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Construction of exact periodic wave and solitary wave solutions to the modified forms of Degasperis–Procesi and Camassa–Holm equations by the generalized G'/G-expansion method

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Abstract

In this work, the exact solutions to the modified forms of Degasperis–Procesi and Camassa–Holm equations are established. The generalized G'/G-expansion method is used to construct solitary wave solutions of nonlinear evolution equations. The generalized G'/G-expansion method presents a wider applicability for handling nonlinear wave equations. It is shown that the G'/G-expansion method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics. The soliton solutions demonstrate the competency of the proposed technique in identifying traveling wave solutions, offering a useful tool for tackling a variety of nonlinear evolution equation. The proposed models have applications in various scientific domains, including condensed matter physics, hydrodynamics, plasma physics, optical fiber communications, ocean engineering, and nonlinear dynamics.

Keywords. The generalized G'/G-expansion method; Modified forms of Degasperis–Procesi and Camassa–Holm equations; Solitary wave solutions; Solitons.

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1. Introduction

In the recent years, the investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. During the past decades, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of the nonlinear ordinary or partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented [4, 32, 58]. For instance, Hirota's bilinear method [24], the inverse scattering transform [3], F-expansion method [2, 47], sine-cosine method [53, 60], homotopy perturbation method [16], homotopy analysis method [1, 17] variational iteration method [21], tanh-function method [6, 25, 54], Bäcklund transformation [30, 40], Exp-function method [9], the improved $tan(\phi)$-expansion method [26, 33, 35, 36], Hirota bilinear method [45, 49, 61], the multiple rogue wave solution [34] and tended sinh-Gordon equation expansion method [5]. Here, we use of an effective method for constructing a range of exact solutions for following nonlinear partial differential equations that first proposed by Wang [52] which a new method called the G'/G-expansion method to look for travelling wave solutions of NLEEs. Zhang et al. [62] examined the generalized G'/G-expansion method and its applications. Also, Bekir [7] used to application of the G'/G-expansion method for nonlinear evolution equations. In this article we explain method which is called the

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G'/G-expansion method to look for travelling wave solutions of nonlinear evolution equations.
The Camassa–Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

has been investigated in the literature in [10–12, 27, 29, 31, 41, 44, 50, 51, 57] and the references therein. However, changing the coefficients 3 and 2 in Eq. (1.1) to 4 and 3, respectively, gives the Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

The CH equation has peaked solitary wave solutions of the form [57]

$$u(x,t) = ce^{-|x-ct|}, \tag{1.3}$$

where c is the wave speed. The name 'peakon', that is, solitary wave with slope discontinuities, was used to single them from general solitary wave solutions since they have a corner at the peak of height c [57]. In this article an application of the proposed method to the modified forms of Degasperis–Procesi and Camassa–Holm equations is illustrated. we will investigate modified forms of the DP and the CH equations given by

