Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-12
DOI:10.22034/cmde.2026.66757.3159

Existence and uniqueness of fuzzy differential equations with jump

Samira Siahmansouri[1], Esmaeil Yousefi[2,*], and Mohammad Gholamianand[3]
[1]Center of Mahan Hedaiat, University of Applied Science and Technology, Tehran, Iran.
[2]Department of Mathematics and computer science, Science and Research branch, Islamic Azad University, Tehran, Iran.
[3]Department of Mathematics and computer science, WT.C., Islamic Azad University, Tehran, Iran.

Abstract

A Fuzzy differential equation with jumps is a type of fuzzy differential equation driven by Liu's process and a fuzzy renewal process which is called a fuzzy differential equation with jumps, to model the sharp drifts embedded in an uncertain dynamic system. Up to now, this paper, we provide and prove a novel existence and uniqueness theorem for the proposed equations under Local Lipschitz and monotone conditions. This result allows us to consider and analyze solutions to a wide range of nonlinear fuzzy differential equations driven by Liu's process and fuzzy renewal process. We will also investigate the stability of solutions to fuzzy differential equation with jumps by theorem.

Keywords. Fuzzy differential equation, Fuzzy renewal process, Liu's process, Credibility space.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Fuzziness is a kind of uncertainty in the real world which has been first investigated by Zadeh [19] through proposing the concept of fuzzy sets via membership functions.

Liu was the first who introduced credibility theory and presented the concept of credibility measure [9]. Credibility measure is a powerful tool for dealing with fuzzy phenomena, to facilitate measuring of fuzzy events that based on normality, monotonicity, self-duality, and maximality axioms.

Furthermore, Liu proposed the concept of fuzzy process [8]. This concept introduces a particular fuzzy process with stationary and independent increment named Liu's process which is just like a stochastic process described by Brownian motion.

Recently, literatures on the Liu process and its applications in other sciences, such as economics and optimal control has been published [20]. Liu inspired by stochastic notions and Ito process introduced fuzzy differential equations [11] which were driven by Liu process for better understand the fuzzy phenomena.

Solving European Pricing Problem in fuzzy environment is one of the most well known application of fuzzy differential equations driven by Liu's process [15]. Regarding to the importance of existence and uniqueness of a solution to fuzzy differential equations driven by Liu process, Liu investigated the existence and uniqueness of solution to the fuzzy differential equations by employing Lipschitz and Linear growth conditions [18]. Afterward, Fei studied the uniqueness of solution to the fuzzy differential equations driven by Liu process with non-Lipschitz coefficients [4].

The main goal of this paper is to provide some weaker conditions to study the existence and uniqueness of solution to the fuzzy differential equations. In this regard, we prove a new existence and uniqueness theorem under the Local Lipschitz and monotone conditions.

The paper is arranged as follow. We will review some basic concepts about credibility theory, fuzzy variable, fuzzy process and Liu process in Section 2. A new existence and uniqueness theorem for fuzzy differential equations

Received: 26 April 2025; Accepted: 13 July 2026.
∗ Corresponding author. Email: Esmaeil.yousefi@iau.ac.ir.

under Local Lipschitz and monotone conditions is proved in Section 3. Section 4 in first we defined fuzzy differential equation(FDE) with jump, then a new theorem for this FDE with monotone condition Finally, We will also investigate the stability of solutions to fuzzy differential equation with jumps by theorem.

2. Fuzzy variable

The emphasis in this section is mainly on introducing some concepts such as credibility measure, credibility space, fuzzy variables, independence of fuzzy variables, expected value, variance, fuzzy process, Liu process, and stopping time.

Suppose that Θ is a non-empty set and $\mathcal{P}$ is the power set of Θ. Each element of $\mathbf{A}$ in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign a number $\mathbf{Cr}\{\mathbf{A}\}$ to each event $\mathbf{A}$ which indicates the credibility that $\mathbf{A}$ will occur. To ensure that the number $\mathbf{Cr}\{\mathbf{A}\}$ has certain mathematical properties which we intuitively expect to have a credibility, we accept the following four axioms [8]:

(1) Axiom (Normality) $\mathbf{Cr}\{\Theta\} = 1$.
(2) Axiom (Monotonicity) $\mathbf{Cr}\{\mathbf{A}\} \leq \mathbf{Cr}\{\mathbf{B}\}$ whenever $\mathbf{A} \subset \mathbf{B}$.
(3) Axiom (Self-Duality) $\mathbf{Cr}\{\mathbf{A}\} + \mathbf{Cr}\{\mathbf{A}^c\} = 1$ for any event $\mathbf{A}$.
(4) Axiom (Maximality) $\mathbf{Cr}\{\mathbf{U}_i \mathbf{A}_i\} = \sup_i \mathbf{Cr}\{\mathbf{A}_i\}$ for any events $\{\mathbf{A}_i\}$ with $\sup\limits_i \mathbf{Cr}\{\mathbf{A}_i\} < 0.5$.

Definition 2.1. [18] The set function $\mathbf{Cr}$ is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

A family $\mathcal{P}$ with these four properties is called a σ-algebra. The pair $(\Theta, \mathcal{P})$ is called a measurable space, and the elements of $\mathcal{P}$ is afterwards called $\mathcal{P}$-measurable sets instead of events.

