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Existence of $T-\vec{p}-\vec{\omega}$ solutions for quasilinear elliptic problem in anisotrpic weighted Sobolev spaces

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Abstract

In this paper, we study the following quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) = f - \sum_{1=1}^{N} D^i F_i, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $f \in L^1(\Omega)$ and $F_i \in L^{p_i'}(\Omega, \omega_i^*)$. We establish the existence of $T-\vec{p}-\vec{\omega}$ solutions, and some regularity results were concluded.

Keywords. Minty's lemma, $T-\vec{p}-\vec{\omega}$ solutions, Anisotropic weighted Sobolev spaces, Quasiliniear elliptic problems.
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1. Introduction

Let Ω be an open subset of $\mathbb{R}^N (N \geq 2)$ with a Lipschitz boundary.
In [9], Boccardo have proved the existence of $T-$solutions for the following elliptic problem

$$\begin{cases} -\text{div } a(x,u,\nabla u) = F, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the right-hand side F belongs to $W^{-1,p'}(\Omega)$ and $a(x,s,\xi)$ is a Carathéodory function satisfying the natural growth, coercivity and large monotonicity conditions by using the L^1-version of Minty's lemma (see also [11]). In the case where the right-hand side F belongs to $W^{-1,p'}(\Omega,\omega^*)$, we refer the reader to [4] (see also [1, 3, 7] for more details about the weighted Sobolev spaces).

The authors in [11] have treated the following quasilinear elliptic problem

$$\begin{cases} -\text{div } a(x,u,\nabla u) = f - \text{div } F, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$. They have proved the existence of entropy solutions. We refer the readers to [17] for $f \in W^{-1,p'}(\Omega)$ and to [18], for the existence of solutions in the sense of distributions with $f(x)$ is a Radon measure. Also to [21], where the authors have established the local and global estimates for degenerated nonlinear elliptic equations with measure data (see also [22]).

In [5], Azroul et al. established the existence of entropy solutions for an anisotropic quasilinear elliptic equation of the form

$$\begin{cases} Au + |u|^{s-1}u = f + \lambda \dfrac{|u|^{p_0-2}u}{|x|^{p_0}}, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

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where $\lambda > 0$, the data $f(x)$ is assumed to be in $L^1(\Omega)$, and A is a Leray-Lions operator acting from $W_0^{1,\vec{p}}(\Omega,\sigma)$ into its dual. For more details, we refer the readers to [10, 20, 25–27] for more details.

Recently, Bouzelmate et al. studied in [12] the following elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i\left(a_i(x,\nabla u)+\phi_i(x,u)\right)=f, & \text{in} \quad \Omega, \\ u=0, & \text{on} \quad \partial\Omega. \end{cases} \tag{1.4}$$

They established the existence and uniqueness of renormalized solutions in anisotropic weighted Sobolev spaces and obtained several regularity results. In the framework of anisotropic weighted Sobolev spaces, the authors of [6] established several fundamental properties and auxiliary lemmas. Furthermore, they investigated the existence of renormalized solutions for a class of strongly nonlinear and non-coercive elliptic equations.

In this paper, we study the existence of $T-\vec{p}-\vec{\omega}$ solution for the following nonlinear elliptic problems

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u)=f-\sum_{1=1}^{N} D^i F_i, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a_i(x,u,\nabla u)$ are a Carathéodory functions satisfying the natural growth condition and the degenerate coercivity condition, but only large monotonicity is assumed. Additionally, we assume that $f(x)\in L^1(\Omega)$ and $F_i\in L^{p_i'}(\Omega,\omega_i^*)$ for $i=1,\dots,N$ (see [13, 14, 28, 30] for more details).

The main difficulty in the study of the problem (1.5) is to prove the existence of $T-\vec{p}-\vec{\omega}$ solutions under the degenerate coercivity and large monotonicity conditions for the quasilinear elliptic equation with measure data μ in the anisotropic weighted Sobolev spaces $W^{1,\vec{p}}(\Omega,\vec{\omega})$.

This paper is organized as follows. In Section 2, we present some definitions, properties, and lemmas related to anisotropic weighted Sobolev spaces. In Section 3, we introduce the main assumptions used throughout the paper. Finally, in Section 4, we prove the existence of $T-\vec{p}-\vec{\omega}$ solutions for our nonlinear and non-coercive problem with measure data by means of Minty's lemma.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with a Lipschitz boundary.
Considering N exponents $p_1,\dots,p_N$ such that $1<p_i<\infty$ for $i=1,\dots,N$. We set

$$\vec{p}=(p_1,\dots,p_N), \qquad \text{and} \qquad D^i u=\frac{\partial u}{\partial x_i} \quad \text{for} \quad i=1,\dots,N,$$

and

$$\underline{p}=\min\{p_1,p_2,\dots,p_N\}, \qquad \text{and} \qquad p_m=\max\{p_1,p_2,\dots,p_N\}.$$

Let $\vec{\omega}(x)=(\omega_0(x),\omega_1(x),\dots,\omega_N(x))$ be a vector of weight functions, such that $\omega_0(x)$ and $\omega_i(x)$ for $i=1,\dots,N$, are measurable and positive almost everywhere in Ω, with $\omega_0\leq\omega_i$ and

$$\omega_0\in L^1(\Omega), \qquad \text{with} \qquad \omega_0^*=\omega_0^{-\frac{1}{\underline{p}-1}}\in L^1(\Omega), \tag{2.1}$$

and

$$\omega_i\in L^1(\Omega), \qquad \text{with} \qquad \omega_i^*=\omega_i^{-\frac{1}{p_i-1}}\in L^1(\Omega), \quad \text{for} \quad i=1,\dots,N. \tag{2.2}$$

