1p}(t,t_p)|_{t=\chi_j}, \tag{2.8}$$

$$\Theta_1(f(s)) = \frac{1}{\nabla_1}\begin{vmatrix} \sum_{j=1}^{\ell-2}\kappa_j K_{1j}(\chi_j) & \rho_1 - \sum_{j=1}^{\ell-2}\kappa_j z_1(\chi_j) \\ \sum_{j=1}^{\ell-2}\varrho_j K_{1j}(\chi_j) & -\sum_{j=1}^{\ell-2}\varrho_j z_1(\chi_j) \end{vmatrix}, \tag{2.9}$$

and

$$\Gamma_1(f(s)) = \frac{1}{\nabla_1}\begin{vmatrix} -\sum_{j=1}^{\ell-2}\kappa_j y_1(\chi_j) & \sum_{j=1}^{\ell-2}\kappa_j K_{1j}(\chi_j) \\ \frac{\rho_1}{v_1(1)} - \sum_{j=1}^{\ell-2}\varrho_j y_1(\chi_j) & \sum_{j=1}^{\ell-2}\varrho_j K_{1j}(\chi_j) \end{vmatrix}, \tag{2.10}$$

then $\omega_1(t)$ is a solution of the 2^{nd}-order ℓ-point ibvp (2.1).

Proof. The proof of this lemma is provided by Lemma 2 in [20]. □

Lemma 2.3. *Let (H1)-(H8) hold.*
Then for $\omega_1(t) \in \mathbb{B}$ with $f(t) \geq 0$, the solution $\omega_1(t)$ of the 2^{nd}-order ℓ-point ibvp (2.1) satisfies $\omega_1(t)$ is positive for $t \in J$.

Proof. Firstly, it is evident that the Green's function $H_1(t,s)$ remains positive for $t, s \in [0,1] \times [0,1]$. Moreover, under the condition $(H6)$, both $\Theta_1(f)$ and $\Gamma_1(f)$ are positive. Furthermore, since $I_{1p}(\omega_1(t_p))$ and $\overline{I}_{1p}(\omega_1(t_p))$ are positive, this leads to the positivity of $W_{1p}(t,t_p)$. Consequently, $\omega_1(t)$ is positive for $t \in [0,1]$. □

Lemma 2.4. *Suppose that (H1)-(H8) are valid. Then the solution $\omega_1(t) \in \mathbb{B}$, of the 2^{nd}-order ℓ-point ibvp (2.1) satisfies the inequality $\omega_1'(t) \geq 0$ for $t \in J$.*

Proof. The proof of this lemma is provided by Lemma 6 in [20]. □

Lemma 2.5. *Under the assumption that (H1)-(H8) are satisfied, we can deduce that for any $t, s \in J$, we get*

$$0 \leq H_1(t,s) \leq \delta_1 H_1(s,s), \tag{2.11}$$

where $\delta_1 = \dfrac{\max\{y_1(1), z_1(0)\}}{\min\{y_1(0), z_1(1)\}}$.

Proof. The proof of this lemma is provided by Lemma 4 in [20]. □

Lemma 2.6. *Assume that the conditions (H1)-(H8) are valid. Let $\alpha \in (0, \frac{1}{2})$. Then for any $t \in [\alpha, 1-\alpha]$ and $s \in J$, we have*

$$H_1(t,s) \geq \gamma_1 H_1(s,s), \tag{2.12}$$

where $\gamma_1 := \dfrac{\min\{y_1(\alpha), z_1(1-\alpha)\}}{\max\{y_1(1), z_1(0)\}}$.

Proof. The proof of this lemma is provided by Lemma 5 in [20]. □

Consider the set K defined as $K = \{\omega_1(t) \in \mathbb{B} : \omega_1(t)$ is nonnegative, nondecreasing, and concave on $J\}$. It follows that K constitutes a cone within $PC^1(J)$.

Lemma 2.7. *Suppose that the conditions (H1)-(H8) are satisfied. And $\omega_1(t) \in K, \quad \alpha \in (0, \frac{1}{2})$. Then,*

$$\min_{t \in [\alpha, 1-\alpha]} \omega_1(t) \geq \alpha \|\omega_1\|_{PC}, \tag{2.13}$$

where $\|\omega_1\|_{PC} = \sup_{t \in J} |\omega_1(t)|$.

Proof. The proof of this lemma is provided by Lemma 7 in [20]. □

Lemma 2.8. *Suppose that the conditions (H1)-(H8) are valid. And $\omega_1(t) \in K$. Then for $M > 0$,*

$$\max_{t \in J} \omega_1(t) \leq M \max_{t \in J} \omega_1^{'}(t), \tag{2.14}$$

where $M = 1 + \dfrac{\tau_2 + \sum_{j=1}^{\ell-2} \kappa_j \chi_j}{\tau_1 - \sum_{j=1}^{\ell-2} \kappa_j}$.

Proof. This Lemma's proof is given in [12]. □

From Lemma 2.8, we obtain

$$\begin{aligned}\|\omega_1\|_{PC^1} &= \max\{\|\omega_1\|_{PC}, \|\omega_1^{'}\|_{PC}\}\\ &= \max\left\{\max_{t\in J}|\omega_1(t)|, \max_{t\in J}|\omega_1^{'}(t)|\right\}\\ &\le \max\left\{M\max_{t\in J}|\omega_1^{'}(t)|, \max_{t\in J}|\omega_1^{'}(t)|\right\}\\ &= M\max_{t\in J}\omega_1^{'}(t).\end{aligned}$$

For brevity, we note that

$$\Theta_i := \Theta_i(h_i(\cdot,\omega_{i+1}(\cdot))), \quad \Gamma_i := \Gamma_i(h_i(\cdot,\omega_{i+1}(\cdot))).$$