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

and

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.5}$$

respectively.
In [41], Mustafa studied Degasperis-Procesi equation. Multi-peakon solutions of Degasperis-Procesi equation was investigated by Lundmark [31]. Shen have obtained new integrable equation with peakon and compactons [50]. Chen have obtained new type of bounded waves for Degasperis–Procesi equation [12]. In [10, 11] have obtained integrable shallow water equation and completely integrable shallow water equation respectively. In [27, 29] Liu and et. al, have studied the Camassa–Holm equation and have obtained peaked wave solutions [29] and peakons [27]. Peakons and periodic cusp waves of the generalized Camassa-Holm equation have been studied by [44]. Also in [51] Tian and et. al, have studied the generalized Camassa-Holm equation and obtained new peaked solitary wave solutions. Boyd [8] derived a perturbation series which converges even at the peakon limit, and gave three analytical representation for the spatially periodic generalization of the peakon, called "coshoidal wave". Cooper and Shepard [14] derived approximate solitary wave solution by using some variational functions. Constantin [13] gave a mathematical description of the existence of interacting solitary waves. Wazwaz [55] derived a solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations. A class of nonlinear fourth order variant of a generalized Camassa-Holm equation was investigated by [56] where have obtained compact and noncompact solutions. Recently, CH equation and some its generalized forms have been studied by many authors, for instance, Wazwaz [57] have obtained new solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations. He [23] derived exact travelling wave solutions of a generalized Camassa-Holm equation using the integral bifurcation method. In [19] an integrable shallow water equation has been discussed with linear and nonlinear dispersion. Also, new integrable equation with peakon solutions has been studied by Degasperis [18]. Explicit solutions of the Camassa-Holm equation have been obtained in citeParkes. Guo [20] obtained periodic cusp wave solutions and single-solitons for the b-equation. Bäcklund transformation has been applied for the modified DGH equation by [59]. In [22] bifurcations of travelling wave solutions for a variant of Camassa-Holm equation investigated by He. In [48] Rui et. al, applied the integral bifurcation method and its application for solving a family of third-order dispersive PDEs. Finally, Liu and Qian [28] have derived peakons and their bifurcation for the generalized Camassa-Holm equation. The recent years were worked on the expansion of research to include optical soliton, soliton crystals, and soliton complexes [15, 37, 39, 42, 46]. During previous decades, many methods were developed to solve nonlinear partial differential equations, including the Kundu-Eckhaus equation via $tan(\phi/2)$-expansion method [38].
The article is organized as follows: In Section 2, first we briefly give the steps of the method and apply the method to solve the nonlinear partial differential equations. In Section 3, the application of the $(\frac{G'}{G})$-expansion method to the modified Degasperis–Procesi equation will be introduced briefly. Also, Section 4 by using the results obtained in

Section 2, the corresponding solutions of the modified Camassa–Holm equation can be obtained. Also a conclusion is given in Section 5. Finally some references are given at the end of this paper.

2. Basic idea of the $(\frac{G'}{G})$-expansion method

Another powerful analytical method is called $(\frac{G'}{G})$-expansion method, we give the detailed description of method which first presented by Wang [52].

Step 1. For a given NLPDE with independent variables $\mathrm{X} = (\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ and dependent variable u:

$$\mathcal{P}(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{tt}, u_{tx}, u_{ty}, u_{tz} ...) = 0, \tag{2.1}$$

can be converted to on ODE

$$\mathcal{M}(u, -cu', u', u', u', u'', ...) = 0, \tag{2.2}$$

which transformation $\xi = \mathrm{x} + \mathrm{y} - \mathrm{ct}$ is wave variable. Also, c is constant to be determined later.

Step 2. we seek its solutions in the more general polynomial form as following

$$\mathrm{u}(\xi) = \mathrm{a}_0 + \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{a}_\mathrm{k} \left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^{\mathrm{k}}, \tag{2.3}$$

where $G(\xi)$ satisfies the second order LODE in the form

$$\mathrm{G}''(\xi) + \lambda \mathrm{G}'(\xi) + \mu \mathrm{G}(\xi) = 0, \tag{2.4}$$

where a_0, $\mathrm{a}_\mathrm{k}(\mathrm{k} = 1, 2, ..., \mathrm{m})$, λ and μ are constants to be determined later, $\mathrm{a}_\mathrm{m} = 0$, but the degree of which is generally equal to or less than $\mathrm{m} - 1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2).

Step 3. Substituting Eq. (2.3) and Eq. (2.4) into Eq. (2.2) with the value of n obtained in Step 1. Collecting the coefficients of $\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^{\mathrm{k}}$ $(\mathrm{k} = 0, \pm 1, \pm 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $\mathrm{a}_0, \mathrm{a}_\mathrm{i}(\mathrm{i} = 1, 2, ..., \mathrm{n}), \lambda, \mathrm{c}$ and μ with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equations in Step 3, then substituting $\mathrm{a}_\mathrm{i}, ..., \mathrm{a}_\mathrm{m}, \mathrm{c}$ and general solutions of Eq. (2.4) into Eq. (2.3) we can obtain a series of fundamental solutions of Eq. (2.1) depending of the solution $\mathrm{G}(\xi)$ of Eq. (2.4).

