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Optimal stochastic Robust controllers based on mean-CVaR sliding function and its application in pairs trading strategy

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Abstract

Robust control of some nonlinear stochastic risky systems, in the presence of uncertainties and undesirable risks, requires the use of suitable and coherent risk measurement tool and efficient filters. In this paper, two new optimal robust stochastic controllers are designed for nonlinear stochastic risk systems and applications in financial engineering. For this purpose, a new weighted conditional sliding function is designed based on a new convex, coherent and differentiable risk measure of the state variables. Also, a new efficient frontier with slope CVaR-Sharp- ratio is designed to optimally estimate the weight of the CVaR measure in the proposed conditional sliding function, based on the efficient frontier analysis approach. Additionally, the adaptive boundary layer width and robust controllers are tuned to increase the performance of the proposed controllers. Also, four theorems are proved and a new algorithm is designed for stability analysis and computational support. Finally, to demonstrate the advantages of the proposed methods, a pair trading stock problem is simulated. The simulation results show that the proposed methods improve portfolio returns and achieve effective CVaR-based risk control.

Keywords. Stochastic sliding mode control, Boundary layer, Conditional value at risk, Efficient frontier, Portfolio optimization.
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1. Introduction

Uncertainty effect reduction is one of the main goals of robust control design. Physical and financial systems inherently involve modeling errors, disturbances, and random fluctuations that may significantly affect performance [52]. Classical approaches such as robust control [35] and adaptive control [26, 27, 42] have been widely adopted to ensure acceptable operation under uncertainty; however, each has limitations. Robust control usually guarantees stability against bounded parametric uncertainty but lacks adaptability in time-varying conditions, while adaptive control is flexible but less robust to unstructured disturbances. Consequently, hybrid schemes combining both robustness and adaptability have been sought to balance performance and resilience [26, 35].

Sliding mode control (SMC) is an analytical robust control approach [6, 54]. In the past few decades, the sliding mode control method has attracted the attentions of researchers due to its many applications in linear and non-linear systems [8]. This approach is a robust control method with two stages: the first stage is designing the sliding surface and the second is defining the control law such that the sliding surface be adsorbent [24, 25, 34].

A notable point in the traditional sliding mode control (SMC) method is that the dynamics of the sliding surface are defined as a deterministic manifold. In practice, many physical and financial systems are subject to random noise and random parameter changes. In such circumstances, the assumption of deterministic sliding surface dynamics may reduce the performance of the designed controller and even cause system instability. The stochastic sliding mode control (SSMC) framework extends conventional SMC to stochastic systems described by Itô stochastic differential equations [15, 32, 47, 49]. Previous studies have established fundamental results on integral SMC for time-delayed systems [37] and robust observer design for Ito stochastic systems [36]. However, phenomena such as strong market

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volatility, correlated asset dynamics, and coherence of risk effects make the application of these methods to risky decision-making problems and financial systems inefficient. Thus, integrating robust stochastic control with modern financial risk theory constitutes a timely and promising research direction.

In financial optimization, systematic quantification of risk is as crucial as maximizing expected return. Early risk modeling rely on variance or standard deviation [29, 41, 45, 50], which fail to capture tail risk or asymmetric distributions. This motivated the development of Value-at-Risk (VaR) [10, 11, 43] and subsequently a coherent and convex risk measure that is called Conditional Value at Risk (CVaR) and proposed by Artzner et al. [3, 4] and further formulated by Rockafellar and Uryasev [40]. CVaR evaluates the expected loss beyond a predefined confidence level and satisfies the coherence and sub-additivity axioms that VaR lacks [18, 48, 51]. Due to these advantages, CVaR plays a central role in modern portfolio optimization and risk management frameworks [5, 17].

Despite these advances, stochastic or robust control frameworks in finance still focus on expectation-based optimization and ignore coherent risk measures such as CVaR due to limitations such as non-derivativeness. Several CVaR-based robust portfolio optimization frameworks have been proposed to address uncertainty in financial returns without considering their effects on the portfolio efficiency frontier (see Sun et al. [46]). However, this gap motivates the present paper.

Recent studies such as Lotfi and Zenios [28] have extended robust Mean-CVaR optimization to handle ambiguity in mean and covariance parameters; however, these approaches do not address dynamic stochastic control settings. To overcome these challenges, this paper develops a stochastic robust control framework that directly integrates the CVaR criterion into the SMC design. Two robust controllers, including a stochastic sliding mode controller and a stochastic boundary layer sliding mode controller with variable boundary layer, are designed based on linear optimization, the CVaR criterion, and a new efficient Sharpe ratio definition. The proposed sliding surface is defined based on the weighted sum of the expected value and the conditional value at risk of the stochastics state variable of the system, through the optimal weight value and a new coherent Markowitz efficient frontier. Also, an adaptive boundary layer width rule and two robust control rules are designed to enhance the performance of the proposed controllers. Additionally, according to the $Mean-CVaR$, efficient frontier and capital market line (CML), an optimal λ value is computed. The stability analysis and convergence are proved based on the Lyapunov stability. Four theorems and one algorithm for analytical facilitating of the presented method are proved and designed. Finally, the suggested control methods is used to control the value of the portfolio that is including a pair of stocks with a stochastic spread approach for pair trading strategy and a risk-free asset. As the simulation results indicate, the proposed controllers increase the value of the wealth and reduce the conditional value at risk measure, effectively. Some comparisons of the proposed controllers are presented in simulation section. The main approaches and the novel ideas of this paper are described in section 2.

In Section 3, two new stochastic sliding mode controllers, based on four theorems and a new algorithm, are designed. In Section 4, a new efficient frontier with slope CVaR-Sharp- ratio is designed to optimally estimate the weight of the CVaR measure in the proposed conditional sliding function, based on the efficient frontier analysis approach. Also, in Section 5, a simulation example for illustrating the advantages of the proposed method is presented. A case study is presented in Section 6. The conclusions of this paper are presented in Section 7. For computational and practical support, in Appendices B and C, the dynamics of pair trading and value of the portfolio and Ornstein-Uhlenbeck stochastic process are expressed, respectively.

2. Problem Statement

Consider the following stochastic system:

$$\begin{aligned} dX_t &= (f(X_t,t) + u_t g(X_t,t))dt + u_t\sigma(X_t,t)dW_t, \\ X(t_0) &= X_0. \end{aligned} \tag{2.1}$$

Where:

- $W(t) = (W_1(t), ..., W_m(t))^T$ is m-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_{t\geq 0}), P)$.
- $X_t = X_t(\omega) \in \mathbb{R}^n (\omega \in \mathbb{R})$ is the state system at time $t \geq 0$.
- $u = \{u_t\}_{t\geq 0}$ is the control policy(an adapted $-\mathbb{F}$ process) in admissible set control $\mathcal{A} \subseteq \mathbb{R}$.

- $f : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$, $g : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times m}$ are defined as drift and diffusion functions.

In the context of the financial pair trading problem, the functions $f(X_t, t), g(X_t, t)$, and $\sigma(X_t, t)$ have the following specific interpretations. The term $f(X_t, t)$ represents the drift term related to the variable component of the wealth dynamics and the risk-free. The function $g(X_t, t)$ denotes the sensitivity of the system to the control input u_t, reflecting the investor's decision on the proportion of wealth allocated between the risky pair portfolio and the risk-free return value. Finally, $\sigma(X_t, t)$ characterizes the stochastic diffusion part of the system, corresponding to the market volatility and variable component of the wealth dynamics process. These specifications link the general stochastic control model to the financial dynamics used later in the simulation of the pair trading strategy.