We note that an n-tuple $(\omega_1(t), \omega_2(t), ..., \omega_n(t))$ is a solution for the 2^{nd}-order ℓ-point ibvp (1.1)'s iterative system if and only if

$$\begin{aligned}\omega_1(t) = &\int_0^1 H_1(t,s_1)v_1(s_1)h_1\Bigg(s_1, \int_0^1 H_2(s_1,s_2)v_2(s_2)h_2\Bigg(s_2, \int_0^1 H_3(s_2,s_3)v_3(s_3)h_3...\\ &\quad ...\ h_{n-1}\Bigg(s_{n-1}, \int_0^1 H_n(s_{n-1},s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n + \sum_{p=1}^k W_{np}(s_{n-1},t_p)\\ &\quad + y_n(s_{n-1})\Theta_n + z_n(s_{n-1})\Gamma_n\Bigg)ds_{n-1} + \sum_{p=1}^k W_{n-1,p}(s_{n-2},t_p)\\ &\quad + y_{n-1}(s_{n-2})\Theta_{n-1} + z_{n-1}(s_{n-2})\Gamma_{n-1}\Bigg)ds_{n-2} + ... + \sum_{p=1}^k W_{3p}(s_2,t_p)\\ &\quad + y_3(s_2)\Theta_3 + z_3(s_2)\Gamma_3\Bigg)ds_2 + \sum_{p=1}^k W_{2p}(s_1,t_p) + y_2(s_1)\Theta_2 + z_2(s_1)\Gamma_2\Bigg)ds_1\\ &\quad + \sum_{p=1}^k W_{1p}(t,t_p) + y_1(t)\Theta_1 + z_1(t)\Gamma_1.\\ \omega_i(t) = &\int_0^1 H_i(t,s)v_i(s)h_i(s,\omega_{i+1}(s))ds + \sum_{p=1}^k W_{ip}(t,t_p) + y_i(t)\Theta_i + z_i(t)\Gamma_i, \quad t\in J,\\ \omega_{n+1}(t) = &\ \omega_1(t),\end{aligned}$$

where

$$\rho_i = \tau_1 z_i(0) - \tau_2 z_i^{'}(0), \tag{2.15}$$

$$v_i(t) = exp\left(\int_0^t d_i(s)ds\right), \ \forall t\in J, \tag{2.16}$$

$$\nabla_i = \begin{vmatrix} -\sum_{j=1}^{\ell-2}\kappa_j y_i(\chi_j) & \rho_i - \sum_{j=1}^{\ell-2}\kappa_j z_i(\chi_j)\\ \frac{\rho_i}{v_i(1)} - \sum_{j=1}^{\ell-2}\varrho_j y_i(\chi_j) & -\sum_{j=1}^{\ell-2}\varrho_j z_i(\chi_j)\end{vmatrix}, \tag{2.17}$$

$$\Theta_i = \frac{1}{\nabla_i}\begin{vmatrix} \sum_{j=1}^{\ell-2}\kappa_j K_{ij}(\chi_j) & \rho_i - \sum_{j=1}^{\ell-2}\kappa_j z_i(\chi_j) \\ \sum_{j=1}^{\ell-2}\varrho_j K_{ij}(\chi_j) & -\sum_{j=1}^{\ell-2}\varrho_j z_i(\chi_j) \end{vmatrix}, \quad \Gamma_i = \frac{1}{\nabla_i}\begin{vmatrix} -\sum_{j=1}^{\ell-2}\kappa_j y_i(\chi_j) & \sum_{j=1}^{\ell-2}\kappa_j K_{ij}(\chi_j) \\ \frac{\rho_i}{v_i(1)} - \sum_{j=1}^{\ell-2}\varrho_j y_i(\chi_j) & \sum_{j=1}^{\ell-2}\varrho_j K_{ij}(\chi_j) \end{vmatrix},$$

$$W_{ip}(t,t_p) = \frac{1}{\rho_i}\begin{cases} y_i(t)v_i(t_p)[z_i^{'}(t_p)I_{ip}(\omega_i(t_p)) + z_i(t_p)\overline{I}_{ip}(\omega_i(t_p))], & t < t_p, \\ z_i(t)v_i(t_p)[y_i^{'}(t_p)I_{ip}(\omega_i(t_p)) + y_i(t_p)\overline{I}_{ip}(\omega_i(t_p))], & t_p \leq t, \end{cases}$$

$$H_i(t,s) = \frac{1}{\rho_i}\begin{cases} y_i(s)z_i(t), & s \leq t, \\ y_i(t)z_i(s), & t \leq s, \end{cases}$$

and

$$K_{ij}(\chi_j) = \int_0^1 H_i(t,s)|_{t=\chi_j} v_i(s)h_i(s,\omega_{i+1}(s))ds + \sum_{p=1}^{k} W_{ip}(t,t_p)|_{t=\chi_j}.$$

For $\omega_1(t) \in K$, let's define the integral operator from K to $\mathbb{B}$ as

$$\begin{aligned}
A\omega_1(t) = \int_0^1 & H_1(t,s_1)v_1(s_1)h_1\Big(s_1, \int_0^1 H_2(s_1,s_2)v_2(s_2)h_2\Big(s_2, \int_0^1 H_3(s_2,s_3)v_3(s_3)h_3 \ldots \\
& h_{n-1}\Big(s_{n-1}, \int_0^1 H_n(s_{n-1},s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n + \sum_{p=1}^{k} W_{np}(s_{n-1},t_p) \\
& + y_n(s_{n-1})\Theta_n + z_n(s_{n-1})\Gamma_n\Big)ds_{n-1} + \sum_{p=1}^{k} W_{n-1,p}(s_{n-2},t_p) \\
& + y_{n-1}(s_{n-2})\Theta_{n-1} + z_{n-1}(s_{n-2})\Gamma_{n-1}\Big)ds_{n-2} + \ldots + \sum_{p=1}^{k} W_{3p}(s_2,t_p) \\
& + y_3(s_2)\Theta_3 + z_3(s_2)\Gamma_3\Big)ds_2 + \sum_{p=1}^{k} W_{2p}(s_1,t_p) + y_2(s_1)\Theta_2 + z_2(s_1)\Gamma_2\Big)ds_1 \\
& + \sum_{p=1}^{k} W_{1p}(t,t_p) + y_1(t)\Theta_1 + z_1(t)\Gamma_1.
\end{aligned} \tag{2.18}$$