We define the weighted Lebesgue space $L^{p_i}(\Omega,\omega_i)$ by

$$L^{p_i}(\Omega,\omega_i)=\left\{u \text{ measurable function such that } \int_\Omega |u(x)|^{p_i}\omega_i(x)\,dx<\infty\right\},$$

endowed with norm

$$\|u\|_{p_i,\omega_i}=\left(\int_\Omega |u(x)|^{p_i}\omega_i(x)dx\right)^{\frac{1}{p_i}}, \quad \text{for} \quad i=1,\dots,N. \tag{2.3}$$

The spaces $(L^{p_i}(\Omega,\omega_i), \|u\|_{p_i,\omega_i})$ are separable and reflexive Banach spaces (we refer the readers to [20]).
We represent the Hölder's type inequality as

$$\int_\Omega uv\,dx \le \|u\|_{p_i,\omega_i}\|v\|_{p_i',\omega_i^*}, \qquad \text{with} \qquad \frac{1}{p_i}+\frac{1}{p_i'}=1, \tag{2.4}$$

and the Young's inequality as

$$\int_\Omega uv\,dx \le \frac{1}{p_i}\int_\Omega |u|^{p_i}\omega_i\,dx + \frac{1}{p_i'}\int_\Omega |v|^{p_i'}\omega_i^*\,dx, \tag{2.5}$$

for any $u \in L^{p_i}(\Omega,\omega_i)$ and $v \in L^{p_i'}(\Omega,\omega_i^*)$.
Now, we define the anisotropic weighted Sobolev space as follows

$$W^{1,\vec{p}}(\Omega,\vec{\omega}) = \left\{u \in L^{\underline{p}}(\Omega,\omega_0), \text{ and } D^i u \in L^{p_i}(\Omega,\omega_i), \quad \text{for} \quad i=1,\dots,N\right\},$$

which is a Banach space with respect to the norm

$$\|u\|_{1,\vec{p},\vec{\omega}} = \|u\|_{\underline{p},\omega_0} + \sum_{i=1}^N \|D^i u\|_{p_i,\omega_i}. \tag{2.6}$$

Under the assumptions (2.1) and (2.2), we have $C_0^\infty(\Omega)$ is a subspace of $W^{1,\vec{p}}(\Omega,\vec{\omega})$, then we define the Sobolev space $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ with respect to the norm (2.6). Moreover, the anisotropic weighted Sobolev spaces $W^{1,\vec{p}}(\Omega,\vec{\omega})$ and $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ are reflexive and separable Banach spaces.

Proposition 2.1. *(see [23]) (The anisotropic weighted Poincaré inequality)*
Let $N \ge 2$, and Ω be an open bounded subset of $I\!R^N$ with Lipschitz boundary. Then for any $u \in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ we have

$$\|u\|_{p_i,\omega_i} \le C\|D^i u\|_{p_i,\omega_i}, \qquad \textit{for} \quad i=1,\dots,N, \tag{2.7}$$

where C is a constant depending only to p_i and Ω.

Proposition 2.2. *The dual of $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ is denoted by $W^{-1,\vec{p'}}(\Omega,\vec{\omega^*})$, where $\vec{\omega^*} = (\omega_0^*,\omega_1^*,\dots,\omega_N^*)$ and $\vec{p'} = (p_1',\dots,p_N')$ with $\omega_0^* = \omega_0^{1-\underline{p}'}$, $\omega_i^* = \omega_i^{1-p_i'}$, and $\dfrac{1}{p_i'}+\dfrac{1}{p_i}=1$, such that*

$$W^{-1,\vec{p'}}(\Omega,\vec{\omega^*}) = \left\{F = F_0 - \sum_{i=1}^N D^i F_i, \quad \textit{with} \quad F_0 \in L^{\underline{p}'}(\Omega,\omega_0^*), \quad \textit{and} \quad F_i \in L^{p_i'}(\Omega,\omega_i^*), \quad \forall i\right\}.$$

Moreover, for any $u \in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ we have

$$\langle F,u\rangle = \sum_{i=0}^N \int_\Omega F_i\, D^i u\,dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p'},\vec{\omega^*}} = \inf\Bigg\{\sum_{i=0}^N \|F_i\|_{p_i',\omega_i^*}, \quad \textit{such that} \qquad F = F_0 - \sum_{i=1}^N D^i F_i,$$
$$\textit{with} \quad F_0 \in L^{\underline{p}'}(\Omega,\omega_0), \quad \textit{and} \quad F_i \in L^{p_i'}(\Omega,\omega_i), \quad \forall i=1,\dots,N\Bigg\}.$$

Definition 2.3. Let $k>0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, giving by

$$T_k(s) = \begin{cases} s, & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|}, & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$T_0^{1,\vec{p}}(\Omega,\vec{\omega}) := \Big\{ u:\Omega \longmapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}), \quad \text{for any} \quad k>0 \Big\}.$$

Proposition 2.4. *Let $u \in T_0^{1,\vec{p}}(\Omega,\vec{\omega})$. For any $i \in \{1,\dots,N\}$, there exists a unique measurable function $v_i : \Omega \longmapsto \mathbb{R}$ such that*

$$\forall k>0 \qquad D^i T_k(u) = v_{i.\chi_{\{|u|<k\}}} \qquad a.e. \ \ in \ \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincide with the standard distributional of u, that is $v_i = D^i u$.