Also, for a finite time horizon $0 = t_0 \le t < \infty$, the following conditions are necessary and sufficient for the existence and uniqueness of the solution of the stochastic differential equation (2.1):

$$\mid a(x,u) \mid + \mid b(x,u) \mid \le K\left(1+ \mid x \mid + \mid u \mid\right), K>0, x,y \in \mathbb{R}^n, \forall u \in \mathcal{A}, \tag{2.2}$$

$$\mid a(x,u) - a(y,u) \mid + \mid b(x,u) - b(y,u) \mid \le K \mid x-y \mid, K>0, \forall x,y \in \mathbb{R}^n, \forall u \in \mathcal{A}, \tag{2.3}$$

$$a(X_t,u_t) = f(X_t,t) + u_t g(X_t,t),\ b(X_t,u_t) = u_t\sigma(X_t,t). \tag{2.4}$$

In addition, suppose that:

$$\int_{t_0=0}^{T} \mathbb{E}\left(u_t\sigma(X_t,t)\right)^2 dt < \infty. \tag{2.5}$$

Note 1. In some stochastic risky systems, such as financial systems, robust controlling these systems in the presence of uncertainties and undesirable risks is a fundamental issue. On the other hand, using appropriate risk measurement tools that are simultaneously convex, coherent and differentiable is a fundamental challenge. In this paper, to overcome these challenges, two new types of robust stochastic controllers based on conditional sliding mode control and conditional sliding mode control with adaptive boundary layer are designed. The following table shows the main approaches and the novel ideas of this paper in overcoming the above issues and challenges.

Note 2. A complete list of the symbols and notations used throughout the paper is given in Appendix D.

3. Mean-CVaR Stochastic Sliding Mode Control (MCSSMC)

In this section, a new sliding mode function based on an conditional sliding function via Mean-CVaR is designed. The main objective is to design a stochastic sliding control law, such that the state of stochastic system (2.1) reaches to the sliding surface $s(t) = 0$. The next subsection presents an optimal estimator of the CVaR measure to find an efficient evaluation of the risk measure part of the proposed conditional sliding function.

3.1. CVaR measure estimation via linear optimization appraoch.

Since the $CVaR$ definition is explicitly involved in the VaR function but VaR is a non-convex measure. Therefore, Pflug suggested the definition of $CVaR$ through an optimization problem as follows [39]:

$$CVaR_\alpha(X) = \min_a \left\{ \frac{1}{(1-\alpha)}\mathbb{E}[a-X]^+ - a \right\}. \tag{3.1}$$

In discrete form $CVaR$ is approximated as:

$$\min_a \left\{ \frac{1}{(1-\alpha)}\mathbb{E}(a-X)^+ - a \right\} \simeq \min_a \left\{ \frac{1}{(1-\alpha)} \sum_{i=1}^{N} (a-x_i)^+ \rho(x_i) - a \right\}, \tag{3.2}$$

where, $\rho(x)$ is the probability density function of random variable X. Now, in an approximation approach for probability density function, equally likely scenarios are used. To solve the optimization problem (3.2), slack variables

TABLE 1. The novelties and contributions of this paper.

Issues and Challenges	Ideas and Novelties	Related Sections
Using a convex, coherent and differentiable risk measurement.	A new feasible linear optimization problem is designed to estimate CVaR(x) by a convex, coherent and differentiable function.	**Subsection(3.1):** Problem (3.3) and Theorem(3.1).
Eliminating the effects of uncertainties and undesirable risks.	A new weighted conditional sliding function is defined to control stochastic nonlinear risky systems according to the measures of risk and returns of the state variables.	**Subsection(3.2):** Equations (3.6)-(3.13) and Definitions (3.2, 3.3).
Robust and stable controller design.	New conditional control gain and new stochastic sliding mode control low are designed.	**Subsection (3.3):** Equation (3.16), Theorems (3.4) and (3.5).
Eliminating the chattering phenomenon of control signal.	Conditional boundary layer sliding mode controller with an adaptive boundary layer width, is designed.	**Subsection (3.4):** Theorems (3.6), Equations ((3.27),(3.28)).
Optimal estimation of weight parameter, λ, in the conditional sliding function.	A new optimal efficient index is defined, which we will call the CVaR-Sharpe ratio index.	**Subsection (4.1-4.2):** Definition (4.1), Equations ((4.10)-(4.12)), Appendix (A) and Optimization Problem (4.13).
Application and implementation.	Algorithm (4.3), Financial Risky Problem Simulation	**Subsection (4.3), Section (5), Section (6)** and **Appendices** (B and C).

$z_i, i = 1, ..., N$ are defined for replacement $(a - x_i)^+$. Therefore, in a computationally approach, the conditions $z_i \geq 0$ and $z_i \geq a - x_i$ are considered as constraints for the following newly designed optimal problems:

$$\begin{aligned} &\min_{a,z} \frac{1}{(1-\alpha)N} \sum_{i=1}^{N} z_i - a, \\ s.t \quad & z_i \geq a - x_i, \\ & z_i \geq 0, \\ & a \in \mathbb{R}. \end{aligned} \tag{3.3}$$

Finally, to calculate CVaR, the linear optimization problem (3.3) should be solved.

Reminding 3.1. According to Farkas lemma, one and only one of the following systems is feasible [2].

$$\begin{cases} AX = b, \\ X \geq 0, \end{cases} \quad (I) \qquad \begin{cases} b^T Y > 0, \\ A^T Y \leq 0. \end{cases} \quad (II) \tag{3.4}$$

In the next theorem, it will be proved that problem (3.3) has a feasible solution, based on the Farkas Lemma.

Theorem 3.1. *(Feasible solution) The linear optimization problem* (3.3) *has a feasible solution always.*

Proof. See appendix A. □

3.2. **Dynamics of the Conditional Sliding Function.** Consider the following conditional sliding function:

$$s(t) = \mathbb{E}[X_t] + \lambda CVaR(X_t). \tag{3.5}$$

Now, according to Theorem (3.1), $s(t)$ in (3.5) can be rewritten as:

$$s(t) = \mathbb{E}[X_t] + \lambda \left(\frac{1}{(1-\alpha)} \left(\mathbb{E}[a^* - X_t]^+ - a^* \right) \right), \tag{3.6}$$

where a^* is an optimal solution to problem (3.3).

Remark 3.2. In the derivation of the time variation dynamics of the sliding function, the optimal value a^* obtained from the $CVaR$ optimization problem (3.3) is considered quasi-static within each infinitesimal time interval. That is, a^* is assumed constant during differentiation but updated at discrete time steps according to the current distribution of X_t. This local stationarity assumption is commonly used in stochastic control and risk-sensitive optimization (see Rockafellar and Uryasev, 2002 [48]) to preserve analytical tractability while still capturing the time-varying nature of the $CVaR$ parameter in numerical implementation.

Since the $s(t)$, in (3.6), depends on CVaR parameters a^* and α, we called it Conditional Sliding Function. Therefore, by differentiating from the sliding function(3.6), we have:

$$\begin{aligned} ds(t) &= d\left(\mathbb{E}[X_t] + \lambda \left(\frac{1}{(1-\alpha)} \mathbb{E}[a^* - X_t]^+ - a^* \right) \right), \\ &= \mathbb{E}[dX_t] + \frac{\lambda}{1-\alpha} \mathbb{E}\left[d(a^* - X_t)^+ \right]. \end{aligned} \tag{3.7}$$

On the other hand, according to (2.1), the dynamics $(a^* - X_t)^+$ can be considered as follows:

$$d(a^* - X_t)^+ = (f((a^* - X_t)^+, t) + u_t g((a^* - X_t)^+, t))dt + u_t \sigma((a^* - X_t)^+, t) dW_t. \tag{3.8}$$

By substituting dX_t and $d(a^* - X_t)^+$ in (3.7), we have:

$$\begin{aligned} ds(t) &= \mathbb{E}\left[(f(X_t, t) + u_t g(X_t, t))\, dt + u_t \sigma(X_t, t) dW_t \right] \\ &\quad + \frac{\lambda}{(1-\alpha)} \mathbb{E}\left[\left(f((a^* - X_t)^+, t) + u_t g((a^* - X_t)^+, t) \right) dt + u_t \sigma((a^* - X_t)^+, t) dW_t \right]. \end{aligned} \tag{3.9}$$

By integrating of both sides of (3.9) on $[t, t + \triangle t]$, one can yield:

$$\begin{aligned} \triangle s(t) =& \mathbb{E}\bigg[\int_t^{t+\triangle t} \left(f(X_s, s) + u_s g(X_s, s) + \frac{\lambda}{(1-\alpha)} (f((a^* - X_s)^+, s) + u_s g((a^* - X_s)^+, s)) \right) ds \bigg] \\ &+ \mathbb{E}\left(\int_t^{t+\triangle t} u_s \left(\sigma(X_s, s) + \frac{\lambda}{(1-\alpha)} \sigma((a^* - X_s)^+, s) \right) dW_s \right). \end{aligned} \tag{3.10}$$

According to the Ito integral property, we have:

$$\mathbb{E}\left(\int_t^{t+\triangle t} \left(\sigma(X_s, s) + \frac{\lambda}{(1-\alpha)} \sigma((a^* - X_s)^+, s) \right) u_s dW_s \right) = 0. \tag{3.11}$$

Now, by dividing both sides in (3.10) by Δt when $\Delta t \to 0$, we have:

$$\dot{s}(t) = \mathbb{E}\big[f(X_t, t) + u_t g(X_t, t) + \frac{\lambda}{(1-\alpha)} \big(f((a^* - X_t)^+, t) + u_t g((a^* - X_t)^+, t) \big) \big]. \tag{3.12}$$

Let:

$$\begin{cases} \widehat{f}_{a^*} = f(X_t, t) + \frac{\lambda}{(1-\alpha)} f((a^* - X_t)^+, t) \\ \widehat{g}_{a^*} = g(X_t, t) + \frac{\lambda}{(1-\alpha)} g((a^* - X_t)^+, t). \end{cases} \tag{3.13}$$

Definition 3.3. Since the $\widehat{f}_{a^*}$ and $\widehat{g}_{a^*}$ depend on CVaR parameters a^* and α, we called it Conditional Drift and Conditional Diffusion.