3. The modified Degasperis–Procesi equation

In this section, we employ the $(\frac{G'}{G})$-expansion method to the modified Degasperis–Procesi equation [57]

$$\mathrm{u_t} - \mathrm{u_{xxt}} + 4\mathrm{u}^2\mathrm{u_x} = 3\mathrm{u_x}\mathrm{u_{xx}} + \mathrm{u}\mathrm{u_{xxx}}. \tag{3.1}$$

Using the wave variables as follow $\xi = \mathrm{x} - \mathrm{ct}$, carries the DP equation Eq. (3.1) into the ODE

$$-\mathrm{cu}' + \mathrm{cu}''' + 4\mathrm{u}^2\mathrm{u}' = 3\mathrm{u}'\mathrm{u}'' + \mathrm{uu}'''. \tag{3.2}$$

Integrating Eq. (3.2) with respect to ξ and considering the zero constants for integration, we obtain

$$\mathrm{c}(\mathrm{u}'' - \mathrm{u}) + \frac{4}{3}\mathrm{u}^3 - \mathrm{uu}'' - (\mathrm{u}')^2 = 0. \tag{3.3}$$

In order to determine value of m, we balance the linear term of the highest order uu'' with the highest order nonlinear term u^3 in Eq. (3.3), we have By using Eq. (2.3) it is easily derived that

$$u^3(\xi) = a_m^3 \left(\frac{G'(\xi)}{G(\xi)}\right)^{3m} + ..., \tag{3.4}$$

$$u_\xi(\xi) = -ma_m \left(\frac{G'(\xi)}{G(\xi)}\right)^{m+1} + ...,$$

$$u_{\xi\xi}(\xi) = m(m+1)a_m \left(\frac{G'(\xi)}{G(\xi)}\right)^{m+2} + ...$$

In order to determine value of m, we balance uu$''$ with u^3 in Eq. (3.3), we have

$$2m + 2 = 3m, \tag{3.5}$$

we find m = 2. We can suppose that the solutions of Eq. (3.1) is of the form

$$\mathrm{u}(\xi) = \mathrm{a}_0 + \mathrm{a}_1 \left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right) + \mathrm{a}_2 \left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^2, \qquad \mathrm{a}_2 \neq 0, \tag{3.6}$$

and therefore

$$u^2(\xi) = a_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 + 2a_1a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (a_1^2 + 2a_0a_2) \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2, \tag{3.7}$$

and

$$\begin{aligned} u_{\xi\xi}(\xi) = 6a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 &+ (2a_1 + 10a_2\lambda) \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \\ &+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) \left(\frac{G'(\xi)}{G(\xi)}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \tag{3.8}$$

Substituting Eqs. (3.6)–(3.8), and by using the well-known software Maple, we obtain the system of following results

$$a_0 = \frac{15\mu}{2},\ a_1 = \frac{15\lambda}{2},\ a_2 = \frac{15}{2},\ c = \frac{5}{2}, \tag{3.9}$$

or

$$a_0 = \frac{-9 \pm \sqrt{15}i}{8} + \frac{15\mu}{2},\ a_1 = \frac{15\lambda}{2},\ a_2 = \frac{15}{2},\ c = \frac{11 \mp \sqrt{15}i}{8}, \tag{3.10}$$

where λ and μ are arbitrary constants.
Substituting Eqs. (3.9) and (3.10) into expression Eq. (3.6), can be written as

$$u(\xi) = \frac{15\mu}{2} + \frac{15\lambda}{2} \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{15}{2} \left(\frac{G'(\xi)}{G(\xi)}\right)^2,\ \xi = x - \frac{15}{2}t, \tag{3.11}$$

or

$$u(\xi) = \frac{-9 \pm \sqrt{15}i}{8} + \frac{15\mu}{2} + \frac{15\lambda}{2} \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{15}{2} \left(\frac{G'(\xi)}{G(\xi)}\right)^2,\ \xi = x - \frac{11 \mp \sqrt{15}i}{8}t. \tag{3.12}$$