Now, we will present some technical lemmas essential to establish our main result.

Lemma 2.5. *(see [6]) Assuming that $\omega_0 \le \omega_i$, and (2.1)$-$(2.2) hold true. Thus, the following continuous and compact embedding are concluded*

- **(i):** *The embedding $L^{p_i}(\Omega,\omega_i) \hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is continuous.*
- **(ii):** *The embedding $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow W_0^{1,\underline{p}}(\Omega,\omega_0)$ is continuous.*
- **(iii):** *The embedding $L^{p_i}(\Omega,\omega_i) \hookrightarrow L^1(\Omega)$ is continuous for $i=1,\dots,N$.*
- **(iv):** *The embedding $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is compact.*
- **(v):** *The embedding $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^q(\Omega)$ is compact for any $1 \le q < \dfrac{N}{N-1}$.*

Remark 2.6. In view of Theorem 2.1 and **(i)** of Lemma 2.5, we have that $\|u\|_{1,\vec{p},\vec{\omega}}$ and $|u|_{1,\vec{p},\vec{\omega}} = \sum_{i=1}^{N} \|D^i u\|_{p_i,\omega_i}$ are two equivalent norms on the Sobolev space $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$.

Lemma 2.7. *(see [2]) For $1<p_i<\infty$, let $g \in L^{p_i}(\Omega,\omega_i)$ and $(g_n)_n$ be a sequence uniformly bounded in $L^{p_i}(\Omega,\omega_i)$. If $g_n(x) \to g(x)$ a. e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^{p_i}(\Omega,\omega_i)$.*

Lemma 2.8. *(see [6]) Let $(u_n)_n$ be a bounded sequence in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$. If $u_n \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$, then $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ for any $k>0$.*

Lemma 2.9. *(see [6]) Let $u \in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ then $T_k(u) \in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$, for all $k>0$. Moreover, we have*

$$T_k(u) \longrightarrow u, \qquad \textit{strongly in} \qquad W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \qquad \textit{as} \qquad k \to \infty.$$

Lemma 2.10. *(see [24], Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

- **(i):** *$u_n \to u$ a.e. in Ω,*
- **(ii):** *$u_n \ge 0$ and $u \ge 0$ a.e. in Ω,*
- **(iii):** $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx,$

then $u_n \to u$ in $L^1(\Omega)$.

3. Essential assumptions

Let us consider the following quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) = f - \sum_{1=1}^{N} D^i F_i, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a_i(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ are Carathéodory function for $i = 1, 2, \dots, N$, (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfy the following conditions:

$$|a_i(x,s,\xi)| \leq \beta \left(K_i(x) + |s|^{\underline{p}/p_i'} \omega_0^{\frac{1}{p_i'}} \omega_i^{\frac{1}{p_i}} + |\xi_i|^{p_i-1}\omega_i \right), \quad \text{for} \quad i = 1, \dots, N, \tag{3.2}$$

where $K_i(\cdot)$ is a non-negative function lying in $L^{p_i'}(\Omega, \omega_i^*)$ and $\beta > 0$.
There exists a positive decreasing function $b(\cdot) : [0,\infty[\longmapsto]0,\infty[$ and a constant $b_0 > 0$ such that

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i}\omega_i(x), \quad \text{with} \quad b(|s|) \geq \frac{b_0}{(1+|s|)^\lambda}, \tag{3.3}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $0 \leq \lambda < \underline{p} - 1$,

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') \geq 0, \qquad \text{for any} \quad \xi_i, \xi_i' \in \mathbb{R}. \tag{3.4}$$

Moreover, we assume that

$$f(x) \in L^1(\Omega), \quad \text{and} \quad F_i(x) \in L^{p_i'}(\Omega, \omega_i^*), \quad \text{for} \quad i = 1, \dots, N. \tag{3.5}$$

4. Main results

Definition 4.1. A measurable function u is called a $T - \vec{p} - \vec{\omega}$ solution of the quasilinear elliptic problem (3.1), if $T_k(u) \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$ and u satisfies the equality

$$\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u) D^i \left(T_k(u) - \varphi\right)\, dx = \int_\Omega f(T_k(u) - \varphi)\, dx + \sum_{i=1}^N \int_\Omega F_i(x) D^i \left(T_k(u) - \varphi\right)\, dx, \tag{4.1}$$

for every $\varphi \in W_0^{1,\vec{p}}(\Omega, \vec{\omega}) \cap L^\infty(\Omega)$.

The main result of this paper is to proved the following existence Theorem.

Theorem 4.2. *Assuming that (2.1)-(2.2) and (3.2)-(3.5) holds true, then the elliptic problem (3.1) has at least one $T - \vec{p} - \vec{\omega}$ solution u.*

Proof of theorem 4.2

The proof of Theorem 4.2 is divided into several steps.

Step 1: Approximate problems.

Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence in $L^1(\Omega) \cap L^\infty(\Omega)$ such that $f_n \to f$ strongly in $L^1(\Omega)$.
We consider the approximate problem of our quasilinear elliptic equation

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^i a_i(x, T_n(u_n), \nabla u_n) = f_n(x) - \sum_{i=1}^N D^i F_i(x), & \text{in} \quad \Omega, \\ u_n = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2}$$

Let us define the operator $A_n : W_0^{1,\vec{p}}(\Omega, \vec{\omega}) \longmapsto W^{-1,\vec{p}'}(\Omega, \vec{\omega}^*)$ by

$$\langle A_n u, v \rangle = \sum_{i=1}^N a_i(x, T_n(u), \nabla u) D^i v \, dx \quad \text{for any} \quad u, v \in W_0^{1,\vec{p}}(\Omega, \vec{\omega}). \tag{4.3}$$

Lemma 4.3. *Let $v \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$. Assuming that (3.2)-(3.4) hold true, then the operator A_n satisfy the following conditions:*

- **(i):** *(The continuous of the operator) The operator A_n is continuous from $W_0^{1,\vec{p}}(\Omega, \vec{\omega})$ into its dual $W^{-1,\vec{p}'}(\Omega, \vec{\omega}^*)$.*
- **(ii):** *(The boundedness of the operator) The operator A_n is bounded.*

(iii): *(The sequential pseudo-monotonicity) Let $(u_k)_k$ be a sequence in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ such that*

$$\begin{cases} u_k \rightharpoonup u, & \text{in} \quad W_0^{1,\vec{p}}(\Omega,\vec{\omega}),\\ A_n u_k \rightharpoonup \chi_n, & \text{in} \quad W^{-1,\vec{p}'}(\Omega,\vec{\omega}^*),\\ \limsup\limits_{k\to\infty}\langle A_n u_k, u_k\rangle \le \langle \chi_n, u\rangle. \end{cases} \tag{4.4}$$

we will show that

$$\chi_n = A_n u, \quad \text{and} \quad \langle A_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle, \quad \text{as} \quad k \to +\infty.$$

(iv): *The operator A_n is coercive in the following sense:*

$$\frac{\langle A_n u, u\rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} \to +\infty, \quad \text{if} \quad \|u\|_{1,\vec{p},\vec{\omega}} \to +\infty, \quad \text{for} \quad u \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}).$$

Proof of Lemma 4.3.
(i) The continuity of the operator A_n :
Let t be a real variable such that $t \to t_0$, we have $a_i(x,s,\xi)$ is a Carathéodory function, thus

$$a_i(x, T_n(u+tv), \nabla(u+tv)) \longrightarrow a_i(x, T_n(u+t_0v), \nabla(u+t_0v)) \quad \text{a.e. in} \quad \Omega,$$

and since $a_i(x, T_n(u+tv), \nabla(u+tv))$ is uniformly bounded in $L^{p_i'}(\Omega,\omega_i)$, then

$$a_i(x, T_n(u+tv), \nabla(u+tv)) \rightharpoonup a_i(x, T_n(u+t_0v), \nabla(u+t_0v)), \quad \text{weakly in} \quad L^{p_i'}(\Omega,\omega_i).$$

Hence, for any $z \in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ we have

$$\begin{aligned} \lim_{t\to t_0}\langle A_n(u+tv), z\rangle &= \lim_{t\to t_0}\sum_{i=1}^N \int_\Omega a_i(x, T_n(u+tv), \nabla(u+tv)) D^i z\, dx \\ &= \sum_{i=1}^N \int_\Omega a_i(x, T_n(u+t_0v), \nabla(u+t_0v)) D^i z\, dx \\ &= \langle A_n(u+t_0v), z\rangle. \end{aligned} \tag{4.5}$$

Therefore, the operator A_n is continuous.
(ii) The operator A_n is bounded :
By using (3.2) and Hölder's type inequality, we have

$$\begin{aligned} |\langle A_n u, v\rangle| &\le \sum_{i=1}^N \int_\Omega |a_i(x, T_n(u), \nabla u)|\, |D^i v|\, dx \\ &\le \beta \sum_{i=1}^N \int_\Omega \left(K_i(x) + |T_n(u)|^{\frac{\underline{p}}{p_i'}} \omega_0^{\frac{1}{p_i'}} \omega_i^{\frac{1}{p_i}} + |D^i u|^{p_i-1}\omega_i \right) |D^i v|\, dx \\ &\le \beta \sum_{i=1}^N \left[\left(\int_\Omega |K_i(x)|^{p_i'} \omega_i^{1-p_i'}\, dx \right)^{\frac{1}{p_i'}} + \left(\int_\Omega |T_n(u)|^{\underline{p}} \omega_0\, dx \right)^{\frac{1}{p_i'}} \right. \\ &\left. + \left(\int_\Omega |D^i u|^{p_i} \omega_i\, dx \right)^{\frac{1}{p_i'}} \right] \left(\int_\Omega |D^i v|^{p_i} \omega_i\, dx \right)^{\frac{1}{p_i}} \\ &\le \beta \sum_{i=1}^N \left(\|K_i(x)\|_{p_i',\omega_i^*} + \|T_n(u)\|_{\underline{p},\omega_0}^{\frac{\underline{p}}{p_i'}} + \|D^i u\|_{p_i,\omega_i}^{p_i-1} \right) \|D^i v\|_{p_i,\omega_i} \\ &\le C_1 \left(1 + \|u\|_{1,\vec{p},\vec{\omega}}^{p_m-1} \right) \|v\|_{1,\vec{p},\vec{\omega}} \qquad \text{for any} \quad u, v \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}), \end{aligned} \tag{4.6}$$

where C_1 is a constant depending on n. Thus, the operator A_n is bounded in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$.
(iii) The operator A_n is pseudo-monotone :