According to (3.13), the equation (3.12) is rewritten as:

$$\dot{s}(t) = \mathbb{E}\left[\widehat{f}_{a^*} + u_t \widehat{g}_{a^*} \right]. \tag{3.14}$$

3.3. Mean-CVaR Stochastic Sliding Mode Control Law.

Definition 3.4. The following inequality is known as sliding condition:

$$\dot{s}sign(s) \leq -\eta, \quad \eta > 0. \tag{3.15}$$

If the sliding condition is satisfied, then all of the system's states reach to the sliding surface in a finite time [7]. According to (3.14), to satisfy the sliding condition, the control rule is defined as follows:

$$u_t = -\frac{\widehat{f}_{a^*}}{\widehat{g}_{a^*}} - \bar{K}(X_t)sign(s(t)), \tag{3.16}$$

where

$$\bar{K}(X_t) = sign(\mathbb{E}(\widehat{g}_{a^*}))K(X_t), \tag{3.17}$$

and the gain control $K(X_t)$ is a positive real function, for all $t \geq 0$.
Now, according to equation (3.16), the necessary condition for satisfying the sliding condition is presented in next theorem.

Theorem 3.5. *Consider the stochastic system* (2.1)*. If the control gain $K(X_t)$ satisfy the following inequality*

$$K(X_t, t) \geq \frac{\eta}{|\mathbb{E}(\widehat{g}_{a^*})|}, \qquad \eta > 0, \tag{3.18}$$

then the mean $-CVaR$ stochastic sliding mode controller (3.16) *guarantees the sliding condition, i.e. $\dot{s}(t)sign(s(t)) \leq -\eta$.*

Proof. Consider the sliding function (3.6) and the dynamics (3.14). Therefore:

$$\begin{aligned} sign(s(t))\dot{s}(t) &= sign(s(t))\mathbb{E}\left[\widehat{f}_{a^*} + u_t\widehat{g}_{a^*}\right] \\ &= sign(s(t))\mathbb{E}\left[\widehat{f}_{a^*} + \left(-\frac{\widehat{f}_{a^*}}{\widehat{g}_{a^*}} - \bar{K}(X_t)sign(s(t))\right)\widehat{g}_{a^*}\right]. \end{aligned} \tag{3.19}$$

If $\mathbb{E}(\widehat{g}_{a^*}) > 0$ then according to equation (3.16), $u_t = -\frac{\widehat{f}_{a^*}}{\widehat{g}_{a^*}} - K(X_t)sign(s(t))$, thus

$$sign(s(t))\dot{s}(t) = sign(s(t))\mathbb{E}\left[\widehat{f}_{a^*} + \left(-\frac{\widehat{f}_{a^*}}{\widehat{g}_{a^*}} - K(X_t)sign(s(t))\right)\widehat{g}_{a^*}\right] \tag{3.20}$$

$$= \mathbb{E}\left[-K(X_t)\widehat{g}_{a^*}\right] = -K(X_t)\mathbb{E}\left[\widehat{g}_{a^*}\right]. \tag{3.21}$$

If $\mathbb{E}(\widehat{g}_{a^*}) < 0$ then $u_t = -\frac{\widehat{f}_{a^*}}{\widehat{g}_{a^*}} + K(X_t)sign(s(t))$. Therefore:

$$sign(s(t))\dot{s}(t) = \mathbb{E}\left[K(X_t)\widehat{g}_{a^*}\right] = K(X_t)\mathbb{E}\left[\widehat{g}_{a^*}\right]. \tag{3.22}$$

According to inequality (3.18) and equations (3.20) and (3.22), one can onclude that in both cases:

$$sign(s(t))\dot{s}(t) \leq -\eta, \tag{3.23}$$

and this completes the proof. □

Theorem 3.6. *Consider the system* (2.1) *and Mean-CVaR stochastic sliding mode control law* (3.16) *where $K(X_t, t)$ is satisfied the inequality* (3.18)*, then the system*(2.1) *is asymptotically stable.*

Proof. Consider the following Lyapunov function:

$$V(t) := \frac{1}{2}s(t)^T s(t). \tag{3.24}$$

It is obvious that $V(t) \geq 0$. Now, $\frac{dV}{dt}$ will be obtained.

$$\frac{dV}{dt} = s(t)^T \dot{s}(t). \tag{3.25}$$

According to Theorem (3.5), $sign(s(t))\dot{s}(t) \leq -\eta$, therefore

$$sign(s(t)) \mid s(t) \mid \dot{s}(t) \leq -\eta \mid s(t) \mid, \tag{3.26}$$

on the other words, $s(t)\dot{s}(t) \leq -\eta \mid s(t) \mid$, thus $\frac{dV}{dt} \leq 0$ and this completes the proof. □

Due to the discontinuous nature of the sign(.) in the sliding function, sliding mode controller can lead to chattering phenomena. In the next subsection a new condotional boundary layer stochastic sliding mode control method is designed, for over coming this challege.

3.4. **Mean-CVaR Boundary Layer Stochastic Sliding Mode Control(MCBLSSMC).** Four methods are proposed to eliminate or reduce chattering: boundary layer [20], adaptive boundary layer [7], higher order SMC or HOSMC [23] and dynamic SMC or DSMC [22]. In this section, a new boundary layer stochastic sliding mode controller with an adaptive boundary layer with $\beta(t)$ is designed. For this purpose, $tanh(\beta(t)s(t))$ is considered as a saturation function in equal control law.
Consider the control rule as follow:

$$u_t = -\frac{\hat{f}_{a^*}}{\hat{g}_{a^*}} - \bar{K}(X_t) tanh(\beta(t)s(t)), \tag{3.27}$$

where $\beta(t)$ is an adaptive boundary layer width and $\bar{K}(X_t)$ is same as (3.18). In the following Theorem, the stability of the system based on the boundary layer sliding mode controller is proved.

Theorem 3.7. *Consider the system* (2.1) *and Mean-CVaR boundary layer stochastic sliding mode control law* (3.27) *where* $\bar{K}(X_t)$ *is satisfied the inequality* (3.18). *If*

$$\dot{\beta}(t) = -\xi^2 \beta(t) \mid s(t) \mid, \tag{3.28}$$

then the system (2.1) *is asymptotically stable.*

Proof. Consider the new the Lyapunov function as follows:

$$V(t) := \frac{1}{2}s(t)^T s(t) + \frac{1}{2\xi^2}\beta(t)^T \beta(t), \quad \xi > 0. \tag{3.29}$$

Clearly, $V(t) \geq 0$. Differentiating $V(t)$ with respect to time yields

$$\frac{dV}{dt} = s^T(t)\dot{s}(t) + \frac{1}{\xi^2}\beta(t)^T \dot{\beta}(t). \tag{3.30}$$

From the system dynamics and the control law, we have

$$\dot{s} = -\bar{K}(X_t) E[\hat{g}_{a^*}] \tanh(\beta s). \tag{3.31}$$

Substituting this and $\dot{\beta} = -\xi^2 \beta |s|$ into $\dot{V}$ gives

$$\dot{V} = -\bar{K}(X_t) E[\hat{g}_{a^*}]\, s \tanh(\beta s) - \beta^2 |s|. \tag{3.32}$$

Since $\text{sign}(s) = \text{sign}(\tanh(\beta s))$, therefore, $s \tanh(\beta s) = |s|\,|\tanh(\beta s)| \geq 0$. According to equations (3.18) and (3.32), one can infer: $\bar{K} \geq \eta / |E[\hat{g}_{a^*}]|$, we obtain

$$\dot{V} \leq -|s|(\eta + \beta^2). \tag{3.33}$$

Hence, $\dot{V} \leq 0$ and it is negative semi-definite. Since $\dot{V} = 0$ only when $s(t) = \beta(t) = 0$. Therefore, the closed-loop system is asymptotically stable. □

Remark 3.8. As one can see, conditional sliding function (3.6) has a weight parameter λ. In this paper, an efficient value of λ will be determined based on a new efficient frontier approach. For this purpose a new effcient frontier with slope CVaR sharp- ratio will be defined in the next section.