Substituting the general solutions of Eq. (2.4) into Eqs. (3.12) and (3.12) we have three types of exact solutions of Eq. (3.1) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$u_1(\xi) = \frac{15}{8}(\lambda^2 - 4\mu) \left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 - \frac{15}{8}\lambda^2 + \frac{15\mu}{2}, \tag{3.13}$$

where $\xi = \mathrm{x} - \frac{15}{2}\mathrm{t}$ and

$$u_2(\xi) = \frac{15}{8}(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 - \frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}i}{8} + \frac{15\mu}{2}, \tag{3.14}$$

where $\xi = \mathrm{x} - \frac{11\mp\sqrt{15}\mathrm{i}}{8}\mathrm{t}$.
When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u_3(\xi) = \frac{15}{8}(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 - \frac{15}{8}\lambda^2 + \frac{15\mu}{2}, \tag{3.15}$$

where $\xi = x - \frac{15}{2}t$ and

$$u_4(\xi) = \frac{15}{8}(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 - \frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}i}{8} + \frac{15\mu}{2}, \tag{3.16}$$

where $\xi = x - \frac{11\mp\sqrt{15}i}{8}t$.

When $\lambda^2 - 4\mu = 0$, we get rational solution

$$\begin{aligned} u_5(\xi) &= \frac{15}{2}\frac{C_2^2}{(C_1 + C_2\xi)^2} - \frac{15}{8}\lambda^2 + \frac{15\mu}{2}, \quad \xi = x - \frac{15}{2}t, \\ u_6(\xi) &= \frac{15}{2}\frac{C_2^2}{(C_1 + C_2\xi)^2} - \frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}i}{8} + \frac{15\mu}{2}, \quad \xi = x - \frac{11 \mp \sqrt{15}i}{8}t. \end{aligned} \tag{3.17}$$

If $C_1 \neq 0, \mathrm{C}_2 = 0, \lambda > 0, \mu = 0$, then Eqs. (3.13) and (3.14) give respectively

$$u_7(\xi) = -\frac{15}{8}\lambda^2 \; sech^2\left[\frac{\lambda}{2}\left(x - \frac{15}{2}t\right)\right], \tag{3.18}$$

$$u_8(\xi) = \frac{-9 \pm \sqrt{15}i}{8} - \frac{15}{8}\lambda^2 \; sech^2\left[\frac{\lambda}{2}\left(x - \frac{11 \mp \sqrt{15}i}{8}t\right)\right].$$

In particular, If $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu > 0$, then Eqs. (3.15) and (3.16) can be written as respectively

$$\mathrm{u}_9(\xi) = \frac{15}{2}\mu \sec^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right], \tag{3.19}$$

$$\mathrm{u}_{10}(\xi) = \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15}{2}\mu \sec^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right].$$

But, if $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu < 0$, then Eqs. (3.13) and (3.14) get respectively

$$\mathrm{u}_{11}(\xi) = \frac{15}{2}\mu \,\mathrm{sech}^2\left[\sqrt{-\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right], \tag{3.20}$$

$$\mathrm{u}_{12}(\xi) = \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15}{2}\mu \,\mathrm{sech}^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}\right)\right],$$

