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Applications of invariant subspace method for solving a coupled KdV system obtained from two-layer fluids

Gaizhu Qu[1,*], Gangwei Wang[2,3,*], and Jina Li[4]

[1]School of Mathematics and Physics, Weinan Normal University, Weinan 714000, China.

[2]School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China.

[3]Center for Differential Equations and Their Applications, Hebei University of Economics and Business, Shijiazhuang 050061, China.

[4]College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China.

Abstract

In this work, a general coupled KdV system obtained from the two-layer fluid dynamic systems are investigated. The governing system and the corresponding invariant subspaces are displayed by adopting the invariant subspace method (ISM). The key point of ISM is to transform coupled partial differential equations (PDEs) into a system of ordinary differential equations (ODEs). By solving the obtained ODE system, the new explicit solutions of the coupled KdV system are derived accordingly. Finally, the figures of several solutions are shown.

Keywords. Coupled KdV system, Invariant subspace, Exponential function, Exact solutions.

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1. Introduction

Nonlinear partial differential equations (NPDEs) [1, 20] are increasingly important because they can always describe a large number of nonlinear phenomena. The exploration of exact solutions of NPDEs has always received considerable attention. And during the past decades, many effective methods have been proposed for solving NPDEs like Lie group method [2, 4, 7, 23, 25], nonlocal symmetry method [24], the tanh method [26], the conditional Lie-Bäcklund symmetry [3, 15, 16, 28], the invariant subspace method (ISM) [6, 12, 17, 22, 29] and so on.

ISM is very powerful in deriving various solutions of scalar or coupled system of NPDEs. Galationov and Svirshchevskii are the first scholars to propose the idea of ISM. More detailed descriptions for this method can be found in their book [6]. Subsequently, numerous works on the applications of ISM have appeared. Recently, this method has also been effectively applied to fractional nonlinear partial differential equations (fNPDEs) [5, 11, 13, 14, 18, 19, 21]. These results are enough to prove the effectiveness of ISM from both the classification perspective and application analysis. Nonlinear coupled dynamical systems are widely present in fields such as physics, biology, and engineering due to their characteristics of nonlinearity and coupling [8, 27].

The main attentio of the current article is to extend ISM to investigate a general coupled KdV system

$$\begin{aligned} &u_t + p_1 v u_x + (p_2 v^2 + p_3 uv + p_4 u_{xx} + p_5 u^2)_x = 0,\\ &v_t + q_1 v u_x + (q_2 u^2 + q_3 uv + q_4 v_{xx} + q_5 v^2)_x \ = 0, \end{aligned} \tag{1.1}$$

where $p_1, p_2, p_3, p_4, p_5, p_6, q_1, q_2, q_3, q_4, q_5, q_6$ are constants. System (1.1) describes a two layer model of the atmospheric dynamical [9]. More recently, researchers have shown much interest in the investigation of (1.1) or some special forms of (1.1) [9, 10]. The authors used the Painlevé to study the integrability and achieved the τ-function and multiple

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∗ Corresponding author. Email: gangwei@hueb.edu.cn.

soliton solutions of (1.1) [9]. If we take $p_1 = p_3 = 0, p_4 = 0, p_2 = -p_5 = 3$ and $q_1 = q_2 = q_5 = 0, q_3 = -6, q_4 = 1$ in system (1.1), then it becomes the coupled KdV system

$$\begin{aligned} &u_t + 6vv_x - 6uu_x + u_{xxx} = 0,\\ &v_t - 6uv_x - 6vu_x + v_{xxx} = 0. \end{aligned} \tag{1.2}$$

Authors obtained analytical and nonsingular complexition solutions of system (1.2) by applying Darboux transformations [10].

In Section 2, we give a short review of ISM. In Section 3, we perform the complete classification of ISs to system (1.1) by utilizing this method. With the help of the results given in Section 3, we will discuss several new explicit solutions of the system (1.1) in Section 4. In the final section, a short summary of the work is presented.

2. Summary of ISM

In this section, we would like to give a brief account of ISM to a nonlinear system. The nonlinear system

$$\begin{aligned} &u_t = G_1\left(x, u, v, u^{(1)}, v^{(1)}, \cdots, u^{(k_1)}, v^{(k_2)}\right),\\ &v_t = G_2\left(x, u, v, u^{(1)}, v^{(1)}, \cdots, u^{(k_1)}, v^{(k_2)}\right), \end{aligned} \tag{2.1}$$

where $u^{(i)} = \frac{\partial^i u(x,t)}{\partial x^i}, v^{(j)} = \frac{\partial^j v(x,t)}{\partial x^j} (i = 1, 2, \cdots, k_1, j = 1, 2, \cdots, k_2)$.

A novel subspace $\mathcal{W}$ is $\mathcal{W} = W^1_{m_1} \times W^2_{m_2}$, where

$$W^1_{m_1} = \mathcal{L}\{f_1(x), \cdots, f_{m_1}(x)\} \equiv \left\{\sum_{j=1}^{m_1} C_j f_j(x)\right\}, W^2_{m_2} = \mathcal{L}\{g_1(x), \cdots, g_{m_2}(x)\} \equiv \left\{\sum_{j=1}^{m_2} D_j g_j(x)\right\}.$$