4. Parameter Estimation in Conditional Sliding Function via Efficient Frontier Indexes

In financial markets, there exists a portfolio with the highest Sharpe ratio and all investors want to have this portfolio in their investment. We are inspired by this for calculating an optimal value for λ in conditional sliding function (3.6). For this purpose, the conditions for the existence of the $Mean - CVaR$ efficient frontier will be presented, in the next subsections. Also, these conditions are considered as two constraints of a linear optimization problem for finding optimal value of λ. Additionally, the modified Sharpe ratio based on $CVaR$ will be defined.

4.1. **Efficient Frontier and Capital Market Line.** The efficient frontier is a set of all optimal portfolios that reach the highest return for a given level of risk or the least risk for a given amount of return. The efficient frontier is often displayed as a curve on a two-dimensional graph, where the coordinates of a plotted point correspond to the expected return and risk of portfolios. Any point below the efficient frontier is a feasible stock portfolio and any point on the efficient frontier represents an optimal stock portfolio.
Let μ_A and σ_A are the mean and standard deviation of a risky assets portfolio, respectively. Also, suppose that r_f is the interest rate of portfolio of a risk-free asset. If a combination of these two portfolios is represented by C, then the linear equation $\mu_C = \frac{(\mu_A - r_f)}{\sigma_A}\sigma_C + r_f$ is called Capital Allocation Line(CAL) [21]. The Capital Allocation Line helps investors for investing [9]. The slope of this line, i.e, $\frac{(\mu_A - r_f)}{\sigma_A}\sigma_C$ is called the Sharp ratio [44]. Obviously every investor chooses a portfolio with a maximum CAL slope or maximum Sharp ratio. The value of the Sharp ratio of a portfolio is maximized when its CAL touches the Markowitz efficient frontier. This portfolio dominates all the other portfolios on the Markowitz efficient frontier. The corresponding capital market allocation line is called Tobin frontier.

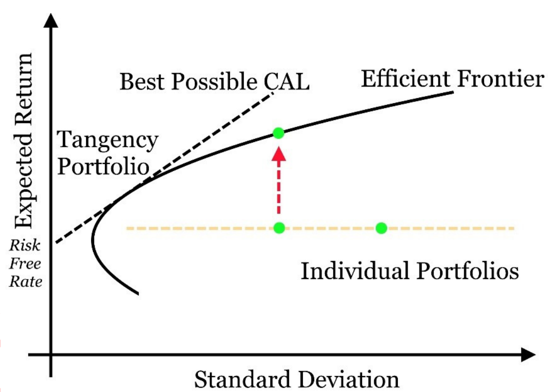

Figure 1. Efficient frontier, capital market line, and tangency portfolio.

4.2. **Optimal Sharpe Ratio based on $CVaR$ measure.** Assume that a portfolio consists of risky and risk-free securities. Suppose that two risky securities are stocks A and B, and their returns and variances are R_A, R_B, σ_A^2 and σ_B^2, respectively. Also, r_f indicates the rate of return of risk-free asset. Let π be the proportion of the wealth invested in the risky asset. Also assume that R_R and R_P are the returns of risky assets and portfolio, respectively. Then we have

$$R_P = \pi R_R + (1 - \pi) r_f. \tag{4.1}$$

Therefore

$$\mathbb{E}[R_P] = \pi \mathbb{E}[R_R] + (1 - \pi) r_f. \tag{4.2}$$

Let ω_A and ω_B, are the weights assigned to stocks A and B, respectively in the completely risky portfolio and $\omega_A + \omega_B = 1$. Thus:

$$R_R = \omega_A R_A + (1-\omega_A)R_B. \tag{4.3}$$

Therefore

$$\mathbb{E}[R_R] = \omega_A \mathbb{E}[R_A] + (1-\omega_A)\mathbb{E}[R_B]. \tag{4.4}$$

Also, we have:

$$CVaR[\alpha, R_P] = CVaR_\alpha(\pi R_R + (1-\pi)r_f) = \pi\left(CVaR[\alpha, R_R] + r_f\right) - r_f, \tag{4.5}$$

with a simple calculation, can conclude the following relation:

$$\pi = \frac{CVaR[\alpha, R_P] + r_f}{CVaR[\alpha, R_R] + r_f}. \tag{4.6}$$

By substituting (4.6) in (4.2), one can obtain:

$$\mathbb{E}[R_P] = \left(\frac{\mathbb{E}[R_R] - r_f}{CVaR[\alpha, R_R] + r_f}\right) CVaR[\alpha, R_P] + \left(\frac{\mathbb{E}[R_R] + CVaR[\alpha, R_R]}{CVaR[\alpha, R_R] + r_f}\right) r_f. \tag{4.7}$$

The traditional Sharpe ratio is defined as follows [21]:

$$SR = \frac{\mathbb{E}[R_R] - r_f}{\sqrt{\sigma_R^2}}. \tag{4.8}$$

Now, according to (4.7), a modified Sharpe ratio based on Conditional Value at Risk is defined as following.

Definition 4.1. If α, R_R and r_f are confidence level, the return of risky-assets and the return of a risk-free asset, respectively then CVaR Sharp ratio is defined as follow:

$$SR_{CVaR} = \frac{\mathbb{E}[R_R] - r_f}{CVaR[\alpha, R_R] + r_f}. \tag{4.9}$$

where $\mathbb{E}[R_R] = \omega_A \mu_A + \omega_B \mu_B$, $CVaR[\alpha, R_R] = k_\alpha \sigma[R_R] - \mathbb{E}[R_R]$ and $\omega_A + \omega_B = 1$. To guarantee the existence of efficient frontiers and the minimum-$CVaR$ portfolio, it is necessary to satisfying the following constraints:

$$\frac{\mathbb{E}[R_R] - r_f}{\sigma[R_R]} < k_\alpha, \tag{4.10}$$

$$\mathbb{E}[R_R] - r_f \geq 0, \tag{4.11}$$

$$k_\alpha > \sqrt{\frac{(\mu_A - \mu_B)^2}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}}, \tag{4.12}$$

where σ_{AB} is the covariance between the returns of two stocks A and B and $k_\alpha = -\frac{\int_{-\infty}^{-z_\alpha} x\phi(x)dx}{1-\alpha}$ (see more details in[1, 19, 31]). According to (4.9)-(4.12) an optimal problem respect to ω_A and ω_B, will be designed as follow:

$$\begin{aligned} &\max_{\omega_A} \frac{Q(\omega_A)}{k_\alpha\sqrt{H(\omega_A)} + G(\omega_A)}, \\ &\quad \frac{\mathbb{E}[R_R] - r_f}{\sigma[R_R]} < k_\alpha, \\ &\quad \mathbb{E}[R_R] - r_f \geq 0. \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} Q(\omega_A) &= \omega_A \mu_A + (1-\omega_A)\mu_B - r_f, \\ H(\omega_A) &= \omega_A^2 \sigma_A^2 + (1-\omega_A)^2 \sigma_B^2 + 2\omega_A(1-\omega_A)\sigma_{AB}, \\ G(\omega_A) &= -\left(\omega_A \mu_A + (1-\omega_A)\mu_B\right) + r_f. \end{aligned}$$

By solving the optimization problem (4.13), the maximum of SR_{CVaR} can be designed and we will set $\lambda = maxSR_{CVaR}$ in (3.5).