Definition 2.2. [18] Let Θ be a nonempty set, $\mathcal{P}$ the power set of Θ, and $\mathbf{Cr}$ a credibility measure. The triple $(\Theta, \mathcal{P}, \mathbf{Cr})$ is called a credibility space.

Let $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A filtration is a family $\{\mathcal{P}_t\}_{t \geq 0}$ of increasing sub-σ-algebras of $\mathcal{P}$ (i.e. $\mathcal{P}_t \subset \mathcal{P}_s \subset \mathcal{P}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathcal{P}_t = \bigcap_{s>t} \mathcal{P}_s$ for all $t \leq 0$. When the credibility space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{P}_0$ contains all $\mathbf{Cr}$-null sets.

We also define $\mathcal{P}_\infty = \sigma(U_{t \geq 0} \mathcal{P}_t)$ (i.e. σ-algebra generated by $U_{t \geq 0} \mathcal{P}_t$.)
$\mathcal{P}$-measurable fuzzy variable are denoted by $\mathbf{L}^p(\Theta, \mathbf{R^d})$ that will be defined later.

A process is called $\mathcal{P}$-adapted, if for all $t \in [0, t]$ the fuzzy variable $x(t)$ is $\mathcal{P}$-measurable.

Definition 2.3. [18] A fuzzy variable is defined as a (measurable) function
$\xi : (\Theta, \mathcal{P}, \mathbf{Cr}) \longrightarrow \mathbf{R}$.

Definition 2.4. [18] Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathbf{Cr}\{\xi \geq r\} \mathrm{dr} - \int_{-\infty}^{0} \mathbf{Cr}\{\xi \leq r\} \mathrm{dr}$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined by $\mathbf{E}[(\xi - e)^2]$.
Let ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have

$$\mathbf{E}[a\xi + b\eta] = a\mathbf{E}[\xi] + b\mathbf{E}[\eta].$$

Definition 2.5. [11] The credibility distribution $\mu(x)$ of a fuzzy variable ξ is defined by

$$\mu(x) = \max\{1, 2\mathbf{Cr}(\xi = x)\}, \ x \in \mathbf{R}.$$

Definition 2.6. [11] A credibility distribution $\mu(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \mu(x) < 1$, and

$$\lim_{x \to -\infty} \mu(x) = 0, \quad \lim_{x \to +\infty} \mu(x) = 1.$$

In addition, the inverse function $\mu^{-1}(\alpha)$ is called the inverse credibility distribution of ξ.

Definition 2.7. [10]. Let $\mathbf{T}$ be an index set and $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A fuzzy process is a function from $\mathbf{T} \times (\Theta, \mathcal{P}, \mathbf{Cr})$ to the set of real numbers.

That is, a fuzzy process $X_t(\theta)$ is a function of two variables such that the function $X_{t^*}(\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $X_t(\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X_t(\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ.

In the following, we use the notation $\mathbf{x}(t)$ instead of $\mathbf{x}_t(\theta)$.
A fuzzy process is essentially a sequence of fuzzy variables indexed by time or space. As one of the most important types of fuzzy processes, the Liu process is defined as follows.

Definition 2.8. [11] A fuzzy process $\mathbf{C}_t$ is said to be a Liu process if

(1) $\mathbf{C}_0 = 0$,
(2) $\mathbf{C}_t$ has stationary and independent increments,
(3) every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value $\mathbf{e}\mathbf{t}$ and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2(1 + \exp(\frac{\pi|x - et|}{\sqrt{6}\sigma t}))^{-1}, \quad -\infty < x < +\infty.$$

The parameters e and σ are called the drift and diffusion coefficients, respectively, in addition, Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Based on Liu process, Liu integral is defined as a fuzzy counterpart of Ito integral as follows.

Definition 2.9. [11] Suppose that $x(t)$ is a fuzzy process and $\mathbf{C}_t$ is a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \ldots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of $x(t)$ with respect to $\mathbf{C}_t$ is

$$\int_a^b x(t) d\mathbf{C}_t = \lim_{\Delta \longrightarrow 0} \sum_{i=1}^{k} x(t_i).(\mathbf{C}_{t_{i+1}} - \mathbf{C}_{t_i})$$

provided that the limit exists almost surely and is a fuzzy variable.

Theorem 2.10. *[3] Let $\mathbf{C}_t$ be a standard Liu process, and $h(t, c)$ a continuously differentiable function. Define $x(t) = h(t, \mathbf{C}_t)$. Then we have the following chain rule*

$$dx(t) = \frac{\partial h}{\partial t}(t, \mathbf{C}_t) dt + \frac{\partial h}{\partial c}(t, \mathbf{C}_t) d\mathbf{C}_t,$$

which is called Liu formula.

Definition 2.11. [18] The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$\mathbf{Cr}\{\bigcap_{i=1}^{m} \{\xi_i \in \mathbf{B}_i\}\} = \min_{1 \le i \le m} \mathbf{Cr}\{\xi_i \in \mathbf{B}_i\}$$

for any sets $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_m$ of real numbers.
Let us define a sequence of credibilistic stopping times.