We have $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$, then $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ for a subsequence still denoted $(u_k)_{k\in\mathbb{N}}$. The sequence $(u_k)_k$ is bounded in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$, and thanks to (3.2) we have $(a_i(x,T_n(u_k),\nabla u_k))_k$ is a uniformly bounded sequence in $L^{p_i'}(\Omega,\omega_i^*)$. Therefore there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega,\omega_i^*)$ such that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i, \quad \text{weakly in} \quad L^{p_i'}(\Omega,\omega_i^*), \quad \text{as} \quad k\to+\infty. \tag{4.7}$$

Thus, for all $v\in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ we obtain

$$\langle \chi_n, v\rangle = \lim_{k\to\infty}\langle A_n u_k, v\rangle = \lim_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx = \sum_{i=1}^N\int_\Omega \varphi_i D^i v\,dx. \tag{4.8}$$

Thanks to (4.4) and (4.8), we get

$$\limsup_{k\to\infty}\langle A_n u_k, u_k\rangle \le \langle\chi_n, u\rangle = \sum_{i=1}^N\int_\Omega \varphi_i D^i u\,dx. \tag{4.9}$$

On the other hand, by (3.4) we have

$$\sum_{i=1}^N\int_\Omega \Big(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla v)\Big)(D^i u_k - D^i v)\,dx \ge 0, \tag{4.10}$$

it follows that

$$\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx \ge \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla v)\left(D^i u_k - D^i v\right)\,dx.$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k)\to T_n(u)$ strongly in $L^{p_i}(\Omega,\omega_i)$ then $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$, and by using (4.7), we get

$$\sum_{i=1}^N\int_\Omega \varphi_i D^i u\,dx \ge \sum_{i=1}^N\int_\Omega \varphi_i D^i v\,dx + \sum_{i=1}^N\int_\Omega a_i(x,T_n(u),\nabla v)\left(D^i u - D^i v\right)\,dx.$$

Hence

$$\sum_{i=1}^N\int_\Omega (\varphi_i - a_i(x,T_n(u),\nabla v))\left(D^i u - D^i v\right)\,dx \ge 0. \tag{4.11}$$

Let $z\in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$, taking $v = u - tz$ for $t>0$, we have

$$\sum_{i=1}^N\int_\Omega \left[\varphi_i - a_i(x,T_n(u),\nabla(u-tz))\right]D^i z\,dx \ge 0,$$

then, by letting t tends to 0, we get

$$\sum_{i=1}^N\int_\Omega (\varphi_i - a_i(x,T_n(u),\nabla u))\,D^i z\,dx \ge 0.$$

Similarly, by taking $v = u - tw$ with $t<0$, then letting t tends to 0 in (4.11), we obtain

$$\sum_{i=1}^N\int_\Omega (\varphi_i - a_i(x,T_n(u),\nabla u))\,D^i z\,dx \le 0.$$

It follows that

$$\sum_{i=1}^N\int_\Omega (\varphi_i - a_i(x,T_n(u),\nabla u))\,D^i z\,dx = 0 \quad \text{for any} \quad z\in W_0^{1,\vec{p}}(\Omega,\vec{\omega}).$$

Consequently, we conclude that

$$\varphi_i = a_i(x, T_n(u), \nabla u) \quad \text{in} \quad L^{p_i'}(\Omega, \omega_i^*). \tag{4.12}$$

According to (4.7) and (4.11), we deduce that $A_n u_k \rightharpoonup \chi_n$ as $k \to \infty$ in $W^{-1,\vec{p'}}(\Omega, \vec{\omega}^*)$. On the other hand, we have

$$\sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u) \Big) (D^i u_k - D^i u)\, dx \geq 0,$$

it implies that

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq & \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u \, dx \\ & + \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_k), \nabla u) \left(D^i u_k - D^i u \right) \, dx. \end{aligned}$$

Thanks to (4.7) and (4.12), we obtain

$$\liminf_{k \to \infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i u \, dx.$$

Having in mind (4.9), we conclude that

$$\lim_{k \to \infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx = \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i u \, dx.$$

Consequently, we deduce that $\langle A_n u_k, u_k \rangle \longrightarrow \langle \chi_n, u \rangle$. Thus, the operator A_n is pseudo-monotone.

(iii) The coercivity of the operator A_n :

Thanks to (3.3), we have

$$\begin{aligned} \langle A_n u, u \rangle & = \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i u \, dx \\ & \geq \frac{b_0}{(1+n)^{\lambda}} \sum_{i=1}^{N} \int_{\Omega} |D^i u|^{p_i} \omega_i \, dx \\ & \geq C_2 \|u\|^{\underline{p}}_{1,\vec{p},\vec{\omega}}. \end{aligned} \tag{4.13}$$

Hence

$$\frac{\langle A_n u, u \rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} \geq C_2 \|u\|^{\underline{p}-1}_{1,\vec{p},\vec{\omega}} \longrightarrow +\infty, \quad \text{as} \quad \|u\|_{1,\vec{p},\vec{\omega}} \to +\infty.$$

which completes the proof of Lemma 4.3.

In view of Lemma 4.3 (see [28], Theorem 8.2), there exists at least one weak solution $u_n \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$ for the problem (4.2), i.e.