4.3. **Mean-CVaR Stochastic Sliding Mode Control Design Algorithm.** In this subsection, to implement facilitation of the proposed methods, a new algorithm is obtained:

Algorithm 1 Mean-CVaR stochastic sliding mode control design algorithm.

Start
Step1. Consider stochastic system dynamics (2.1) for the given parameters α, η, $\beta(1)$ and ξ.
Step2. Solve the optimal problem (4.13) and determine optimal λ.
Repeat the following steps for $t \in (0, T]$:
Step3. Put $Y_t = log(X_{t+1}/X_t)$.
Step4. According to (2.1) and (A.1) calculate $CVaR_\alpha(Y_t)$ and $z = (a^* - Y_t)^+$.
Step5. Set $\widehat{f}_{a^*} = f(Y_t, t) + \frac{\lambda}{(1-\alpha)} f((a^* - Y_t)^+, t)$ and $\widehat{g}_{a^*} = g(Y_t, t) + \frac{\lambda}{(1-\alpha)} g((a^* - Y_t)^+, t)$.
Step6. Compute $s(t)$ by using (3.6).
Step7. Choose $K(Y_t, t) \geq \dfrac{\eta}{|\mathbb{E}[\widehat{g}_{a^*}]|}$.
Step8. Compute $\beta(t)$ from (3.28).
Step9. Calculate u_t by using (3.16) for implementing MCSSMC method or (3.27) for implementing MCBLSSMC method.
End

Remark on the determination of λ: As stated in Step 2 of the proposed algorithm, λ is obtained from the optimization problem (4.13). Solving (4.13) yields two optimal quantities: (i) the optimal portfolio weights (ω_A^*, ω_B^*) defining the tangency point on the Mean-CVaR efficient frontier, and (ii) the maximum value of the objective function SR^*_{CVaR}. According to solutions of this optimization problem directly the weighting parameter λ in the sliding surface is assigned as follows:

$$\lambda = SR^*_{\text{CVaR}} = \max SR_{\text{CVaR}}.$$

Hence, the sliding function coefficient λ numerically reflects the optimal trade-off between return and conditional risk, ensuring that the control law inherits the same efficiency condition as the portfolio optimization model.

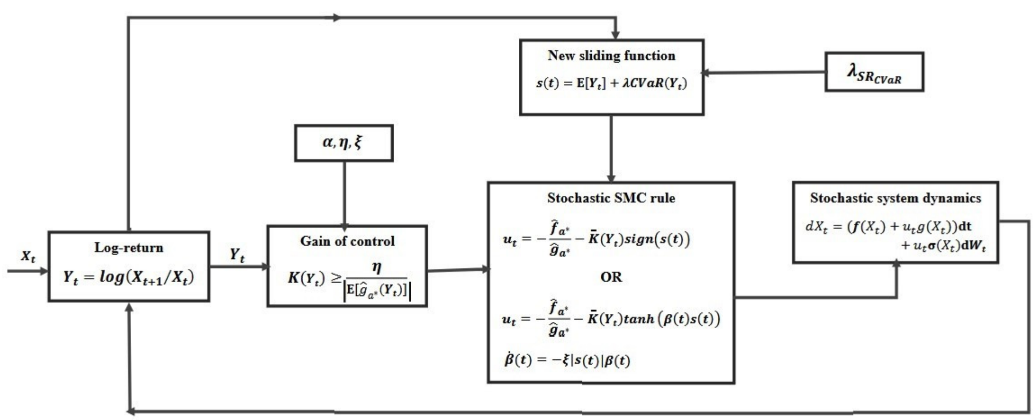

FIGURE 2. Diagram of Mean-CVaR Stochastic Sliding Mode Controllers (MCSSMC and MCBLSSMC).

In Figure 2, the diagram of the suggested controllers performance is presented.
For illustrating the advantages of the proposed method a simulation example is presented in the next section.

5. Simulation Example

Consider the wealth dynamics equation [33]:

$$dX(t) = \left(r_f X(t) + u_t(\kappa(\theta - Spr(t)) + \frac{1}{2}\gamma^2 + \rho\sigma\gamma)X(t)\right) dt + \gamma u(t)X(t)dW(t). \tag{5.1}$$

In this section, the objective is the control of the asset allocation problem for the pairs trading portfolio by using the stochastic sliding mode control methods, presented in this paper. Let u_t and X_t denote control policy and state of the financial system (5.1), respectively. The parameters of the problem are selected as follows:

$$\begin{aligned}
&T = 1, \quad dt = \frac{1}{251}, \quad \mu = 0.3, \quad \sigma = 0.1, \quad r_f = 0.05\\
&A_0 = 100, \quad B_0 = 50, \quad \alpha = 0.95, \ \lambda_{SR_{CVaR}} = 1.6996\\
&\theta = 1, \ \kappa = 5, \ \gamma = 0.5, \ \eta = 0.5, \ \rho = 0.19.
\end{aligned} \tag{5.2}$$

According to Appendix A, the dynamics equation for log-return is:

$$dY(t) = \left(r_f + u_t(\kappa(\theta - Spr(t)) + \frac{1}{2}\gamma^2 + \rho\sigma\gamma)\right) dt + \gamma u(t)dW(t). \tag{5.3}$$

Now, by comparing the dynamics of (2.1) and (5.3) and also according to (3.13), $\widehat{f}_{a^*}$ and $\widehat{g}_{a^*}$ are defined as follows:

$$\widehat{f}_{a^*} = \left(1 + \frac{\lambda}{1-\alpha}\right) r_f, \tag{5.4}$$

$$\widehat{g}_{a^*} = \left(1 + \frac{\lambda}{1-\alpha}\right)\left(\kappa(\theta - spr(t)) + \frac{\gamma^2}{2} + \rho\sigma\gamma\right). \tag{5.5}$$

Since $Spr_0 < \theta$, according to the Appendix C, $\mathbb{E}[\widehat{g}_{a^*}]$ is a positive number, always. Now, according to $MCSSMC$ and $MCBLSSMC$ methods and the theorems (3.5) and (3.7), the control signals are expressed as follows:

$$u_t = -\frac{r_f}{\left(\kappa(\theta - spr(t)) + \frac{\gamma^2}{2} + \rho\sigma\gamma\right)} - \bar{K}(Y_t)sign(s(t)), \tag{5.6}$$

$$u_t = -\frac{r_f}{\left(\kappa(\theta - spr(t)) + \frac{\gamma^2}{2} + \rho\sigma\gamma\right)} - \bar{K}(Y_t)tanh(\beta(t)s(t)). \tag{5.7}$$

According to (3.6) and (4.7), $\lambda = \lambda_{SR_{CVaR}}$. It is worth noting that the dynamics of(5.1) is based on the wealth. Now, based on the presented methods and Algorithm1 this problem is simulated (see Appendix A). First, according to the dynamics of (A-2) to (A-4), the stock price processes A and B and the spread are simulated during the one trading year (251 days) are simulated. The results are shown in Figure 3.

Let $X_0 = 1000\$$, the simulation results of the wealth evaluation and MCSSMC policy are shown in Figures 4(a) and 4(b), for a given sample path. As one can see, the value of the portfolio is increasing. Also, for this sample path, it increases from 1000$ to 1445.782$ in one year, which represents a return of 44.5782%(with risk 29.5885%). While, investment in the risk-free asset has only a 5% interest rate. The proposed $MCSSMC$ pairs trading control policy is simulated in Figure 4.b. According to the control signal, the proportion of wealth that is allocated to investment in stocks is determined at any time.