Definition 2.12. [10] A fuzzy variable $\tau : \Theta \to [0, \infty]$ (it may take the value ∞) is called an $\{\mathcal{P}_t\}$-stopping time (or simply, stopping time) if $\{\theta : \tau(\theta) \le t\} \in \mathcal{P}_t$ for any $t \ge 0$

$$\begin{cases} \tau_k = \inf\{t \ge 0 : |x(t)| \ge k\}, \\ \sigma_1 = \inf\{t \ge 0 : w(x(t)) \ge 2\varepsilon\}, \\ \sigma_{2i} = \inf\{t \ge \sigma_{2i-1} : w(x(t)) \le \varepsilon\}, & i = 1, 2, \ldots, \\ \sigma_{2i+1} = \inf\{t \ge \sigma_{2i} : w(x(t)) \ge 2\varepsilon\}, & i = 1, 2, \ldots, \end{cases}$$

where throughout this paper we set $\inf \phi = \infty$.

Definition 2.13. [10] If $X = \{X_t\}_{t\geq 0}$ is a measurable process and τ is a stopping time, then $\{X_{\tau\wedge t}\}_{t\geq 0}$ is called a stopped process of X.
There are several useful inequalities for fuzzy variables, such as Hölder inequality and Chebyshev inequality. In the sequens, we introduce generalized inequalities for fuzzy variables.

Theorem 2.14. *[**Hölder**'s I=inequality [4]] Let $\mathbf{p}$ and $\mathbf{q}$ be two positive real numbers with $\frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}} = 1$, ξ and η be independent fuzzy variables with*

$$\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ] \leq +\infty \ \textit{and}\ \mathbf{E}\ [\ |\eta|^{\mathbf{q}}\ ] \leq +\infty.$$

We have

$$\mathbf{E}\ [\ |\xi\eta|\ ] \leq \sqrt[\mathbf{p}]{\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ]}\ \sqrt[\mathbf{q}]{\mathbf{E}\ [\ |\eta|^{\mathbf{q}}\ ]}.$$

Theorem 2.15. *[**Chebychev**'s inequality] Let $\xi : \theta \to \mathbf{R}^n$ be a fuzzy variable such that $\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ] \leq +\infty$ for some $\mathbf{p}$, $0 \leq \mathbf{p} \leq \infty$.*

Then Chebychev's inequality:

$$\mathbf{Cr}[\ |\xi| \geq \lambda\ ] \leq \frac{1}{\lambda^{\mathbf{p}}}\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ],\ \textit{for all}\ \lambda \geq 0.$$

Proof. Put $\mathbf{A} = \{w \mid\ |\xi(w)| \geq \lambda\}$. Then

$$\int_\theta |\xi(w)|^{\mathbf{p}} \mathrm{d}\mathbf{Cr}_w \geq \int_A |\xi(w)|^{\mathbf{p}} \mathrm{d}\mathbf{Cr}_w \geq \lambda^p \mathbf{Cr}_A.$$

Before, ending this section it is essential to introduce some symbols that are used in next sections. □

Notation 1: $\mathbf{L}^{\mathbf{p}}(\theta, \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued fuzzy variables ξ with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.
Notation 2: $\ell^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued $\mathcal{P}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^{\mathbf{p}} \mathrm{d}t < \infty$ almost surely.
Notation 3: $M^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ the family of processes $\{f(t)\}_{a\leq t\leq b}$ in $\ell^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ such that $\int_a^b |f(t)|^{\mathbf{p}} dt < \infty$.
Notation 4: $\ell^{\mathbf{p}}(\mathbf{R}_+, \mathbf{R}^{\mathbf{d}})$ the family of processes $\{f(t)\}_{t\geq 0}$ such that for every $T > 0$, $\{f(t)\}_{a\leq t\leq T} \in \ell^{\mathbf{p}}([0,T], \mathbf{R}^{\mathbf{d}})$.

In this section, the required tools to prove our proposed theorem for existence and uniqueness, such as Lemma's and inequalities our mentioned.

Lemma 2.16. *(**Burkholder -Davis-Gundy** inequality for* Liu process*):*
Let $g \in \ell^p(\mathbf{R}_+, \mathbf{R}^{d\times m})$. Define for $t > 0$,

$$x(t) = \int_0^t g(s) d\mathbf{C}_s \ \textit{and}\ A(t) = \int_0^t |g(s)|^2 ds.$$

Then,

$$\mathbf{E}|A(t)| \leq \mathbf{E}(\sup_{0\leq s\leq t} |x(s)|^2) \leq 4\mathbf{E}|A(t)| \tag{2.1}$$

Proof. Without lose of generality, consider $x(t)$ and $A(t)$ are bounded. Otherwise, for each integer $n > 1$, define the stopping time

$$\tau_n = \inf\{t \geq 0 : |x(t)| \vee A(t) \geq n\}.$$

If (2.1) can be state by the stopped processes $x(t\wedge\tau_n)$ and $A(t\wedge\tau_n)$, then the general case follows by letting $n \to \infty$. Moreover, for simplicity we set

$$x^*(t) = \sup_{0\leq s\leq t} |x(s)|.$$

Consider, the following inequality

$$E|x(t)|^2 \leq E\int_0^t |g(s)|^2 ds = E(A(t)) \tag{2.2}$$

then by the Doob martingale inequality[3], we get,

$$E|x^*(t)| \leq 4E|x(t)|^2$$

by substituting this into (2.2) yields,

$$E|x^*(t)| \le 4E(A(t))$$

which is the right-hand-side inequality of (2.1). In order to show the left-hand-side one, we set

$$y(t) = \int_0^t g(s)ds.$$

Then,

$$\mathbf{E}|y(t)|^2 = \mathbf{E}\int_0^t |g(s)|^2 ds = \mathbf{E}\int_0^t dA(s) = \mathbf{E}|A(t)|. \tag{2.3}$$