$f_1(x), \cdots, f_{m_1}(x)$ are linearly independent and $g_1(x), \cdots, g_{m_2}(x)$ are linearly independent too. If $\mathbb{G} = (G_1, G_2)$ fulfills $W^1_{m_1} \times W^2_{m_2} \to W^1_{m_1} \times W^2_{m_2}$, namely,

$$\begin{aligned} &G_1\left[\sum_{i=1}^{m_1} C_i f_i(x), \sum_{j=1}^{m_2} D_j g_j(x)\right] = \sum_{i=1}^{m_1} \Phi(C_1, \cdots, C_{m_1}, D_1, \cdots, D_{m_2}) f_i(x),\\ &G_2\left[\sum_{i=1}^{m_1} C_i f_i(x), \sum_{j=1}^{m_2} D_j g_j(x)\right] = \sum_{j=1}^{m_2} \Psi(C_1, \cdots, C_{m_1}, D_1, \cdots, D_{m_2}) g_j(x), \end{aligned}$$

then $\mathbb{G}$ is said to admit the subspaces $W^1_{m_1} \times W^2_{m_2}$. Then the solution of system (2.1) is of the following:

$$u(x,t) = \sum_{i=1}^{m_1} C_i(t) f_i(x), v(x,t) = \sum_{j=1}^{m_2} D_j(t) g_j(x).$$

where $C_i(t)$ and $D_j(t)$ satisfy the following system of ODEs

$$\begin{aligned} &\frac{\mathrm{d}C_i(t)}{\mathrm{d}t} = \Phi_i\left(C_1(t), \cdots, C_{m_1}(t), D_1(t), \cdots, D_{m_2}(t)\right),\\ &\frac{\mathrm{d}D_j(t)}{\mathrm{d}t} = \Psi_j\left(C_1(t), \cdots, C_{m_1}(t), D_1(t), \cdots, D_{m_2}(t)\right). \end{aligned}$$

Generally, $W^1_{m_1}, W^2_{m_2}$ is generated by

$$\begin{aligned} &L^1[y] = y^{(m_1)} + a_{m_1-1} y^{(m_1-1)} + \cdots + a_1 y' + a_0 y = 0,\\ &L^2[z] = z^{(m_2)} + b_{m_2-1} z^{(m_2-1)} + \cdots + b_1 z' + b_0 z = 0. \end{aligned} \tag{2.2}$$

If the maximal dimension of ISs is evaluated, a complete classification and a set of explicit solutions of the governing system can be easily derived. And More detailed results of the maximal dimension theorems can be found in Refs. [17, 29].

3. Invariant subspace classification of (1.1)

In this section, we perform all the invariant subspace classification of (1.1). Firstly, we need to rewrite system (1.1) in the following form

$$
\begin{aligned}
&u_t = -p_1 v u_x - (p_2 v^2 + p_3 uv + p_4 u_{xx} + p_5 u^2)_x \equiv G_1[u,v],\\
&v_t = -q_1 v u_x - (q_2 u^2 + q_3 uv + q_4 v_{xx} + q_5 v^2)_x \equiv G_2[u,v],
\end{aligned} \tag{3.1}
$$

the operator (G_1, G_2) preserves an invariant subspace $W^1_{m_1} \times W^2_{m_2}$ determined by the following ODEs

$$
\begin{aligned}
&L^1[y] = y^{(m_1)} + a_{m_1-1} y^{(m_1-1)} + \cdots + a_1 y' + a_0 y = 0,\\
&L^2[z] = z^{(m_2)} + b_{m_2-1} z^{(m_2-1)} + \cdots + b_1 z' + b_0 z = 0.
\end{aligned} \tag{3.2}
$$

With the help of the results on maximal dimensions of invariant subspaces [12], we can see that the following cases for (m_1, m_2) should be discussed respectively: (2,2),(3,2),(3,3),(4,2),(4,3). We first discuss the case of $(m_1, m_2) = (2, 2)$. In this case, we have the invariant condition

$$
\begin{aligned}
&(D^2 G_1 + a_1 D G_1 + a_0 G_1)_{M_1 \cap M_2} = 0,\\
&(D^2 G_2 + b_1 D G_2 + b_0 G_2)_{M_1 \cap M_2} = 0,
\end{aligned} \tag{3.3}
$$

where M_1, M_2 indicate that the equations $L^1[u] = 0, L^2[v] = 0$ and their differential consequences with respect to variable x, a_0, a_1, b_0, b_1 are determined constants.

Substituting these expressions for G_1, G_2 in (3.1) into (3.3) and replacing u_{xx}, v_{xx} by $-a_1 u_x - a_0 u, -b_1 v_x - b_0 v$ respectively, we acquire the following results:

$$
\begin{aligned}
&[(a_1 + b_1)p_1 + (a_1 + 3b_1)p_3]u_x v_x + 4a_1 p_5 u_x^2 - 2(a_1 - 3b_1)p_2 v_x^2 + 6a_0 p_5 u u_x + b_0(p_1 + 3p_3) v u_x\\
&[2a_0(p_1 + p_3) + p_3(b_0 + a_1 b_1 - b_1^2)]u v_x + 2(a_1 b_1 - a_0 + 4b_0 - b_1^2)p_2 v v_x + (a_1 - b_1) b_0 p_3 uv\\
&+ 2(a_1 - b_1) b_0 p_2 v^2 = 0,\\
&[2a_1 q_1 + (3a_1 + b_1)q_3]u_x v_x + 2(3a_1 - b_1)q_2 u_x^2 + 4b_1 q_5 v_x^2 + 2(a_1 b_1 + 4a_0 - b_0 - a_1^2)q_2 u u_x\\
&+ [(a_0 + a_1 b_1 - a_1^2)q_1 + (a_0 + 2b_0 + a_1 b_1 - a_1^2)q_3] v u_x + a_0(2q_1 + 3q_3) u v_x + 6b_0 q_5 v v_x\\
&- a_0(a_1 - b_1)(q_1 + q_3)uv - 2(a_1 - b_1)a_0 q_2 u^2 = 0.
\end{aligned}
$$

