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Chelyshkov operational matrix of fractional integration for solving Volterra integral equations of the third kind

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Abstract

A numerical method is presented for solving third-kind linear Volterra integral equations with smooth ($\alpha = 0$) and weakly singular ($0 < \alpha < 1$) kernels. Using the operational matrix of fractional integration with Chelyshkov polynomials, the integral equation is transformed into a system of linear algebraic equations whose solution provides an accurate approximation. Residual error estimates are derived to assess the accuracy of the proposed method. Three numerical examples illustrate its efficiency, precision, and practical applicability.

Keywords. Chelyshkov polynomials, Riemann–Liouville fractional integral, Cordial integral equations, Third-kind integral equations.

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1. Introduction

Volterra integral equations of the third kind arise in various areas of applied mathematics and engineering. A general form of such equations is

$$A(t)u(t) = f(t) + \int_a^t k(t,x)u(x)\,dx, \quad t \in [a,T],$$

where $A(t)$ is a continuous function on $[a,T]$ and vanishes at least at one point in this interval. In this work, we consider the special case

$$t^{\beta}u(t) = f(t) + \int_0^t (t-x)^{-\alpha}k(t,x)u(x)\,dx, \quad t \in I := [0,1], \tag{1.1}$$

where $\alpha \in [0,1)$, $\beta > 0$, $f \in C(I)$, and $k \in C(\Delta)$, with $\Delta = \{(t,x) : 0 \le x \le t \le 1\}$. For the case $\alpha + \beta \ge 1$, we assume that

$$k(t,x) = x^{\alpha+\beta-1}h(t,x), \tag{1.2}$$

where $h(t,x) \in C(\Delta)$ and $h(0,0) = 0$.

The Volterra integral equation of the third kind (1.1) can be equivalently written as

$$u(t) = t^{-\beta}f(t) + (V_{k,\beta,\alpha}u)(t), \quad t \in (0,1], \tag{1.3}$$

where $V_{k,\beta,\alpha}$ denotes the Volterra integral operator

$$(V_{k,\beta,\alpha}u)(t) = \int_0^t t^{-\beta}(t-x)^{-\alpha}k(t,x)u(x)\,dx, \quad t \in I. \tag{1.4}$$

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Under the above conditions, it has been established in [2, 24, 25] that the operator $V_{k,\beta,\alpha}$ defines a compact cordial Volterra integral operator on $C(I)$. In contrast, for $\alpha+\beta \geq 1$ with $h(0,0) \neq 0$, the operator is noncompact. The existence, uniqueness, and regularity of solutions to (1.1) have been discussed in [2, 24, 25].

The study of integral equations of the third kind dates back to the early 1900s [2]. Pioneering contributions were made by Picard and Hilbert, who worked primarily on Fredholm-type formulations, and later by Evans, who extended the theory to Volterra-type problems [12, 13, 19]. A distinctive feature of these equations is the multiplicative factor on the left-hand side, which introduces analytical challenges that are not present in first- and second-kind models. Growing interest in this subject has been motivated by its applications in areas such as heat transfer processes, control theory, and other applied sciences [2]. A variety of numerical approaches have been proposed, including uniform and graded collocation techniques, as well as multistep extensions [1, 14, 20, 21, 23, 24]. A spectral Legendre-Galerkin method for solving (1.1)-(1.2) has been proposed in [10]. Nemati and coauthors employed operational matrices constructed from hat functions [16] and later developed wavelet-based schemes [17]. In addition, Allaei et al. investigated the existence and uniqueness of solutions for various ranges of the parameters α and β [2].

Many studies have developed various numerical methods for solving Volterra integral equations using operational matrices constructed from different polynomial bases such as Bernoulli, Euler, Legendre, Chelyshkov and sigmoidal functions [3–9, 15, 18]. Compared with classical polynomial bases such as Legendre and Chebyshev polynomials, Chelyshkov polynomials provide convenient operational matrix representations while preserving orthogonality on the interval $[0, 1]$. These properties make them effective for the numerical treatment of fractional and weakly singular integral equations [11, 22]. Building upon these studies, the present work introduces a novel scheme based on Chelyshkov polynomial expansions and the operational matrix of fractional integration, which efficiently reduces third-kind Volterra integral equations—including those with singular or weakly singular kernels—into systems of linear algebraic equations, enabling highly accurate and computationally efficient approximations.

The rest of the paper is organized as follows. Section 2 provides a brief overview of Chelyshkov polynomials and their approximation properties. Section 3 outlines the proposed computational approach for solving equation (1.1), in which the unknown function and related terms are approximated by Chelyshkov expansions, and the operational matrix of Riemann–Liouville fractional integration is employed to transform the problem into a system of algebraic equations. Section 4 presents an analysis of the residual errors to assess the accuracy and convergence behavior of the proposed method. Section 5 reports several numerical examples that verify the accuracy and efficiency of the method. Finally, Section 6 concludes the paper.

2. Chelyshkov Polynomials and Function Approximation

In this section, Chelyshkov polynomials, also known as alternative Legendre polynomials (ALPs), are introduced, and their application in function approximation is discussed. The definitions and properties are adapted from Bazm and Hosseini [8].

2.1. Chelyshkov Polynomials.

Definition 2.1 ([8]). Let m be a fixed positive integer. The set of Chelyshkov polynomials on the interval $[0, 1]$ is defined by

$$\mathcal{C}_m = \{C_{mi}(x)\}_{i=0}^{m}, \quad C_{mi}(x) = \sum_{j=i}^{m} \rho_{i,j} x^j, \quad i = 0, 1, \dots, m,$$

where the coefficients are given by

$$\rho_{i,j} = (-1)^{j-i} \binom{m-i}{j-i} \binom{m+j+1}{m-i}.$$

The Chelyshkov polynomials form an orthogonal system on the interval $[0, 1]$ with respect to the weight function $w(x) = 1$, i.e.,

$$\int_0^1 C_{mi}(x)\, C_{mj}(x)\, dx = \frac{\delta_{ij}}{i+j+1}, \quad i, j = 0, 1, \dots, m, \tag{2.1}$$

where δ_{ij} denotes the Kronecker delta. Unlike standard families of orthogonal polynomials, each member of $\mathcal{C}_m$ has the same degree m.