Remark 2.9. We verify that the operator A defined in (2.18) preserves the impulsive conditions. Let $\omega = A\omega_1$. Since the integral terms are continuous, the jumps at t_p depend only on $W_{ip}(t,t_p)$. Using the limits from (2.18), we have:

$$\begin{aligned}
\Delta\omega_i|_{t=t_p} &= W_{ip}(t_p^+,t_p) - W_{ip}(t_p^-,t_p) \\
&= \frac{v_i(t_p)}{\rho_i}\left[I_{ip}(z_i(t_p)y_i'(t_p) - y_i(t_p)z_i'(t_p)) + \overline{I}_{ip}(z_i(t_p)y_i(t_p) - y_i(t_p)z_i(t_p))\right].
\end{aligned}$$

Since the coefficient of $\overline{I}_{ip}$ is zero and the coefficient of I_{ip} equals $\rho_i/v_i(t_p)$ (from the Wronskian relation), we obtain $\Delta\omega_i|_{t=t_p} = I_{ip}$.
Differentiating $W_{ip}(t,t_p)$ with respect to t, we get

$$W_{ip}'(t,t_p) = \frac{v_i(t_p)}{\rho_i}\begin{cases} y_i'(t)[z_i'(t_p)I_{ip} + z_i(t_p)\overline{I}_{ip}], & t < t_p, \\ z_i'(t)[y_i'(t_p)I_{ip} + y_i(t_p)\overline{I}_{ip}], & t > t_p. \end{cases}$$

Calculating the jump at $t = t_p$,

$$\begin{aligned}\Delta\omega_i'|_{t=t_p} &= W_{ip}'(t_p^+, t_p) - W_{ip}'(t_p^-, t_p)\\ &= \frac{v_i(t_p)}{\rho_i}\left[I_{ip}(z_i'(t_p)y_i'(t_p) - y_i'(t_p)z_i'(t_p)) + \overline{I}_{ip}(z_i'(t_p)y_i(t_p) - y_i'(t_p)z_i(t_p))\right].\end{aligned}$$

The coefficient of I_{ip} is zero. The term $z_i'y_i - y_i'z_i$ equals $-\rho_i/v_i(t_p)$. Thus, we get $\Delta\omega_i'|_{t=t_p} = -\overline{I}_{ip}$. This confirms that A satisfies both impulsive conditions correctly.

Lemma 2.10. *Let (H1)–(H8) be satisfied. Then $A(K) \subset K$ is completely continuous.*

Proof. Notice from (H1)-(H8) and Lemmas 2.3, 2.4, and the definition of A that, for $\omega_1(t) \in K$, $A\omega_1(t) \geq 0$, $(A\omega_1)'(t) \geq 0$ *and* $A\omega_1(t)$ is concave on J. Thus, $A(K) \subset K$. Additionally, using the *Arzela-Ascoli* theorem it can be shown that the operator A is completely continuous. □

3. Main Results

In this section, the following four functionals fixed point theorem [1] will be used in this part to obtain adequate conditions for the existence of at least one positive solution to the iterative system of 2^{nd}-order ℓ-point ibvp (1.1).

Let ζ and λ be nonnegative continuous concave functionals on K and let η and μ be nonnegative continuous convex functionals on K. The sets are defined as follows for positive numbers z, ν, ς and Z,

$$\begin{aligned}&D(\zeta, \eta, z, Z) = \{\omega(t) \in K : z \leq \zeta(\omega(t)), \eta(\omega(t)) \leq Z\},\\ &E(\lambda, \nu) = \{\omega(t) \in D(\zeta, \eta, z, Z) : \nu \leq \lambda(\omega(t))\},\\ &F(\mu, \varsigma) = \{\omega(t) \in D(\zeta, \eta, z, Z) : \mu(\omega(t)) \leq \varsigma\}.\end{aligned} \tag{3.1}$$

Lemma 3.1. *(Four Functionals Fixed Point Theorem, [1]) Assume that K is a cone in a real Banach space B, ζ and λ are nonnegative continuous concave functionals on K, η and μ are nonnegative continuous convex functionals on K and there exist positive numbers z, ν, ς and Z, such that*

$$A : D(\zeta, \eta, z, Z) \to K,$$

is a completely continuous operator and $D(\zeta, \eta, z, Z)$ is a bounded set. If

(i) $\{\omega(t) \in E(\lambda, \nu) : \eta(\omega(t)) < Z\} \cap \{\omega(t) \in F(\mu, \varsigma) : z < \zeta(\omega(t))\} \neq \emptyset$;
(ii) $\zeta(A\omega(t)) \geq z$ for all $\omega(t) \in D(\zeta, \eta, z, Z)$ with $\zeta(\omega(t)) = z$ and $\varsigma < \mu(A\omega(t))$;
(iii) $\zeta(A\omega(t)) \geq z$ for all $\omega(t) \in F(\mu, \varsigma)$ with $\zeta(\omega(t)) = z$;
(iv) $\eta(A\omega(t)) \leq Z$ for all $\omega(t) \in D(\zeta, \eta, z, Z)$ with $\eta(\omega(t)) = Z$ and $\lambda(A\omega(t)) < \nu$;
(v) $\eta(A\omega(t)) \leq Z$ for all $\omega(t) \in E(\lambda, \nu)$ with $\eta(\omega(t)) = Z$

then A has a fixed point $\omega(t) \in D(\zeta, \eta, z, Z)$.

Now the relevant fixed points of A belonging to the cone K are investigated. The notations that follow are provided for the convenience.

