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Analyzing lumps, manifold periodic, interactions and rogue wave solutions for Klein-Gordon equation

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Abstract

In present article, we discourse the problem of solving the non-linear Klein-Gordon equation to understand the wave dynamics in numerous physical systems. This famous mathematical model has significant practical applications, particularly in optics, quantum mechanics, solid state physics and cosmology. The purpose of this work is to analyze various lumps, breathers and interaction solutions to the renowned non-linear Klein-Gordon equation (KGE) via Hirota bilinear method. In particular, we obtain lump solution, lump one stripe, lump two stripe, lump periodic solution, manifold periodic solution, rogue wave solution, Ma breather and its relating rogue wave and Kuznetsov-Ma breather and its relating rogue wave for the proposed model. Furthermore, we have also shown interaction between lump, periodic and kink-wave. To calculate these solutions, the symbolic software Mathematica is used. The obtained solutions are demonstrated by generating two-dimensional, three-dimensional, contour and density graphs. The results obtained appear as a noteworthy participation to the scientific world, providing a wide range of applications in many fields of science and technology. Hirota bilinear method appears as one of the most effective tools for constructing such solutions to various non-linear partial differential equations. This technique is authentic and simple for calculating several solutions to the proposed mathematical model. The study highlights the remarkable features of the applied method and its ability to handle various other mathematical models.

Keywords. Hirota bilinear method, Klein-Gordon equation, Lumps, Manifold periodic solution, Rogue wave solution.
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1. Introduction

It is well known that most of the phenomena arising in the fields of mathematics [60], physics [18, 50], ecology [21], and engineering [2, 14, 33] are described by partial differential equations. Also, various problems appearing in electrostatics [19, 20], plasma physics [1, 26], chemical kinetics [45] and fluid dynamics [3, 36, 56] can be modeled through non-linear partial differential equations. Non-linear KGE is an example of such partial differential equations which appears as an essential model of quantum mechanics [25, 34, 59]. This equation is of great importance in solid-state physics [6]. The KGE, related to the Schrödinger equation, is a relativistic wave equation. This equation is of order two both in time and space. The purpose of present article is to study the solutions of KGE using Hirota bilinear method. Many researchers studied analytical solutions of non-linear KGE by using different methods. In the year 2022, using clique polynomials, Kumbinarasaiah constructed an operational matrix of integration and generated an innovative method to get an approximate solution for the KGE [27]. Sayed Saifullah investigated the nonlinear KGE with Caputo fractional derivative [52]. The novel approximate solutions for the nonlinear KGE were presented by Necdet Bildik and Sinan Deniz in 2020. These solutions were developed by using newly constructed optimal perturbation iteration technique and perturbation iteration method [7]. Keskin and Oturanc proposed the reduced differential transform method (RDTM) for finding the solutions of partial differential equations. It is an iterative method derived by using the Taylor series solution of differential equations. This method has been used successfully to get the solutions of numerous nonlinear partial differential equations. In 2020, Wayinhareg Gashaw Belayeh implemented the reduced

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differential transform technique for obtaining the solution of two-dimensional nonlinear KGE with quadratic and cubic nonlinearities dependent on suitable initial conditions [8].

In recent years, the interest in the KGE with various potentials has raised because of its probable significance in hadron physics [13], cosmology [43] and high-energy physics. By applying the Feshbach-Villars approach, Megías illustrated, under the effect of an external potential, the solutions of the KGE for bosons [44]. By applying the extended sinh Gordon expansion technique and the generalized Kudryashov approach, Alrazi Abdeljabbar studied the rogue wave, dark and bright solitonic character solutions of the quadratic nonlinear KGE [4]. Traveling waves characteristically keep a constant speed during their diffusion. There has been a remarkable focus on traveling wave solutions in recent years. These waves have been detected in sevsral areas of science, such as in burning following a chemical reaction. In the field of mathematical biology, the impulses occuring in nerve fibers appear as traveling waves, and the conservation laws relevant to topics in fluid dynamics illustrate shock profiles as traveling waves. By applying the qualitative theory of planar dynamical systems, Meraa Arab computed the traveling wave solutions to the nonlinear KGE [5]. For the nonlinear KGE having weak nonlinearity, Yue Feng studied the error analysis for longer time of the 4th order compact finite difference approaches [16].

For a test particle with an electric charge, having motion in an external electro-magnetic field in a spacetime manifold on the isotropic hydrosphere, the algebra of the symmetry operators for the KGE is significant. Jinxing Liu developed a numerical as well as an analytical technique for computing the solutions of a family of fractional nonlinear and linear KGEs appearing in relativistic, quantum and classical mechanics [32]. Kholmetskii illustrated the KGE for a charged particle without spin in the existence of an electromagnetic field and focused on its familiar inadequacy, associated to the presence of solutions with a negative probability density [28]. Pablo Rabán studied the stability of topological solitary waves and pulses in one-dimensional non-linear Klein-Gordon systems [51]. Jiahua Fang presented a novel iterative procedure to generate the approximate solutions for the KGE as well as sine-Gordon equation (SGE). This technique was developed by the combination of the homotopy perturbation method (HPM) and the Mohand transform [17]. Ross Parker considered, both for hard and soft symmetric potentials, the spectral stability and the existence of multi-breather structures in the discrete KGE [47].