On the other hand, the integration by parts formula yields,

$$\begin{aligned} x(t) &= \int_0^t dx(s) + \int_0^t x(s)ds \\ &= y(t) + \int_0^t x(s)ds. \end{aligned}$$

Therefore,

$$|y(t)| \le |x(t)| + \int_0^t |x(s)|ds \le 2x^*(t).$$

Here, by substituting this into (2.3) one sees that,

$$\mathbf{E}|A(t)| \le 4\mathbf{E}[|x^*(t)|^2].$$

This implies,

$$\frac{1}{4}\mathbf{E}|A(t)| \le \mathbf{E}[|x^*(t)|^2].$$

Finally, the proof is complete. □

The integral inequalities of Gronwall type have been widely applied in the theory of ordinary differential equations, stochastic differential equations, and fuzzy differential equations to prove the results on existence, uniqueness, and stability.

Lemma 2.17. *[**Gronwall**'s inequality for* Liu process*] Let* $\mathrm{T} > 0$*,* $c \le 0.$*, and* $u(.)$ *be a credibility measurable bounded nonnegative function on* $[0, \mathrm{T}]$*, and* $v(.)$ *be a nonnegative integrable function on* $[0, \mathrm{T}]$*. If*

$$u(t) \le c + \int_0^t v(s)u(s)ds\ forall 0 \le t \le T, \tag{2.4}$$

then

$$u(t) \le c\exp(\int_0^t v(s)ds)\ forall 0 \le t \le T. \tag{2.5}$$

Proof. First, we may assume that $c > 0$. Set

$$z(t) = c + \int_0^t v(s)u(s)ds\ for 0 \le t \le T.$$

Then $u(t) \le z(t)$. Moreover, by the chain rule of classical calculus, we have

$$\log(z(t)) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)}ds \le \log(c) + \int_0^t v(s)ds.$$

This implies

$$z(t) \le c\exp(\int_0^t v(s)ds)\ for 0 \le t \le T.$$

According to $u(t) \le z(t)$, the proof is complete. □

3. Fuzzy differential equation

Throughout this paper, we consider the fuzzy differential equations

$$dx(t) = f(x(t),t)dt + g(x(t),t)d\mathbf{C}_t \tag{3.1}$$

where $\mathbf{C}_t$ is a standard Liu process and f, g are some given functions. $x(t)$ is the solution to the (3.1) which is a fuzzy process in the sense of Liu.
By the definition of fuzzy differential, this equation is equivalent to the following fuzzy integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} g(x(s),s)d\mathbf{C}_s. \tag{3.2}$$

Furthermore, let us state the following conditions that are needed in the process our proof.
(I) Local Lipschitz condition: For each integer $n \geq 1$, there exists a positive constant number $\mathbf{L}_n$ such that

$$|f(x(t),t) - f(y(t),t)|^2 \vee |g(x(t),t) - g(y(t),t)|^2 \leq \mathbf{L}_n|x(t) - y(t)|^2,$$

for those $x(t), y(t) \in \mathbf{R}^n$ with $|x(t)| \vee |y(t)| \leq n$.
(II) Linear growth condition: There exists a positive number $\mathbf{L}$ such that

$$|f(x(t),t)|^2 \vee |g(x(t),t)|^2 \leq \mathbf{L}(1 + |x(t)|^2),$$

(III) Monotone condition: there exists a positive constant $\mathbf{K}$ such that

$$x(t)^T f(x(t),t) + \frac{1}{2}|g(x(t),t)| \leq \mathbf{K}(1 + |x(t)|^2)$$

for all $x(t) \in \mathbf{R}^n$.
The following Remark proves the exact solution to Equation (3.1) under the monotone condition **(III)**.

Remark 3.1. Assume the monotone condition **(III)**, there exists a positive constant $\mathbf{G}$ such that the solution of (2.1) satisfies

$$\mathbf{E}(\sup_{t_0 \leq t \leq T} |x(t)|^2) \vee \mathbf{E}(\sup_{t_0 \leq t \leq T} |y(t)|^2) \leq \mathbf{G}, \tag{3.3}$$

where $\mathbf{G} = \mathbf{G}(T, K, x_0)$ is a constant independent of h, ($h = \frac{1}{m}$ be a given step size with integer $m \geq 1$ and $T = Nh$).