The vector of Chelyshkov polynomials can be expressed in matrix form as

$$C_m(x) = [C_{m0}(x), C_{m1}(x), \ldots, C_{mm}(x)]^T = CX_m(x), \tag{2.2}$$

where

$$X_m(x) = [1, x, x^2, \ldots, x^m]^T,$$

and $C = [c_{ij}]_{i,j=0}^m$ is an upper-triangular matrix with entries

$$c_{ij} = \begin{cases} \rho_{i,j}, & j \geq i, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.3}$$

Using the orthogonality relation (2.1), the dual matrix corresponding to $C_m(x)$ is diagonal:

$$G_m = \int_0^1 C_m(x) C_m^T(x)\, dx = \operatorname{diag}\left\{1, \frac{1}{3}, \ldots, \frac{1}{2m+1}\right\}. \tag{2.4}$$

Lemma 2.2. *[8] Let $C_{mi}(x)$ and $C_{mj}(x)$ denote the ith and jth Chelyshkov polynomials, respectively, and $v_1 = \min\{i, j\}$ and $v_2 = \max\{i, j\}$. Then the product of $C_{mi}(x)$ and $C_{mj}(x)$ is a polynomial of degree at most $2m$ that can be written as*

$$C_{2m}^{(i,j)}(x) := C_{mi}(x) C_{mj}(x) = \sum_{n=i+j}^{2m} \sigma_n^{(i,j)} x^n,$$

where

$$\sigma_n^{(i,j)} = \sum_{r=\max\{n-m, v_2\}}^{\min\{n-v_1, m\}} \rho_{v_2,r}\, \rho_{v_1,n-r}.$$

Lemma 2.3. *[8] Let n be a nonnegative integer, then*

$$\int_0^1 x^n C_{mi}(x)\, dx = \sum_{r=i}^{m} \frac{\rho_{i,r}}{n+r+1}, \qquad i = 0, 1, \ldots, m.$$

Lemmas 2.2 and 2.3 together yield the following result.

Lemma 2.4. *[8] Let $C_{mi}(x)$, $C_{mj}(x)$ and $C_{mk}(x)$ be the ith, jth and kth Chelyshkov polynomials, respectively. If, according to Lemma 2.2, we define*

$$C_{mi}(x) C_{mj}(x) = \sum_{n=i+j}^{2m} \sigma_n^{(i,j)} x^n,$$

then

$$\int_0^1 C_{mi}(x) C_{mj}(x) C_{mk}(x)\, dx = \sum_{n=i+j}^{2m} \sigma_n^{(i,j)} \sum_{r=k}^{m} \frac{\rho_{k,r}}{n+r+1}.$$

We now introduce the operational matrix of Riemann–Liouville fractional integration of order α associated with Chelyshkov polynomials.

Definition 2.5. The Riemann-Liouville fractional integral of order α of a function $f(t)$ is defined as

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-x)^{\alpha-1} f(x)\, dx, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases} \tag{2.5}$$

where $\Gamma(\alpha)$ denotes the Gamma function, given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

For monomials, the Riemann-Liouville fractional integral of order α admits the closed-form expression

$$I^\alpha\, t^r = \frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}\, t^{r+\alpha}, \quad \alpha \geq 0,\ r > -1.$$

Theorem 2.6. *[8] Let $C_m(x)$ denote the Chelyshkov vector defined in (2.2), and let $\alpha \geq 0$. Then*

$$I^\alpha C_m(x) \approx P^{(\alpha)} C_m(x), \tag{2.6}$$

where $P^{(\alpha)} = \big[p_{ij}^{(\alpha)}\big]_{i,j=0}^{m}$ is the Chelyshkov operational matrix of fractional integration of order α in the Riemann-Liouville sense, with entries

$$p_{ij}^{(\alpha)} = (2j+1) \sum_{r=i}^{m} \frac{r!\,\rho_{i,r}}{\Gamma(r+1+\alpha)} \sum_{s=j}^{m} \frac{\rho_{j,s}}{\alpha+r+s+1}.$$

The notation $\approx$ in (2.6) indicates that the vector on the right-hand side approximates the vector on the left-hand side.

The notation "$\approx$" denotes the approximation obtained through truncation in the Chelyshkov polynomial basis.

Theorem 2.7. *[8] Consider an arbitrary vector $U = [u_0, u_1, \ldots, u_m]^T \in \mathbb{R}^{m+1}$. Then, the product of the Chelyshkov vector with its transpose acting on U can be approximated as*

$$C_m(x) C_m^T(x) U \approx \tilde{U} C_m(x), \tag{2.7}$$

where $\tilde{U} = [\tilde{u}_{ik}]_{i,k=0}^{m}$ denotes the Chelyshkov operational matrix of the product, with entries given by

$$\tilde{u}_{ik} = (2k+1) \sum_{j=0}^{m} u_j\, \eta_{ijk},$$

and

$$\eta_{ijk} = \int_0^1 C_{mi}(x) C_{mj}(x) C_{mk}(x)\, dx. \tag{2.8}$$

The coefficients η_{ijk} in (2.8) are computed using Lemma 2.4. The symbol $\approx$ is defined as in Theorem 2.6.