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i v \, dx = \int_{\Omega} f_n(x) v \, dx + \sum_{i=1}^{N} \int_{\Omega} F_i(x) D^i v \, dx, \tag{4.14}$$

for any $v \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$.

Step 2: Weak convergence of truncations.
Let $k \geq 1$, by choosing $T_k(u_n)$ as a test function for the approximate problem (4.2), we get

$$\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = \int_\Omega f_n T_k(u_n) \, dx + \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} F_i(x) D^i u_n \, dx. \tag{4.15}$$

According to (3.3) and (3.5), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} b(|u_n|) |D^i u_n|^{p_i} \omega_i \, dx \\
&\leq k \|f\|_{L^1(\Omega)} + \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |F_i(x)| \, |D^i u_n| \, dx \\
&\leq k \|f\|_{L^1(\Omega)} + C_1 \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} \frac{|F_i(x)|^{p_i'}}{b(|u_n|)^{p_i'-1}} \omega_i^* \, dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} b(|u_n|) |D^i u_n|^{p_i} \omega_i \, dx \\
&\leq k \|f\|_{L^1(\Omega)} + C_2 k^{\lambda(\underline{p}'-1)} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |F_i(x)|^{p_i'} \omega_i^* \, dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} b(|u_n|) |D^i u_n|^{p_i} \omega_i \, dx.
\end{aligned} \tag{4.16}$$

It follows that

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} b(|u_n|) |D^i u_n|^{p_i} \omega_i \, dx \quad &\leq k \|f\|_{L^1(\Omega)} + C_2 k^{\lambda(\underline{p}'-1)} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |F_i(x)|^{p_i'} \omega_i^* \, dx \\
&\leq C_3 (k + k^{\lambda(\underline{p}'-1)}).
\end{aligned} \tag{4.17}$$

Having in mind that $\lambda(\underline{p}' - 1) < 1$, we obtain

$$\frac{b_0}{2(1+k)^\lambda} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |D^i u_n|^{p_i} \omega_i \, dx \leq C_4 k. \tag{4.18}$$

Thus, we conclude that

$$\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |D^i u_n|^{p_i} \omega_i \, dx \leq C_5 k^{\lambda+1} \quad \text{for any} \quad k \geq 1. \tag{4.19}$$

with C_5 is a positive constant that does not depend on k and n. Moreover, we get

$$\begin{aligned}
\|T_k(u_n)\|_{1,p(\cdot),\vec{\omega}} \quad &= \|T_k(u_n)\|_{\underline{p},\omega_0} + \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{p_i,\omega_i} \\
&= \Big(\int_\Omega |T_k(u_n)|^{\underline{p}} \omega_0 \, dx \Big)^{\frac{1}{\underline{p}}} + \sum_{i=1}^{N} \Big(\int_\Omega |D^i T_k(u_n)|^{p(x)} \omega_i \, dx \Big)^{\frac{1}{p_i}} \\
&\leq C_6 k + C_7 \sum_{i=1}^{N} \Big(k^{\lambda+1} \Big)^{\frac{1}{p_i}} \\
&\leq C_8 k, \qquad \text{for any} \quad k \geq 1.
\end{aligned} \tag{4.20}$$

Hence, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega, \vec{\omega})$, and there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $\eta_k \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k, & \text{weakly in} \quad W_0^{1,\vec{p}}(\Omega, \vec{\omega}), \\ T_k(u_n) \to \eta_k, & \text{strongly in} \quad L^{\underline{p}}(\Omega, \omega_0) \quad \text{and a.e in} \quad \Omega. \end{cases} \tag{4.21}$$

On the other hand, since $p_m = \max\{p_1, p_2, \dots, p_N\}$, then there exists $i_0 \in \{1, 2, \dots, N\}$ such that $p_m = p_{i_0}$, and thanks to (4.19) and Poincaré inequality, we get

$$
\begin{aligned}
k \text{ meas}\left(\{|u_n| > k\}\right) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \le \int_\Omega |T_k(u_n)|\, dx \le C_5 \|T_k(u_n)\|_{p_{i_0},\omega_{i_0}} \\
&\le C_5 C_p \|D^{i_0} T_k(u_n)\|_{p_{i_0},\omega_{i_0}} \\
&= C_5 C_p \Big(\int_\Omega |D^{i_0} T_k(u_n)|^{p_{i_0}} \omega_{i_0}\, dx \Big)^{\frac{1}{p_m}} \\
&\le C_5 C_p \Big(\sum_{i=i}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i} \omega_i\, dx \Big)^{\frac{1}{p_m}} \\
&\le C_6 k^{\frac{\lambda+1}{p_m}}.
\end{aligned}
\tag{4.22}
$$

Thus

$$
\text{meas}\ \left(\{|u_n| > k\}\right) \le \frac{C_6 k^{\frac{\lambda+1}{p_m}}}{k} \longrightarrow 0, \qquad \text{as} \quad k \to \infty. \tag{4.23}
$$

Moreover, we have for every $\delta > 0$,

$$
\text{meas}\{|u_n - u_m| > \delta\} \ \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.
$$

Let $\varepsilon > 0$, in view of (4.23), we choose $k = k(\varepsilon)$ large enough such that

$$
\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3}, \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3}. \tag{4.24}
$$