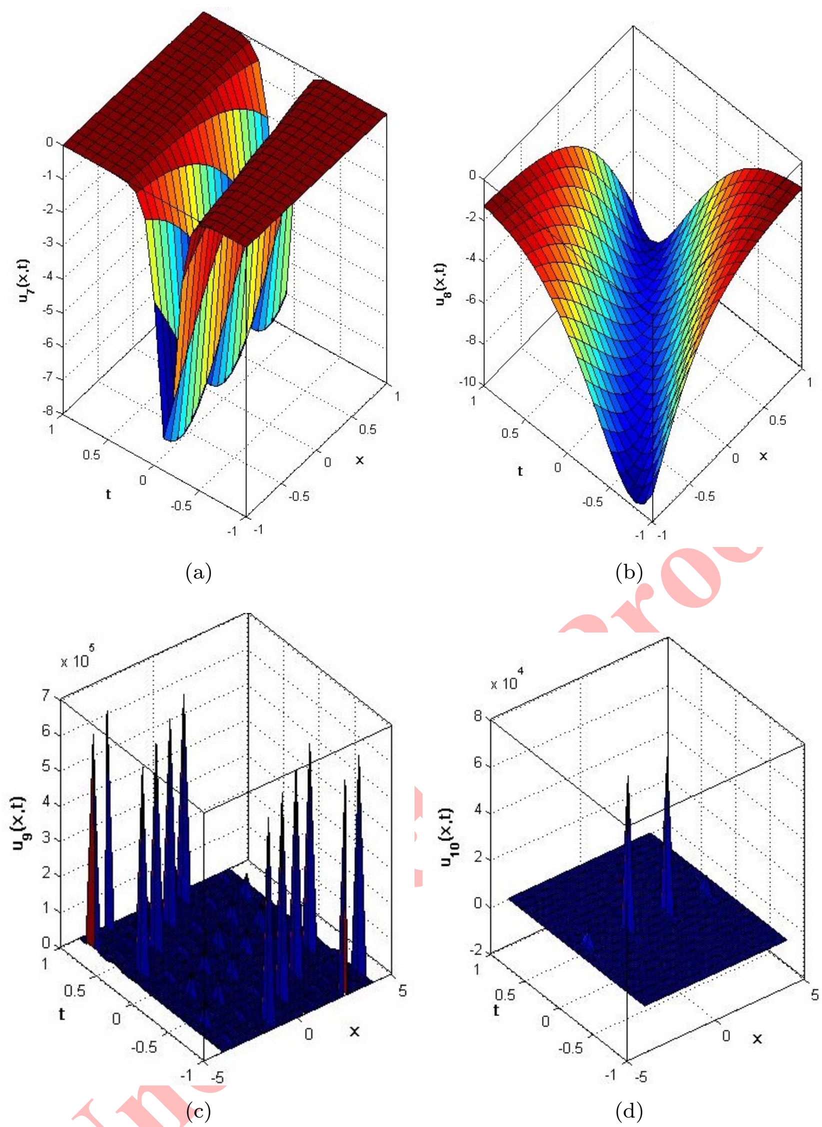


FIGURE 1. The periodic and solitary solutions (a) u_7, (b) u_8, (c) u_9, and (d) u_{10} position at $\lambda = 2$ and $\mu = 2$ for Eqs. (3.18) and (3.19), respectively.

which are the exact solutions of discussing equation. Can be seen that the results are the same, with comparing results [57].