2.2. **Function Approximation.** Let $\mathbb{V} = L^2(0,1)$, equipped with the standard L^2-norm and inner product:

$$\|g\|_{L^2} = \Big(\int_0^1 g^2(x)\, dx \Big)^{1/2}, \qquad \langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

We define the finite-dimensional subspace

$$\mathbb{V}_m = \operatorname{span}\{C_{m0}(x), C_{m1}(x), \ldots, C_{mm}(x)\},$$

where $C_{mk}(x)$ are the Chelyshkov polynomials. Being finite-dimensional, $\mathbb{V}_m$ is closed in $\mathbb{V}$. Therefore, for any $u \in \mathbb{V}$, there exists a unique best approximation $\bar{u}_m \in \mathbb{V}_m$ satisfying

$$\|u - \bar{u}_m\|_{L^2} \leq \|u - g\|_{L^2}, \qquad \forall g \in \mathbb{V}_m.$$

This best approximation can be represented as

$$\bar{u}_m(x) = \sum_{j=0}^{m} u_j C_{mj}(x),$$

with coefficients

$$u_j := \frac{\langle u, C_{mj} \rangle}{\langle C_{mj}, C_{mj} \rangle} = (2j+1) \langle u, C_{mj} \rangle. \tag{2.9}$$

Equivalently, in matrix form,

$$u(x) \approx \bar{u}_m(x) = C_m^T(x)U,$$

where

$$C_m(x) = [C_{m0}(x), C_{m1}(x), \dots, C_{mm}(x)]^T, \qquad U = [u_0, u_1, \dots, u_m]^T.$$

Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate kernel function. We approximate $k(t,x)$ in terms of the Chelyshkov basis as

$$k(t,x) \approx \bar{k}_m(t,x) = C_m^T(t)\, K\, C_m(x),$$

where $K = [K_{rs}] \in \mathbb{R}^{(m+1)\times(m+1)}$ is the coefficient matrix with entries

$$K_{rs} := \frac{\langle\langle k, C_{mr}\rangle, C_{ms}\rangle}{\langle C_{mr}, C_{mr}\rangle\, \langle C_{ms}, C_{ms}\rangle} = (2r+1)(2s+1)\, \langle\langle k, C_{mr}\rangle, C_{ms}\rangle. \tag{2.10}$$

3. Method of solution

Consider equation (1.1). We approximate all functions involved using Chelyshkov polynomials as

$$t^\beta = A^T C_m(t), \quad u(t) = U^T C_m(t), \quad f(t) = F^T C_m(t), \quad k(t,x) = C_m^T(t) K C_m(x), \tag{3.1}$$

where

$$A = [a_0, a_1, \dots, a_m]^T, \quad F = [f_0, f_1, \dots, f_m]^T,$$

are vectors of known coefficients determined from (2.9), $U = [u_0, u_1, \dots, u_m]^T$ denotes the unknown coefficient vector, and $K = [k_{ij}]_{i,j=0}^m$ is obtained from (2.10).

Substituting (3.1) into (1.1) yields

$$U^T C_m(t) C_m^T(t) A = F^T C_m(t) + C_m^T(t) K \int_0^t (t-x)^{-\alpha} C_m(x) C_m^T(x) U\, dx. \tag{3.2}$$

Using Eq. (2.7), Eq. (3.2) becomes

$$U^T \tilde{A} C_m(t) = F^T C_m(t) + C_m^T(t) K \tilde{U} \int_0^t (t-x)^{-\alpha} C_m(x)\, dx. \tag{3.3}$$

Applying the operational matrix representation (2.6) to the fractional integral term, we obtain a matrix representation of the original integral equation as

$$\begin{aligned} U^T \tilde{A} C_m(t) &= F^T C_m(t) + C_m^T(t) K \tilde{U} \int_0^t (t-x)^{(1-\alpha)-1} C_m(x)\, dx \\ &= F^T C_m(t) + C_m^T(t) K \tilde{U} \left(\Gamma(1-\alpha) I^{1-\alpha} C_m(t)\right) \\ &= F^T C_m(t) + \Gamma(1-\alpha) C_m^T(t) K \tilde{U} P^{(1-\alpha)} C_m(t) \\ &= F^T C_m(t) + \Gamma(1-\alpha) C_m^T(t) Q C_m(t), \end{aligned} \tag{3.4}$$

where $Q = K\tilde{U}P^{(1-\alpha)}$. Using the dual matrix G_m defined in (2.4), we obtain

$$C_m^T(t) Q C_m(t) = \hat{Q} C_m(t),$$

where the row coefficient vector $\hat{Q} \in \mathbb{R}^{1\times(m+1)}$ depends on U and is computed as

$$\hat{Q} = \left(\int_0^1 C_m^T(t)\, Q\, C_m(t)\, C_m^T(t)\, dt\right) G_m^{-1}.$$

After rearranging the resulting terms, Equation (3.4) reduces to

$$\left(U^T \tilde{A} - F^T - \Gamma(1-\alpha)\hat{Q}\right) C_m(t) = 0. \tag{3.5}$$

Using the orthogonality properties of Chelyshkov polynomials, the problem reduces to the following algebraic system for the unknown coefficient vector U:

$$U^T \tilde{A} - F^T - \Gamma(1-\alpha)\hat{Q} = 0,$$

whose solution determines the unknown coefficient vector U. Consequently, the approximate solution to equation (1.1) is obtained as

$$u_m(t) = U^T C_m(t), \quad t \in I.$$

4. Residual error estimate

This section estimates the residual function associated with the Chelyshkov polynomial approximation introduced in Section 3 and establishes convergence rates for the method. The analysis relies on the L^∞-error bounds of the L^2-projection onto the Chelyshkov basis.