Taking the coefficients of u_x, v_x, u, v to be zero and solving the algebraic system for $a_0, a_1, b_0, b_1, p_i, q_i (i = 1, 2, 3, 4, 5)$, then we obtain all the classifying coupled KdV system, the corresponding linear ODEs, and the invariant subspaces $W^1_2 \times W^2_2$ listed in Table 1, where eight functions $f^s_{r_1}(x), g^s_{r_2}(x) (r_1 = 1, r_2 = 1, 2, 3, s = 1, 2)$ are given as follows:

$$
\begin{aligned}
&b_0 > 0, f_1^1(x) = \cos\left(2\sqrt{b_0}x\right), f_1^2(x) = \sin\left(2\sqrt{b_0}x\right), g_1^1(x) = \cos\left(\sqrt{b_0}x\right), g_1^2(x) = \sin\left(\sqrt{b_0}x\right),\\
&b_0 < 0, f_1^1(x) = \exp\left(2\sqrt{-b_0}x\right), f_1^2(x) = \exp\left(-2\sqrt{-b_0}x\right),\\
&b_0 > 0, f_1^1(x) = \cos\left(2\sqrt{b_0}x\right), f_1^2(x) = \sin\left(2\sqrt{b_0}x\right),\\
&g_1^1(x) = \cos\left(\sqrt{b_0}x\right), g_1^2(x) = \sin\left(\sqrt{b_0}x\right);\\
&b_0 < 0, f_1^1(x) = \exp\left(2\sqrt{-b_0}x\right), f_1^2(x) = \exp\left(-2\sqrt{-b_0}x\right),\\
&g_1^1(x) = \exp\left(\sqrt{-b_0}x\right), g_1^2(x) = \exp\left(-\sqrt{-b_0}x\right);\\
&a_0 > 0, f_2^1(x) = \cos\left(\sqrt{a_0}x\right), f_2^2(x) = \sin\left(\sqrt{a_0}x\right), g_2^1(x) = \cos\left(\frac{1}{2}\sqrt{a_0}x\right), g_2^2(x) = \sin\left(\frac{1}{2}\sqrt{a_0}x\right),\\
&g_3^1(x) = \cos\left(\sqrt{2a_0}x\right), g_3^2(x) = \sin\left(\sqrt{2a_0}x\right);\\
&a_0 < 0, f_2^1(x) = \exp\left(\sqrt{-a_0}x\right), f_2^2(x) = \exp\left(-\sqrt{-a_0}x\right),
\end{aligned}
$$

$$g_2^1(x) = \exp\left(\frac{1}{2}\sqrt{-a_0}x\right), g_2^2(x) = \exp\left(-\frac{1}{2}\sqrt{-a_0}x\right),$$
$$g_3^1(x) = \exp\left(2\sqrt{-a_0}x\right), g_3^2(x) = \exp\left(-2\sqrt{-a_0}x\right).$$

After the same computations, we get classifications of coupled KdV system, the corresponding linear ODEs, and the invariant subspaces $W^1_{m_1} \times W^2_{m_2}$ for $(n_1, n_2) = (3,2), (3,3), (4,2), (4,3)$ presented in Table 2, where the unknown functions $f_r^s(x), g_r^s(x) (r = 3 \cdots 8, s = 1 \cdots 4)$ are given as below:

$$f_3^1(x) = \exp\left[\frac{1}{m_1}\left(\sqrt{3}I - 1\right)\left(\frac{1}{3}a_2^2 + 7b_1^2 - 3a_2b_1\right) - \frac{1}{12}\left(\sqrt{3}I + 1\right)m_1 - \frac{1}{3}a_2\right]x,$$
$$f_3^2(x) = \exp\left[\frac{1}{m_1}\left(\sqrt{3}I + 1\right)\left(-\frac{1}{3}a_2^2 - 7b_1^2 + 3a_2b_1\right) + \frac{1}{12}\left(\sqrt{3}I - 1\right)m_1 - \frac{1}{3}a_2\right]x,$$
$$f_3^3(x) = \exp\left[\frac{2}{m_1}\left(\frac{1}{3}a_2^2 + 7b_1^2 - 3a_2b_1\right) - \frac{1}{6}m_1 - \frac{1}{3}a_2\right]x,$$
$$m_1 = \left[-4(a_2 - 3b_1)(2a_2^2 - 21a_2b_1 - 54b_1) + 12\sqrt{\triangle_1}\right]^{1/3},$$
$$\triangle_1 = -3b_1\left[a_2^2b_1(9a_2^2 + 805b_1^2 - 150a_2b_1) + b_1^3(1372b_1^2 - 1764a_2b_1 - 972)\right.$$
$$\left.+a_2^2(8a_2^2 - 132a_2b_1 - 108b_1 + 576b_1^2) + a_2b_1(648 - 756b_1)\right],$$
$$f_4^1(x) = \exp\left[\frac{1}{6m_2}\left(\sqrt{3}I - 1\right)(a_2^2 - 12b_0) - \frac{1}{6}\left(\sqrt{3}I + 1\right)m_2 - \frac{1}{3}a_2\right]x,$$
$$f_4^2(x) = \exp\left[\frac{1}{6m_2}\left(\sqrt{3}I + 1\right)(12b_0 - a_2^2) + \frac{1}{6}\left(\sqrt{3}I - 1\right)m_2 - \frac{1}{3}a_2\right]x,$$
$$f_4^3(x) = \exp\left[\frac{1}{3m_2}(a_2^2 - 12b_0) + \frac{1}{3}m_2 - \frac{1}{3}a_2\right]x,$$
$$m_2 = \left[a_2(72b_0 - a_2^2) + 6\sqrt{-3b_0(a_2^4 - 44a_2^2b_0 - 16b_0^2)}\right]^{1/3},$$
$$f_5^1(x) = \exp\left[\frac{1}{m_3}\left(\sqrt{3}I - 1\right)\left(\frac{1}{3}a_2^2 - a_1\right) - \frac{1}{12}\left(\sqrt{3}I + 1\right)m_3 - \frac{1}{3}a_2\right]x,$$
$$f_5^2(x) = \exp\left[\frac{1}{m_3}\left(\sqrt{3}I + 1\right)\left(a_1 - \frac{1}{3}a_2^2\right) + \frac{1}{12}\left(\sqrt{3}I - 1\right)m_3 - \frac{1}{3}a_2\right]x,$$
$$f_5^3(x) = \exp\left[\frac{2}{m_3}\left(\frac{1}{3}a_2^2 - a_1\right) + \frac{1}{6}m_3 - \frac{1}{3}a_2\right]x,$$
$$m_3 = \left(36a_1a_2 - 108a_0 - 8a_2^3 + 12\sqrt{12a_1^3 - 3a_1^2a_2^2 - 54a_0a_1a_2 + 81a_0^2 + 12a_0a_2^2}\right)^{1/3},$$