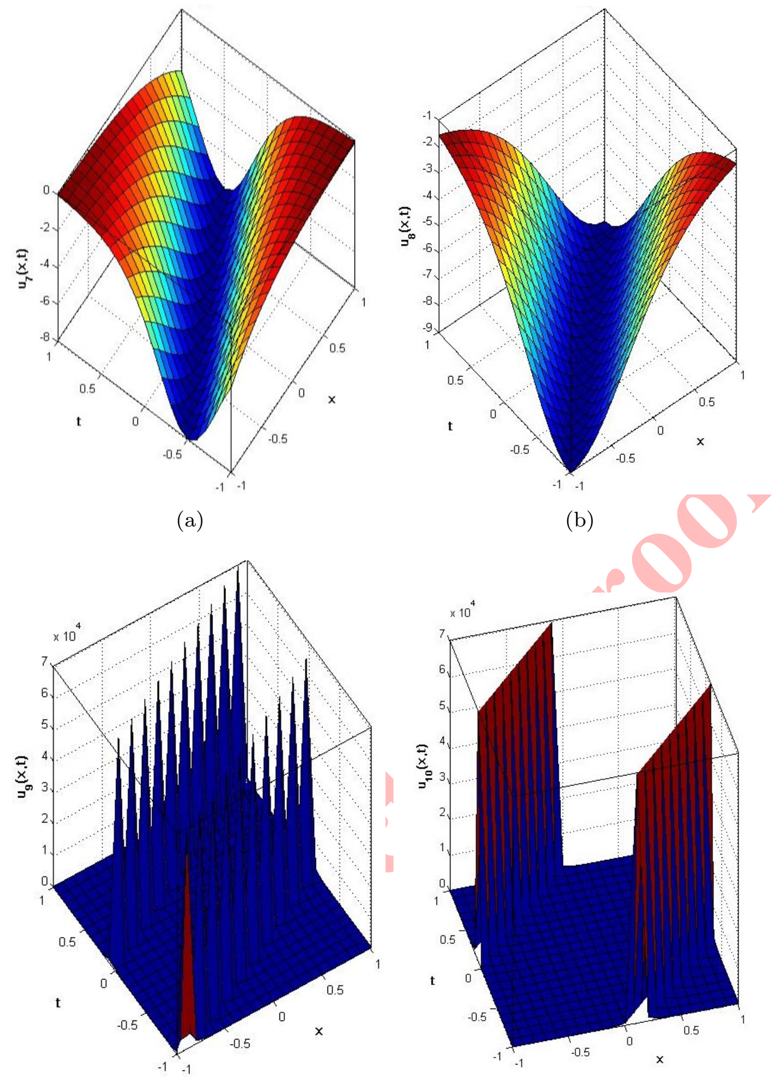
4. The modified Camassa–Holm equation

We next consider the modified Camassa–Holm equation [57] as

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx}. \tag{4.1}$$

Using the wave variables as follow $\xi = x - ct$, carries the CH equation Eq. (4.1) into the ODE

$$-cu' + cu''' + 43u^2 u' = 2u'u'' + uu'''. \tag{4.2}$$



(a) (b) (c) (d)

FIGURE 2. The periodic and solitary solutions (a) u_7, (b) u_8, (c) u_9, and (d) u_{10} position at $\lambda = 2$ and $\mu = 2$ for Eqs. (4.18) and (4.19), respectively.

Integrating Eq. (4.2) with respect to ξ and considering the zero constants for integration, we obtain

$$c(u'' - u) + u^3 - uu'' - \frac{1}{2}(u')^2 = 0. \tag{4.3}$$

In order to determine value of m, we balance the linear term of the highest order uu'' with the highest order nonlinear term u^3 in Eq. (4.3), we have By using Eq. (2.3) it is easily derived that

$$u^3(\xi) = a_m^3 \left(\frac{G'(\xi)}{G(\xi)}\right)^{3m} + ..., \tag{4.4}$$

$$u_\xi(\xi) = -ma_m \left(\frac{G'(\xi)}{G(\xi)}\right)^{m+1} + ...,$$

$$u_{\xi\xi}(\xi) = m(m+1)a_m \left(\frac{G'(\xi)}{G(\xi)}\right)^{m+2} + ...$$

In order to determine value of m, we balance uu'' with u^3 in Eq. (4.3), we have

$$2m + 2 = 3m, \tag{4.5}$$

we find $m = 2$. We can suppose that the solutions of Eq. (4.1) is of the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0, \tag{4.6}$$

and therefore

$$u^2(\xi) = a_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 + 2a_1a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (a_1^2 + 2a_0a_2) \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2 \tag{4.7}$$

and

$$\begin{aligned} u_{\xi\xi}(\xi) = {} & 6a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \\ & + (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) \left(\frac{G'(\xi)}{G(\xi)}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \tag{4.8}$$

Substituting Eqs. (4.6)–(4.8), and by using the well-known software Maple, we obtain the system of following results

$$a_0 = 8\mu, \qquad a_1 = 8\lambda, \qquad a_2 = 8, \qquad c = 2, \tag{4.9}$$

or

$$a_0 = 8\mu - 1, \qquad a_1 = 8\lambda, \qquad a_2 = 8, \qquad c = 1, \tag{4.10}$$

where λ and μ are arbitrary constants.
Substituting Eqs. (4.9) and (4.10) into expression Eq. (4.6), can be written as

$$u(\xi) = 8\mu + 8\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) + 8 \left(\frac{G'(\xi)}{G(\xi)}\right)^2, \quad \xi = x - 2t, \tag{4.11}$$

or

$$u(\xi) = 8\mu - 1 + 8\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) + 8 \left(\frac{G'(\xi)}{G(\xi)}\right)^2, \quad \xi = x - t. \tag{4.12}$$