For $i = 1, 2, ...n$,

$$\overline{\Theta}_i := \frac{1}{\nabla_i}\begin{vmatrix} \sum_{j=1}^{\ell-2}\kappa_j\left[\int_0^1 H_i(\chi_j, s)v_i(s)ds + \frac{1}{\rho_i}\sum_{p=1}^{k}C_i(t_p)v_i(t_p)\right] & \rho_i - \sum_{j=1}^{\ell-2}\kappa_j z_i(\chi_j)\\ \sum_{j=1}^{\ell-2}\varrho_j\left[\int_0^1 H_i(\chi_j, s)v_i(s)ds + \frac{1}{\rho_i}\sum_{p=1}^{k}C_i(t_p)v_i(t_p)\right] & -\sum_{j=1}^{\ell-2}\varrho_j z_i(\chi_j)\end{vmatrix},$$

$$\overline{\Gamma}_i := \frac{1}{\nabla_i}\begin{vmatrix} -\sum_{j=1}^{\ell-2}\kappa_j y_i(\chi_j) & \sum_{j=1}^{\ell-2}\kappa_j\left[\int_0^1 H_i(\chi_j, s)v_i(s)ds + \frac{1}{\rho_i}\sum_{p=1}^{k}C_i(t_p)v_i(t_p)\right]\\ \frac{\rho_i}{v_i(1)} - \sum_{j=1}^{\ell-2}\varrho_j y_i(\chi_j) & \sum_{j=1}^{\ell-2}\varrho_j\left[\int_0^1 H_i(\chi_j, s)v_i(s)ds + \frac{1}{\rho_i}\sum_{p=1}^{k}C_i(t_p)v_i(t_p)\right]\end{vmatrix},$$

where

$$C_i(t_p) := \max\{y_i(1), z_i(0)\}y_i^{'}(t_p) + y_i(1)z_i(0).$$

It also appears that

$$\Theta_i \leq \overline{\Theta}_i \max\{h_i(\cdot, \omega_{i+1}(\cdot)), I_{ip}(\omega_i(t_p)), \overline{I}_{ip}(\omega_i(t_p))\}, \quad i = 1, 2, ...n,$$

and

$$\Gamma_i \leq \overline{\Gamma}_i \max\{h_i(\cdot, \omega_{i+1}(\cdot)), I_{ip}(\omega_i(t_p)), \overline{I}_{ip}(\omega_i(t_p))\} \quad i = 1, 2, ...n.$$

For the purpose of simplicity, we use the notations for $i = 1, 2, ...n$,

$$\Omega_i = \int_{\alpha}^{1-\alpha} \gamma_i H_i(s, s)v_i(s)ds,$$

$$\Lambda_i = \int_0^1 \delta_i H_i(s, s)v_i(s)ds + \frac{1}{\rho_i}\sum_{p=1}^{k} C_i(t_p)v_i(t_p) + y_i(1)\overline{\Theta}_i + z_i(0)\overline{\Gamma}_i,$$

$$\overline{\Lambda}_1 = \frac{1}{\rho_1}\int_0^1 y_1^{'}(0)z_1(s)v_1(s)ds + \frac{1}{\rho_1}\sum_{p=1}^{k}[y_1^{'}(0)(z_1(t_p) + z_1^{'}(t_p))]v_i(t_p) + y_1^{'}(0)\overline{\Theta}_1 + z_1^{'}(0)\overline{\Gamma}_1,$$

$$L = \frac{\tau_1 - \sum_{j=1}^{\ell-2}\kappa_j}{\tau_2 + \sum_{j=1}^{\ell-2}\kappa_j\chi_j},$$

and define the maps

$$\zeta(\omega_1(t)) = \min_{t\in[\alpha,1-\alpha]}\omega_1(t) = \omega_1(\alpha), \quad \mu(\omega_1(t)) = \max_{t\in J}\omega_1(t) = \omega_1(1), \quad \eta(\omega_1(t)) = \lambda(\omega_1(t)) = \max_{t\in J}\omega_1^{'}(t) = \omega_1^{'}(0).$$

Moreover, assume that $D(\zeta, \eta, z, Z)$, $E(\lambda, \nu)$, $F(\mu, \varsigma)$ are defined by (3.1).

Theorem 3.2. *Suppose that (H1)-(H8)are satisfied. Additionally, assume that there exist constants z, ν, ς and Z with*

$$\max\left\{\frac{z}{\alpha}, \frac{L+1}{L}Z\right\} \leq \varsigma, \quad \max\left\{\frac{L+1}{L}\nu, \frac{L+1}{L\alpha+1}z\right\} < Z,$$

such that $\frac{z}{\Omega_i} \leq \frac{Z}{W_i} \leq \frac{\varsigma}{\Lambda_i}$ for $i = 1, ..., n$ where $W_i = \max\{\Lambda_i, \overline{\Lambda}_1\}$. In addition, the function h_i satisfies the following conditions:

(a) $h_i(t, \omega_{i+1}(t)) \geq \frac{z}{\Omega_i}$ *for* $(t, \omega_{i+1}(t)) \in [\alpha, 1-\alpha] \times [z, \varsigma]$,

(b) $h_i(t, \omega_{i+1}(t)) \leq \frac{\varsigma}{\Lambda_i}$ *for* $(t, \omega_{i+1}(t)) \in [0, 1] \times [0, \varsigma]$,

(c) $h_i(t, \omega_{i+1}(t)) \leq \frac{Z}{W_i}$ *for* $(t, \omega_{i+1}(t)) \in [0, 1] \times [z, ZM]$,

there exists an n-tuple $(\omega_1(t), \omega_2(t), ..., \omega_n(t))$ satisfying the iterative system of 2^{nd}-order ℓ-point ibvp (1.1) with $\omega_i(t) > 0$ such that

$$\min_{t\in[\alpha,1-\alpha]}\omega_i(t) \geq z, \quad \max_{t\in J}\omega_i(t) \leq Z,$$

for $i = 1, 2, ..., n$ on J.