$$b_1 > 0, f_6^1(x) = \cos\left(\sqrt{b_1x}\right), f_6^2(x) = \sin\left(\sqrt{b_1x}\right),$$
$$f_6^3(x) = \cos\left(2\sqrt{b_1x}\right), f_6^4(x) = \sin\left(2\sqrt{b_1x}\right),$$
$$f_8^1(x) = \cos\left(\frac{1}{2}\sqrt{b_1x}\right), f_8^2(x) = \sin\left(\frac{1}{2}\sqrt{b_1x}\right),$$

$$f_8^3(x) = \cos\left(\frac{3}{2}\sqrt{b_1x}\right), f_8^4(x) = \sin\left(\frac{3}{2}\sqrt{b_1x}\right);$$
$$b_1 < 0, f_6^1(x) = \exp\left(\sqrt{-b_1x}\right), f_6^2(x) = \exp\left(-\sqrt{-b_1x}\right),$$

$$f_6^3(x) = \exp\left(2\sqrt{-b_1 x}\right), f_6^4(x) = \exp\left(-2\sqrt{-b_1 x}\right),$$

$$f_8^1(x) = \exp\left(\frac{1}{2}\sqrt{-b_1 x}\right), f_8^2(x) = \exp\left(-\frac{1}{2}\sqrt{-b_1 x}\right),$$

$$f_8^3(x) = \exp\left(\frac{3}{2}\sqrt{-b_1 x}\right), f_8^4(x) = \exp\left(-\frac{3}{2}\sqrt{-b_1 x}\right);$$

$$b_1 = 0, f_6^1(x) = x, f_6^2(x) = x^2;$$

$$f_1^3(x) = \exp\left[-\frac{1}{2}(a_3 - \sqrt{a_3^2 - 4a_2 - 16b_0})\right]x, f_1^4(x) = \exp\left[-\frac{1}{2}(a_3 + \sqrt{a_3^2 - 4a_2 - 16b_0})\right]x;$$

$$b_0 > 0, f_7^1(x) = \cos\left(\frac{1}{2}\sqrt{b_0}x\right), f_7^2(x) = \sin\left(\frac{1}{2}\sqrt{b_0}x\right), f_7^3(x) = \cos\left(\frac{3}{2}\sqrt{b_0}x\right),$$

$$f_7^4(x) = \sin\left(\frac{3}{2}\sqrt{b_0}x\right);$$

$$b_0 < 0, f_7^1(x) = \exp\left(\frac{1}{2}\sqrt{-b_0}x\right), f_7^2(x) = \exp\left(-\frac{1}{2}\sqrt{-b_0}x\right), f_7^3(x) = \exp\left(\frac{3}{2}\sqrt{-b_0}x\right),$$

$$f_7^4(x) = \exp\left(-\frac{3}{2}\sqrt{-b_0}x\right).$$

4. Explicit solutions and figures of system (1.1)

In Sec.3, it is shown that the classifying invariant subspaces of system (1.1) are obtained. According to the theory of ISM, the solutions of system (1.1) are in the form $u(x,t) = \sum_{i=1}^{m_1} C_i(t) f_i(x), v(x,t) = \sum_{j=1}^{m_2} D_j(t) g_j(x)$. In this section, we determine the exact solutions of system (1.1) by applying the classification results in Section 3. We set out three typical types and give the figures of these exact solutions.