At the atomic scale, the events seen could not be sufficiently illustrated by using the classical theories of physics. Therefore, a novel abstract framework was construted for physics. The name proposed for this new theory in physics is "Quantum mechanics". Schrödinger, in his study for an equation illustrating the de Broglie waves, considered the KGE as a quantum wave equation. The wave function is vital in relativistic quantum mechanics, since it has all the information illustrating a quantum system. In recent years, how to find the solution of the KGE has become the topic of remarkable interest. Khalid Reggab studied the semi-inverse variational technique to compute the solution of the KGE for perturbed and harmonic Coulomb potentials [48]. In 2023, Cardona-García studied the solution to the KGE by finite element technique [10]. Natalia Kolkovska studied the non-existence of global solutions to Klein-Gordon equations with variable coefficients power-type nonlinearities [31]. Buyang Li studied a second-order low-regularity correction of lie splitting for the semi linear Klein-Gordon equation [35]. For finding the approximate solution relevant to the extended n-term local fractional KGE, the efficiency of the local fractional reduced differential transformation technique was illustrated by Nikhil Sharma in 2023. In this study, local fractional derivative and fractional complex transform was used to investigate the n-term KGEs and Cantor sets. The presented technique, together with the existence of the solutions illustrated through some examples, presented a strong mathematical tool for finding the solution to fractional linear differential equations [54].

Fractional calculus, a theory including integeration and differentiation of fractional order [9], has been effectively used in numerous areas of Science and engineering [55]. These arbitrary order operators appear as powerful and effective mathematical means to demonstrate the scientific models of various daily life probelms. In recent times, many researchers have described that fractional order operators illustrate a more accurate interpretation of various daily life phenomena in many fields like biology [24, 57], medicine [29] and engineering [53]. In 2024, Mohammad Partohaghighi studied numerical approximation of the fractional KGE with discrete Chebyshev polynomials [46]. Kachhia and Prajapati studied solutions of non-linear time fractional KGE applying composite fractional derivatives [30]. Lixia Wang studied multiple solutions for nonhomogeneous Klein-Gordon equation with sign-changing potential coupled with born-infeld theory [11]. Focusing on multi-solitons for the Klein-Gordon equation, Gong Chen established their conditional asymptotic stability [12]. Lijia Han studied the standing wave solutions of Klein-Gordon equation with logarithmic nonlinearity [22]. Murat Uzunca presented energy-preserving linearly implicit integrators for the nonlinear KGE, based on polarization of the polynomial functions [58].

In recent times, lumps, interactions and rogue wave solutions have played a significant role to describe the wave properties for suitable nonlinear partial differential equations. Lumps, appearing as significant non-linear wave structures, are seen in numerous domains of science and technology. Lump waves can be illustrated analytically as a limit structure of solitonic character wave and travel carrying high energy as compared to any other soliton. The (1+1)-dimensional KGE [37] can be written as

$$u_{tt} - u_{xx} + qu + su^3 = 0. \tag{1.1}$$

In above equation, $u = u(x,t)$ is the real-valued unknown function. x and t are the independent variables with real values and q, s are real constants. The present article has the following structure: Section 1 contains introduction. In Section 2, lump solution for the proposed equation is given. Moving to Section 3, lump 1-stripe solution is computed. Lump 2-stripe solution of proposed equation is evaluated in Section 4. In Section 5, lump periodic solution is provided. In Section 6, manifold periodic solution is presented. Section 7, presents rogue wave solution of proposed model. Moving to Section 8, interaction between lump, periodic and 1-kink wave is computed. Ma-breather and its relating rogue wave is discussed in Section 9. Moving to Section 10, Kuznetsov-Ma breather and its related roguewave is discussed. The computed results are interpreted in detail by drawing graphs in Section 11. Finally, conclusion of the study is provided in Section 12.

2. Lump solution

To get the billinear form of Eq. (1.1), the following transformation [49] will be used:

$$u(x,t) = 2b(\ln f)_x + m. \tag{2.1}$$

Above transformation gives,

$$\begin{aligned} &mqf^3 + m^3sf^3 + 2bqf^2f_x + 6bm^2sf^2f_x + 4bf_xf_t{}^2 \\ &- 2bff_{tt}f_x + 12b^2msff_x{}^2 - 4bf_x{}^3 + 8b^3sf_x{}^3 \\ &- 4bff_tf_{xt} + 2bf^2f_{xtt} + 6bff_xf_{xx} - 2bf^2f_{xxx} = 0. \end{aligned} \tag{2.2}$$

To compute the lump solution, the following form of f [49] will be used:

$$f = \beta_1{}^2 + \beta_2{}^2 + a_7, \tag{2.3}$$

where $\beta_1 = a_1x + a_2t + a_3$, $\beta_2 = a_4x + a_5t + a_6$ and $a_i(1\leq i\leq 7)$ are some parameters. By substituting f into Eq. (2.2) and putting the coefficients of the variables t and x equal to zero, we get the solution.
When $a_3 = a_6 = 0$, then

$$a_2 = -2\sqrt{-\left(a_1^2\left(2b^2s-1\right)\right)} - ia_5,\ a_4 = -ia_1,\ m = 0,\ q = 0,\ a_7 = 0,\ a_1 = a_1,\ a_5 = a_5. \tag{2.4}$$

These parameters generate the lump solution

$$u_1(x,t) = \frac{2b\left(\frac{2a_5\left(\frac{a_5x}{\sqrt{8b^2s-1}}+t\left(-2\sqrt{-\frac{a_5^2(2b^2s-1)}{8b^2s-1}}-ia_5\right)\right)}{\sqrt{8b^2s-1}} - \frac{2ia_5\left(a_5t-\frac{ia_5x}{\sqrt{8b^2s-1}}\right)}{\sqrt{8b^2s-1}}\right)}{\left(\frac{a_5x}{\sqrt{8b^2s-1}}+t\left(-2\sqrt{-\frac{a_5^2(2b^2s-1)}{8b^2s-1}}-ia_5\right)\right)^2 + \left(a_5t-\frac{ia_5x}{\sqrt{8b^2s-1}}\right)^2}. \tag{2.5}$$

3. Lump one stripe solution

To compute the lump one stripe solution of the proposed equation, the subsequent form of f [49] will be used:

$$f={\beta_1}^2+{\beta_2}^2+a_7+b_1\exp\left(k_2t+k_1x\right), \tag{3.1}$$

where $\beta_1=a_1x+a_2t+a_3$, $\beta_2=a_4x+a_5t+a_6$. k_1, k_2, $a_i(1{\le}i{\le}7)$ and b_1 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of t, exponential functions and x equal to zero, we get the solution.
When $a_3=a_6=0$, then

$$a_1=-ia_4,\ a_2=-ia_5,\ k_2=-\sqrt{k_1^2+2q},\ m=-bk_1,\ s=-\frac{q}{b^2k_1^2},\ a_4=a_4,\ a_5=a_5,\ a_7=a_7. \tag{3.2}$$