Proof. By Liu'formula and condition **(III)**, we can derive that for $t \in [t_0, T]$

$$|x(t)|^2 = |x_0|^2 + \int_{t_0}^{t} [2(x^T f(x(s),s)) + \frac{1}{2}|g(x(s),s)|^2]ds + \int_{t_0}^{t} 2(x^T g(x(s),s))d\mathbf{C}_s,$$

we have

$$1 + |x(t)|^2 \leq 1 + |x_0|^2 + 2k \int_{t_0}^{t} [1 + |x(s)|^2]ds + 2 \int_{t_0}^{t} 2(x^T g(x(s),s))d\mathbf{C}_s,$$

we know

$$\mathbf{E}\sup(1 + |x(t)|^2) \leq (1 + |x_0|^2) + \mathbf{E}\sup \int_{t_0}^{t} 2k[1 + |x(s)|^2]ds + \mathbf{E}\sup|\int_{t_0}^{t} 2(x^T g(x(s),s))d\mathbf{C}_s|. \tag{3.4}$$

By the Lemma 2.17 (Burkholder -Davis-Gundy inequality for *Liu process*) to show that

$$\begin{aligned}
\mathbf{E}\sup|\int_{t_0}^{t} 2(x(s)^T g(x(s),s))d\mathbf{C}_s| &\leq 2\mathbf{E}(\int_{t_0}^{t} 4(|x(s)|^2|g(x(s),s)|^2)ds)^{\frac{1}{2}} \\
&\leq 2\mathbf{E}(\sup(1+|x(t)|^2)\int_{t_0}^{t} 4|g(x(s),s)|^2ds)^{\frac{1}{2}} \\
&\leq 3\mathbf{E}(\sup(1+|x(t)|^2)\int_{t_0}^{t} 4k(1+|x(t)|^2)ds)^{\frac{1}{2}} \\
&\leq 0.5\mathbf{E}\sup(1+|x(t)|^2)+\frac{9}{2}\mathbf{E}\int_{t_0}^{t} 4k(1+|x(t)|^2)ds. \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.4) and using the Hölder inequality

$$\mathbf{E}\sup(1+|x(t)|^2) \leq [2(1+|x(0)|^2)+40k(\mathbf{E}\int_{t_0}^{t}(1+|x(t)|^2)ds)],$$

so we obtain

$$\mathbf{E}\sup(1+|x(t)|^2) \leq [2(1+|x(0)|^2)+80k(\int_{t_0}^{t}\mathbf{E}\sup(1+|x(t)|^2)ds)].$$

By the Gronwall inequality, we must get

$$\mathbf{E}\sup(|x(t)|) \leq \mathbf{G}$$

where $\mathbf{G}=(T,k,x_0)$ is a constant independent of h. Similarly, we can show that $\mathbf{E}\sup(|y(t)|) \leq \mathbf{G}$. □

Theorem 3.2. ***(Existence and Uniqueness of solution)*** *Assume the locally Lipschitz condition (**I**) holds, but the linear growth condition (**II**) is replaced with the monotone condition (**III**). Then there is a unique solution $x(t)$ to equation (3.1) in $M^2((t_0,T],\mathbf{R}^d)$.*

Proof. By using from truncation procedure. For each $n \geq 1$, define the truncation function

$$f_n(x(t),t)=\begin{cases} f(x(t),t), & |x(t)| \leq n \\ f(\frac{nx(t)}{|x(t)|},t), & |x(t)| > n \end{cases}$$

$$g_n(x(t),t)=\begin{cases} g(x(t),t), & |x(t)| \leq n \\ g(\frac{nx(t)}{|x(t)|},t), & |x(t)| > n, \end{cases}$$

then f_n and g_n satisfy Lipschitz condition. So that, equation

$$x_n(t)=x(0)+\int_{t_0}^{t} f_n(x_n(s),s)ds+\int_{t_0}^{t} g_n(x_n(s),s)d\mathbf{C}_s, \quad t_0 leqt \leq T \qquad (3.6)$$

Due to Remark 3.1, $x(t)$ is a unique solution to equation (3.1) in $\mathbf{M}^2((t_0,T],\mathbf{R}^d)$. In addition, $x_n(t) \in \mathbf{M}^2((t_0,T],\mathbf{R}^d)$. Of course, $x_{n+1}(t)$ is the unique solution of equation

$$x_{n+1}(t)=x(0)+\int_{t_0}^{t} f_{n+1}(x_{n+1}(s),s)ds+\int_{t_0}^{t} g_{n+1}(x_{n+1}(s),s)d\mathbf{C}_s, \quad t_0 \leq t \leq T,$$

and $x_{n+1}(t) \in \mathbf{M}^2((t_0, T], \mathbf{R}^d)$. Define the stopping time $\tau_n = T \wedge \inf\{t \in [t_0, T] : |(x_n)_t| \geq n\}$. By taking the expectation, and by the Hölder inequality, we have