Proof. It is clear that the existence of a positive solution to the iterative system of 2^{nd}-order ℓ-point ibvp (1.1) is equivalent to the existence of fixed points of the operator A. Hence, it is required to prove that the operator A satisfies the four-functionals fixed-point theorem. This will show that there is a fixed point of A.

We begin by showing that $D(\zeta,\eta,z,Z)$ is bounded and $A : D(\zeta,\eta,z,Z) \to K$ is completely continuous. For all $\omega_1(t) \in D(\zeta,\eta,z,Z)$, with the help of Lemma 2.8, we obtain that

$$\|\omega_1\|_{PC^1} \leq M \max_{t\in J} \omega_1^{'}(t) = M\eta(\omega_1(t)) \leq MZ.$$

This implies that $D(\zeta,\eta,z,Z)$ is a bounded set. It is obvious that $A : D(\zeta,\eta,z,Z) \to K$ is completely continuous according to Lemma 2.10.

Let

$$\omega_0(t) = \frac{Z}{L+1}(Lt+1).$$

Clearly, $\omega_0(t) \in K$. By direct computations, we can conclude that

$$\zeta(\omega_0(t)) = \omega_0(\alpha) = \frac{Z}{L+1}(L\alpha+1) > z,$$

$$\eta(\omega_0(t)) = \omega_0^{'}(0) = \frac{Z}{L+1}L < Z,$$

$$\lambda(\omega_0(t)) = \eta(\omega_0(t)) = \frac{Z}{L+1}L \geq \nu,$$

$$\mu(\omega_0(t)) = \omega_0(1) = \frac{Z}{L+1}(L+1) = Z \leq \varsigma.$$

Thus,

$$\omega_0(t) \in \{\omega(t) \in E(\lambda,\nu) : \eta(\omega(t)) < Z\} \cap \{\omega(t) \in F(\mu,\varsigma) : z < \zeta(\omega(t))\}.$$

This yields that the condition (i) in Lemma 3.1 is satisfied.

We verify condition (ii) of Lemma 3.1. Let $\omega_1(t) \in D(\zeta,\eta,z,Z)$ with $\zeta(\omega_1(t)) = z$ and $\varsigma < \mu(A\omega_1(t))$. We need to prove that $\zeta(A\omega_1(t)) \geq z$. Recall that $\zeta(\omega_1(t)) = \omega_1(\alpha)$. Using the definition of operator A and neglecting the non-negative impulsive and boundary terms (since $W_{1p}, \Theta_1, \Gamma_1 \geq 0$), we write:

$$\zeta(A\omega_1(t)) = (A\omega_1)(\alpha) \geq \int_0^1 H_1(\alpha,s)v_1(s)h_1(s,\omega_2(s))ds.$$

Although Lemma 2.6 is valid for $s \in [0,1]$, the lower bound condition on the nonlinear term h_i given in assumption (a) is specified for the interval $[\alpha, 1-\alpha]$. To utilize assumption (a) rigorously, we restrict the integration interval to $[\alpha, 1-\alpha]$. Since the integrand is non-negative, this restriction preserves the inequality direction:

$$\zeta(A\omega_1(t)) \geq \int_\alpha^{1-\alpha} H_1(\alpha,s)v_1(s)h_1(s,\omega_2(s))ds.$$

For $t=\alpha$ and any $s \in [0,1]$, Lemma 2.6 ensures $H_1(\alpha,s) \geq \gamma_1 H_1(s,s)$. Furthermore, for $s \in [\alpha, 1-\alpha]$, assumption (a) guarantees that $h_1(s,\omega_2(s)) \geq \frac{z}{\Omega_1}$. Substituting these bounds into the integral:

$$\begin{aligned}\zeta(A\omega_1(t)) &\geq \int_\alpha^{1-\alpha} \gamma_1 H_1(s,s)v_1(s)\frac{z}{\Omega_1}ds\\ &= \frac{z}{\Omega_1}\int_\alpha^{1-\alpha} \gamma_1 H_1(s,s)v_1(s)ds.\end{aligned}$$

By the definition of $\Omega_1 = \int_\alpha^{1-\alpha} \gamma_1 H_1(s,s)v_1(s)ds$, we obtain:

$$\zeta(A\omega_1(t)) \geq \frac{z}{\Omega_1}\cdot\Omega_1 = z.$$

Thus, condition (ii) is satisfied.

Let $\omega_1(t) \in F(\mu, \varsigma)$ with $\zeta(\omega_1(t)) = z$. Now, we can show that $\omega_2(t) \in [z, \varsigma]$ for $t \in [\alpha, 1-\alpha]$. For all $(s_{n-1}, \omega_1) \in [\alpha, 1-\alpha] \times [z, \varsigma]$, by using the the assumption (a) in Theorem 3.2, we obtain

$$\begin{aligned}
&\int_0^1 H_n(s_{n-1}, s_n)v_n(s_n)h_n(s_n, \omega_1(s_n))ds_n + \sum_{p=1}^{k} W_{np}(s_{n-1}, t_p) + y_n(s_{n-1})\Theta_n + z_1(s_{n-1})\Gamma_n \\
&\quad \geq \int_0^1 H_n(s_{n-1}, s_n)v_n(s_n)h_n(s_n, \omega_1(s_n))ds_n \\
&\quad \geq \int_\alpha^{1-\alpha} \gamma_n H_n(s_n, s_n)v_n(s_n)h_n(s_n, \omega_1(s_n))ds_n \\
&\quad \geq \frac{z}{\Omega_n}\int_\alpha^{1-\alpha} \gamma_n H_n(s_n, s_n)v_n(s_n)ds_n \\
&\quad = z
\end{aligned}$$

and for all $(s_{n-1}, \omega_1) \in [0, 1] \times [z, \varsigma]$, by using the assumption (b) in Theorem 3.2 and the condition (H5), we have