These parameters generate lump one stripe solution

$$u_2(x,t)=-bk_1+\frac{2b\left(2a_4\left(a_5t+a_4x\right)-2ia_4\left(-ia_5t-ia_4x\right)+b_1k_1e^{k_1x-t\sqrt{k_1^2+2q}}\right)}{\left(a_5t+a_4x\right)^2+\left(-ia_5t-ia_4x\right)^2+a_7+b_1e^{k_1x-t\sqrt{k_1^2+2q}}}. \tag{3.3}$$

4. Lump two stripe solution

To compute the lump two stripe solution of the proposed equation, the subsequent form of f [49] will be used:

$$f={\beta_1}^2+{\beta_2}^2+a_7+b_1\exp\left(k_2t+k_1x\right)+b_2\exp\left(k_2t+k_1x\right), \tag{4.1}$$

where $\beta_1=a_1x+a_2t+a_3$, $\beta_2=a_4x+a_5t+a_6$. k_1, k_2, $a_i(1{\le}i{\le}7)$, b_1 and b_2 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of t, exponential functions and x equal to zero, we get the solution.
When $a_3=a_6=0$, then

$$a_1=ia_4,\ a_2=ia_5,\ k_2=\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)},\ m=-bk_1,\ q=-b^2k_1^2s,\ a_4=a_4,\ a_5=a_5,\ a_7=a_7. \tag{4.2}$$

These parameters generate lump two stripe solution

$$u_3(x,t)=-bk_1+\frac{2b\left(2a_4\left(a_5t+a_4x\right)+2ia_4\left(ia_5t+ia_4x\right)+b_1k_1e^{t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}+k_1x}+b_2k_1e^{t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}+k_1x}\right)}{\left(a_5t+a_4x\right)^2+\left(ia_5t+ia_4x\right)^2+a_7+b_1e^{t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}+k_1x}+b_2e^{t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}+k_1x}}.$$

5. Lump periodic solution

To compute the lump periodic solution, the subsequent form of f [49] will be used:

$$f={\beta_1}^2+{\beta_2}^2+a_7+b_1\cos\left(p_1t+k_1x\right), \tag{5.1}$$

where $\beta_1=a_1x+a_2t+a_3$, $\beta_2=a_4x+a_5t+a_6$. p_1, $a_i(1{\le}i{\le}7)$ and k_1 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of t, cos functions and x equal to zero, we get the solution.
When $a_3=a_6=m=0$, then

$$a_1=ia_4,\ a_5=-ia_2,\ q=4b^2k_1^2s,\ p_1=-\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)},\ a_7=0,\ a_2=a_2,\ a_4=a_4. \tag{5.2}$$

These parameters generate lump periodic solution

$$u_4(x,t)=\frac{2b\left(2a_4\left(a_4x-ia_2t\right)+2ia_4\left(a_2t+ia_4x\right)-b_1k_1\sin\left(k_1x-t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}\right)\right)}{\left(a_4x-ia_2t\right)^2+\left(a_2t+ia_4x\right)^2+b_1\cos\left(k_1x-t\sqrt{-\left(k_1^2\left(2b^2s-1\right)\right)}\right)}. \tag{5.3}$$

6. Manifold periodic solution

To compute the manifold periodic solution, the subsequent form of f [49] will be used:

$$f = \exp\left(\beta_1\left(-d_1\right)\right) + h_1 \exp\left(\beta_1 d_1\right) + h_2 \cos\left(\beta_2 d_2\right), \tag{6.1}$$

where $\beta_1 = a_1 x + a_2 t + a_3$, $\beta_2 = a_4 x + a_5 t + a_6$. d_1, d_2, $a_i (1 \leq i \leq 7)$, h_1 and h_2 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of t, cos functions and x equal to zero, we get the solution.
When $m = a_6 = 0$, then

$$\begin{gathered} a_2 = \frac{a_1 \sqrt{-\left(a_4^2\left(2b^2 s - 1\right)\right)}}{a_4},\ a_5 = \sqrt{-\left(a_4^2\left(2b^2 s - 1\right)\right)},\ d_1 = \frac{i\sqrt{q}}{2a_1 b\sqrt{s}}, \\ d_2 = \frac{\sqrt{q}}{2a_4 b\sqrt{s}},\ h_1 = 0,\ a_1 = a_1,\ a_3 = a_3,\ a_4 = a_4,\ a_7 = a_7. \end{gathered} \tag{6.2}$$

These parameters generate manifold periodic solution

$$u_5(x,t) = \frac{2b\left(-\frac{h_2\sqrt{q}\sin\left(\frac{\sqrt{q}\left(t\sqrt{-\left(a_4^2\left(2b^2 s-1\right)\right)}+a_4 x\right)}{2a_4 b\sqrt{s}}\right)}{2b\sqrt{s}} - \frac{i\sqrt{q}\exp\left(-\frac{i\sqrt{q}\left(\frac{a_1 t\sqrt{-\left(a_4^2\left(2b^2 s-1\right)\right)}}{a_4}+a_1 x+a_3\right)}{2a_1 b\sqrt{s}}\right)}{2b\sqrt{s}}\right)}{h_2\cos\left(\frac{\sqrt{q}\left(t\sqrt{-\left(a_4^2\left(2b^2 s-1\right)\right)}+a_4 x\right)}{2a_4 b\sqrt{s}}\right) + \exp\left(-\frac{i\sqrt{q}\left(\frac{a_1 t\sqrt{-\left(a_4^2\left(2b^2 s-1\right)\right)}}{a_4}+a_1 x+a_3\right)}{2a_1 b\sqrt{s}}\right)}. \tag{6.3}$$