Theorem 4.1. *Let $f \in C^{m+1}([0,1])$ and let*

$$f_m(t) = C_m^T(t)F, \tag{4.1}$$

be the L^2-projection of f onto

$$\mathbb{V}_m = \operatorname{span}\{C_{m0}, C_{m1}, \ldots, C_{mm}\}.$$

Define

$$c_m := \sup_{v\in\mathbb{V}_m,\, v\neq 0} \frac{\|v\|_\infty}{\|v\|_{L^2}} < \infty. \tag{4.2}$$

Then the following estimates hold:

(i) *The L^∞-error for the Chelyshkov projection satisfies*

$$\|f - f_m\|_\infty \le (1 + 2c_m)\, E_m(f)_\infty, \tag{4.3}$$

where

$$E_m(f)_\infty := \inf_{q\in\mathbb{V}_m} \|f - q\|_\infty, \tag{4.4}$$

is the best uniform approximation error from $\mathbb{V}_m$.

(ii) *The constant c_m satisfies the upper bound*

$$c_m \le \max_{x\in[0,1]} \left(C_m^T(x) G_m^{-1} C_m(x)\right)^{1/2} \le (m+1)M_m, \tag{4.5}$$

where

$$M_m := \max_{0\le j\le m} \|C_{mj}\|_\infty = \max_{0\le j\le m} \max_{x\in[0,1]} |C_{mj}(x)|$$

denotes the maximum sup-norm of the Chelyshkov basis functions.

(iii) *Using Jackson's inequality for functions in $C^{m+1}([0,1])$, one has*

$$E_m(f)_\infty \le \frac{\|f^{(m+1)}\|_\infty}{(m+1)!\, 2^{2m+1}}. \tag{4.6}$$

Consequently,

$$\|f - f_m\|_\infty \le (1 + 2c_m)\, \frac{\|f^{(m+1)}\|_\infty}{(m+1)!\, 2^{2m+1}}. \tag{4.7}$$

Moreover, the same right-hand side bounds $\|f - f_m\|_2$.

Proof. **Proof of part (i):** For any $q \in \mathbb{V}_m$,

$$f - f_m = (f - q) + (q - f_m),$$

and taking norms gives

$$\|f - f_m\|_\infty \leq \|f - q\|_\infty + \|q - f_m\|_\infty. \tag{4.8}$$

Since $q - f_m \in \mathbb{V}_m$, the equivalence of norms implies

$$\|q - f_m\|_\infty \leq c_m \|q - f_m\|_{L^2}.$$

Since f_m is the L^2-projection of f onto $\mathbb{V}_m$, we have

$$\|f - f_m\|_{L^2} \leq \|f - q\|_{L^2} \leq \|f - q\|_\infty,$$

which yields

$$\begin{aligned}\|q - f_m\|_\infty &\leq c_m \big(\|q - f\|_{L^2} + \|f - f_m\|_{L^2}\big) \\ &\leq 2c_m \|f - q\|_\infty.\end{aligned} \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$\|f - f_m\|_\infty \leq (1 + 2c_m)\|f - q\|_\infty.$$

Taking the infimum over $q \in \mathbb{V}_m$ gives (4.3).

Proof of part (ii): Since $\{C_{m0}, C_{m1}, \dots, C_{mm}\}$ forms a basis for $\mathbb{V}_m$, for any $v \in \mathbb{V}_m$ with $v \neq 0$, there exists a unique nonzero vector $a = [a_0, a_1, \dots, a_m]^T \in \mathbb{R}^{m+1}$ such that

$$v(x) = \sum_{k=0}^{m} a_k C_{mk}(x) = C_m^T(x) a.$$

Then

$$\|v\|_{L^2}^2 = a^T G_m a, \qquad \|v\|_\infty = \max_{x\in[0,1]} |C_m^T(x) a|.$$

For any fixed $x \in [0, 1]$, apply the Cauchy–Schwarz inequality in $\mathbb{R}^{m+1}$:

$$|C_m^T(x)a| = \left|\big(G_m^{-1/2} C_m(x)\big)^T \big(G_m^{1/2} a\big)\right| \leq \|G_m^{-1/2} C_m(x)\|_2 \, \|G_m^{1/2} a\|_2.$$

Since $\|G_m^{1/2} a\|_2 = (a^T G_m a)^{1/2} = \|v\|_{L^2}$, we obtain

$$\frac{|C_m^T(x)a|}{\|v\|_{L^2}} \leq \big(C_m^T(x) G_m^{-1} C_m(x)\big)^{1/2}.$$

Taking the maximum over $x \in [0, 1]$ gives

$$\frac{\|v\|_\infty}{\|v\|_{L^2}} \leq \max_{x\in[0,1]} \big(C_m^T(x) G_m^{-1} C_m(x)\big)^{1/2}.$$

Now, using the fact that the diagonal entries of G_m^{-1} are $(2k + 1)$ for $k = 0, \dots, m$, we expand

$$C_m^T(x) G_m^{-1} C_m(x) = \sum_{k=0}^{m} (2k + 1) C_{mk}^2(x).$$

Let $M_m := \max_{0\leq k\leq m} \|C_{mk}\|_\infty$. Then, for every $x \in [0, 1]$,

$$\sum_{k=0}^{m} (2k + 1) C_{mk}^2(x) \leq \sum_{k=0}^{m} (2k + 1) M_m^2 = M_m^2 \sum_{k=0}^{m} (2k + 1) = (m + 1)^2 M_m^2.$$

Therefore,

$$\max_{x\in[0,1]} \big(C_m^T(x) G_m^{-1} C_m(x)\big)^{1/2} \leq (m + 1) M_m.$$

Since the inequality

$$\frac{\|v\|_\infty}{\|v\|_{L^2}} \le (m+1)M_m$$

holds for every nonzero $v \in \mathbb{V}_m$, taking the supremum over all such v yields

$$c_m = \sup_{v\in\mathbb{V}_m,\, v\neq 0} \frac{\|v\|_\infty}{\|v\|_{L^2}} \le (m+1)M_m.$$

Proof of part (iii): Let $p_m(x)$ be the interpolating polynomial of f at the shifted Chelyshev nodes x_l, $l = 0, \dots, m$. Then, for any $x \in [0, 1]$,

$$f(x) - p_m(x) = \frac{f^{(m+1)}(\xi_x)}{(m+1)!} \prod_{l=0}^{m} (x - x_l),$$

where $\xi_x \in [0, 1]$.