$$\begin{aligned}
&\int_0^1 H_n(s_{n-1}, s_n)v_n(s_n)h_n(s_n, \omega_1(s_n))ds_n + \sum_{p=1}^{k} W_{np}(s_{n-1}, t_p) + y_n(s_{n-1})\Theta_n + z_1(s_{n-1})\Gamma_n \\
&\quad \leq \Bigg[\int_0^1 \delta_n H_n(s_n, s_n)v_n(s_n)ds_n + \frac{1}{\rho_n}\sum_{p=1}^{k}[\max\{y_n(1), z_n(0)\}y_n^{'}(t_p) + y_n(1)z_n(0)]v_n(t_p) \\
&\quad + y_n(s_{n-1})\overline{\Theta}_n + z_1(s_{n-1})\overline{\Gamma}_n\Bigg] . \max\left\{h_n(s_n, \omega_1(s_n)), I_{np}(\omega_n(t_p)), \overline{I}_{np}(\omega_n(t_p))\right\} \\
&\quad \leq \varsigma.
\end{aligned}$$

Therefore, we have $\omega_n(t) \in [z, \varsigma]$ for $t \in [\alpha, 1-\alpha]$. By continuing with this bootstrapping argument, we can establish that $\omega_2(t) \in [z, \varsigma]$ for all t belonging to the interval $[\alpha, 1-\alpha]$.

For all $\omega_1(t) \in F(\mu, \varsigma)$ with $\zeta(\omega_1(t)) = z$, by using the assumption (a) in Theorem 3.2, we get

$$\begin{aligned}
\zeta(A\omega_1(t)) &= (A\omega_1)(\alpha) \\
&= \int_0^1 H_1(\alpha, s)v_1(s)h_1(s, \omega_2(s))ds + \sum_{p=1}^{k} W_{1p}(\alpha, t_p) + \dots \\
&\geq \int_\alpha^{1-\alpha} H_1(\alpha, s)v_1(s)h_1(s, \omega_2(s))ds \\
&\geq \int_\alpha^{1-\alpha} \gamma_1 H_1(s, s)v_1(s)\frac{z}{\Omega_1}ds \quad \text{(by Lemma 2.6 and assumption (a))} \\
&= \frac{z}{\Omega_1}\int_\alpha^{1-\alpha} \gamma_1 H_1(s, s)v_1(s)ds \\
&= \frac{z}{\Omega_1} \cdot \Omega_1 = z.
\end{aligned}$$

Hence, the condition (iii) in Lemma 3.1 holds.

For all $\omega_1(t) \in D(\zeta, \eta, z, Z)$ with $\eta(\omega_1(t)) = Z$ and $\lambda(A\omega_1(t)) < \nu$, we can write

$$\eta(A\omega_1(t)) = \max_{t\in J} A\omega_1^{'}(t) = \lambda(A\omega_1(t)) < \nu < \frac{L}{L+1}Z < Z.$$

Thus, the condition (iv) in Lemma 3.1 holds.

Let $\omega_1(t) \in E(\lambda,\nu)$ with $\eta(\omega_1(t)) = Z$. Now, we can show that $\omega_2(t) \in [z,\varsigma]$ for $t \in J$. For all $(s_{n-1},\omega_1) \in [0,1]\times[z,ZM]$, by using the assumption (a) in Theorem 3.2, we have

$$\begin{aligned}
&\int_0^1 H_n(s_{n-1},s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n + \sum_{p=1}^{k} W_{np}(s_{n-1},t_p) + y_n(s_{n-1})\Theta_n + z_1(s_{n-1})\Gamma_n\\
&\quad \geq \int_0^1 H_n(s_{n-1},s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n\\
&\quad \geq \int_\alpha^{1-\alpha} \gamma_n H_n(s_n,s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n\\
&\quad \geq \frac{z}{\Omega_n}\int_\alpha^{1-\alpha} \gamma_n H_n(s_n,s_n)v_n(s_n)ds_n\\
&\quad = z
\end{aligned}$$

and for all $(s_{n-1},\omega_1) \in [0,1]\times[z,ZM]$, by using the assumption (c) in Theorem 3.2 and the condition (H5), we obtain

$$\begin{aligned}
&\int_0^1 H_n(s_{n-1},s_n)v_n(s_n)h_n(s_n,\omega_1(s_n))ds_n + \sum_{p=1}^{k} W_{np}(s_{n-1},t_p) + y_n(s_{n-1})\Theta_n + z_1(s_{n-1})\Gamma_n\\
&\quad \leq \Bigg[\int_0^1 \delta_n H_n(s_n,s_n)v_n(s_n)ds_n + \frac{1}{\rho_n}\sum_{p=1}^{k}[\max\{y_n(1),z_n(0)\}y_n^{'}(t_p) + y_n(1)z_n(0)]v_n(t_p)\\
&\quad + y_n(s_{n-1})\overline{\Theta}_n + z_1(s_{n-1})\overline{\Gamma}_n\Bigg].\max\left\{h_n(s_n,\omega_1(s_n)), I_{np}(\omega_n(t_p)), \overline{I}_{np}(\omega_n(t_p))\right\}\\
&\quad \leq \Lambda_n \frac{Z}{W_n}\\
&\quad \leq Z\\
&\quad \leq ZM.
\end{aligned}$$

Therefore, we have $\omega_n(t) \in [z,ZM]$ for $t \in [0,1]$. Continuing with this bootstrapping argument, we can conclude that $\omega_2(t) \in [z,ZM]$ for all t in the set J.