7. Rogue wave solution

To compute the rogue wave solution, the subsequent form of f [49] will be used:

$$f = {\beta_1}^2 + {\beta_2}^2 + a_7 + b_1 \cosh\left(p_1 t + k_1 x\right), \tag{7.1}$$

where $\beta_1 = a_1 x + a_2 t + a_3$, $\beta_2 = a_4 x + a_5 t + a_6$. p_1, $a_i (1 \leq i \leq 7)$, b_1 and k_1 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of t, hyperbolic functions and x equal to zero, we get the solution.
When $a_3 = a_6 = m = 0$, then

$$a_1 = i a_4,\ a_2 = i a_5,\ a_7 = 0,\ s = \frac{k_1^2 - p_1^2}{2b^2 k_1^2},\ q = 2\left(p_1^2 - k_1^2\right),\ a_4 = a_4,\ a_5 = a_5. \tag{7.2}$$

These parameters generate rogue wave solution

$$u_6(x,t) = \frac{2b\left(2a_4\left(a_5 t + a_4 x\right) + 2i a_4\left(i a_5 t + i a_4 x\right) + b_1 k_1 \sinh\left(k_1 x + p_1 t\right)\right)}{\left(a_5 t + a_4 x\right)^2 + \left(i a_5 t + i a_4 x\right)^2 + b_1 \cosh\left(k_1 x + p_1 t\right)}. \tag{7.3}$$

8. Interaction between lump, periodic and 1-kink wave

We suppose f in the subsequent form [49]

$$f = {\beta_1}^2 + {\beta_2}^2 + a_7 + b_1 \cosh\left(p_1 t + k_1 x\right) + b_2 \exp\left(k_4 t + k_3 x\right), \tag{8.1}$$

where $\beta_1 = a_1 x + a_2 t + a_3$, $\beta_2 = a_4 x + a_5 t + a_6$. b_1, b_2, $a_i (1 \leq i \leq 7)$, p_1, k_1, k_3 and k_4 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients of x, exponential functions, hyperbolic functions and t equal to zero, we get the solution.
When $a_3 = a_6 = a_7 = 0$, then

$$a_4 = 0,\ a_5 = 0,\ a_2 = -a_1,\ q = 0,\ s = 0,\ m = 2bk_4,\ k_1 = -p_1,\ k_3 = -k_4,\ a_1 = a_1,\ a_5 = a_5. \tag{8.2}$$

These parameters generate the solution

$$u_7(x,t) = \frac{2b\left(2a_1\left(a_1x - a_1t\right) - b_2k_4e^{k_4t-k_4x} - b_1p_1\sinh\left(p_1t - p_1x\right)\right)}{\left(a_1x - a_1t\right)^2 + b_2e^{k_4t-k_4x} + b_1\cosh\left(p_1t - p_1x\right)} + 2bk_4. \tag{8.3}$$

9. Ma-breather and its relating rogue wave

The subsequent form of f [49] will be assumed:

$$a_1 + b_1\exp\left(2\left(k_1t + k_2\right)\right) + \exp\left(k_1t + k_2\right)\exp\left(-ip_1x\right) + \exp\left(ip_1x\right) + 1, \tag{9.1}$$

where b_1, a_1, p_1, k_1 and k_2 are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients appearing with exponential functions equal to zero, we get

$$b_1 = 0,\ b = \frac{\sqrt{k_1^2 + 4p_1^2}}{2\sqrt{2}p_1\sqrt{s}},\ m = 0,\ q = \frac{1}{2}\left(k_1^2 + 4p_1^2\right),\ a_1 = -1,\ k_1 = k_1,\ k_2 = k_2,\ p_1 = p_1. \tag{9.2}$$

These parameters generate the solution

$$u_8(x,t) = \frac{\sqrt{k_1^2 + 4p_1^2}\left(ip_1e^{ip_1x} - ip_1e^{k_1t+k_2-ip_1x}\right)}{\sqrt{2}p_1\sqrt{s}\left(e^{k_1t+k_2-ip_1x} + e^{ip_1x}\right)}. \tag{9.3}$$

10. Kuznetsov-Ma breather and its relating roguewave

We suppose f in the succeeding form [49]

$$a_1\cos\left(p\left(b_1t + x\right)\right) + a_2\cos\left(p\left(x - b_1t\right)\right) + \exp\left(-p_1\left(x - b_1t\right)\right), \tag{10.1}$$

where a_1, p_1, b_1, a_2 and p are some particular parameters. By substituting f into Eq. (2.2) and putting the coefficients appearing with exponential and cos functions equal to zero, we get

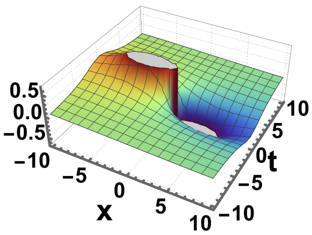
$$a_1 = 0,\ p_1 = \frac{i\sqrt{q}}{b\sqrt{s}},\ p = 0,\ m = \frac{i\sqrt{q}}{\sqrt{s}},\ b_1 = \sqrt{1 - 2b^2s},\ a_2 = a_2. \tag{10.2}$$

These parameters generate the solution

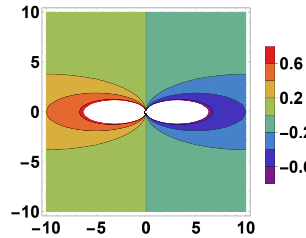
$$u_9(x,t) = \frac{i\sqrt{q}}{\sqrt{s}} - \frac{2i\sqrt{q}e^{-\frac{i\sqrt{q}\left(x - t\sqrt{1-2b^2s}\right)}{b\sqrt{s}}}}{\sqrt{s}\left(a_2 + e^{-\frac{i\sqrt{q}\left(x - t\sqrt{1-2b^2s}\right)}{b\sqrt{s}}}\right)}. \tag{10.3}$$