Using standard Chebyshev node estimates, we obtain

$$\begin{aligned} |f(x) - p_m(x)| &\le \frac{\|f^{(m+1)}\|_\infty}{(m+1)!} \max_{x\in[0,1]} \left| \prod_{l=0}^{m} (x - x_l) \right| \\ &\le \frac{\|f^{(m+1)}\|_\infty}{(m+1)!\, 2^{2m+1}}. \end{aligned}$$

Since $\mathbb{V}_m$ contains all polynomials of degree at most m, we conclude

$$E_m(f)_\infty \le \|f - p_m\|_\infty \le \frac{\|f^{(m+1)}\|_\infty}{(m+1)!\, 2^{2m+1}}.$$

Substituting into (4.3) gives (4.7). Since $\|g\|_2 \le \|g\|_\infty$ on $[0, 1]$, the same bound applies to $\|f - f_m\|_2$. □

Remark 4.2. The constant c_m quantifies the equivalence between the L^∞- and L^2-norms on the finite-dimensional space $\mathbb{V}_m$. Its magnitude depends on the polynomial degree m and generally increases with m, reflecting the growing conditioning effects of the chosen basis.

Theorem 4.3. *Let $u \in C^{m+1}([0, 1])$ be the exact solution to (1.1), and let*

$$u_m(t) = C_m^T(t)U$$

be the approximate solution obtained by the method in Section 3. Define the residual

$$r_m(t) = t^\beta u_m(t) - f(t) - \int_0^t (t - x)^{-\alpha} k(t, x) u_m(x)\, dx, \quad t \in [0, 1]. \tag{4.10}$$

Then, using the constant c_m defined in Theorem 4.1, there exists a constant $C_m > 0$ such that

$$|r_m(t)| \le C_m \frac{\|u^{(m+1)}\|_\infty}{(m+1)!\, 2^{2m+1}}, \quad t \in [0, 1],$$

where

$$C_m = (1 + 2c_m) \left(1 + \frac{\mu}{1-\alpha}\right), \quad \mu = \sup_{(t,x)\in\Delta} |k(t, x)|.$$

Proof. Substituting $f(t)$ from (1.1) into (4.10) gives

$$r_m(t) = t^\beta (u_m(t) - u(t)) + \int_0^t (t - x)^{-\alpha} k(t, x)(u(x) - u_m(x))\, dx.$$

Hence,

$$|r_m(t)| \leq t^{\beta}\|u-u_m\|_{\infty} + \mu\|u-u_m\|_{\infty}\int_0^t (t-x)^{-\alpha}dx$$
$$= \left(t^{\beta} + \frac{\mu t^{1-\alpha}}{1-\alpha}\right)\|u-u_m\|_{\infty} \leq \left(1+\frac{\mu}{1-\alpha}\right)\|u-u_m\|_{\infty}.$$

Finally, applying Theorem 4.1 part (iii) gives

$$|r_m(t)| \leq (1+2c_m)\left(1+\frac{\mu}{1-\alpha}\right)\frac{\|u^{(m+1)}\|_{\infty}}{(m+1)!\,2^{2m+1}}, \quad t\in[0,1].$$

□

5. Numerical examples

This section presents three illustrative examples to further demonstrate the efficiency, robustness, and accuracy of the proposed scheme. These examples have also been investigated in earlier studies, enabling a direct comparison between our numerical results and those previously reported in the literature.

Let

$$e_m(t) = u(t) - u_m(t), \quad t\in[0,1],$$

where $u(t)$ and $u_m(t)$ denote the exact and approximate solutions, respectively. To evaluate the numerical performance, we compute the maximum error and the corresponding experimental convergence order as

$$\|e_m\|_{\infty} = \max\{|e_m(t_j)| : 1\leq j\leq m\}, \quad p_m = \frac{\log\left(\|e_m\|_{\infty}/\|e_{2m}\|_{\infty}\right)}{\log(2)},$$
$$\|r_m\|_{\infty} = \max\{|r_m(t_j)| : 1\leq j\leq m\}, \quad q_m = \frac{\log\left(\|r_m\|_{\infty}/\|r_{2m}\|_{\infty}\right)}{\log(2)},$$

with the grid points $t_j = j/m$, and also the L^2-error as

$$\|e_m\|_2 = \left(\int_0^1 |e_m(t)|^2\,dt\right)^{1/2}.$$

Notation. NBFs denotes the number of basis functions used in the numerical implementation to obtain the results.

All numerical experiments were performed on a personal computer equipped with an Intel Core i7-8650U processor (base frequency 1.90 GHz, turbo up to 2.11 GHz) and 16 GB of RAM, and the algorithms were implemented in *Mathematica* 14.1.

Example 5.1. In this example, we consider a Volterra integral equation of the third kind in the form (1.1):

$$tu(t) = \frac{6}{7}t^3\sqrt{t} + \int_0^t \frac{1}{2}u(x)dx, \quad t\in[0,1]. \tag{5.1}$$

The equation (5.1) corresponds to (1.1) with $\beta=1$, $\alpha=0$, $f(t)=\frac{6}{7}t^3\sqrt{t}$, and $k(t,x)=1/2$. It can be verified that $u(t)=t^{\frac{5}{2}}$ is the exact solution. Moreover, it follows from [2, Remark 3.5] that $u(t)=t^{\frac{5}{2}}$ is the unique solution of (5.1) in $C^2([0,1])$.