For all $\omega_1(t) \in E(\lambda,\nu)$ with $\eta(\omega_1(t)) = Z$, by using the assumption (c) in Theorem 3.2 and the condition (H5), we can write

$$\begin{aligned}
\eta(A\omega_1(t)) &= \max_{t\in J} A\omega_1^{'}(t)\\
&= A\omega_1^{'}(0)\\
&= \frac{1}{\rho_1}\int_0^1 y_1^{'}(0)z_1(s)v_1(s)h_1(s,\omega_2(s))ds + \frac{1}{\rho_1}\sum_{p=1}^{k}[y_1^{'}(0)(z_1(t_p)+z_1^{'}(t_p))v_1(t_p)] + y_1^{'}(0)\Theta_1 + z_1^{'}(0)\Gamma_1\\
&\leq \left[\frac{1}{\rho_1}\int_0^1 y_1^{'}(0)z_1(s)v_1(s)ds + \frac{1}{\rho_1}\sum_{p=1}^{k}[y_1^{'}(0)(z_1(t_p)+z_1^{'}(t_p))v_1(t_p)] + y_1^{'}(0)\overline{\Theta}_1 + z_1^{'}(0)\overline{\Gamma}_1\right]\\
&\quad .\max\left\{h_1(s,\omega_2(s)), I_{1p}(\omega_1(t_p)), \overline{I}_{1p}(\omega_1(t_p))\right\}\\
&= \overline{\Lambda}_1\frac{Z}{W_1}\\
&\leq W_1\frac{Z}{W_1}\\
&= Z.
\end{aligned}$$

This implies that the condition (v) in Lemma 3.1 is satisfied.

As a consequence, all conditions of Lemma 3.1 are satisfied. Hence, A has a fixed point $\omega_1(t) \in D(\zeta, \eta, z, Z)$. In conclusion, by setting $\omega_{n+1}(t) = \omega_1(t)$, we obtain a positive solution $(\omega_1(t), \omega_2(t), ..., \omega_n(t))$ for the iterative system of 2^{nd}-order ℓ-point ibvp (1.1) given iteratively by

$$\omega_r(t) = \int_0^1 H_r(t,s)v_r(s)h_r(s, \omega_{r+1}(s))ds + \sum_{p=1}^{k} W_{rp}(t, t_p) + y_r(t)\Theta_r + z_r(t)\Gamma_r, \quad r = n, n-1, ..., 1,$$

such that

$$\min_{t\in[\alpha,1-\alpha]} \omega_i(t) \geq z, \quad \max_{t\in J} \omega_i(t) \leq Z,$$

for $i = 1, 2, ..., n$ on J. Theorem 3.2 is proved. □

Example 3.3. In the iterative system of 2^{nd}-order ℓ-point ibvp (1.1), assume that $n = \ell = 3,\ p = 1,\ d_i(t) = 4,\ e_i(t) = -2$ for $i = 1, 2, 3,\ \tau_1 = \tau_3 = 2, \tau_2 = \tau_4 = 1, \chi_1 = \frac{1}{3}, \kappa_1 = 1,\ \varrho_1 = 2$ and $t_1 = \frac{1}{2}$ i.e.,

$$\begin{cases} \omega_i''(t) + 4\omega_i'(t) - 2\omega_i(t) + h_i(t, \omega_{i+1}(t)) = 0,\ t \in J = [0,1], t \neq \dfrac{1}{2}, i = 1,2,3, \\ \triangle\omega_i(\frac{1}{2}) = I_{i1}(\omega_i(\frac{1}{2})), \\ \triangle\omega_i'(\frac{1}{2}) = -\overline{I}_{i1}(\omega_i(\frac{1}{2})), \\ 2\omega_i(0) - \omega_i'(0) = \omega_i(\frac{1}{3}), \\ 2\omega_i(1) + \omega_i'(1) = 2\omega_i(\frac{1}{3}) \end{cases} \tag{3.2}$$

where

$$h_1(t, \omega_2(t)) = \begin{cases} 400\omega_2 + \dfrac{3}{5}, & \omega_2 \in \left[0, \dfrac{1}{1000}\right], \\ 1, & \omega_2 \in \left[\dfrac{1}{1000}, 70\right], \\ \dfrac{1}{5}\omega_2 - 13, & \omega_2 \in [70, 80], \end{cases}$$

$$h_2(t, \omega_3(t)) = \begin{cases} 600\omega_3 + \dfrac{3}{5}, & \omega_3 \in \left[0, \dfrac{1}{1000}\right], \\ \dfrac{6}{5}, & \omega_3 \in \left[\dfrac{1}{1000}, 70\right], \\ \dfrac{23}{100}\omega_3 - \dfrac{149}{10}, & \omega_3 \in [70, 80], \end{cases}$$

$$h_3(t, \omega_1(t)) = \begin{cases} 1, & \omega_1 \in [0, 70], \\ \dfrac{1}{4}\omega_1 - \dfrac{33}{2}, & \omega_1 \in [70, 80], \end{cases}$$

$$I_{11}(\omega_1(t_1)) = \frac{1}{e^{\omega_1(t_1)+2}},\ I_{21}(\omega_2(t_1)) = \frac{1}{2^{\omega_2(t_1)+3}},\ I_{31}(\omega_3(t_1)) = \frac{1,2}{2^{\ln[3+\omega_3(t_1)]+1}},$$

and

$$\overline{I}_{11}(\omega_1(t_1)) = \frac{1}{e^{\omega_1(t_1)}},\ \overline{I}_{21}(\omega_2(t_1)) = \frac{1}{2^{\omega_2(t_1)}},\ \overline{I}_{31}(\omega_3(t_1)) = \frac{2,8}{2^{\ln[3+\omega_3(t_1)]}}.$$

Let y_i and z_i denote the unique solution of boundary value problem:

$$\begin{cases} y_i''(t) + 4y_i'(t) - 2y_i(t) = 0, \quad t \in J, \\ y_i(0) = 1, \qquad y_i'(0) = 2 \end{cases} \tag{3.3}$$

and

$$\begin{cases} z_i''(t) + 4z_i'(t) - 2z_i(t) = 0, & t \in J, \\ z_i(1) = 1, \qquad z_i'(1) = -2 \end{cases} \tag{3.4}$$