$$\begin{aligned}
\mathbf{E}|x_{n+1}(t) - x_n(t)|^2 \leq\ & 2\mathbf{E}|\int_{t_0}^{t} [f_{n+1}(x_{n+1}(s), s)]ds - \int_{t_0}^{t} [f_n(x_n(s), s)]ds|^2 \\
& + 2\mathbf{E}|\int_{t_0}^{t} [g_{n+1}(x_{n+1}(s), s)]d\mathbf{C}s - \int_{t_0}^{t} [g_n(x_n(s), s)]d\mathbf{C}_s|^2 \\
\leq\ & 2(t - t_0)\mathbf{E}\int_{t_0}^{t} |f_{n+1}(x_{n+1}(s), s) - f_n(x_n(s), s)|^2 ds \\
& + \mathbf{E}\int_{t_0}^{t} |g_{n+1}(x_{n+1}(s), s) - g_n(x_n(s), s)|^2 ds \\
\leq\ & 4(t - t_0)\mathbf{E}\int_{t_0}^{t} [|f_{n+1}(x_{n+1}(s), s) - f_{n+1}(x_n(s), s)|^2 \\
& + |f_{n+1}(x_n(s), s) - f_n(x_n(s), s)|^2]ds \\
& + 4\mathbf{E}\int_{t_0}^{t} [|g_{n+1}(x_{n+1}(s), s) - g_{n+1}(x_n(s), s)|^2 \\
& + |g_{n+1}(x_n(s), s) - g_n(x_n(s), s)|^2]ds.
\end{aligned}$$

For $t_0 \leq t \leq \tau_n$, getting

$$\begin{aligned}
& f_{n+1}(x_n(s), s) = f_n(x_n(s), s) = f(x_n(s), s), \\
& g_{n+1}(x_n(s), s) = g_n(x_n(s), s) = g(x_n(s), s),
\end{aligned}$$

again by substituting $x_{n+1}(t_0 + s) = x_n(t_0 + s) = x(s), s \in (t_0, 0]$, one gets that

$$\begin{aligned}
& \mathbf{E}(\sup_{t_0 \leq r \leq t} |x_{n+1}(t) - x_n(t)|^2) \\
& \quad \leq 4(t - t_0)\mathbf{E}\int_{t_0}^{t} |f_{n+1}(x_{n+1}(s), s) - f_{n+1}(x_n(s), s)|^2 ds \\
& \quad + 4\mathbf{E}\int_{t_0}^{t} |g_{n+1}(x_{n+1}(s), s) - g_{n+1}(x_n(s), s)|^2 ds \\
& \quad + 4(t - t_0 + 1)\mathbf{E}\int_{t_0}^{t} K_n|(x_{n+1}(s), s) - (x_n(s), s)|^2 ds \\
& \quad \leq 4(t - t_0 + 1)K_n \int_{t_0}^{t} \mathbf{E}(\sup_{t_0 \leq r \leq s} |(x_{n+1}(s), s) - (x_n(s), s)|^2) ds.
\end{aligned}$$

From the Gronwall inequality, one sees that

$$\mathbf{E}(\sup_{t_0 \leq s \leq t} |x_{n+1}(t) - x_n(t)|^2) = 0, \quad t_0 \leq t \leq \tau,$$

so this means that for $t_0 \leq t \leq \tau_n$, we always have

$$x_n(t) = x_{n+1}(t). \tag{3.7}$$

We give that τ_n is increasing, that is as $n \to \infty, \tau_n \uparrow T$ a.s. By the monotone (**III**) condition, for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$ such that $\tau_n = T$ as $n \geq n_0$. Now define $x(t)$ by $x(t) = x_{n_0}(t), t \in [t_0, T]$. In order to verify that $x(t)$ is the solution of (3.1). By (4.1), $x(t \wedge \tau_n) = x_n(t \wedge \tau_n)$, and by (3.6), it follows that

$$\begin{aligned} x(t \wedge \tau_n) &= \xi(0) + \int_{t_0}^{t \wedge \tau_n} f_n(x(s), s) ds + \int_{t_0}^{t \wedge \tau_n} g_n(x(s), s) d\mathbf{C}_s \\ &= \xi(0) + \int_{t_0}^{t \wedge \tau_n} f(x(s), s) ds + \int_{t_0}^{t \wedge \tau_n} g(x(s), s) d\mathbf{C}_s. \end{aligned}$$

Upon letting $n \to \infty$ then yields

$$x(t \wedge \tau_n) = \xi(0) + \int_{t_0}^{t \wedge \tau_n} f(x(s), s) ds + \int_{t_0}^{t \wedge \tau_n} g(x(s), s) d\mathbf{C}_s$$

that is

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x(s), s) ds + \int_{t_0}^{t} g(x(s), s) d\mathbf{C}_s.$$

We can see $x(t)$ is the solution of (3.1), and $x(t) \in \mathbf{M}^2((t_0, T], \mathbf{R}^d)$. Therefor, the existence is complete. □

4. Main result

in this section in first we defined fuzzy differential equation(FDE) with jump, then a new theorem for this FDE with monotone condition

Definition 4.1. Let X_t be a fuzzy process and C_t be a fuzzy Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$; the mesh is written as

$$\triangle = \max_{1 \leq i \leq k} \mid ti + 1 - t_i \mid$$

Then the Liu integral of X_t is defined by

$$\int_a^b X_t dC_t = \lim_{\triangle \to 0} \sum_{i=1}^{k} X_{t(i)} (C_t(i+1) - C_{t(i)})$$

provided that the limit exists almost surely and is finite.

Definition 4.2. Let $\gamma_1, \gamma_2, \cdots$ be iid positive fuzzy variables. Define $S_0 = 0$ and $S_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for $n \geq 1$: Then the fuzzy process

$$N_t = max_{n \geq 0} \{ n \mid S_n \leq t \}$$

is called a fuzzy renewal process.