We have applied the present method with different values of m to equation (5.1), and the corresponding results are presented in Tables 1-2 and displayed in Figures 1–2. Table 1 reports the maximum error and convergence order, and provides a comparison with the spline collocation method in piecewise polynomial spaces [2] and with the operational matrix method based on an adjustment of hat functions [16]. Table 2 reports the L^2-error, and provides a comparison with the first-kind Chebyshev wavelets method [17]. Figures 1 and 2 display the graphs of the error and residual functions, respectively.

TABLE 1. The maximum error and the estimated convergence order corresponding to Example 5.1.

Method	NBFs	$\|e_m\|_\infty$	p_m	$\|r_m\|_\infty$	q_m
Proposed Method	m				
	3	1.67×10^{-3}	8.06	6.67×10^{-4}	6.01
	6	6.27×10^{-5}	4.63	1.03×10^{-5}	6.19
	12	2.54×10^{-6}	–	1.41×10^{-7}	–
Adjusted hat functions method[16]	n				
	3	1.13×10^{-3}	2.5		
	6	2.00×10^{-4}	2.5		
	12	3.54×10^{-5}	2.5		
	24	6.26×10^{-6}	2.5		
Collocation method-Chebyshev points ($m = 2$) [1]	N				
	32	7.80×10^{-4}	1.92		
	64	2.05×10^{-4}	1.96		
	128	5.18×10^{-5}	1.98		
	256	1.30×10^{-5}	1.99		

TABLE 2. The L^2-error corresponding to Example 5.1.

Method	NBFs	$\|e_m\|_2$
Proposed Method	m	
	4	5.92×10^{-4}
	8	1.65×10^{-5}
	10	5.08×10^{-6}
	12	1.92×10^{-6}
First-kind Chebyshev wavelets method[17]	$2^{k-1}(M+1)$	
	$k = 1, M = 3$	1.28×10^{-3}
	$k = 2, M = 3$	1.84×10^{-4}
	$k = 2, M = 4$	4.04×10^{-5}
	$k = 2, M = 5$	1.29×10^{-5}
	$k = 3, M = 3$	2.71×10^{-5}
	$k = 3, M = 4$	5.97×10^{-6}
	$k = 3, M = 5$	1.91×10^{-6}
	$k = 4, M = 3$	4.02×10^{-6}

Example 5.2. We consider the equation

$$t^{\frac{3}{2}}u(t) = f(t) + \int_0^t \frac{x}{\sqrt{2\pi}}(t-x)^{-\frac{1}{2}}u(x)dx, \quad t \in [0,1], \tag{5.2}$$

where

$$f(t) = t^{\frac{33}{10}}\left(1 - \frac{\Gamma(\frac{19}{5})}{\sqrt{2\pi}\Gamma(\frac{43}{10})}\right).$$

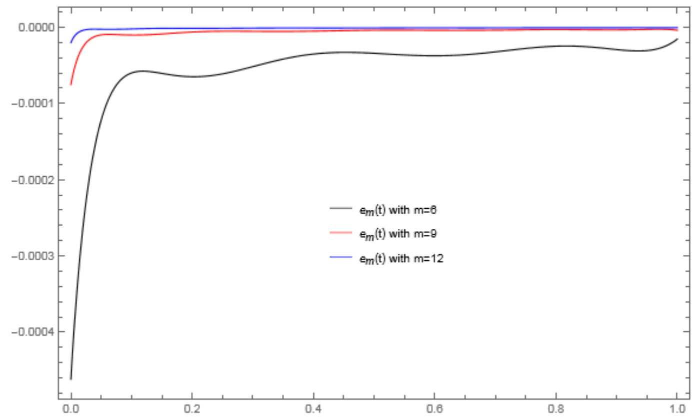

FIGURE 1. Error function $e_m(t)$ for Example 5.1 with $m = 6, 9, 12$.

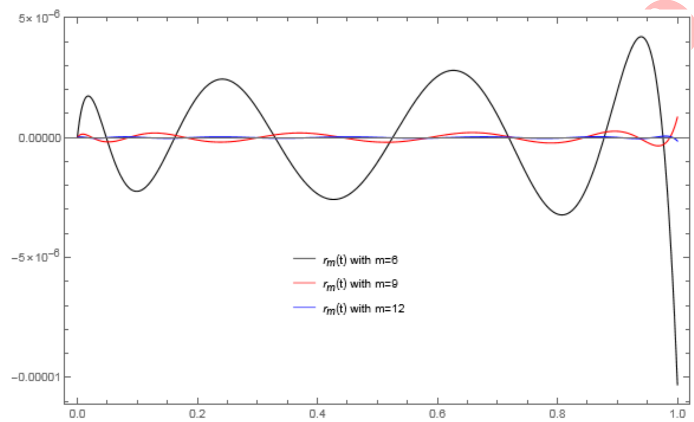

FIGURE 2. Residual function $r_m(t)$ for Example 5.1 with $m = 6, 9, 12$.

Equation (5.2) is of the form (1.1) with $\beta = 3/2$ and $\alpha = 1/2$ and $k(t, x) = x/(\sqrt{2}\pi)$. The function $u(t) = t^{\frac{9}{5}}$ satisfies (5.2) exactly. We have $\alpha + \beta > 1$ and the kernel admits the representation (1.2) with $h(t, x) = 1/(\sqrt{2}\pi)$. According to [2, Theorem 4.4], the cordial operator $V_{k,\frac{3}{2},\frac{1}{2}}$ associated with (5.2) is non-compact on $C([0, 1])$, and its spectrum is given by

$$\sigma_{C[0,1]}\left(V_{k,\frac{3}{2},\frac{1}{2}}\right) = \{0\} \cup \left\{ \frac{1}{\sqrt{2}\pi} \mathrm{B}\big(2 + \lambda, \tfrac{1}{2}\big) : \ \lambda \in \mathbb{C}, \ \mathrm{Re}(\lambda) \geq 0 \right\}.$$

It can be verified that $g(t) = t^{-\beta} f(t) \in C^1([0, 1])$ and $1 \notin \sigma_{C[0,1]}(V_{k,\frac{3}{2},\frac{1}{2}})$, and therefore [2, Theorem 4.7] ensures that $u(t) = t^{\frac{9}{5}}$ is the unique solution to (5.2) in $C^1([0, 1])$.