Example 4.1. The following system

$$\begin{aligned}
&u_t - \frac{3}{2}p_3 v u_x + (p_2 v^2 + p_3 uv + p_4 u_{xx})_x = 0,\\
&v_t + q_4 u_{xxx} = 0.
\end{aligned} \tag{4.1}$$

From the above results, we find the following invariant subspace of is $W_2^1 \times W_2^2 = \mathcal{L}\{e^{-b_1 x}, e^{-2b_1 x}\} \times \mathcal{L}\{1, e^{-b_1 x}\}$. Which means that system (4.1) has exact solutions of exponential functions:

$$\begin{aligned}
&u(x,t) = C_1(t)e^{-b_1 x} + C_2(t)e^{-2b_1 x},\\
&v(x,t) = D_1(t) + D_2(t)e^{-b_1 x},
\end{aligned}$$

where the functions $C_i(t)(i = 1,2)$ and $D_j(t)(j = 1,2)$ are determined by the nonlinear ODE system:

$$\begin{aligned}
&C_1' = -\frac{1}{2}b_1(p_3 C_1 D_1 - 2b_1^2 p_4 C_1 - 4p_2 D_1 D_2),\\
&C_2' = -\frac{1}{2}b_1(2p_3 C_2 D_1 - 4p_2 D_2^2 - p_3 C_1 D_2 - 16b_1^2 p_4 C_2),\\
&D_1' = 0,\\
&D_2' = q_4 b_1^3 D_2.
\end{aligned}$$

TABLE 1. System (1.1) and the corresponding linear ODEs.

No.	System (1.1)	ODEs (1.2)	ISs (1.2)
1	$u_t = p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y'' + a_1y' + a_0y = 0$	$\mathcal{L}\{e^{\frac{1}{2}(\sqrt{a_1^2-4a_0}-a_1)x}, e^{-\frac{1}{2}(\sqrt{a_1^2-4a_0}+a_1)x}\}$
	$v_t = -(q_4v_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
2	$u_t = -p_1vu_x - p_4u_{xxx}$	$y'' + a_1y' = 0$	$\mathcal{L}\{1, e^{-a_1x}\}$
	$v_t = q_3vu_x - (q_3uv + q_4v_{xx})_x$	$z'' - a_1z' = 0$	$\mathcal{L}\{1, e^{a_1x}\}$
3	$u_t = -p_4u_{xxx}$	$y'' + a_1y' = 0$	$\mathcal{L}\{1, e^{-a_1x}\}$
	$v_t = -q_1vu_x - (q_3uv + q_4v_{xx})_x$	$z'' + b_1z' + \frac{1}{4}(b_1^2 - a_1^2)z = 0$	$\mathcal{L}\{e^{\frac{1}{2}(a_1-b_1)x}, e^{-\frac{1}{2}(a_1+b_1)x}\}$
4	$u_t = -p_4u_{xxx}$	$y'' + 4b_0y = 0$	$\mathcal{L}\{f_1^1(x), f_1^2(x)\}$
	$v_t = \frac{3}{2}q_3vu_x - (q_3uv + q_4v_{xx})_x$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
5	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y'' + a_0y = 0$	$\mathcal{L}\{f_2^1(x), f_2^2(x)\}$
	$v_t = \frac{3}{2}q_3vu_x - (q_3uv + q_4v_{xx})_x$	$z'' + \frac{1}{4}a_0z = 0$	$\mathcal{L}\{g_2^1(x), g_2^2(x)\}$
6	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y'' + a_1y' + \frac{2}{9}a_1^2y = 0$	$\mathcal{L}\{e^{-\frac{1}{3}a_1x}, e^{-\frac{2}{3}a_1x}\}$
	$v_t = -q_4v_{xxx}$	$z'' + \frac{1}{3}a_1z' = 0$	$\mathcal{L}\{1, e^{-\frac{1}{3}a_1x}\}$
7	$u_t = -p_4u_{xxx}$	$y'' + a_1y' = 0$	$\mathcal{L}\{1, e^{-a_1x}\}$
	$v_t = 3q_3vu_x - (q_2u^2 + q_3uv + q_4v_{xx})_x$	$z'' + 3a_1z' + 2a_1^2z = 0$	$\mathcal{L}\{e^{-a_1x}, e^{-2a_1x}\}$
8	$u_t = -p_1vu_x - (p_2v^2 + p_3uv + p_4u_{xx} + p_5u^2)_x$	$y'' = 0$	$\mathcal{L}\{1, x\}$
	$v_t = -q_1vu_x - (q_2u^2 + q_3uv + q_4v_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
9	$u_t = -(p_4u_{xx} + p_5u^2)_x$	$y'' = 0$	$\mathcal{L}\{1, x\}$
	$v_t = -q_1vu_x - q_4v_{xxx}$	$z'' + b_1z' + b_0z = 0$	$\mathcal{L}\{e^{\frac{1}{2}(\sqrt{b_1^2-4b_0}-b_1)x}, e^{-\frac{1}{2}(\sqrt{b_1^2-4b_0}+b_1)x}\}$
10	$u_t = 2p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y'' + a_1y' = 0$	$\mathcal{L}\{1, e^{-a_1x}\}$
	$v_t = 2q_3vu_x - (q_3uv + q_4v_{xx})_x$	$z'' + a_1z' = 0$	$\mathcal{L}\{1, e^{-a_1x}\}$
11	$u_t = \frac{a_1+3b_1}{a_1+b_1}p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y'' + a_1y' + \frac{1}{4}(a_1^2 - b_1^2)y = 0$	$\mathcal{L}\{e^{-\frac{1}{2}(a_1+b_1)x}, e^{-\frac{1}{2}(a_1-b_1)x}\}$
	$v_t = -q_4v_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
12	$u_t = \frac{3}{2}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y'' + 3b_1y' + 2b_1^2y = 0$	$\mathcal{L}\{e^{-b_1x}, e^{-2b_1x}\}$
	$v_t = -q_4v_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
13	$u_t = 3p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y'' + a_0y = 0$	$\mathcal{L}\{f_2^1(x), f_2^2(x)\}$
	$v_t = -(q_2u^2 + q_4v_{xx})_x$	$z'' + 4a_0z = 0$	$\mathcal{L}\{g_3^1(x), g_3^2(x)\}$