Figure 5.a shows that the suggested control (5.6) guarantees reaching behavior of the proposed sliding function. The conditional gain control by the $Mean - CVaR$ stochastic SMC method is shown in Figure 5(b). The proposed MCBLSSMC pairs trading control policy and sliding function are simulated in Figures (6(a) and 6(b)). According to these simulation results the proposed controller (5.7) guarantees the reaching sliding function reaching behavior without any chattering phenomenon in control signal (return and risk are 52.6525% and 30.9793%, respectively). The proposed MCBLSSMC pairs trading policy is simulated in Figure 6(a). As can be seen, the control signal determine the proportions of wealth that are allocated to investment in stocks, without any chattering. Figure 6(b) shows that the suggested control guarantees the sliding function reaching behavior. Now, (5.7) two proposed MCSSMC and

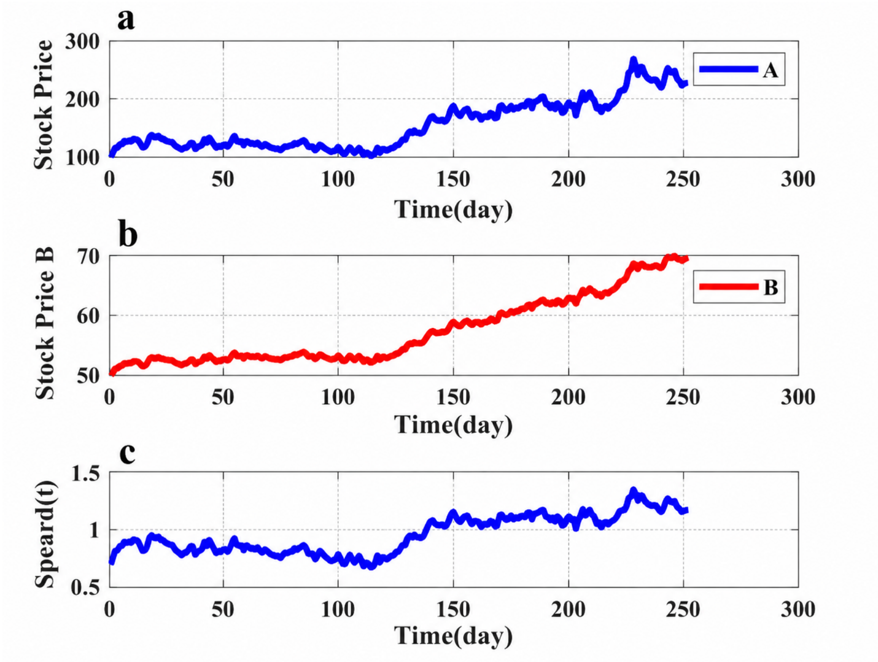

FIGURE 3. Time series of stocks A, B and spread between stocks.

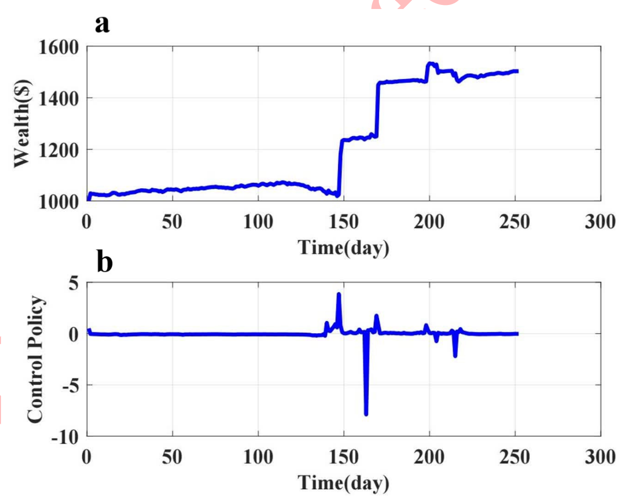

FIGURE 4. Wealth evolution and Pairs-trading strategy by the MCSSMC method.

MCBLSSMC methods can be compared with themselves and also with the stochastic optimal control (SOC) in [33]. For this purpose, in MCBLSSMC and SOC methods the values of parameters are selected similarly to (5.2) and also let, $(\beta(1) = 40, \xi = 45, \eta = 65)$. Pairs-trading strategy and wealth evolution by stochastic optimal control method [33] are shown in Figures 7(a) and 7(b), respectively. Figure 7(a) shows that the value of wealth increases from 1000\$ to 1380\$, which represents a return of 38%. In Table 2, the performances comparison of the proposed MCSSMC, MCBLSMC methods and SOC are presented. As can be seen, the return performances of MCSSMC and MCBLSSMC

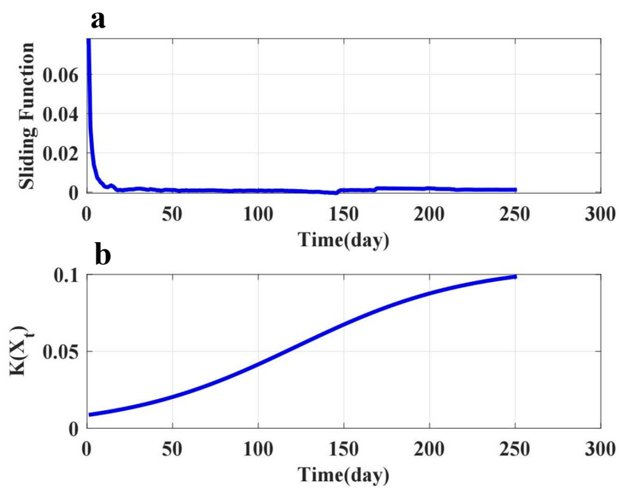

FIGURE 5. The sliding function and conditional gain by the MCSSMC method.

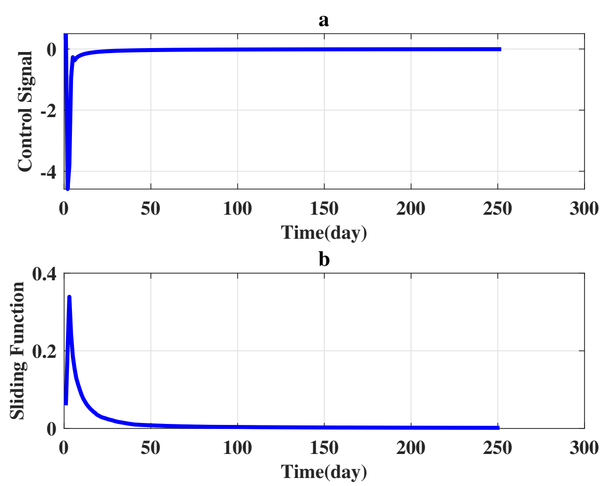

FIGURE 6. Control signal and $s(t)$ by the MCBLSSMC method.

methods are better than SOC method, but the control amplitude of the MCSSMC is bigger than the control amplitude of the SOC method. See details in Table 2.

TABLE 2. Performances comparison of the proposed methods (MCSSMC and MCBLSSMC) with SOC in [33].

Method	Amplitude of Control Signal	Return of Wealth
SOC [33]	[-4.6987,0.5297]	38 %
MCSSMC	[-8.0342,3.9851]	44.5782%
MCBLSSMC	[-4.5795,0.50]	52.6525 %

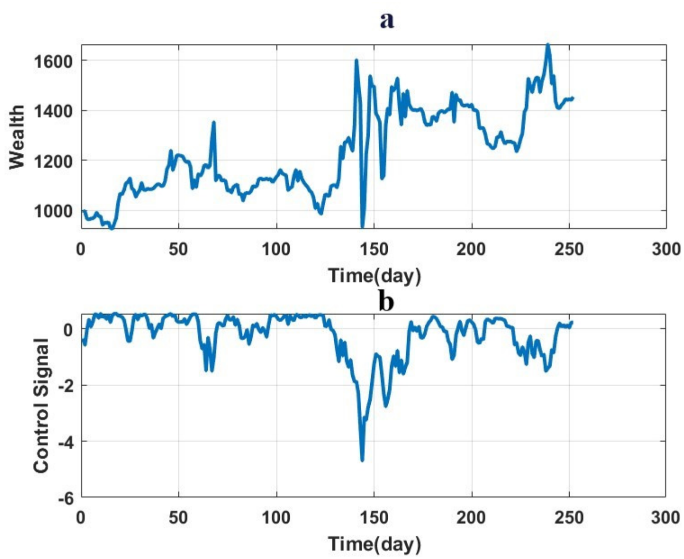

FIGURE 7. Wealth evolution and Pairs-trading strategy by the SOC method [33].

TABLE 3. Estimated values of pair trading parameters for Behshahr and Margarine.

Parameter	μ	σ	κ	θ	γ	ρ
Estimated value	1.1504	0.9714	1.2417	1.0326	1.0046	-0.9346

TABLE 4. Performances comparison of the proposed methods MCSSMC and MCBLSSMC.