On the other hand, thanks to (4.21) we have

$$
T_k(u_n) \to \eta_k, \quad \text{strongly in} \quad L^{\underline{p}}(\Omega, \omega_0), \quad \text{and a.e in} \quad \Omega.
$$

Hence, $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$
\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \le \frac{\varepsilon}{3} \qquad \text{for all} \qquad n, m \ge n_0(k, \delta, \varepsilon). \tag{4.25}
$$

By combining (4.24) and (4.25), we deduce that: for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$
\text{meas}\{|u_n - u_m| > \delta\} \le \varepsilon \qquad \text{for any} \qquad n, m \ge n_0.
$$

Thus, the sequence $(u_n)_n$ is a Cauchy in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Furthermore, thanks to (4.21), we conclude that

$$
\begin{cases}
T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W_0^{1,\vec{p}}(\Omega, \vec{\omega}), \\
T_k(u_n) \to T_k(u) & \text{strongly in} \quad L^{\underline{p}}(\Omega, \omega_0) \quad \text{and a.e in} \quad \Omega.
\end{cases}
\tag{4.26}
$$

Step 3: Passage to the limit.
Let $\varphi \in W_0^{1,\vec{p}}(\Omega, \vec{\omega}) \cap L^\infty(\Omega)$, by using $T_k(u_n - \varphi)$ as a test function for the approximate problem (4.2), we obtain

$$
\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx = \int_\Omega f_n(x) T_k(u_n - \varphi) dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u_n - \varphi) dx. \tag{4.27}
$$

Thanks to (3.4), we have

$$
\sum_{i=1}^{N} \int_\Omega \left(a_i(x, T_n(u_n), \nabla u_n) - a_i(x, T_n(u_n), \nabla \varphi) \right) \left(D^i u_n - D^i \varphi \right) . \chi_{\{|u_n - \varphi| \le k\}}\, dx \ge 0. \tag{4.28}
$$

Thus

$$
\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla \varphi) D^i T_k(u_n - \varphi)\, dx \le \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\, dx. \tag{4.29}
$$

Hence

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla\varphi)D^iT_k(u_n-\varphi)dx \le \int_{\Omega} f_n(x)T_k(u_n-\varphi)dx + \sum_{i=1}^{N}\int_{\Omega} F_i(x)D^iT_k(u_n-\varphi)dx. \tag{4.30}$$

Now, we pass to the limit on each terms of the inequality (4.30).
By choosing $M = k + \|\varphi\|_\infty$, we get $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le M\}$. Thus, we deduce that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla\varphi)D^iT_k(u_n-\varphi)\,dx = \sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi).\chi_{\{|u_n-\varphi|\le k\}}\,dx \tag{4.31}$$

In view of (4.26) and Lebesgue's dominated convergence theorem, we have $a_i(x,T_M(u_n),\nabla\varphi)$ converges to $a_i(x,T_M(u),\nabla\varphi)$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$. Moreover, since $D^iT_M(u_n)$ converges to $D^iT_M(u)$ weakly in $L^{p_i}(\Omega,\omega_i)$, we can deduce

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi).\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\qquad=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi).\chi_{\{|u-\varphi|\le k\}}\,dx\\
&\qquad=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla\varphi)D^iT_k(u-\varphi)\,dx.
\end{aligned} \tag{4.32}$$

Moreover, we have $f_n(x)\to f(x)$ strongly in $L^1(\Omega)$, and since $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^\infty(\Omega)$, we get

$$\lim_{n\to\infty}\int_{\Omega} f_n(x)T_k(u_n-\varphi)\,dx = \int_{\Omega} f(x)T_k(u-\varphi)\,dx, \tag{4.33}$$

Furthermore, since $F_i(x)\in L^{p_i'}(\Omega,\omega_i^*)$, and $D^iT_M(u_n)\rightharpoonup D^iT_M(u)$ converge weakly in $L^{p_i}(\Omega,\omega_i)$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} F_i(x)D^iT_k(u_n-\varphi)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} F_i(x)\left(D^iT_M(u_n)-D^i\varphi\right)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega} F_i(x)\left(D^iT_M(u)-D^i\varphi\right)\chi_{\{|u-\varphi|\le k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega} F_i(x)D^iT_k(u-\varphi)\,dx.
\end{aligned} \tag{4.34}$$

Consequently, according to (4.32) – (4.34), we conclude that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla\varphi)D^iT_k(u-\varphi)\,dx \le \int_{\Omega} f(x)T_k(u-\varphi)\,dx + \sum_{i=1}^{N}\int_{\Omega} F_i(x)D^iT_k(u-\varphi)\,dx, \tag{4.35}$$

for any $\varphi\in W_0^{1,\vec{p}}(\Omega,\vec{\omega})\cap L^\infty(\Omega)$.

Step 4: The Minty Lemma.
We now introduce the following lemma, which will be proven later. This lemma is considered as an L^1-version of Minty's lemma in the anisotropic weighted Sobolev spaces.

Lemma 4.4. *Let u be a measurable function such that $T_k(u_n)$ belongs to $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ for every $k>0$. Then the following statements are equivalent :*

Assertion 1:

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla\varphi)D^iT_k(u-\varphi)\,dx \le \int_{\Omega} fT_k(u-\varphi)\,dx + \sum_{i=1}^{N}\int_{\Omega} F_iD^iT_k(u-\varphi)\,dx,$$

for every φ in $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$.