Substituting the general solutions of Eq. (2.4) into Eqs. (4.12) and (4.12) we have three types of exact solutions of Eq. (4.1) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$\begin{aligned} u_1(\xi) = {} & 2(\lambda^2 - 4\mu) \left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)} \right)^2 - 2\lambda^2 + 8\mu, \\ & \xi = x - 2t, \end{aligned} \tag{4.13}$$

and

$$u_2(\xi) = 2(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 - 2\lambda^2 + 8\mu - 1,$$
$$\xi = x - t. \tag{4.14}$$

When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u_3(\xi) = 2(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 - 2\lambda^2 + 8\mu,$$
$$\xi = x - 2t, \tag{4.15}$$

and

$$u_4(\xi) = 2(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 - 2\lambda^2 + 8\mu - 1,$$
$$\xi = x - t. \tag{4.16}$$

When $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_5(\xi) = 2\frac{C_2^2}{(C_1 + C_2\xi)^2} - 2\lambda^2 + 8\mu, \qquad \xi = x - 2t, \tag{4.17}$$

$$u_6(\xi) = 2\frac{C_2^2}{(C_1 + C_2\xi)^2} - 2\lambda^2 + 8\mu - 1, \qquad \xi = x - t.$$

If $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda > 0, \mu = 0$, then Eqs. (4.13) and (4.14) give respectively

$$u_7(\xi) = -2\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}(\mathrm{x} - 2\mathrm{t})\right], \tag{4.18}$$

$$u_8(\xi) = -1 - 2\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}(\mathrm{x} - 2\mathrm{t})\right].$$

In particular, If $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu > 0$, then Eqs. (4.15) and (4.16) give respectively

$$u_9(\xi) = 8\mu \sec^2[\sqrt{\mu}(\mathrm{x} - 2\mathrm{t})], \tag{4.19}$$

$$u_{10}(\xi) = -1 + 8\mu \sec^2[\sqrt{\mu}(\mathrm{x} - \mathrm{t})].$$

But, if $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu < 0$, then Eqs. (4.13) and (4.14) get respectively

$$\mathrm{u}_{11}(\xi) = 8\mu \operatorname{sech}^2[\sqrt{-\mu}(\mathrm{x} - 2\mathrm{t})], \tag{4.20}$$

$$\mathrm{u}_{12}(\xi) = -1 + 8\mu \operatorname{sech}^2[\sqrt{-\mu}(\mathrm{x} - \mathrm{t})],$$

which are the exact solutions of discussing equation. Can be seen that the results are the same, with comparing results [57].

The obtained analytical solutions by the generalized $(\frac{G'}{G})$-expansion method are investigated. To show the properties of the solutions for the modified forms of Degasperis–Procesi and Camassa–Holm equations, we take some solutions, as illustrative samples and draw theirs plots (see Figs. 1 and 2).

5. Conclusion

The generalized G'/G-expansion method was successfully used to establish periodic wave and solitary wave solutions. The obtained results was complemented the useful works of others for this important equations. The generalized G'/G-expansion method was an useful method for finding travelling wave solutions of nonlinear evolution equations. These exact solutions were included three types hyperbolic function solution, trigonometric function solution and rational solution. The generalized G'/G-expansion method was more powerful in searching for exact solutions of NLPDEs. Some of these results were in agreement with the results reported specially by [57]. Also, new results are formally developed in this article. It can be concluded that the this method was a very powerful and efficient technique in finding exact solutions for wide classes of problems. The result will significantly enhance nonlinear science and serve as a valuable resource for further research. In the future, this article will also help the researchers to use this technique on higher-dimensional nonlinear fractional models and to the system of coupled equations.

Conflict of Interest

The authors declare that they have no conflict of interest.

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