11. Graphical representation

Visually alluring illustrations of the analytical solutions to Klein-Gordon equation have been shown in this section. These obtained solutions are computed by the Hirota bilinear method. It is necessary to place emphasis on the statement that the particular profiles and properties of obtained solutions may vary depending on the nonlinearities and the values of parameters involved in the equation. Our findings unveil the newness of our results, because they have not been described in any previously available material. To demonstrate the wave profiles, we have generated two-dimensional (2D), three-dimensional (3D) and their concerned contour and density graphs. After setting the values of parameters involved in the concerned equations, we employ a large number of graphical illustrations, each showing a different type of the solution. The graphs in our representation unveil elegantly these computed solutions in 2D and 3D, giving a complete analysis of the wave profiles. The concerned contour graphs which we obtain additionally strengthen the clearness of our solutions to proposed model, making it simpler to understand their complicated properties. It is necessary to highlight that the novelty of our technique is represented by the uniqueness of the findings, making the path for new innovations into the features of these proposed equations. Our findings will participate to the betterment of research work in this field, providing a wide range of applications in many fields of science and technology. Moreover, the versatile nature of our technique permits us to comprehend and recognize the properties of the results with accuracy, giving remarkable perceptions into the fundamental dynamics of the proposed



(A)



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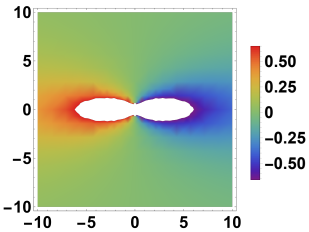

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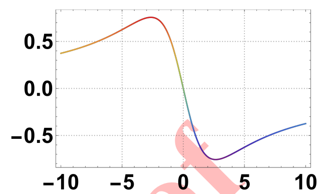


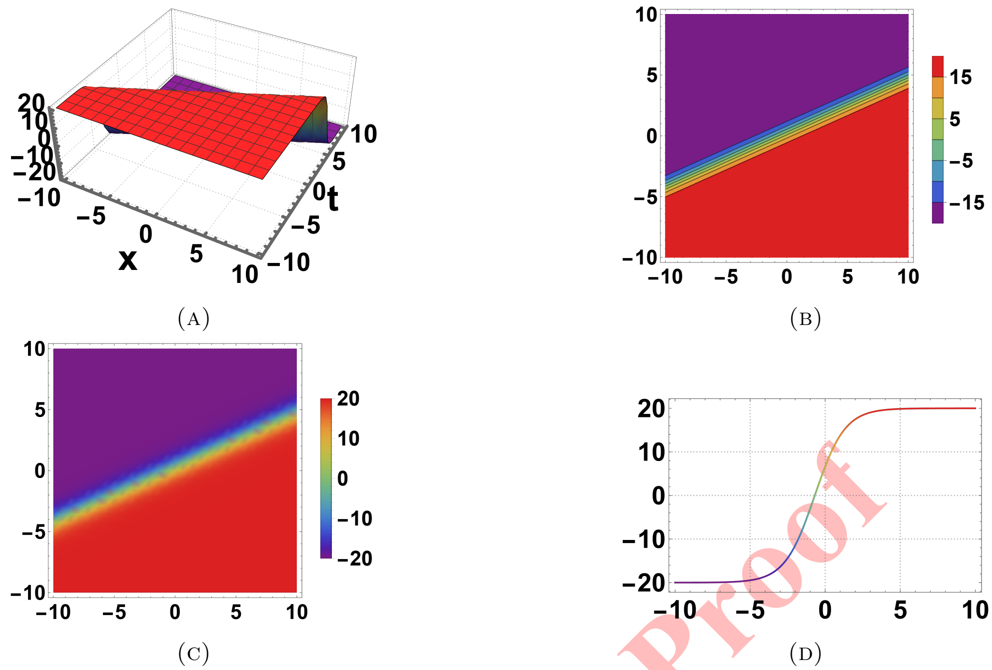
(D)

FIGURE 1. Graphical representation of $u_1(x,t)$ with different values of parameters $b = -2,\ a_2 = 2,\ a_5 = 10,\ s = 1$, and $q = 2$.

model. Our results appear as a noteworthy participation to the scientific world, providing innovative and interesting results that were not explored.