, respectively. When the differential equations are solved, $y_i(t)$ and $z_i(t)$ solutions are found as

$$y_i(t) = \left(\frac{\sqrt{6}}{3} + \frac{1}{2}\right) e^{(-2+\sqrt{6})t} + \left(-\frac{\sqrt{6}}{3} + \frac{1}{2}\right) e^{(-2-\sqrt{6})t},$$

and

$$z_i(t) = \frac{1}{2} e^{(2-\sqrt{6})(1-t)} + \frac{1}{2} e^{(2+\sqrt{6})(1-t)},$$

respectively. We set $\alpha = \frac{1}{3}$. As a result of simple calculations, we get $\rho_i = 276.477$, $\gamma_i = 0.033$, $\delta_i = 43.109$, $\nabla_i = -543.666$, $\overline{\Theta}_i = 3.635, \overline{\Gamma}_i = 0.033$, $L = \frac{3}{4}$, $M = \frac{7}{3}$, $\Omega_i = 0.00239$, $\Lambda_i = 22.424$, $\overline{\Lambda}_1 = 13.02$, and

$$H_i(t,s) = \frac{1}{276.477} \begin{cases} y_i(s) z_i(t), & s \le t, \\ y_i(t) z_i(s), & t \le s, \end{cases}$$

for $i = 1,2,3$. Furthermore, it is evident that the conditions (H1)-(H8) have been fulfilled. We choose $z = \frac{1}{1000}$, $\varsigma = 80$, $\nu = 10$ and $Z = 30$. It is easy to check that

$$\max\left\{\frac{z}{\alpha}, \frac{L+1}{L} Z\right\} = \max\left\{\frac{3}{1000}, 70\right\} \le 80 = \varsigma,$$

$$\max\left\{\frac{L+1}{L}\nu, \frac{L+1}{L\alpha+1} z\right\} = \max\left\{\frac{70}{3}, \frac{7}{5000}\right\} < 30 = Z,$$

and

$$\frac{z}{\Omega_i} = 0.5 \le \frac{Z}{W_i} = 1.337 \le \frac{\varsigma}{\Lambda_i} = 3.567, \text{ for } i = 1,2,3,$$

$$h_i(t, \omega_{i+1}(t)) \ge \frac{z}{\Omega_i} = 0.5, \text{ for } (t, \omega_{i+1}(t)) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{1000}, 80\right],$$

$$h_i(t, \omega_{i+1}(t)) \le \frac{Z}{W_i} = 1.337, \text{ for } (t, \omega_{i+1}(t)) \in [0,1] \times \left[\frac{1}{1000}, 70\right],$$

and

$$h_i(t, \omega_{i+1}(t)) \le \frac{\varsigma}{\Lambda_i} = 3.567, \text{ for } (t, \omega_{i+1}(t)) \in [0,1] \times [0, 80].$$

For $i = 1,2,3$ and $\omega_i(t_1) \in [0, \infty)$;

- $I_{11}(\omega_1(t_1)) \le 0,135 \le 1,337 = \frac{R}{W_1}$,
- $I_{21}(\omega_2(t_1)) \le 0,125 \le 1,337 = \frac{R}{W_2}$,
- $I_{31}(\omega_3(t_1)) \le 0,280 \le 1,337 = \frac{R}{W_3}$,
- $\overline{I}_{11}(\omega_1(t_1)) \le 1 \le 1,337 = \frac{R}{W_1}$,
- $\overline{I}_{21}(\omega_2(t_1)) \le 1 \le 1,337 = \frac{R}{W_2}$,
- $\overline{I}_{31}(\omega_3(t_1)) \le 1,307 \le 1,337 = \frac{R}{W_3}$.

As a result, all of Theorem 3.2's criteria are satisfied. Hence, by setting $\omega_4(t) = \omega_1(t)$, we obtain a positive solution $(\omega_1(t), \omega_2(t), \omega_3(t))$ of the iterative system of 2^{nd}-order ℓ-point ibvp (3.2) given iteratively by

$$\omega_r(t) = \int_0^1 H_r(t,s) v_r(s) h_r(s, \omega_{r+1}(s)) ds + W_{r1}\left(t, \frac{1}{2}\right) + y_r(t)\Theta_r + z_r(t)\Gamma_r, \qquad r = 1,2,3,$$

such that

$$\max_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} \omega_i(t) \geq \frac{1}{1000}, \quad \max_{t\in J} \omega_i(t) \leq 30,$$

for $i = 1, 2, 3$ on J.

4. Conclusions

This article deals with the problem (1.1), which has a general form and has not been studied before. We extended and enhanced existing findings in the literature to a certain degree, and our findings could potentially make a valuable contribution to the literature in this field. The focus of our research was to examine the existence of positive solutions for iterative system of the 2^{nd}-order ℓ-point ibvp, as depicted in equation (1.1). For this, firstly we considered the case of $i = 1$ in the iterative system of the 2^{nd}-order ℓ-point ibvp (1.1). So, we gave the solution $\omega_1(t)$ of the 2^{nd}-order ℓ-point ibvp (2.1). Then, we found $\omega_n(t)$, since $\omega_1(t)$ was known. Continuning this argument, we obtained $\omega_{n-1}(t)$, then $\omega_{n-2}(t)$ etc. and finally $\omega_2(t)$. As a result, the solution $(\omega_1(t), ..., \omega_n(t))$ for the iterative system of 2^{nd}-order ℓ-point ibvp (1.1) was obtained. After determining the solution for iterative system of the 2^{nd}-order ℓ-point ibvp (1.1), we used the four functional fixed point theorem to set criteria on the existence of the solution. Afterwards, an example was created to demonstrate the applicability of our results.

Acknowledgments

The first author was granted a fellowship by the Scientific and Technological Research Council of Turkey (TUBITAK-2211-A). The authors would like to thank the editor and the reviewers for their helpful comments and suggestions.

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