Now we consider the above ODE system with $D_1(t) = 0$ and we get the solution

$$\begin{aligned} u(x,t) &= c_1e^{-b_1(x-b_1^2p_4t)} + \frac{p_2d^2}{b_1^2(q_4-4p_4)}e^{-2b_1(x-b_1^2q_4t)} + \frac{c_1dp_3}{2b_1^2(q_4-7p_4)}e^{-b_1[2x-b_1^2(p_4+q_4)t]} \\ &\quad + c_2e^{-2b_1(x-4b_1^2p_4t)}, \\ v(x,t) &= de^{-b_1(x-b_1^2q_4t)}, \end{aligned} \tag{4.2}$$

where c_1, c_2, d are arbitrary constants. The profile of exact solution for (4.1) is displayed in Figure 1.

Example 4.2. The following system

$$\begin{aligned} &u_t - 2p_3vu_x + (p_3uv + p_4u_{xx})_x = 0, \\ &v_t - 2q_3vu_x + (q_3uv + q_4v_{xx})_x = 0. \end{aligned} \tag{4.3}$$

We consider the following invariant subspace $W_2^1 \times W_2^2 = \mathcal{L}\{1, e^{-a_1x}\} \times \mathcal{L}\{1, e^{-a_1x}\}$ of system (4.2). This suggests that system (4.3) has exact solutions of exponential functions:

$$\begin{aligned} u(x,t) &= C_1(t) + C_2(t)e^{-a_1x}, \\ v(x,t) &= D_1(t) + D_2(t)e^{-a_1x}, \end{aligned}$$

where the functions $C_i(t)(i = 1, 2)$ and $D_j(t)(j = 1, 2)$ are determined by the nonlinear ODE system:

$$\begin{aligned} C_1' &= 0, \\ C_2' &= -a_1(p_3C_2D_1 - C_1D_2 - a_1^2p_4C_2), \\ D_1' &= 0, \end{aligned}$$

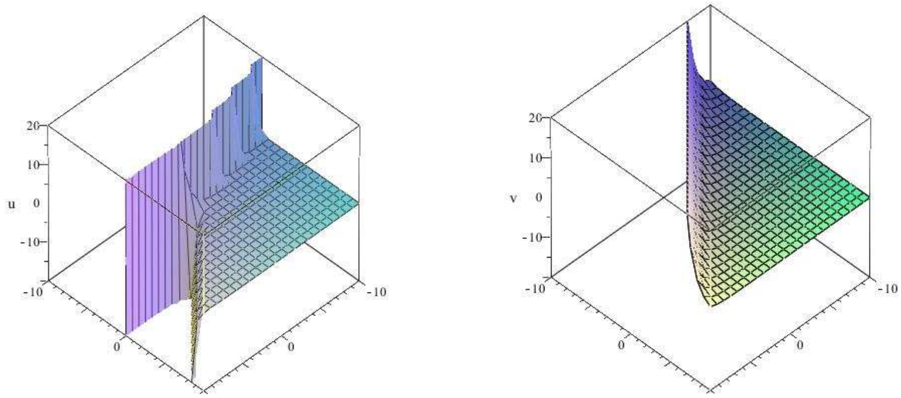

FIGURE 1. The exact solution of (4.1) with $b_1 = -1, c_1 = c_2 = p_4 = q_4 = d = 1, p_2 = 2, p_3 = 4$.

$$D_2' = -a_1(q_3C_2D_1 - q_3C_1D_2 - a_1^2q_4D_2).$$

Let $C_1(t) = c_1, D_1(t) = d_1$, we get the solution

$$u(x,t) = c_1 + c_2e^{-a_1x+\frac{1}{2}(K^2-2d_1p_3+a_1Kt)},$$
$$v(x,t) = d_1 + \frac{d_2}{2c_1p_3}[K^2 - 2a_1^2p_4 + e^{\frac{1}{2}(K^2-2d_1p_3+a_1Kt)}]e^{-a_1x}, \tag{4.4}$$

where $a_1^2(p_4 + q_4) + c_1q_3 + d_1p_3 = K^2 > 0$. The profile of exact solution for system (4.3) is described in Figure 2.

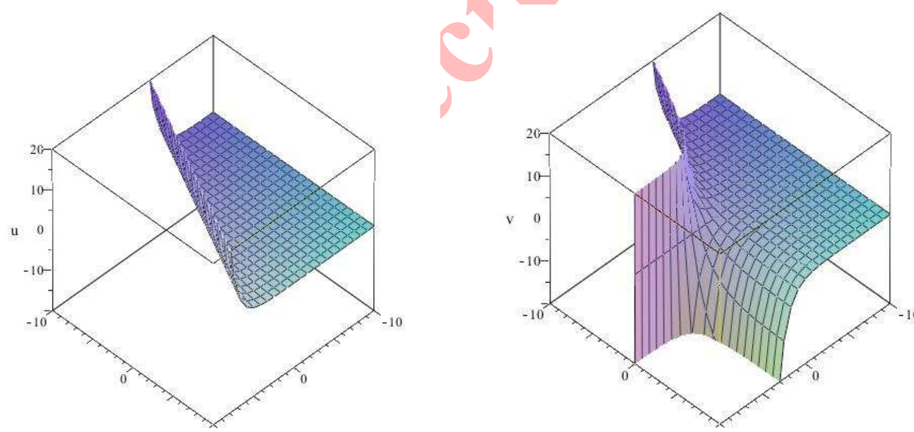

FIGURE 2. The exact solution for system (4.3) with $c_1 = c_2 = d_1 = d_2 = p_4 = K = 1, a_1 = -1, p_3 = \frac{1}{2}$.