The proposed method has been applied to (5.2) for several values of m, and the numerical results are reported in Tables 3 and 4 and displayed in Figures 3 and 4. Table 3 presents the maximum error and observed convergence order, and Table 4 presents the L^2-error and includes comparisons with the first-kind Chebyshev wavelets method [17]. The behavior of the error and residual functions for different values of m is illustrated in Figures 3 and 4.

Example 5.3. We study the integral equation

$$t^{\frac{2}{3}} u(t) = f(t) + \int_0^t \frac{\sqrt{3}}{3\pi} x^{\frac{1}{3}} (t - x)^{-\frac{2}{3}} u(x) dx, \quad t \in [0, 1], \tag{5.3}$$

TABLE 3. The maximum error and the estimated convergence order corresponding to Example 5.2.

m	$\|e_m\|_\infty$	p_m	$\|r_m\|_\infty$	q_m
3	1.67×10^{-3}	3.01	1.60×10^{-3}	3.51
6	2.07×10^{-4}	3.00	2.00×10^{-4}	3.49
12	1.81×10^{-5}	–	1.78×10^{-5}	–

TABLE 4. The L^2-error corresponding to Example 5.2.

Method	NBFs	$\|e_m\|_2$
Proposed Method	m	
	4	1.72×10^{-3}
	8	1.62×10^{-4}
	10	7.30×10^{-5}
	12	3.76×10^{-5}
First-kind Chebyshev wavelets method[17]	$2^{k-1}(M+1)$	
	$k=1, M=3$	9.70×10^{-4}
	$k=2, M=3$	2.27×10^{-4}
	$k=2, M=4$	8.04×10^{-5}
	$k=2, M=5$	3.57×10^{-5}
	$k=3, M=3$	5.43×10^{-5}

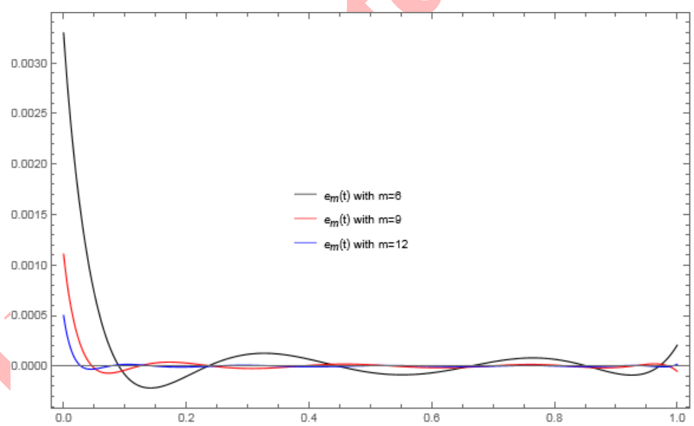

FIGURE 3. Error function $e_m(t)$ for Example 5.2 with $m = 6, 9, 12$.

with

$$f(t) = t^{\frac{47}{12}} \left(1 - \frac{\Gamma(\frac{1}{3})\Gamma(\frac{55}{12})}{\pi\sqrt{3}\Gamma(\frac{59}{12})} \right).$$

Equation (5.3) is of the form (1.1) with $\beta = \alpha = 2/3$ and $k(t,x) = x^{1/3}\sqrt{3}/(3\pi)$. The function $u(t) = t^{\frac{13}{4}}$ satisfies (5.3) exactly. We have $\alpha + \beta > 1$ and the kernel admits the representation (1.2) with $h(t,x) = \sqrt{3}/(3\pi)$. According to [2, Theorem 4.4], the cordial operator $V_{k,\frac{2}{3},\frac{2}{3}}$ associated with (5.3) is non-compact on $C([0,1])$, and its spectrum is

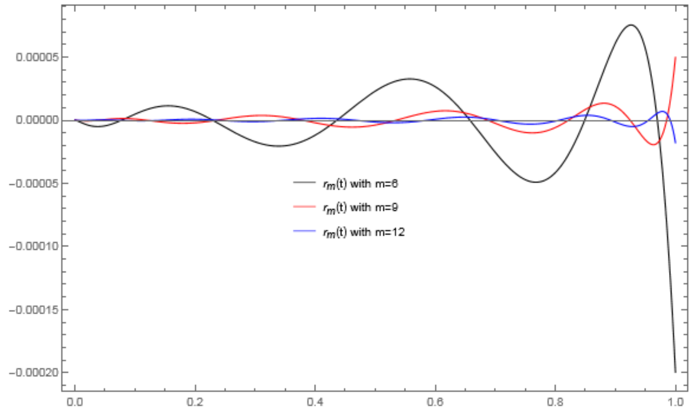

FIGURE 4. Residual function $r_m(t)$ for Example 5.2 with $m = 6, 9, 12$.

given by

$$\sigma_{C[0,1]}\left(V_{k,\frac{2}{3},\frac{2}{3}}\right) = \{0\} \cup \left\{ \frac{\sqrt{3}}{3\pi} \mathrm{B}\left(\tfrac{4}{3}+\lambda, \tfrac{1}{3}\right) : \ \lambda \in \mathbb{C}, \ \mathrm{Re}(\lambda) \geq 0 \right\}.$$

It can be verified that $g(t) = t^{-\beta} f(t) \in C^3([0,1])$ and $1 \notin \sigma_{C[0,1]}(V_{k,\frac{2}{3},\frac{2}{3}})$, and therefore [2, Theorem 4.7] ensures that $u(t) = t^{\frac{13}{4}}$ is the unique solution to (5.3) in $C^3([0,1])$.