Method	Amplitude of Control Signal	Return of Wealth
MCSSMC	[-0.2899, 3.1914]	20.6120%
MCBLSSMC	[-0.9612 , 0.5000]	20.9648 %

6. Case study: application of stochastic sliding mode control to pair trading in the Iranian stock market

In this section, we examine the effectiveness of the stochastic sliding mode control approach for pair trading using real data from two assets. The data pertain to two Iranian companies in the food and beverage industry, excluding the sugar sector. The companies under study are Behshahr Industrial Company (symbol: G-beshahr) and Market Margarine Company (symbol: G-marg). Daily data for these companies were used for the period from March 27, 2010, to April 14, 2013 [16]. As indicated in [33], the model parameters were estimated using the Maximum Likelihood Estimation (MLE) method, and their values are presented in Table 3. Using the parameters in Table 3 and assuming $r_f = 0.18(\mu_A = 0.7531, \sigma_A = 0.5349, \sigma_{AB} = 0.0948)$, the optimal value of λ is first calculated according to the procedure described in Section 4.2 ($\lambda_{SR_{CVaR}} = 0.8782$). Subsequently, the path of the stock pair, spread, portfolio strategy, and wealth dynamics are simulated over a six-month period from April 14, 2013, to October 18, 2013. The corresponding trends are illustrated in Figure 8.

Figures 8-12 illustrate the time series of stock prices, spread dynamics, wealth evolution and control signals over a six-month testing period time (April 2013-October 2013). As can be seen, both controllers exhibit stable and consistent performance under real market conditions: the sliding function converges toward zero, and the control signal in the MCBLSSMC method remains smooth without chattering. Numerical results in Table 4 confirm that

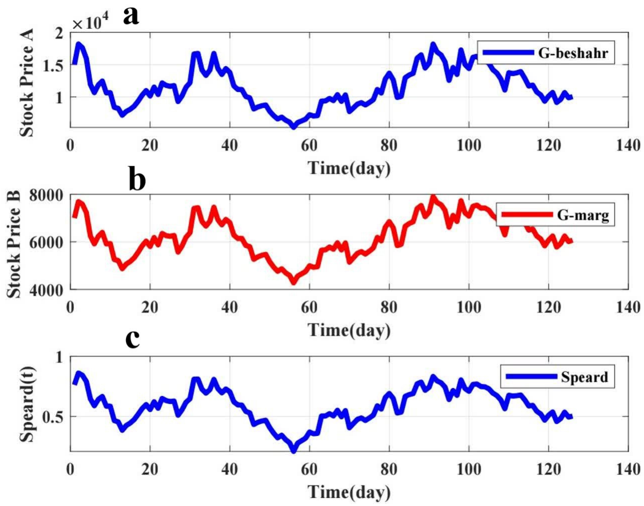

FIGURE 8. Time series of stocks G-beshahr, G-marg, and spread between stocks.

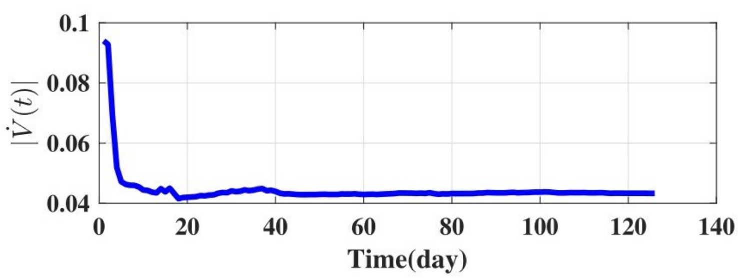

FIGURE 9. $|\dot{V}(t)|$ by the MCSSMC method.

both controllers effectively increase the portfolio wealth (by 20.61% and 20.96%, respectively) while maintaining small control amplitudes. These findings demonstrate that the proposed control framework maintains robustness and efficiency when applied to real financial data.

7. Conclusion

This paper proposed two optimal stochastic robust controllers, namely MCSSMC and MCBLSSMC, based on a Mean–CVaR conditional sliding function for portfolio optimization. By embedding a coherent and differentiable CVaR measure into the sliding mode control framework, the proposed approach effectively tunes a trade off between return maximization and risk control task in stochastic financial systems. Numerical simulations demonstrated that both controllers outperform the benchmark stochastic optimal control method in terms of portfolio growth and risk reduction. In particular, the boundary layer controller (MCBLSSMC) significantly mitigates control chattering while achieving higher returns with smaller control amplitudes. To further validate the practical applicability of the proposed framework, a real-market case study based on historical data from the Iranian stock market was conducted. The empirical results confirm that the proposed controllers maintain stability, robustness, and performance under real trading conditions, leading to consistent wealth growth without excessive control effort. Future research directions include extending the method to multi-asset portfolios, incorporating transaction costs and market frictions, and developing adaptive or learning-based mechanisms for online parameter tuning in high frequency financial market.

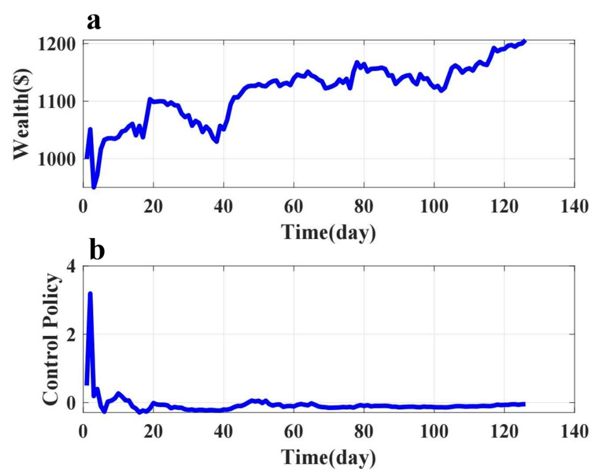

Figure 10. Wealth evolution and Pairs-trading strategy by the MCSSMC method.

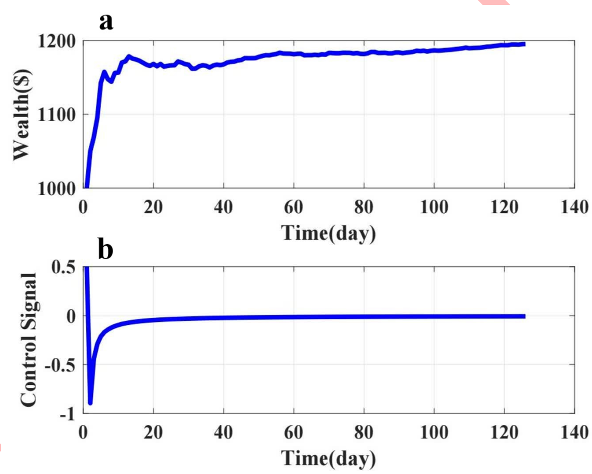

Figure 11. Wealth evolution and Pairs-trading strategy by the MCBLSSMC method.

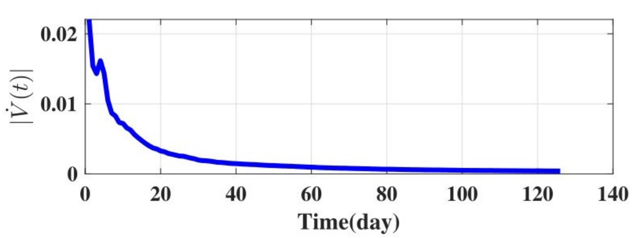

Figure 12. $|\dot{V}(t)|$ by the MCBLSSMC method

Appendix A. Proof of Theorem 3.1

The linear optimization problem (3.3) can be modified to a standard form, as follows. Let $a = a^+ - a^-$, therefore

$$
\begin{aligned}
&\min_{z,a^+,a^-,s} \ \frac{1}{(1-\alpha)N}\sum_{i=1}^{N} z_i - a^+ + a^-,\\
&z_1 + 0z_2 + \cdots + 0z_N - a^+ + a^- - s_1 + 0s_2 + \cdots + 0s_N = -x_1,\\
&0z_1 + z_2 + \cdots + 0z_N - a^+ + a^- + 0s_1 - s_2 + \cdots + 0s_N = -x_2,\\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
&0z_1 + 0z_2 + \cdots + z_N - a^+ + a^- + 0s_1 + 0s_2 + \cdots - s_N = -x_N.
\end{aligned} \tag{A.1}
$$