Definition 4.3. Let X_t be a fuzzy process and N_t be a fuzzy renewal process. Then the fuzzy integral of X_t with respect to N_t on the interval $[a, b]$ is defined by

$$\int_a^b X_t dN_t = \sum_{a \leq t \leq b} X_t (N_t - N_{t(-)})$$

provided that the sum exists almost surely and is finite.

In this section, at first, we will introduce the concept of fuzzy differential equation with jumps, and give the analytic solutions for two special types of fuzzy differential equations.

Definition 4.4. Suppose C_t is a fuzzy Liu process, N_t is a fuzzy renewal process, and f, g and h are some given real functions. Then

$$dx(t) = f(x(t), t) dt + g(x(t), t) d\mathbf{C}_t + h(x(t), t) d\mathbf{N}_t \tag{4.1}$$

is called a fuzzy differential equation with jumps. A solution is a fuzzy process X_t that satisfies (4.1) identically in t.

Remark 4.5. The fuzzy differential equation (4.1) is equivalent to the fuzzy integral equation

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)d\mathbf{C}_s + \int_0^t h(x(t), t)d\mathbf{N}_t x((\gamma), 0) = x(0). \tag{4.2}$$

In this section, we will give a sufficient condition for a fuzzy differential equation having a unique solution.

Theorem 4.6. *The fuzzy differential equation with jumps (4.1) has a unique solution if the coefficients $f(x,t)$ and $g(x,t)$ satisfy the linear growth condition Lipschitz condition and monotone condition for some constants L, and $h(x,t)$ is a real-valued function.*

Proof. For a Lipschitz continuous sample path $C_t(\gamma)$, consider the differential equation

$$dX_t(\gamma) = f(x_t(\gamma), t)dt + g(x_t(\gamma), t)d\mathbf{C}_t(\gamma) + h(x_t(\gamma), t)d\mathbf{N}_t(\gamma) \tag{4.3}$$

Given any time u, we will prove that the differential equation (4.3) has a unique solution on the interval $[0, u]$ Assume that N_t is a fuzzy renewal process generated by iid positive fuzzy variables $\xi_1, \xi_2, \cdots$. Write $S_i = \xi_1 + \xi_2 + \cdots + \xi_i$ for $i \geq 1$. Without loss of generality, we assume $N_u(\gamma) = n$, i.e., $S_n(\gamma) \leq u < S_{n+1}(\gamma)$ Since $dN_t(\gamma) = 0$ for any $t \in [0, S_1(\gamma))$, the differential equation (4.3) degenerates to

$$dX_t(\gamma) = f(x_t(\gamma), t)dt + g(x_t(\gamma), t)d\mathbf{C}_t(\gamma)$$

which has a unique and continuous solution in $[0, S_1(\gamma))$ by Theorem 3.2. So the limit

$$\lim_{t\to(S_1(\gamma))} X_t(\gamma)$$

exists, and

$$X_{S1}(\gamma) = \lim_{t\to(S_1(\gamma))} X_t(\gamma) + h(S_1(\gamma), \lim_{t\to(S_1(\gamma))} X_t(\gamma)).$$

Since $dN_t(\gamma) = 0$ on the interval $[S_1(\gamma), S_2(\gamma))$, the differential equation (4.3) degenerates to

$$dX_t(\gamma) = f(x_t(\gamma), t)dt + g(x_t(\gamma), t)d\mathbf{C}_t(\gamma)$$

with an initial value $X_{S1}(\gamma)$, which has a unique and continuous solution in $[S_1(\gamma), S_2(\gamma))$ by Theorem 3.2. So the limit

$$\lim_{t\to(S_2(\gamma))} X_t(\gamma)$$

exists, and

$$X_S 2(\gamma) = \lim_{t\to(S_2(\gamma))} X_t(\gamma) + h(S_2(\gamma), \lim_{t\to(S_2(\gamma))} X_t(\gamma)).$$

Repeating the process, we obtain that $dN_t(\gamma) = 0$ on the interval $[S_n(\gamma), u]$, and the differential equation (4.3) degenerates to

$$dX_t(\gamma) = f(x_t(\gamma), t)dt + g(x_t(\gamma), t)d\mathbf{C}_t(\gamma)$$

with an initial value $X_{S_n}(\gamma)$, which has a unique solution on the interval $[S_n(\gamma), u]$ by Theorem 3.2. Thus we prove that the differential equation (4.3) has a unique solution on the interval $[0, u]$ Since almost all the sample paths of a fuzzy Liu process C_t are Lipschitz continuous, the fuzzy differential equation with jumps (4.1) has a unique solution. □

5. Stability Theorem for fuzzy differential equation with jumps

Regarding the fact that if a physical examination is repeated twice with the same conditions, how much measurements are conducted carefully, the initial imposed conditions will not be exactly the same. it is expected that the results of two physical examinations have a litter difference which each other. This means that very small changes in initial conditions are expected to cause only small changes in answer or in other words, the answer of mathematical model is stable.In this section, we will propose a concept of stability for a fuzzy differential equation with jumps in the sense of fuzzy measure. A sufficient condition will also be derived for a fuzzy differential equation with jumps being stable.