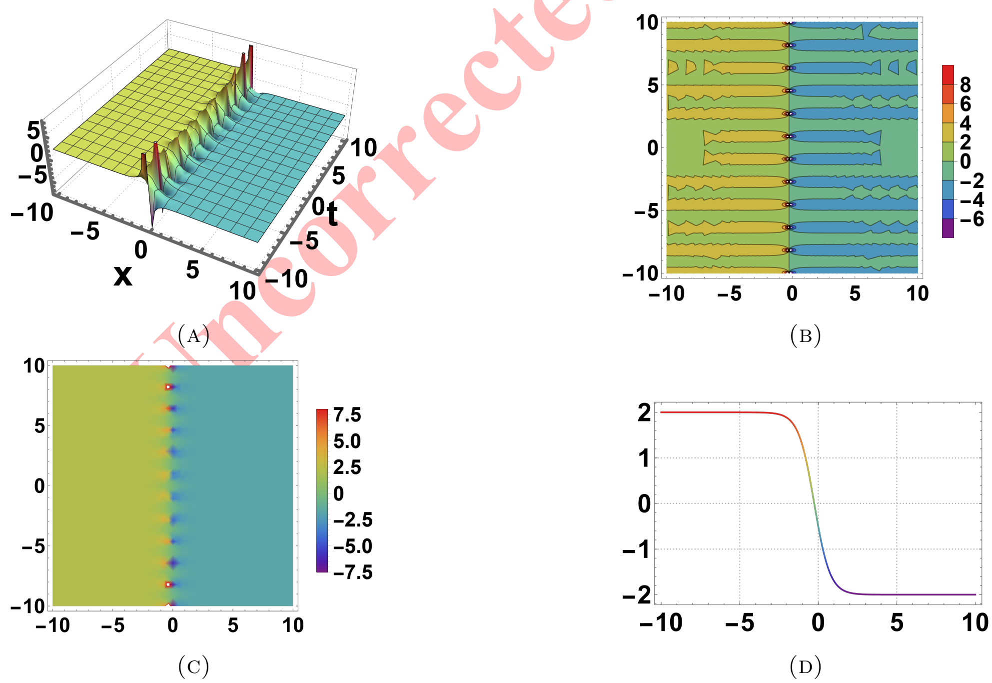
Each figure shown in this section presents a different type of solution and these obtained solutions possess many practical applications across various engineering and scientific fields. For example, lump solutions illustrated in Figures 1–4 find applications in physics. chemistry, biology, finance, oceanographic engineering, capillary flow and nonlinear optics. Rogue waves, as illustrated in Figure 6, are unpredictably greater amplitude displacements from a tranquil background and breathers, as shown in Figures 8 and 9, are pulsating modes. Initially, the rogue waves were observed by sailors in the oceans and by researchers in the fluid mechanics but are now being observed in optics and other domains also. In recent years, the rogue wave has become one of the remarkable topics in various domains of the science such as plasma, ocean dynamics, Bose-Einstein condensate and finance. Besides having the high amplitude, greater than the double of the wave present at the background, rogue waves also possess the features of unpredictability and instability. In order to avoid the damage to ships, caused due to the rogue waves produced in ocean, the learning of the rogue waves shows a significant role in daily life. Breathers, both in discrete and continuous settings, are frequently discussed in various areas of science such as Josephson junctions, Bose-Einstein condensates and optical lattices.

The article presents a comprehensive overview of the several solutions for the Klein-Gordon equation, comprising lumps, manifold periodic, interactions and rogue wave solutions. The wave solutions obtained are of great significance for illustrating the rich wave structures inherent in the nonlinear model equation under consideration. Crucially, the gained solutions are not merely mathematically rigorous, they also possess physical interpretations. In addition, the research explores exact solutions for the Klein-Gordon equation that are not previously reported in the existing literature which shows the novelty of this work. The previous work of many researchers on such solutions for different mathematical models using different techniques elegantly illustrate the importance of this research. Wen-Xiu Ma first introduced the sum-of-squares ansatz to address the significance of Lump solutions to the Kadomtsev–Petviashvili equation in detail [38]. In 2018, interaction solutions involving arbitrary functions were beautifullly presented by Wen-Xiu Ma [39]. Several types of combined solutions such as lumps, periodic waves, kink solutions, mixed lump-kink solutions, and single traveling wave solutions have also been studied in earlier works [40, 41, 61, 62]. More recently, the sum-of-squares ansatz has also been applied to spatially symmetric generalized nonlinear wave equations [15, 23, 42]. The potential extension of the algorithm adopted in the present work represents an intriguing direction for the future research.



(A) (B) (C) (D)

FIGURE 2. Graphical representation of $u_2(x,t)$ with different values of parameters $b = 20$, $a_4 = 4$, $a_5 = 2$, $k_1 = 1$, $q = 2$, $a_7 = 1$ and $b_1 = 2$.



(A) (B) (C) (D)

FIGURE 3. Graphical representation of $u_3(x,t)$ with different values of parameters $b = -1$, $a_7 = 3$, $s = 2$, $a_4 = 2$, $a_5 = 10$, $k_1 = 2$, $b_1 = 2$ and $b_2 = 3$.

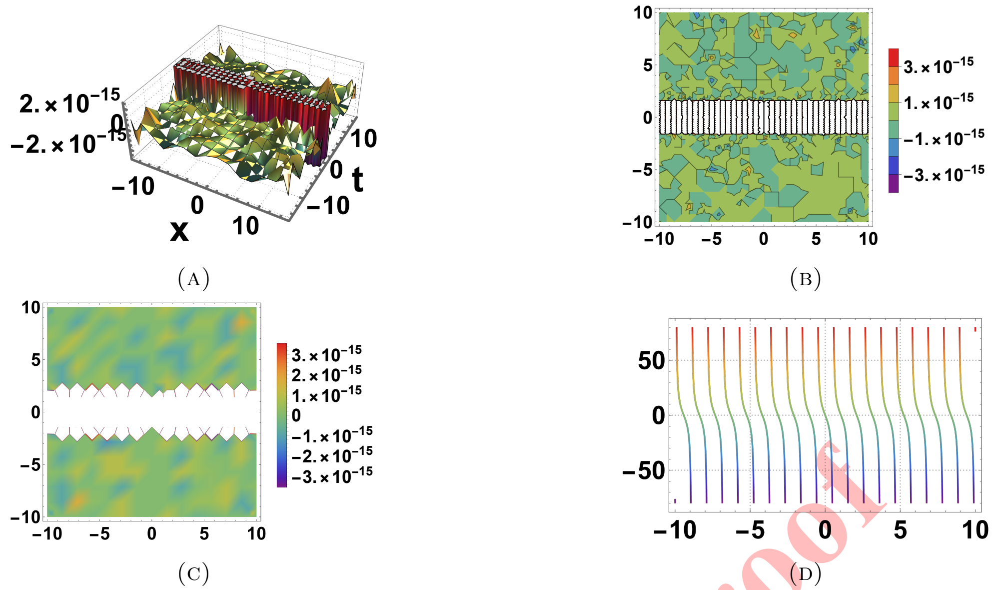


FIGURE 4. Graphical representation of $u_4(x,t)$ with different values of parameters $b = 2,\ b_1 = 1,\ k_1 = 3,\ a_2 = 3,\ a_4 = 1,\ a_7 = 2$ and $s = 2$.