Based on the (A.1), matrix A and vectors X, Y, and $\mathbf{b}$ are:

$$
A = \begin{bmatrix}
1 & 0 & \cdots & 0 & -1 & 1 & -1 & 0 & \cdots & 0\\
0 & 1 & \cdots & 0 & -1 & 1 & 0 & -1 & \cdots & 0\\
0 & 0 & \cdots & 0 & -1 & 1 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & 1 & -1 & 1 & 0 & 0 & \cdots & -1
\end{bmatrix}_{N\times(2N+2)}, X = \begin{bmatrix} z_1\\ z_2\\ \vdots\\ z_N\\ a^+\\ a^-\\ s_1\\ s_2\\ \vdots\\ s_N \end{bmatrix}_{(2N+2)}, Y = \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_N \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -x_1\\ -x_2\\ \vdots\\ -x_N \end{bmatrix}.
$$

Therefore, the problem (A.1) is in the form (3.4-I). Suppose the system (3.4-II) has a feasible solution. Thus, the following relations are inferred:

$$-\left(x_1y_1 + x_2y_2 + \cdots + x_Ny_N\right) > 0, \tag{A.2}$$

$$y_i \le 0, \quad i = 1, 2, ..., N, \tag{A.3}$$

$$-\left(y_1 + y_2 + \cdots + y_N\right) \le 0, \tag{A.4}$$

$$\left(y_1 + y_2 + \cdots + y_N\right) \le 0, \tag{A.5}$$

$$-y_i \le 0, \quad i = 1, 2, ..., N. \tag{A.6}$$

Considering inequalities (A.2)-(A.6) together, the set of feasible solutions for the linear programming (3.4-II) becomes empty. Hence, according to Farkas lemma, (3.4-I) is feasible.

Appendix B. Dynamics of Paired Stock Prices, Spread and Wealth Value

Let $M(t)$ be a risk-free asset rate of return r_f. Therefore:

$$dM(t) = r_f M(t)dt. \tag{A-1}$$

Also, let $A(t)$ and $B(t)$ be the prices of the stock pair A and B at time t, respectively. Assume that, the stock price of B follows a geometric Brownian motion. Therefore

$$dB(t) = \mu B(t)dt + \sigma B(t)dZ(t). \tag{A-2}$$

Where $\{Z(t)\}$ is a standard Brownian motion and μ and σ are drift and diffusion of stock B, respectively. Let Spr denote the price spread between stocks A and B at time t, which is defined as follows:

$$Spr(t) = ln(A(t)) - ln(B(t)). \tag{A-3}$$

Assume that, the spread follows an Ornstein-Uhlenbeck process of the following form [33, 53]:

$$dSpr(t) = \kappa(\theta - Spr(t))dt + \gamma dW(t). \tag{A-4}$$

Where $\mathbb{E}[Spr(t)] = \theta$ is the equilibrium level which the process returns. Notic that $\kappa > 0$, $\gamma > 0$ and $\{W(t)\}$ are the rate of the mean- reversion, the volatility of the price spread and a standard Brownian motion different from $\{Z(t)\}$, respectively. Let $X(t)$ be the value of a self-financing pairs-trading portfolio. According to [33], we have:

$$dX(t) = X(t)\left\{\left(u(t)\left(\kappa(\theta - Spr(t)) + \frac{1}{2}\gamma^2 + \rho\sigma\gamma\right) + r_f\right)dt. + \gamma u(t)dW(t)\right\}. \tag{A-5}$$

Where $u(t)$ and $\widehat{u}(t) = 1 - u(t)$ denote the portfolio weights for stock A and B at time t, respectively. Now, we set $dY(t) = \frac{dX(t)}{X(t)}$. Therefore, we have:

$$dY(t) = \left\{\left(u(t)\left(\kappa(\theta - Spr(t)) + \frac{1}{2}\gamma^2 + \rho\sigma\gamma\right) + r_f\right)dt + \gamma u(t)dW(t)\right\}. \tag{A-6}$$

Appendix C. Ornstein-Uhlenbeck Stochastic Process

The Ornstein-Uhlenbeck stochastic process is described as follows:

$$dSpr(t) = \kappa(\theta - Spr(t))dt + \gamma dW(t). \tag{B-1}$$

$Spr(t)$: The value of the process at time t.
κ : The rate of reversion to the mean (speed of mean reversion).
θ : The long-term mean towards which the process reverts.
$\gamma > 0$: The volatility (standard deviation of the process).
dW: A Wiener process or Brownian motion, representing the stochastic component.

Expected Value Calculation of Ornstein-Uhlenbeck process

To determine the expected value of the OU process, we can consider the following:

$$F(spr, t) = e^{\kappa t}spr, \quad F_{spr} = e^{\kappa t}, \quad F_t = \kappa e^{\kappa t}spr.$$

Now, we apply Ito's formula:

$$dF = d(e^{\kappa t}spr) = e^{\kappa t}[\kappa\theta dt] + \gamma e^{\kappa t}dW_t. \tag{B-2}$$

By integrating of (B-2), we have:

$$e^{\kappa t}Spr(t) = Spr_0 + \theta(e^{\kappa t} - 1) + \gamma\int_0^t e^{\kappa s}dW_s. \tag{B-3}$$

therefore:

$$\mathbb{E}[Spr(t)] = Spr(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}). \tag{B-4}$$

Now, we consider $\mathbb{E}(\widehat{g}_{a^*}) = \theta - \mathbb{E}[Spr(t)]$. By substituting the expression for $E[Spr(t)]$:

$$\mathbb{E}(\widehat{g}_{a^*}) = (\theta - Spr(0))e^{-\kappa t}. \tag{B-5}$$

Interpretation of $g(t)$

From this expression, three cases may occur:

- When $Spr(0) < \theta$, then $\mathbb{E}(\widehat{g}_{a^*})$ will be positive.
- When $Spr(0) = \theta$, there is no difference between initial value and the mean.
- When $Spr(0) > \theta$, then $\mathbb{E}(\widehat{g}_{a^*})$ will be negative.

Appendix D. List of Symbols

For the convenience of the reader, the principal symbols and abbreviations used in this paper are summarized in Table 5.

TABLE 5. List of symbols used in the paper.

Symbol	Definition
Ω	Sample space of the probability space.
$\mathcal{F}$	Sigma-algebra associated with the probability space.
$\{\mathcal{F}_t\}_{t\geq 0}$	Filtration representing the evolution of available information over time.
P	Probability measure defined on $(\Omega, \mathcal{F})$.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$	Complete filtered probability space.
$W(t) = (W_1(t), \ldots, W_m(t))^T$	m-dimensional Brownian motion defined on a filtered probability space.
$X_t \in \mathbb{R}^n$	State variable (state process) of the stochastic dynamic system at time t.
u_t	Control input at time t.
$u = \{u_t\}_{t\geq 0}$	Admissible control policy.
$A \subseteq \mathbb{R}$	Admissible control set.
$f(X_t, t)$	Drift function of the system dynamics.
$g(X_t, t)$	Control gain/sensitivity function.
$\sigma(X_t, t)$	Diffusion coefficient representing stochastic volatility.
$a(X_t, u_t)$	Drift coefficient defined by $a(X_t, u_t) = f(X_t, t) + u_t g(X_t, t)$.
$b(X_t, u_t)$	Diffusion coefficient defined by $b(X_t, u_t) = u_t\sigma(X_t, t)$.
R_R	Return of the risky portfolio.
R_P	Return of the total portfolio.
$R_A,\ R_B$	Returns of assets A and B, respectively.
$\omega_A,\ \omega_B$	Portfolio weights of assets A and B, respectively.
$\mu_A,\ \mu_B$	Expected returns of assets A and B, respectively.
r_f	Risk-free interest rate (risk-free return).
π	Allocation weight between risky and risk-free assets.
σ_{AB}	Covariance between the returns of assets A and B.
$\sigma[R_R]$	Standard deviation of the risky portfolio return.
$E[\cdot]$	Expectation operator.
α	Confidence level used in the CVaR measure.
$\mathrm{CVaR}[\alpha, R_R]$	Conditional Value-at-Risk of the risky portfolio return.
$\mathrm{CVaR}[\alpha, R_P]$	Conditional Value-at-Risk of the total portfolio return.
k_α	Constant coefficient in the analytical representation of CVaR.
z_α	α-quantile of the standard normal distribution.
$\phi(x)$	Probability density function of the standard normal distribution.
SR	Traditional Sharpe ratio.
SR_{CVaR}	CVaR-based Sharpe ratio.

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