Definition 5.1. Let x_t and y_t be two solutions of the fuzzy differential equation with jumps

$$dx(t) = f(x(t),t)dt + g(x(t),t)d\mathbf{C}_t + h(x(t),t)d\mathbf{N}_t \tag{5.1}$$

with different initial values x_0 and y_0, respectively. Then the fuzzy differential equation (5.1) is said to be stable if for any given number $\epsilon > 0$, there exists a real number δ such that

$$\mathbf{Cr}\{|x_t - y_t| > \epsilon\} \leq \epsilon$$

holds for any $t \geq 0$ provided $|x_0 - y_0| \leq \delta$.

Theorem 5.2. *The fuzzy differential equation with jumps* (5.1) *is stable if the coefficients* $f(w,t)$ *and* $g(w,t)$ *satisfy* **I** *and* **II**.

Proof. Assume that $w(t)$ and $y(t)$ are two solutions of the fuzzy differential equation with jump (5.1) with different initial values $w(0)$ and $y(0)$, respectively. Then

$$\begin{aligned} w(t) &= w_0 + \int_0^t f(w(s),s)ds + \int_0^t g(w(s),s)d\mathbf{C}_s + \int_0^t h(w(s),s)d\mathbf{N}_s., \\ y(t) &= y_0 + \int_0^t f(y(s),s)ds + \int_0^t g(y(s),s)d\mathbf{C}_s + \int_0^t h(y(s),s)d\mathbf{N}_s., \end{aligned}$$

For a Lipschitz continuous sample path $\mathbf{C}_{t_\theta}$, we have

$$\begin{aligned} |w(t,\theta) - y(t,\theta)| \leq |w(0) - y(0)| &+ \int_0^t \mathbf{L}_1(s)|(w(s,\theta)) - (y(s,\theta))|ds \\ &+ \int_0^t \mathbf{L}_1(s)\mathbf{K}(s,\theta)|(w(s,\theta)) - (y(s,\theta))|d\mathbf{C}(s,\theta) \\ &+ \int_0^t \mathbf{L}_2(s)|(w(s,\theta)) - (y(s,\theta))|d\mathbf{N}(s,\theta). \\ = |w(0) - y(0)| &+ \int_0^t \mathbf{L}(s)(1 + \mathbf{K}(\theta))|w(s,\theta) - y(s,\theta)ds \\ &+ \int_0^t \mathbf{L}_2(s)|(w(s,\theta)) - (y(s,\theta))|d\mathbf{N}(s,\theta). \end{aligned}$$

where $\mathbf{K}(\theta)$ is the Lipschitz constant of $C(t,\theta)$. Assume that N_t is a fuzzy renewal process generated by iid positive fuzzy variables $\xi_1, \xi_2, \cdots$. Write $S_0 = 0$ and $S_i = \xi_1 + \xi_2 + \cdots + \xi_i$ for $i \geq 1$. Thus we have

$$|w(t,\theta) - y(t,\theta)| \leq |w(0) - y(0)| + \exp(1 + \mathbf{K}(\theta)) \int_0^\infty \mathbf{L}_1(s)ds + \sum_{i=1}^{\infty} \mathbf{L_2}(S(i,\theta))$$

Thus for any given number $\epsilon > 0$, we have

$$\mu\{|x_t - y_t| > \epsilon\}\mu\{|w(0) - y(0)| + \exp(1 + \mathbf{K}(\theta)) \int_0^\infty \mathbf{L}_1(s)ds + \sum_{i=1}^{\infty} \mathbf{L}_2(S(i)) > \epsilon\}.$$

where $\mathbf{K}$ is a fuzzy variable generated by the Lipschitz constants of the fuzzy Liu process C_t. Since $\mathbf{L}_2(t)$ is an integrable and monotone decreasing function on $[0,\infty)$, we have by the Grown wall's equality.
For any given $\varepsilon > 0$, there exists a real number $\mathbf{H}$ such that

$$\begin{aligned} &\mathbf{Cr}\{\theta|\mathbf{K}(\theta) \leq \mathbf{H}\} \geq 1 - \varepsilon \\ &\delta = \exp(-(1 + \mathbf{H}) \int_\tau^\infty \mathbf{L}(s)ds)\varepsilon. \end{aligned}$$

Then $|w(t,\theta)-y(t,\theta)| \le \varepsilon$ for any positive real number t provided that $|x(0)-y(0)| \le \delta$ and $\mathbf{K}(\theta) \le \mathbf{H}$.
It means $\mathbf{Cr}\{|w(t)-y(t)| > \varepsilon\} < \varepsilon$ for any $t \le 0$ as long as $|w(0)-y(0)| \le \delta$.
In other words,

$$\lim_{|w_0-y_0|\to 0} \mathbf{Cr}\{|x(t)-y(t)| > \varepsilon\} = 0$$

so the fuzzy differential equation with jumps is stable. □

6. Conclusions

The determination of existence and uniqueness is a cornerstone in the theoretical analysis of fuzzy differential equations (FDEs) with jumps. In this paper, we have established a new existence and uniqueness theorem for the considered class of equations under local Lipschitz and monotonicity conditions. By refining the analytical approach and strengthening the underlying conditions, our results provide a more rigorous mathematical framework for these systems. These findings not only address current theoretical challenges but also offer a robust foundation for future investigations into the qualitative behavior and stability of fuzzy dynamical systems with jump discontinuities.

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