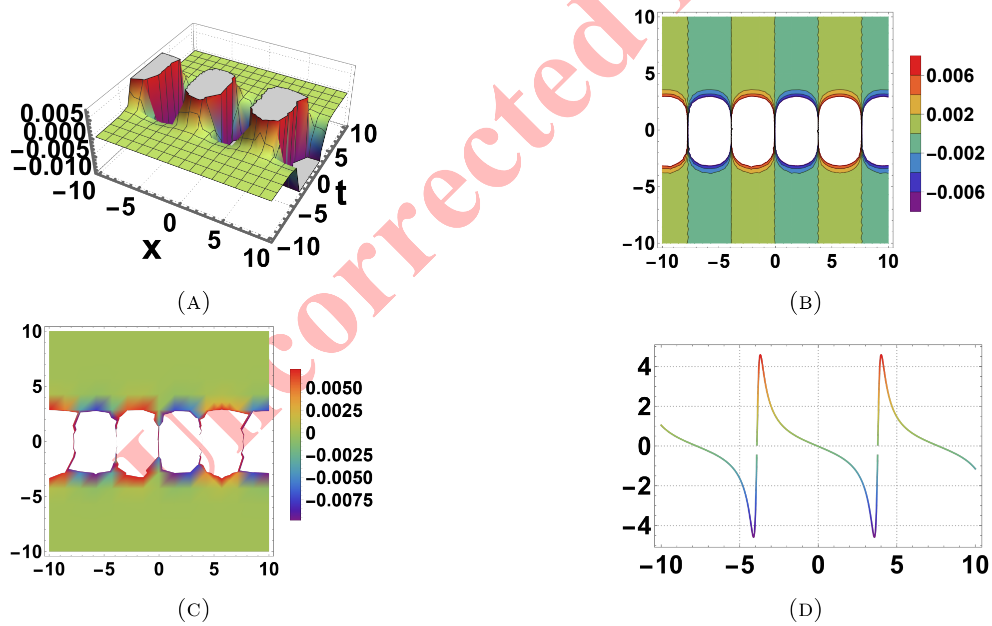


FIGURE 5. Graphical representation of $u_5(x,t)$ with different values of parameters $b = 1,\ q = 2,\ s = 3,\ a_1 = 3,\ a_3 = 2,\ a_4 = 1$ and $h_2 = 10$.

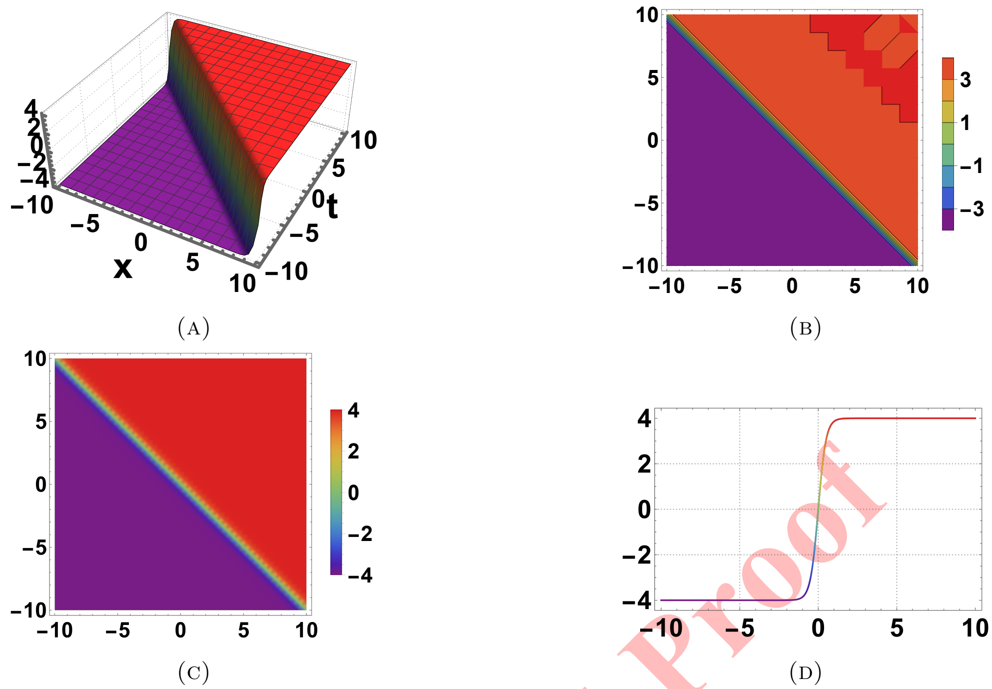


FIGURE 6. Graphical representation of $u_6(x,t)$ with different values of parameters $b=1,\ p_1=2,\ a_4=2,\ a_5=1,\ a_7=2,\ k_1=2$ and $b_1=10$.

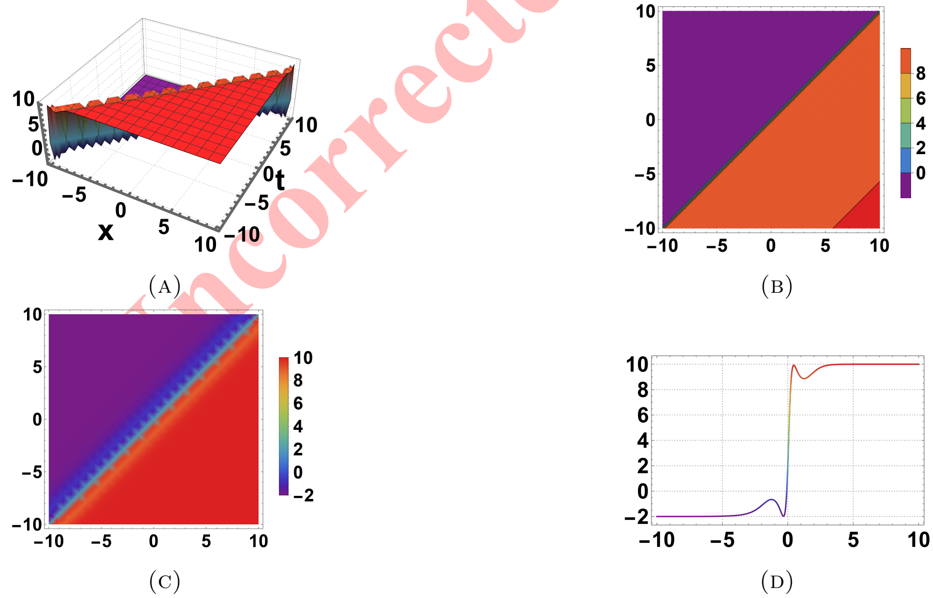


FIGURE 7. Graphical representation of $u_7(x,t)$ with different values of parameters $b=1,\ a_1=3,\ k_4=2,\ p_1=3,\ b_1=1$ and $b_2=1$.