Example 4.3. The following equation

$$u_t - p_3vu_x + (p_2v^2 + p_3uv + p_4u_{xx})_x = 0,$$
$$v_t - 2q_5vv_x = 0. \tag{4.5}$$

From the above results, we find the following invariant subspace of is $W_3^1 \times W_2^2 = \mathcal{L}\{1, x, e^{-a_1x}\} \times \mathcal{L}\{1, x\}$. Which means that has exact solutions of exponential functions:

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)e^{-a_1x},$$

TABLE 2. System (1.1) and the corresponding linear ODEs.

No.	System(1.1)	ODEs (1.2)	ISs (1.2)
1	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y''' + 4b_0y' = 0$	$\mathcal{L}\{1, f_1^1(x), f_1^2(x)\}$
	$v_t = \frac{3}{2}q_3vu_x - (q_3uv + q_4u_{xx})_x$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
2	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y''' + a_2y'' + (3a_2 - 7b_1)b_1y' + 2(3b_1 - a_2)b_1^2y = 0$	$\mathcal{L}\{f_3^1(x), f_3^2(x), f_3^3(x)\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
3	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y''' + a_2y'' + 4b_0y' - 4a_2b_0y = 0$	$\mathcal{L}\{f_4^1(x), f_4^2(x), f_4^3(x)\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
4	$u_t = -p_1vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y''' = 0$	$\mathcal{L}\{1, x, x^2\}$
	$v_t = \frac{3}{2}q_3vu_x - (q_3uv + q_4u_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
5	$u_t = p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y''' + a_2y'' + a_1y' + a_0y = 0$	$\mathcal{L}\{f_5^1(x), f_5^2(x), f_5^3(x)\}$
	$v_t = -(q_4u_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
6	$u_t = \frac{3}{2}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y''' + 3b_1y'' + 2b_1^2y' = 0$	$\mathcal{L}\{1, e^{-b_1x}, e^{-2b_1x}\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
7	$u_t = p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y''' + a_2y'' = 0$	$\mathcal{L}\{1, x, e^{-a_2x}\}$
	$v_t = -(q_4u_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
8	$u_t = 2p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y''' + b_0y' = 0$	$\mathcal{L}\{1, g_1^1(x), g_1^2(x)\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
9	$u_t = 2p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y''' + b_1y' = 0$	$\mathcal{L}\{1, f_6^1(x), f_6^2(x)\}$
	$v_t = 2q_3vu_x - (q_3uv + q_4v_{xx})_x$	$z''' + b_1z' = 0$	$\mathcal{L}\{1, f_6^1(x), f_6^2(x)\}$
10	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y^{(4)} + a_3y''' + a_2y'' + 4a_3b_0y' + 4(a_2 - 4b_0)b_0y = 0$	$\mathcal{L}\{f_1^1(x), f_1^2(x), f_1^3(x), f_1^4(x)\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
11	$u_t = -p_1vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} = 0$	$\mathcal{L}\{1, x, x^2, x^3\}$
	$v_t = -(q_4u_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
12	$u_t = p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} + a_3y''' + a_2y'' = 0$	$\mathcal{L}\{1, x, e^{-\frac{1}{2}(a_3-\sqrt{a_3^2-4a_2})x}, e^{-\frac{1}{2}(a_3+\sqrt{a_3^2-4a_2})x}\}$
	$v_t = -(q_4u_{xx} + q_5v^2)_x$	$z'' = 0$	$\mathcal{L}\{1, x\}$
13	$u_t = \frac{5}{3}p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y^{(4)} + \frac{5}{2}b_0y'' + \frac{9}{16}b_0^2y = 0$	$\mathcal{L}\{f_7^1(x), f_7^2(x), f_7^3(x), f_7^4(x)\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_0z = 0$	$\mathcal{L}\{g_1^1(x), g_1^2(x)\}$
14	$u_t = \frac{5}{4}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} + 10b_1y''' + 35b_1^2y'' + 50b_1^3y' + 24b_1^4y = 0$	$\mathcal{L}\{e^{-b_1x}, e^{-2b_1x}, e^{-3b_1x}, e^{-4b_1x}\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
15	$u_t = \frac{3}{2}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} + 2b_1y''' - b_1^2y'' - 2b_1^3y' = 0$	$\mathcal{L}\{1, e^{b_1x}, e^{-b_1x}, e^{-2b_1x}\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
16	$u_t = \frac{4}{3}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} + 6b_1y''' + 11b_1^2y'' + 6b_1^3y' = 0$	$\mathcal{L}\{1, e^{-b_1x}, e^{-2b_1x}, e^{-3b_1x}\}$
	$v_t = -q_4u_{xxx}$	$z'' + b_1z' = 0$	$\mathcal{L}\{1, e^{-b_1x}\}$
17	$u_t = \frac{5}{3}p_3vu_x - (p_2v^2 + p_3uv + p_4u_{xx})_x$	$y^{(4)} = 0$	$\mathcal{L}\{1, x, x^2, x^3\}$
	$v_t = -q_4u_{xxx}$	$z''' = 0$	$\mathcal{L}\{1, x, x^2\}$
18	$u_t = \frac{5}{3}p_3vu_x - (p_3uv + p_4u_{xx})_x$	$y^{(4)} + \frac{5}{2}b_1y'' + \frac{9}{16}b_1^2y = 0$	$\mathcal{L}\{f_8^1(x), f_8^2(x), f_8^3(x), f_8^4(x)\}$
	$v_t = -q_4u_{xxx}$	$z''' + b_1z' = 0$	$\mathcal{L}\{1, f_6^1(x), f_6^2(x)\}$
19	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y^{(4)} + 5b_1y'' + 4b_1^2y = 0$	$\mathcal{L}\{f_6^1(x), f_6^2(x), f_6^3(x), f_6^4(x)\}$
	$v_t = -q_4u_{xxx}$	$z''' + b_1z' = 0$	$\mathcal{L}\{1, f_6^1(x), f_6^2(x)\}$
20	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y^{(4)} + \frac{10}{3}b_2y''' + \frac{35}{9}b_2^2y'' + \frac{50}{27}b_2^3y' + \frac{8}{27}b_2^4y = 0$	$\mathcal{L}\{e^{-\frac{4}{3}b_2x}, e^{-\frac{2}{3}b_2x}, e^{-\frac{1}{3}b_2x}, e^{-b_2x}\}$
	$v_t = -q_4u_{xxx}$	$z''' + b_2z'' + \frac{2}{9}b_2^2z' = 0$	$\mathcal{L}\{1, e^{-\frac{1}{3}b_2x}, e^{-\frac{2}{3}b_2x}\}$
22	$u_t = -(p_2v^2 + p_4u_{xx})_x$	$y^{(4)} + \frac{10}{3}b_2y''' + \frac{20}{9}b_2^2y'' - \frac{40}{27}b_2^3y' - \frac{32}{27}b_2^4y = 0$	$\mathcal{L}\{e^{-\frac{4}{3}b_2x}, e^{-\frac{2}{3}b_2x}, e^{-2b_2x}, e^{\frac{2}{3}b_2x}\}$
	$v_t = -q_4u_{xxx}$	$z''' + b_2z'' - \frac{1}{9}b_2^2z' - \frac{1}{9}b_2^3z = 0$	$\mathcal{L}\{e^{-b_2x}, e^{-\frac{1}{3}b_2x}, e^{\frac{1}{3}b_2x}\}$