The proposed method has been applied to (5.3) for several values of m, and the numerical results are reported in Tables 5 and 6 and displayed in Figures 5 and 6. Table 5 presents the maximum error and observed convergence order, and includes comparisons with the spline collocation method in piecewise polynomial spaces [2] and the operational matrix approach based on adjusted hat functions [16]. Table 6 reports the L^2-error, and provides a comparison with the first-kind Chebyshev wavelets method [17]. The behavior of the error and residual functions for different values of m is illustrated in Figures 5 and 6.

TABLE 5. The maximum error and the estimated convergence order corresponding to Example 5.3.

Method	NBFs	$\|e_m\|_\infty$	p_n	$\|r_m\|_\infty$	q_n
Proposed Method	m				
	3	3.67×10^{-3}	2.91	3.00×10^{-3}	1.91
	6	9.11×10^{-4}	2.10	1.03×10^{-4}	2.03
	12	2.12×10^{-4}	–	1.95×10^{-4}	–
Adjusted hat functions method[16]	n				
	3	3.82×10^{-3}	3.25		
	6	4.02×10^{-4}	3.25		
	12	4.22×10^{-5}	3.25		
Collocation method-Normal points ($m = 3$) [1]	N				
	4	3.26×10^{-2}	–		
	8	1.08×10^{-2}	1.59		
	16	3.03×10^{-3}	1.83		
	32	7.80×10^{-4}	1.96		
	64	2.05×10^{-4}	1.98		

TABLE 6. The L^2-error corresponding to Example 5.3.

Method	NBFs	$\|e_m\|_2$
Proposed Method	m	
	4	4.10×10^{-4}
	8	7.01×10^{-5}
	10	4.25×10^{-5}
	12	2.98×10^{-5}
	16	2.06×10^{-6}
First-kind Chebyshev wavelets method[17]	$2^{k-1}(M+1)$	
	$k=1, M=3$	1.29×10^{-3}
	$k=2, M=3$	1.12×10^{-4}
	$k=2, M=4$	6.33×10^{-6}
	$k=2, M=5$	1.24×10^{-6}
	$k=3, M=3$	9.78×10^{-6}

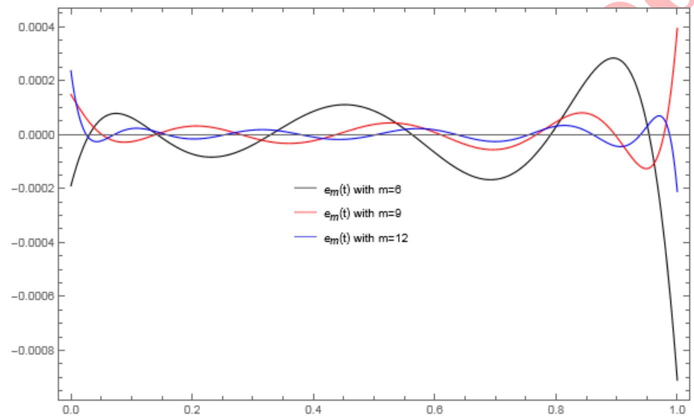

FIGURE 5. Error function $e_m(t)$ for Example 5.3 with $m = 6, 9, 12$.

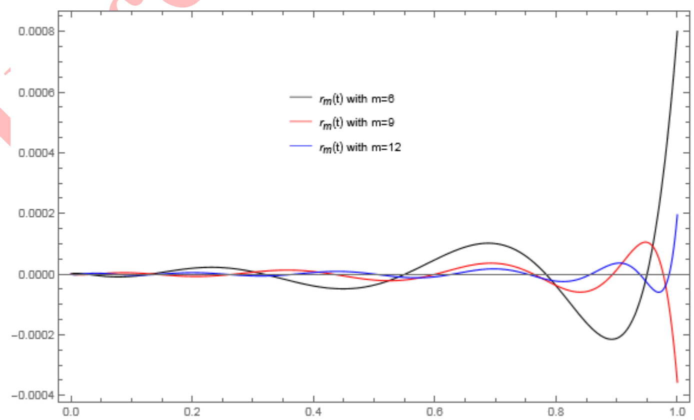

FIGURE 6. Residual function $r_m(t)$ for Example 5.3 with $m = 6, 9, 12$.

6. Conclusion

A numerical scheme has been proposed for solving third-kind Volterra integral equations with both smooth and weakly singular kernels. The method employs Chelyshkov polynomials and their operational matrix of fractional integration to systematically reduce the integral equation to a system of linear algebraic equations.

The results from three illustrative examples demonstrate that the method achieves high accuracy even with a moderate number of basis functions. Both the maximum error and the L^2-error decrease rapidly as the polynomial degree increases, confirming the fast convergence rate predicted by the theoretical residual error estimates. The observed convergence orders in the examples show that the method maintains its efficiency for equations with different types of singular behavior.

Furthermore, the numerical experiments indicate that the residual errors remain small across all examples, highlighting the robustness and stability of the approach. Comparisons with previously reported methods illustrate that the proposed scheme provides competitive or superior accuracy with relatively low computational cost.

Overall, the presented method offers a reliable and practical tool for solving third-kind Volterra integral equations, including those with challenging singular kernels, and can be effectively applied in various computational contexts.

Future research may extend the present approach to nonlinear problems and multidimensional integral equations.

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