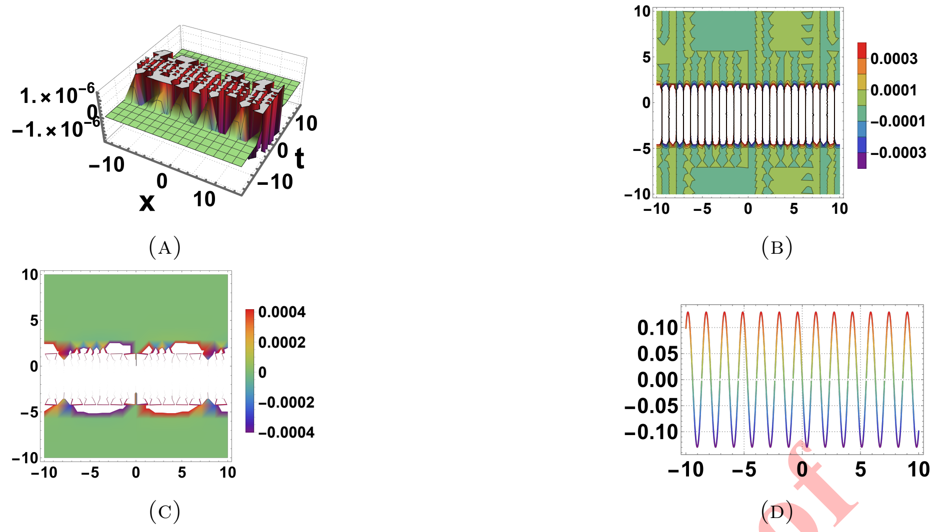


Figure 8. Graphical representation of $u_8(x,t)$ with different values of parameters $p_1 = 2,\ k_1 = 3,\ s = 1,\ k_2 = 4$ and $a_1 = 10$.

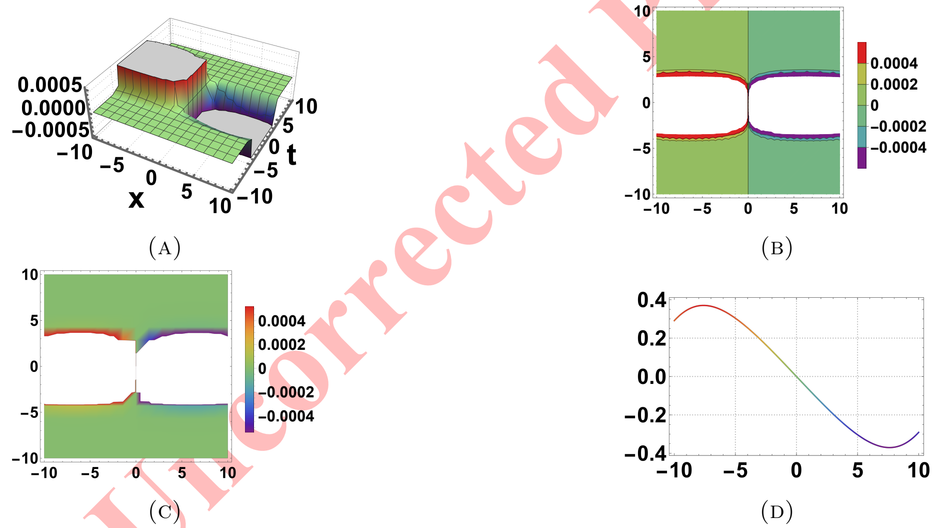


Figure 9. Graphical representation of $u_9(x,t)$ with different values of parameters $b = 5,\ p = 1,\ a_2 = 2,\ b_1 = 3,\ s = 2$ and $q = 3$.

12. Conclusion

In this paper, Klein-Gordon equation is solved using Hirota bilinear method and choosing appropriate polynomial function. The study obtained numerous solutions to proposed model which include lump solution, lump one stripe solution, lump two stripe solution and lump periodic solution. We also computed manifold periodic solution and examined it in detail. We studied and presnted rogue wave solutions in detail. Working on interaction solutions, we illustrated interaction among lump, periodic and 1-kink wave. We also studied Ma-breather and its relating rogue wave and kuznetsov-Ma breather and its relating rogue wave. We also generated 2D, 3D and their associated contour and density graphs for all computed soutions. The accuracy of our findings is remarkably increased through a combination of computational work and graphical interpretations. Hirota bilinear method is a strong tool for effectively studying such solutions to nonlinear partial differential equations. Remarkably, the obtained solutions illustrated in our study surpass that of previous work, thereby communicating appreciated innovations to the scientific world without any compromise on authenticity. This work, in terms of future goals, puts the basis for numerous potential areas of research.

Conflict of interest

The authors declare no conflict of interest.

Availability of data

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests

The authors have no relevant financial or non-financial interests to disclose.

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Ethics approval and consent to participate

Not Applicable.

Consent for publication

All authors have agreed and have given their consent for the publication of this research paper.

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Author Contribution

Jamila Habib: Conceptualization, Methodology, Software, Writing - original draft, Visualization, Investigation, Writing-review and editing.

Jamshad Ahmad: Resources, acquisition, Supervision, Writing - review and editing, Validation.

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