$$v(x,t) = D_1(t) + D_2(t)x,$$

where the functions $C_i(t)(i = 1,2,3)$ and $D_j(t)(j = 1,2)$ are determined by the nonlinear ODE system:

$$\begin{aligned}
C_1' &= -2p_2D_1D_2 - p_3C_1D_2,\\
C_2' &= -2p_2D_2^2 - p_3C_2D_2,\\
C_3' &= -2p_2D_2^2 - p_3C_2D_2,\\
D_1' &= -2q_5D_1D_2,\\
D_2' &= -2q_5D_2^2.
\end{aligned}$$

Then we get the solution

$$u(x,t) = \frac{2p_2}{(2q_5 - p_3)(2q_5t + d_2)}(d_1 + x) + (2q_5t + d_2)^{-\frac{p_3}{2q_5}}(c_1 + c_2x + c_3e^{a_2x - a_2^3p_4t}),$$

$$v(x,t) = \frac{1}{2q_5 t + d_2}(d_1 + x), \tag{4.6}$$

The profile of exact solution for system (4.5) is presented in Figure 3.

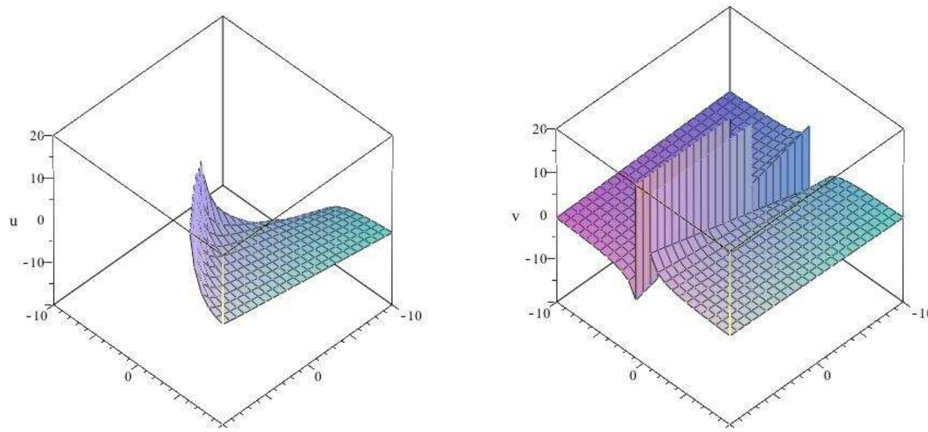

FIGURE 3. The exact solution for system (4.5) with $c_1 = c_2 = c_3 = p_2 = p_3 = p_4 = q_5 = d_1 = a_2 = 1, d_2 = 0$.

5. Conclusions

The intention of this paper is to find invariant subspace classification of system (1.1) and construct exact solutions of (1.1). As a very powerful method, ISM is in essence a part of Lie's reduction method, the reason is that its key point is to convert a nonlinear system to a system of ODEs. By computing the obtained ODE system, the separable solutions of system (1.1) are derived. It is stressed that solutions (4.2), (4.4), and (4.6) obtained in our paper are not appeared in former references, which encourages us to do further research on other nonlinear PDEs in the future.

Data Availability Statement

No datasets were generated or analysed during the current study.

Conflict of Interests/Competing Interests

No potential conflict of interest was reported by the authors.

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