Assertion 2:

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-\varphi)\, dx = \int_\Omega f T_k(u-\varphi)\, dx + \sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u-\varphi)\, dx,$$

for every φ in $W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$.

Proof of Lemma 4.4.
(Assertion 2) $\Longrightarrow$ (Assertion 1)
Thanks to (3.4), we have

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-\varphi)\, dx &= \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla \varphi) D^i T_k(u-\varphi)\, dx \\ &+ \sum_{i=1}^{N} \int_\Omega (a_i(x,u,\nabla u) - a_i(x,u,\nabla\varphi)) D^i T_k(u-\varphi)\, dx \\ &\geq \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla \varphi) D^i T_k(u-\varphi)\, dx.\end{aligned}$$

Hence, assertion 1 is confirmed.
(Assertion 1) $\Longrightarrow$ (Assertion 2)
Taking $h \geq k > 0$ and $\lambda \in]-1,1[$. Let $v \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$, by choosing $\varphi = T_h(u - \lambda T_k(u-v)) \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$ as a test function in (Assertion 1), we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla T_h(u-\lambda T_k(u-v))) D^i T_k(u - T_h(u-\lambda T_k(u-v)))\, dx \\ &\leq \int_\Omega f(x) T_k(u - T_h(u-\lambda T_k(u-v)))\, dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u - T_h(u-\lambda T_k(u-v)))\, dx.\end{aligned} \tag{4.36}$$

Concerning the first term on the left-hand side of (4.36), since $a(x,u,0)=0$ and $\{|u-\lambda T_k(u-v)| \leq h\} \longrightarrow \Omega$ as $h \to \infty$, we conclude that

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla T_h(u-\lambda T_k(u-v))) D^i T_k(u - T_h(u-\lambda T_k(u-v)))\, dx \\ &= \lambda \sum_{i=1}^{N} \int_{\{|u-\lambda T_k(u-v)|\leq h\}} a_i(x,u,\nabla(u-\lambda T_k(u-v))) D^i T_k(u-v)\, dx \\ &\longrightarrow \lambda \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla(u-\lambda T_k(u-v))) D^i T_k(u-v)\, dx, \quad \text{as} \quad h \to \infty.\end{aligned} \tag{4.37}$$

On the other hand, we have

$$\lim_{h\to\infty} \int_\Omega f T_k(u - T_h(u-\lambda T_k(u-v)))\, dx = \lambda \int_\Omega f T_k(u-v)\, dx, \tag{4.38}$$

and

$$\lim_{h\to\infty} \sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u - T_h(u-\lambda T_k(u-v)))\, dx = \lambda \sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u-v)\, dx. \tag{4.39}$$

According to (4.36) – (4.39), we deduce that

$$\lambda \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla(u-\lambda T_k(u-v))) D^i T_k(u-v)\, dx \leq \lambda \int_\Omega f T_k(u-v)\, dx + \lambda \sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u-v)\, dx. \tag{4.40}$$

By dividing both sides of the above inequality by $\lambda > 0$ and then letting $\lambda \to 0^+$, we get

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx \leq \int_\Omega f(x) T_k(u-v)\,dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u-v)\,dx. \tag{4.41}$$

By doing the same for $\lambda < 0$ then letting $\lambda \to 0^-$, we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx \geq \int_\Omega f(x) T_k(u-v)\,dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u-v)\,dx. \tag{4.42}$$

Combining (4.41) and (4.42), we conclude that

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx = \int_\Omega f(x) T_k(u-v)\,dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u-v)\,dx, \tag{4.43}$$

for any $v \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$, which completes the proof of the Lemma 4.4.
Consequently, in view of (4.35) and Lemma 4.4, we conclude the proof of Theorem 4.2.

Example 4.5. Assuming that ω_0 and ω_i verifying the conditions (2.1)$-$(2.2). Let $f(x) \in L^1(\Omega)$ and $F_i(x) \in L^{p_i'}(\Omega, \omega_i^*)$ for $i = 1,\dots,N$. We consider the following Carathéodory functions

$$a_i(x,s,\xi) = \frac{b_0|\xi_i|^{p_i-2}\xi_i\omega_i}{(1+|s|)^\lambda}, \qquad \text{for} \quad i = 1,\dots,N.$$

It is clear that the functions $a_i(x,s,\xi)$ verify the conditions (3.2) $-$ (3.5). In view of the Theorem 4.2, the quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i\Big(\frac{b_0|D^i u|^{p_i-2}D^i u\omega_i}{(1+|u|)^\lambda}\Big) = f(x) - \sum_{1=1}^{N} D^i F_i(x), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.44}$$

has at least one entropy solution $v \in T_0^{1,\vec{p}}(\Omega,\vec{\omega})$, i.e.

$$\sum_{i=1}^{N} \int_\Omega \frac{b_0|D^i u|^{p_i-2}D^i u D^i T_k(u-v)\omega_i}{(1+|u|)^\lambda}\,dx = \int_\Omega f(x) T_k(u-v)\,dx + \sum_{i=1}^{N} \int_\Omega F_i(x) D^i T_k(u-v)\,dx, \tag{4.45}$$

for any $v \in W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^\infty(\Omega)$.

5. Conclusion

In this work, we have studied the existence of $T - \vec{p} - \vec{\omega}$ solutions for a class of quasilinear degenerated elliptic problems, with large monotonicity condition, in anisotropic weighted Sobolev spaces, but the existence of solutions in the sense of distributions for this type of problems remains an open question.

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